LEARNING EQUIVALENCE RELATIONS ON POLISH SPACES

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Abstract. We investigate natural variations of behaviourally correct learning and explanatory learning—two learning paradigms studied in algorithmic learning theory—that allow us to "learn" equivalence relations on Polish spaces. We give a characterization of the learnable equivalence relations in terms of their Borel complexity and show that the behaviourally correct and explanatory learnable equivalence relations coincide both in uniform and non-uniform versions of learnability and provide a characterization of the learnable equivalence relations in terms of their Borel complexity. We also show that the set of uniformly learnable equivalence relations is $\mathbf{\Pi}^1_1$-complete in the codes and study the learnability of several equivalence relations arising naturally in logic as a case study.

§1. Introduction. A common scenario in algorithmic learning theory can be viewed in terms of the following game played in infinitely many stages:

> Fix a countable sequence of languages $(l_i)_{i\in\omega}$, $l_i \subseteq \omega$ (possibly with repetitions) and an index i_0. At each stage s, an *informant* presents a *learner* with a word $w_s \in l_{i_0}$ such that $\{w_s : s \in \omega\} = l_{i_0}$ and the learner has to make a hypothesis $h_s \in \omega$, guessing which language they are presented with.

This way of presenting the language l_{i_0} is often referred to as *learning from text*. There are several ways to define winning conditions for this learning game. The first one, giving the learning paradigm often referred to as *explanatory* learning or *learning in the limit* is due to Gold [8]:

> The learner wins the game if after finitely many stages they stabilize on a correct hypothesis, i.e., if there exists $h = \lim_s h_s$ and $l_h = l_{i_0}$.

Another learning paradigm, *behaviourally correct* (BC) learning, is due to Feldman [6]:

> The learner wins the game if there is a finite stage s such that at all stages $t > s$, $l_{h_t} = l_{i_0}$.

Notice that we do not require that the learner guesses the correct index, they just need to identify the language up to equality.

Recently, Fokina, Kötzing, and San Mauro [7] adapted the explanatory paradigm to learning of the isomorphism relation on countable structures. As above, we can

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view this as a game where a sequence of countable structures $(\mathcal{A}_i)_{i\in\omega}$ is fixed, and the learner, presented with finite fragments of a structure $\mathcal{B}$, needs to hypothesize to which of the $\mathcal{A}_i\mathcal{B}$ is isomorphic. The difference to classical explanatory learning is that the informant may play an isomorphic copy of one of the structures $\mathcal{A}_i$, or a structure not isomorphic to any of the structures in the sequence. Bazhenov, Fokina, and San Mauro [3] characterized learnable sequences of structures as those that have Σ^{in}_2 quasi Scott sentences in the infinitary logic $L_{\omega_1\omega}$. That is, sentences φ_i of the form $\bigvee_{i\in\omega}\exists\bar{x}\bigwedge_{j\in\omega}\forall\bar{y}\psi_{i,j}(\bar{x},\bar{y})$ where the $\psi_{i,j}$ are quantifier-free such that $\mathcal{A}_i\models\varphi_i$ and if $\mathcal{A}_j\not\cong\mathcal{A}_i$, then $\mathcal{A}_j\not\models\varphi_i$. A result connecting this paradigm with descriptive set theory is due to Bazhenov, Cipriani, and San Mauro [2]. They showed that a sequence of structures is learnable if and only if the isomorphism problem on the associated class is continuously reducible to E_0, the eventual equality relation on infinite binary strings.

Motivated by the above work, we take a different approach to connecting algorithmic learning theory with descriptive set theory. Our aim is to study the following question:

> What does it mean for an equivalence relation on a Polish space to be learnable?

Fokina, Kötzing, and San Mauro's learning paradigm already gives some insight into an equivalence relation on a Polish space. After all, the space of countable structures in a relational vocabulary can be viewed as a closed subspace of 2^ω and thus admits a natural Polish topology. However, learnability in their paradigm is a local property, only giving information on the countable set of isomorphism classes obtained from the sequence $(\mathcal{A}_i)_{i\in\omega}$.

Our approach is global, aiming to give information into an equivalence relation on the whole space. We introduce and study both uniform and non-uniform versions of explanatory learning and behaviourally correct learning of equivalence relations on Polish spaces. Our main results show that both the uniform and non-uniform versions of these learning paradigms are equivalent (Theorems 2 and 11). The proof of this result for the non-uniform version is relatively simple, while the proof of the uniform result is quite involved, using recent results of Lecomte [13] on Borel equivalence relations and one of our other main theorems, that an equivalence relation is uniformly (explanatory) learnable if and only if it is $\mathbf{\Sigma}^0_2$. All of the above results are covered in Section 2.

In Section 3 we show that the complexity of the set of learnable $\mathbf{\Pi}^0_2$ equivalence relations on 2^ω is $\mathbf{\Pi}^1_1$-complete in the codes. In Section 4 we analyse the learnability of various equivalence relations that arise naturally in computability and model theory. At last, we give an example of a natural Π^1_1 subset of the natural numbers that is neither Σ^1_1 nor Π^1_1-complete.

The proofs of our results use techniques from several areas of mathematical logic. Our results in Section 2 rely on Cohen forcing and results in the theory of Borel equivalence relations, in Section 3 we use techniques from higher recursion theory and effective descriptive set theory, and most proofs in Section 4 will seem familiar to computability theorists. We assume that the reader has some familiarity with these subjects.

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All our proofs are written assuming that the underlying Polish space is Cantor space 2^ω—the space of infinite binary strings (or sequences) under the product topology. However, the theorems are stated for general (effective) Polish spaces. Our proofs can be easily modified to hold in this setting.

§2. Notions of learnability.

2.1. Non-uniform learnability. The following definition presents our attempt to obtain a non-uniform generalization of the explanatory learning framework. We adopt the following notational convention: if $\vec{x}$ is used to denote a sequence, then x_i denotes its ith column, i.e., $\vec{x} = (x_i)_{i\in\omega}$. If x is used to denote a sequence, then $x(i)$ means its ith element. If $\vec{y}$ is a sequence of infinite strings and x is an infinite string, we write $x^\frown\vec{y}$ to denote $(y, x_0, x_1, \ldots)$

DEFINITION 1. Let E be an equivalence relation on a Polish space X and assume ω is equipped with the discrete topology. We say that E is *non-uniformly learnable* if for every $\vec{x} \in X^\omega$, there are continuous functions $l_n(\vec{x}) : X \to \omega$ such that for every $x \in X$, if xEx_i for some $i \in \omega$, then $L(x, \vec{x}) = \lim l_n(\vec{x}, x)$ exists and $xEx_{L(x,\vec{x})}$. We call the partial function L a learner, and write $L(x, \vec{x}, n)$ for $l_n(\vec{x}, x)$.

If $x \not E x_i$ for any $x_i \in \vec{x}$, then the behavior of a potential learner L is not specified. It might not be defined, or its value might be any index. We say that a learner L which is defined in this case gives *false positives*. The following result is reminiscent of a characterization of explanatory learning of structures given in [3].

PROPOSITION 1. *Let E be an equivalence relation on a Polish space X. Then the following are equivalent.*

(1) *E is non-uniformly learnable.*
(2) *For every $\vec{x} \in X^\omega$, there are sets $S_i \in \mathbf{\Sigma}^0_2(X)$ such that for every $i \in \omega$, $[x_i]_E \subseteq S_i$ and $S_i \cap S_j = \emptyset$ if $x_i \not E x_j$.*

Furthermore, E is non-uniformly learnable by learners not giving false positives if and only if $[x]_E$ is $\mathbf{\Sigma}^0_2$ for every $x \in X$.

PROOF. (2) $\implies$ (1). Given $\vec{x}$, let $(S_i)_{i\in\omega}$ be a countable sequence of $\Sigma^0_2(p_i)$ sets covering $[x_i]_E$ such that if $x_i \not E x_j$, then $S_i \cap S_j = \emptyset$ and let $(R_i)_{i\in\omega}$ be recursive relations such that

$$x \in S_i \iff \exists n \forall m\ R_i(x, p_i, n, m).$$

Then, for any $x \in X$ and $s \in \omega$ define $L(x, \vec{x}, s)$ to be the minimal $i < s$ such that for some $n < s$ and all $m < s$, $R_i(x, p_i, n, m)$ holds, if such i exists and $L(x, \vec{x}, s) = s$, otherwise. From the disjointness of the S_i we get that if j is least such that $x\ E\ x_j$, then $L(x, \vec{x}) = j$. Also, for every s, the function $L(-, \vec{x}, s)$ is recursive in $\bigoplus p_i$ and hence continuous. Thus, L is a learner learning $\vec{x}$ with respect to E. Furthermore, if $[x]_E$ is $\mathbf{\Sigma}^0_2$ for every $x \in X$, then we may assume in the above proof that $S_i = [x_i]_E$. In that case, for fixed x such that for every i, $x \notin [x_i]_E$, $L(x, \vec{x}) \uparrow$ and thus L does not give false positives.

(1) $\implies$ (2). Let L be a learner for $\vec{x}$ and note that for all i the sets $L(x, \vec{x}, n)^{-1}(i)$ are clopen. Then let $C_{n,i} = \{x : (\forall m > n) L(x, \vec{x}, m) = i\}$, define $D_i = \bigcup_n C_{n,i}$ and

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$S_i = \bigcup_{x_i E x_j} D_j$. Then the S_i are $\mathbf{\Sigma}^0_2$ and by the definition of non-uniform learnability $[x_i]_E \subseteq S_i$. If L learns $\vec{x}$ without giving false positives, then clearly $S_i = [x_i]_E$. ⊣

We now turn our attention to a non-uniform adaptation of behaviorally correct learning.

DEFINITION 2. Let E be an equivalence relation on a Polish space X and assume ω is equipped with the discrete topology. We say that E is *non-uniformly* BC*-learnable* if for every $\vec{x} = (x_i)_{i\in\omega} \in X^\omega$ there are continuous functions $l_n : X^\omega \times X \to \omega$ such that for $x \in X$, if $x \; E \; x_i$ for some $i \in \omega$, then for almost every n, $x \; E \; x_{l_n(\vec{x},x)}$.

The observant reader might immediately notice that non-uniform BC-learnability and non-uniform learnability coincide. In order to obtain a learner for $\vec{x}$ from a BC-learner, we just have to fix some more information.

THEOREM 2. *An equivalence relation E is non-uniformly learnable if and only if it is non-uniformly* BC*-learnable.*

PROOF. Clearly, any non-uniformly learnable equivalence relation is non-uniformly BC-learnable. On the other hand, say that $(l_n)_{n\in\omega}$ gives a BC learner for the sequence $\vec{x}$ and associate with every i, the set $e_i = \{j : x_j \; E \; x_i\}$. Now, define a function L by $L(x, \vec{x}, n) = \min j[j \in e_{l_n(x,\vec{x})}]$. We claim that L is a learner for E on $\vec{x}$. Clearly, L is continuous on every n. Say that $x \; E \; x_i$, then for almost every n, $l_n(\vec{x}, x) \in e_i$. So, for almost every n, $L(x, \vec{x}, n) = \min j[j \in e_i]$ and thus L learns E on $\vec{x}$. ⊣

Recall that an equivalence relation is *countable* if all its equivalence classes are countable. Countable Borel equivalence relations play an important role in descriptive set theory and have seen much attention in the past decades, see [12] for a summary of these developments. Unfortunately, non-uniform learnability does not provide information when it comes to countable equivalence relations.

PROPOSITION 3. *Every countable equivalence relation on a Polish space is non-uniformly learnable by learners not giving false positives.*

PROOF. Let E be a countable equivalence relation. Given $\vec{x}$, let $y_{i,j}$ be the j-th element in the enumeration of the equivalence class for x_i and for every n, define the continuous function

$$l_n(x, \vec{x}) = \begin{cases} (\min i < n)[(\exists j < n) \; y_{i,j} \restriction n = x \restriction n] & \text{if such } i \text{ exists} \\ n & \text{otherwise} \end{cases}.$$

It is not hard to see that the so-defined sequence of functions defines a learner giving no false positives. ⊣

2.2. Uniform learnability.

DEFINITION 3. Let E be an equivalence relation on a Polish space X and assume ω is equipped with the discrete topology. We say that E is *uniformly learnable*, or just *learnable*, if there are continuous functions $l_n : X^\omega \times X \to \omega$ such that for every $x \in X$ and $\vec{x} = (x_i)_{i\in\omega} \in X^\omega$, if $x \; E \; x_i$ for some $i \in \omega$, then $L(x, \vec{x}) = \lim l_n(\vec{x}, x)$ exists and $x \; E \; x_{L(x,\vec{x})}$. We call the partial function L a *learner* for E and write $L(x, \vec{x}, n)$ for $l_n(\vec{x}, x)$.

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DEFINITION 4. A learner $L = \lim l_n$ is a-computable for $a \in 2^\omega$ if there is an a-recursive function $f : \omega \to \omega$ such that $\Phi^a_{f(n)} = l_n$ where $(\Phi_i)_{i\in\omega}$ is a standard enumeration of Turing operators.

Clearly, every learner is a-computable for some $a \in 2^\omega$. The main result of this section is a characterization of the uniformly learnable equivalence relations in terms of their Borel complexity. Our proof will rely on an analysis of the Cohen forcing relation. For a concise overview of forcing both in the computability theory and set theory setting we suggest [4]. Let us quickly summarize the main points. The forcing partial orders we will use are $\mathbb{P}$ given by the set of finite binary strings ordered by extension and its products $\mathbb{P}^n$ and $\mathbb{P}^{<\omega} = \bigcup_{n\in\omega} \mathbb{P}^n$. The ordering on the latter sets is defined as the lexicographic ordering induced by the ordering on $\mathbb{P}$. Given $p \in \mathbb{P}$ we let $[\![p]\!]$ denote the basic open set of all infinite extensions of p, and likewise for $\mathbb{P}^{<\omega}$ and $\mathbb{P}^n$. The definition of our forcing relation is standard, where $p \Vdash \varphi(\dot{g})$ if for every generic $x \in [\![p]\!]$, $\varphi(x)$. The syntactic definition of our forcing relations are standard and can be found in any textbook on the topic. The only thing of note is that we use the strong forcing relation, i.e., for a property of the type $\exists n\varphi(\dot{g}, n)$ we have that $p \Vdash \exists n\varphi(\dot{g}, n)$ if and only if $\exists n_0 \leq |p|$ such that $p \Vdash \varphi(\dot{g}, n_0)$. Furthermore, recall that an element y is n-generic relative to x if it decides every Σ^0_n statement about x and that a property holds for sufficiently generic elements y relative to x, if it holds for all n-generics relative to x for some $n \in \omega$.

PROPOSITION 4. *Let E be an equivalence relation learnable by an a-computable learner. If for every $x \in 2^\omega$ there exists $z \in 2^\omega$, 1-generic relative to $x \oplus a$, such that $x\ E\ z$, then there exists $p \in 2^{<\omega}$ such that for all $y \in [\![p]\!]$ we have $x\ E\ y$.*

PROOF. Suppose that E is learnable by an a-computable learner L. Fix a real x and $\vec{g}$, a sequence of mutually 1-generics relative to $x \bigoplus a$. By learnability, if there is $g \in \vec{g}$ such that $x\ E\ g$, then $L(x, \vec{g})$ converges. Suppose that there exists k and n_0 such that for all $m > n_0$ we have $L(x, \vec{g}, m) = k$ and fix such k and n_0. We will argue that then there exists a finite $\vec{p}$ with $p_i \in 2^{<\omega}$ that already forces the convergence. Consider a set C consisting of finite $\vec{h}$ with $h_i \in 2^{<\omega}$ such that for some $n > n_0$ the learner diverges from k, i.e., $L(x, \vec{h}, n) \neq k$. This set is recursively enumerable in x and a and $\vec{g}$ has to either meet or avoid it. If no finite $\vec{p}$ forces the convergence, then every initial segment of $\vec{g}$ can be extended to an element of C. But then $\vec{g}$ cannot avoid C, contradicting the definition of n_0.

So take $\vec{p}$ such that $\vec{p} \Vdash (\forall m > n_0)L(x, \dot{\vec{g}}, m) = k$ and assume without loss of generality that $k < |\vec{p}|$. Then for any $y \in [\![p_k]\!]$, $L(x, p_1, \dots, y, \dots, p_{|\vec{p}|}) = k$, since by the use principle for continuous functions if there exists $q \succeq p_k$ such that $L(x, p_1, \dots, q, \dots, p_{|\vec{p}|}, m) = k$ for any m, then $L(x, p_1, \dots r, \dots, p_{|\vec{p}|}, m) = k$ for all $r \succeq q$. ⊣

COROLLARY 5. *If E is an equivalence relation learnable by an a-computable learner and the E-equivalence class of x is countable, then x is not E-equivalent to any y which is 1-generic relative to $x \oplus a$.*

THEOREM 6. *Let E be an equivalence relation. Then E is learnable by an a-computable learner if and only if E is $\Sigma^0_2(a)$.*

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PROOF. ($\Rightarrow$) Suppose that E is learnable by an a-computable learner L. Take arbitrary x and y such that $x\ E\ y$ and let $\vec{g}$ be a countable sequence of reals mutually sufficiently generic relative to $x \oplus y \oplus a$. We analyze the behavior of $L(x, y^\frown\vec{g})$ and $L(y, x^\frown\vec{g})$. Since $x\ E\ y$, in both cases, the learner has to converge to either 0 or, if $\vec{g}$ contains a g_i such that $g_i\ E\ x$, some other index. Hence, we have two possibilities to consider.

Case 1: $L(x, y^\frown\vec{g})$ or $L(y, x^\frown\vec{g})$ converges to 0. Suppose without loss of generality that the latter is the case. Fix n_0 such that for all $m > n_0$ the value of $L(y, x^\frown\vec{g}, m)$ doesn't change. Then, by the mutual genericity of $\vec{g}$ relative to $x \oplus y \oplus a$ there exists a finite $\vec{p} \in 2^{<\omega^{<\omega}}$ such that

$$\vec{p} \Vdash (\forall m > n_0)(L(y, x^\frown\dot{\vec{g}}, m)\downarrow \implies L(y, x^\frown\dot{\vec{g}}, m) = 0).$$

By the definability of forcing we get that x and y satisfy

$$\exists\vec{p}, n_0(\forall\vec{q} \leq \vec{p})(\forall m > n_0)(L(y, x^\frown\vec{q}, m)\downarrow \implies L(y, x^\frown\vec{q}, m) = 0). \qquad (*)$$

By the same reasoning, if $L(x, y^\frown\vec{g})$ converges to 0, then the following is satisfied by x and y

$$\exists\vec{p}, n_0(\forall\vec{q} \leq \vec{p})(\forall m > n_0)(L(x, y^\frown\vec{q}, m)\downarrow \implies L(x, y^\frown\vec{q}, m) = 0). \qquad (\dagger)$$

Case 2: $L(x, y^\frown\vec{g}) = j_1$ and $L(y, x^\frown\vec{g}) = j_2$ for some indices $j_1, j_2 \neq 0$. Again, by similar genericity arguments as in Case 1, this is forced by some finite $\vec{p}_1 = p_{1,0}, p_{1,1}, \ldots$, respectively, $\vec{p}_2 = p_{2,0}, p_{2,1}, \ldots$. Note that then for any $h_1 \succ p_{1,j_1}$, $h_2 \succ p_{2,j_2}$,

$$L(x, y^\frown(g_0, \ldots, g_{j_1-1}, h_1, g_{j_1+1}, \ldots)) = j_1$$

and

$$L(y, x^\frown(g_0, \ldots, g_{j_2-1}, h_2, g_{j_2+1}, \ldots)) = j_2.$$

Now, fix $h \succ p_{1,j_1}$ and $\vec{h} \succ p_{2,j_2}$ (i.e., every element of $\vec{h}$ extends p_{2,j_2}) mutually sufficiently generic relative to a and look at $L(h, \vec{h})$. By transitivity of E it will stabilize to some k, and as $h, \vec{h}$ are mutually generic, this is forced. Hence, x and y satisfy

$$\begin{aligned}\exists n_0\exists\vec{p}_1, \vec{p}_2\exists j_1, j_2(\forall n > n_0)(\forall\vec{q} \leq \vec{p}_1)\left(L(x, y^\frown\vec{q}, n)\downarrow \implies L(x, y^\frown\vec{q}, n) = j_1\right)\\ \wedge(\forall\vec{q} \leq \vec{p}_2)\left(L(y, x^\frown\vec{q}, n)\downarrow \implies L(y, x^\frown\vec{q}, n) = j_2\right)\\ \wedge\exists k(\exists r \leq p_{1,j_1})(\exists\vec{r} \leq p_{2,j_2})(\forall n > n_0)\\ (\forall q \leq r)(\forall\vec{q} \leq \vec{r})\left(L(q, \vec{q}, n)\downarrow \implies L(q, \vec{q}, n) = k\right). \qquad (\ddagger)\end{aligned}$$

We claim that the disjunction of Eq. ($*$), Eq. ($\dagger$), and Eq. ($\ddagger$) defines E. As argued above, if $x\ E\ y$, then one of the disjuncts is satisfied. On the other hand, if $x \not E y$, then the only way Eq. ($*$), Eq. ($\dagger$), and Eq. ($\ddagger$) can be satisfied is if L produces false positives. The following claim disposes of this possibility in case the learner gives false positives in the third conjunct of Eq. ($\ddagger$). The other cases follow by a similar proof.

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CLAIM 6.1. *If $g, \vec{g}$ is a sequence of mutually sufficiently generics relative to a and $L(g, \vec{g}) = k$ is forced by $p, \vec{p}$, then $L(g, \vec{g})$ does not produce false positives for any $h, h \succ p, \vec{p}$.*

PROOF. Towards a contradiction assume that $L(h, \vec{h}) = k$ is a false positive where without loss of generality $k < |\vec{p}|$. By definition, this is only possible if $h \not E h_i$ for all $i \in \omega$. As $L(h, \vec{h}) = k$ is forced by $p, \vec{p}$ we have that $L(h, (h_0, \ldots h_{|\vec{p}|}, h, h_{|\vec{p}|+1}, \ldots)) = k$. But, by reflexivity, $h \; E \; h$ and thus $h \; E \; h_k$, contradicting our assumption. ⊣

($\Leftarrow$) Suppose that there is an $X \in 2^\omega$ and an X-computable formula φ such that for all x, y we have $x \; E \; y$ if and only if $\exists n \forall m \; \varphi(x, y, n, m)$. The construction of an X-computable learner for E given input $x, \vec{y}$ is now straightforward. The learner works with an enumeration $(a_1, b_1), (a_2, b_2) \ldots$ of all pairs of natural numbers. Given (a_i, b_i), the learner outputs a_i until it finds a witness for a failure of the universal formula, i.e., until it finds m such that $\neg\varphi(x, \vec{y}(a_i), b_i, m)$. If this happens, the learner proceeds to the next pair. Now, if $x \; E \; y$, then for some a_i and b_i we have $\forall m \; \varphi(x, \vec{y}(a_i), b_i, m)$ and the learner converges on a_i. ⊣

Recall that an equivalence relation $E \subseteq X^2$ is *reducible* to an equivalence relation $F \subseteq Y^2$, if there is a function $f : X \to Y$ such that for all $x_1, x_2 \in X$, $x_1 \; E \; x_2$ if and only if $f(x_1) \; F \; f(x_2)$. The relation E is *Borel (continuously) reducible* to F if f is Borel (continuous).

Theorem 6 is in stark contrast to the result by Bazhenov, Cipriani, and San Mauro [2] that a countable class of structures is explanatory learnable if and only if it is continuously reducible to E_0, the eventual equality relation on 2^ω. While E_0 is $\mathbf{\Sigma}^0_2$ by its standard definition

$$x \; E_0 \; y \iff \exists n (\forall m > n) x(m) = y(m),$$

there are many $\mathbf{\Sigma}^0_2$ equivalence relations that are not Borel reducible to E_0. An important example is the equivalence relation E_∞, the shift action of F_2—the free group on 2 generators—on 2^{F_2}, which is a Borel-complete countable equivalence relation and thus vastly more complicated than E_0 from the point of view of Borel reducibility [5].

Notice that the definition of the learner from a $\mathbf{\Sigma}^0_2$ description of E in the proof of Theorem 6 produces a learner that does not give false positives, and that we can get $\mathbf{\Sigma}^0_2$ definitions of learnable equivalence relations regardless if their learners produce false positives or not. Thus, we obtain the following.

COROLLARY 7. *An equivalence relation E is learnable if and only if it is learnable without false positives.*

Let Γ be a class of equivalence relations and say that an equivalence relation $E \in \Gamma$ is Borel (continuous) Γ-complete if every $F \in \Gamma$ Borel (continuously) reduces to E. One of the most tantalizing questions about a class of equivalence relations Γ is whether there is a Borel (continuous) Γ-*complete* object. We thus ask the following.

QUESTION 1. *Does there exist a Borel (continuous) learnable complete equivalence relation*?

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By Theorem 6 one could equivalently ask whether there is a $\mathbf{\Sigma}^0_2$-complete equivalence relation. It turns out that this is a difficult and long-standing open problem in descriptive set theory [13].

On the other hand, given a Polish space X, recall that $A \subseteq X$ is K_σ if A is a union of compact subsets of X. Rosendal [18] showed that there exists a Borel complete K_σ equivalence relation. One example of a complete K_σ equivalence relation is the oscillation relation O [18] which can be defined on 2^ω by

$$x \; O \; y \iff$$
$$\exists N \forall n, m \left((\forall j \in (n, n+m))x(j) = 0 \implies |\{j \in (n, n+m) : y(j) = 1\}| < N\right).$$
$$\wedge \left((\forall j \in (n, n+m))y(j) = 0 \implies |\{j \in (n, n+m) : x(j) = 1\}| < N\right)$$

Remark. The oscillation equivalence relation is usually defined on the space of infinite increasing sequences of natural numbers $[\omega]^\omega$. Our definition of O above is not equivalent, as we have one additional equivalence class consisting of all finite strings. However, the function mapping $(a_0, \dots)$ to the characteristic function of the associated set $\{a_i : i \in \omega\}$ is a continuous embedding of the oscillation equivalence relation on $[\omega]^\omega$ into O.

Since every $\mathbf{\Sigma}^0_2$ relation on a compact space is K_σ we get the following.

Proposition 8. *The oscillation equivalence relation O is Borel complete among learnable equivalence relations on 2^ω.*

2.3. Uniform BC-learnability.

Definition 5. Let E be an equivalence relation on a Polish space X and assume ω is equipped with the discrete topology. We say that E is *(uniformly)* BC-*learnable*, or just *BC*-learnable, if there are continuous functions $l_n : X^\omega \times X \to \omega$ such that for every $x \in X$ and $\vec{x} = (x_i)_{i\in\omega} \in X^\omega$, if xEx_i for some $i \in \omega$, then for almost all n, $xEx_{l_n(\vec{x},x)}$. We call the function $L(x, \vec{x}, n) = l_n(\vec{x}, x)$ a BC-*learner* for E.

The rest of this section is devoted to showing that the BC learnable and learnable equivalence relations coincide. This will follow from a series of lemmas.

Lemma 9. *If E is* BC-*learnable, then every equivalence class is $\mathbf{\Sigma}^0_2$.*

Proof. Suppose that E is BC-learnable and that $[x]_E$ is not open. Pick $y \in [x]_E \setminus int([x]_E)$, where $int([x]_E)$ denotes the interior of $[x]_E$. Then for every n, there is $b_n \in [\![y \restriction n]\!] \setminus [x]_E$. Fix such a sequence $(b_n)_{n\in\omega}$. Now, given a BC-learner L and $z \in 2^\omega$ run the following procedure defined in stages.

At each stage s we will run $L(y, (z, a^{s-1}_1, a^{s-1}_2, \dots), s)$. Here, the a^s_j are either prefixes of y or they are equal to some element $b_n \in 2^\omega$ from the sequence defined above. To start, let $a^{-1}_j = \emptyset$ for all j and $i_{-1} = 0$.

Assume we have defined a^{s-1}_j for all j and $i_{s-1} = L(y, (z, a^{s-1}_1, a^{s-1}_2, \dots), s-1)$, and suppose that k is largest such that for all $j < k$, a^{s-1}_j is equal to some b_n. For all $j < k$, set $a^s_j = a^{s-1}_j$. Then let $(c_k, \dots)$ be the lexicographical minimal sequence such that $a^{s-1}_j \preceq c_j \prec y$ for all $j \geq k$ and for some i_s,

$$L(y, (z, a^s_0, \dots, a^s_{k-1}, c_k, \dots), s) \downarrow = i_s.$$

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Now, if $i_s = i_{s-1} = 0$ or $s = 0$, then let $a_j^s = c_j$ for all $j > k$. Otherwise, for all $j > k$, with $c_j \neq \emptyset$ or $j \leq i_0$, let a_j^s be b_{n_j} where n_j is such that $c_j = y \upharpoonright n_j$. Notice that there are only finitely many $c_j \neq \emptyset$ as $L(-,-,s)$ is continuous and the sequence $(c_k, \dots)$ was lexicographical minimal. This finishes the description.

Say that $a_i = \lim_s a_i^s$ for all i, and consider $L(y, (z, a_0, \dots))$. We claim that $\lim_s L(y, (z, a_0, \dots), s) = 0$ if and only if $z \in [y]_E = [x]_E$. Suppose $z \in [y]_E$, then there is a least stage s_0 such that for all $s > s_0$, $L(y, (z, a_0, \dots), s) = i$ implies $a_i \; E \; y$. Suppose that at stage $s_1 > s_0$, the learner outputs $i \neq 0$. Then at stage $s_1 + 1$, we declare $a_i = b_n \notin [y]_E$ contradicting the learnability of E. On the other hand, if $\lim_s L(y, (z, a_0, \dots), s) = 0$, then there is a least s_0 such that for all $s > s_0$, $L(y, (z, a_0, \dots), s) = 0$. Hence, for some large enough t, $a_t = y$ and thus $z \in [y]_E$.

To see that $[x]_E$ is $\mathbf{\Sigma}^0_2$, let R be the binary predicate recursive in $y \oplus \bigoplus b_n$ such that $R(z, s) = i$ if and only if $L(y, (z, a_s^0, \dots), s) = i$. Then $z \in [x]_E$ if and only if $\exists m (\forall n > m) R(z, n) = 0$. ⊣

The definition of the E-classes in Lemma 9 is inherently non-uniform. We proceed by showing that for all but countably many classes we can obtain a uniform definition, thus obtaining that E is Borel.

LEMMA 10. *Let E be* BC*-learnable, then E is Borel.*

PROOF. Suppose that E is an equivalence relation on 2^ω. Then there are only countably many classes that have non-empty interior and only countably many classes that contain elements with only finitely many non-zero bits. We call these classes *exceptional* as those are the classes for which we will need non-uniform definitions.

Let $[x]_E$ be a non-exceptional class, then the procedure given in Lemma 9 can be turned uniform by choosing $y = x$ and $b_n = (x \upharpoonright n)^\frown 0^\infty$. Hence, we can modify the recursive binary predicate given in the proof of Lemma 9 to obtain a recursive ternary predicate R' so that for x, y in a non-exceptional class, $x \; E \; y$ if and only if the procedure from Lemma 9 returns 0 in the limit on x and y, i.e.,

$$x \; E \; y \iff \exists m (\forall n > m) R'(x, y, n) = 0.$$

To see that E is Borel note that $x \; E \; y$ if and only if x and y are in the same exceptional class or neither are exceptional and $x \; E \; y$ via the procedure R'. This gives a Borel definition as required. ⊣

While the definition given in Lemma 10 is Borel it is not $\mathbf{\Sigma}^0_2$, as saying that neither class is exceptional is $\mathbf{\Pi}^0_2$. However, combining Lemma 10 with recent results of Lecomte [13] we are now ready to show that the BC-learnable and learnable equivalence relations are the same.

THEOREM 11. *An equivalence relation E is* BC*-learnable if and only if E is learnable if and only if E is $\mathbf{\Sigma}^0_2$.*

Lecomte [13] exhibited a finite set of non $\mathbf{\Sigma}^0_2$ equivalence relations that are minimal and form an antichain under continuous reducibility. Furthermore, every Borel equivalence relation that is not $\mathbf{\Sigma}^0_2$ continuously embeds one of these relations. Our proof of Theorem 11 will show that none of Lecomte's relations are BC-learnable. Before we proceed with the proof let us recall Lecomte's result in more detail.

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The five equivalence relations can be defined as follows. Let INF refer to the class of characteristic functions of all infinite subsets of ω. That is, $\mathrm{INF} = \{x \in 2^\omega : \forall n(\exists m > n)x(m) = 1\}$. All the relations below are defined on 2^ω.

- $x \sim_0 y \iff (x, y \in \mathrm{INF}) \vee (x = y)$
- $x \sim_1 y \iff (x, y \in \mathrm{INF}) \vee (x, y \notin \mathrm{INF})$
- $x \sim_3 y \iff (x = y) \vee (x, y \in \mathrm{INF} \wedge (\forall n > 0)x(n) = y(n))$
- $x \sim_4 y \iff x \sim_3 y \vee (x(0) = y(0) = 1 \wedge x, y \notin \mathrm{INF})$
- $x \sim_5 y \iff x \sim_3 y \vee (x(0) = y(0) \wedge x, y \notin \mathrm{INF})$.

THEOREM 12 (Lecomte [13]). *Let E be a Borel equivalence relation. Then exactly one of the following holds*:

- *E is $\mathbf{\Sigma}^0_2$.*
- *there is $i \in \{0, 1, 3, 4, 5\}$ and an injective continuous reduction from $\sim_i$ to E.*

PROOF OF THEOREM 11. Since by definition every learnable equivalence relation is also BC-learnable, we only have to deal with the other implication. Our proof proceeds by showing that each of Lecomte's equivalence relations is not BC-learnable. Since BC-learnability is preserved by continuous reductions, this implies that the BC-learnable Borel equivalence relations are precisely the learnable Borel equivalence relations. Using Lemma 10 we can then infer Theorem 11 in its full generality.

It remains to show that none of the $\sim_i$, $i \in \{0, 1, 3, 4, 5\}$ are BC-learnable. In all five cases, we achieve that by exploiting the fact that INF has no $\mathbf{\Sigma}^0_2$ definition. The argument are analogous. We assume that $\sim_i$ is BC-learnable and use this assumption to obtain a $\mathbf{\Sigma}^0_2$ definition of INF, which gives a contradiction.

Suppose that $\sim_0$ is BC-learnable by L. We claim that the following is a $\mathbf{\Sigma}^0_2$ definition of INF:

$$x \in \mathrm{INF} \iff \exists m(\forall n > m)L(x, (1^\infty, f_1, f_2, \ldots), n) = 0.$$

where $f_1, f_2, \ldots$ is an enumeration of all elements of 2^ω with finitely many ones. Note that x is either finite and identical to some f_i, or infinite and $\sim_0$-equivalent to 1^∞. Hence, if L is a BC-learner for $\sim_0$, then L converges on 0 if x is infinite and to some $k \neq 0$ otherwise. A similar argument shows that the above formula gives a $\mathbf{\Sigma}^0_2$ definition for $\sim_1$ if we assume that L is a BC-learner for $\sim_1$.

Similarly, if we assume that L is a BC-learner for $\sim_i$, $3 \leq i \leq 5$, then the following formula gives a $\mathbf{\Sigma}^0_2$ definition of INF:

$$x \in \mathrm{INF} \iff \exists m(\forall n > m)L(0x, (1x, 0f_1, 0f_2, \ldots), n) = 0. \qquad \dashv$$

2.4. Borel learnability. In modern descriptive set theory Borel functions arguably play a more prominent role than continuous functions. The reason for this is the following classical change of topology theorem. If $f : (X, \sigma) \to Y$ is Borel, then there is a Polish topology $\tau \supset \sigma$ such that the Borel sets generated by τ and σ are the same and $f : (X, \tau) \to Y$ is continuous; see [11, Chapter 13] for a proof.

The following modification of uniform learnability might thus look more natural to a descriptive set theorist.

DEFINITION 6. Let E be an equivalence relation on a Polish space X and assume ω is equipped with the discrete topology. We say that E is *Borel learnable*, if there

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are Borel functions $l_n : X^\omega \times X \to \omega$ such that for $x \in X$ and $\vec{x} = (x_i)_{i\in\omega} \in X^\omega$, if $x\ E\ x_i$ for some $i \in \omega$, then $L(\vec{x}, x) = \lim l_n(\vec{x}, x)$ exists and $x\ E\ x_{L(\vec{x},x)}$. We refer to the partial function L as a *Borel learner* for E.

Combining the above change of topology theorem with the boldface version of Theorem 6 we immediately obtain that an equivalence relation E on a Polish space (X, σ) is Borel learnable if and only if there is $\tau \supseteq \sigma$ such that E is $\mathbf{\Sigma}^0_2$ in $(X, \tau)^2$. Following Louveau [15], we say that such an equivalence relation is *potentially* $\mathbf{\Sigma}^0_2$ and thus get the following.

THEOREM 13. *An equivalence relation E is Borel learnable if and only if E is potentially $\mathbf{\Sigma}^0_2$.*

One can also consider the notion of non-uniform Borel learnability by adapting Definition 1 in the obvious way. We then get the following from another classical change of topology theorem.

PROPOSITION 14. *Let E be an equivalence relation on a Polish space X. Then the following are equivalent.*

(1) *E is non-uniformly Borel learnable.*
(2) *For every $\vec{x} \in X^\omega$, there are sets S_i, $S_i \in Borel(X)$ such that for every $i \in \omega$, $[x_i]_E \subseteq S_i$ and $S_i \cap S_j = \emptyset$ if $x_i \not E x_j$.*

Furthermore, E is non-uniformly learnable by learners not giving false positives if and only if $[x]$ is Borel for every $x \in X$.

PROOF. Suppose we have a learner L learning $\vec{x} \in X^\omega$. Using the change of topology theorem mentioned above, we get a topology τ on X such that the learner L is continuous. Hence, by Proposition 1, there are $(S_i)_{i\in\omega}$ separating the equivalence classes $[x_i]_E$ such that every S_i is $\mathbf{\Sigma}^0_2$ in τ. But then as the Borel sets in τ and the original topology are the same, the S_i are Borel in the original topology on X.

On the other hand, recall another classical change of topology theorem: If $(A_i)_{i\in\omega}$ is a countable sequence of Borel sets on X, then there is a refinement τ of X generating the same Borel sets, such that every A_i is clopen, see again [11, Chapter 13]. Hence, given S_i as in Item 2 we get a topology τ with all S_i clopen and a learner L for $\vec{x}$ that is τ-continuous, and hence Borel. ⊣

§3. Complexity of learnability. For the proofs of the following results, we rely on Louveau's separation theorem and the Spector–Gandy theorem. Recall that HYP denotes the hyperarithmetic, or equivalently Δ^1_1, subsets of the natural numbers. We say that Y is a $\Sigma^0_\alpha(\mathrm{HYP})$ subset of a recursive Polish space X if there is $x \in \mathrm{HYP}$ such that $Y \in \Sigma^0_\alpha(x)$. Louveau's separation theorem says that if X is a recursive Polish space and A_0, A_1 are two disjoint Σ^1_1 sets that are $\mathbf{\Sigma}^0_\alpha$ separated for a recursive ordinal α, then there is a $\Sigma^0_\alpha(\mathrm{HYP})$ set separating A_0 from A_1 [14]. In particular, if X is Δ^1_1 and $\mathbf{\Sigma}^0_\alpha$, then X is $\Sigma^0_\alpha(\mathrm{HYP})$. To see this, just take $A_0 = X$ and $A_1 = \bar{X}$.

We will combine this with the classic Spector–Gandy theorem. The version most suitable for our purpose says that a set $X \subseteq 2^\omega$ is Π^1_1 if and only if there is a recursive predicate R such that

$$x \in X \iff (\exists z \in \Delta^1_1(x))R(x, z).$$

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We suggest [4] for a detailed discussion of this theorem. The following result is a consequence of Louveau's separation theorem and Theorem 6.

LEMMA 15. *Let E be a learnable Δ^1_1 equivalence relation. Then there is a hyperarithmetical learner learning E.*

PROOF. Recall from Theorem 6 that an equivalence relation is learnable by an x-computable learner if and only if it is $\Sigma^0_2(x)$. Now by Louveau's separation theorem $\Delta^1_1 \cap \mathbf{\Sigma}^0_2 = \Sigma^0_2(\mathrm{HYP})$. Thus E being learnable and Δ^1_1 implies that E is $\Sigma^0_2(\mathrm{HYP})$ and so it is learnable by a hyperarithmetical learner. ⊣

We can represent Borel equivalence relations on a Polish space X using their Borel codes. Recall that a *Borel code* is a labeled tree $T \in \omega^{<\omega}$ where the leaf nodes are labeled by (the codes) for elements of a fixed subbase of X and internal nodes are labeled by $\cup$ or $\cap$, indicating that we take unions or intersections of the sets described by the subtree rooted at that node. We can view labeled trees as elements of 2^ω and thus can talk about the complexity of the set of Borel equivalence relations with some property P. As every learnable equivalence relation is Borel by Theorem 6 we can talk about the complexity of learnable equivalence relations in the codes.

THEOREM 16. *The set of learnable equivalence relations on Cantor space is Π^1_1 in the codes.*

PROOF. Note that T is a code for a learnable equivalence relation if and only if (1) T is well-founded, (2) its labeling is correct, and (3) there is a learner learning E, the equivalence relation represented by T. The first statement is Π^1_1, the second arithmetical, and by the relativization of Lemma 15, we have that E is learnable if and only if there exists a learner hyperarithmetic in T. By the Spector-Gandy theorem this is a Π^1_1 statement. ⊣

We will establish that the set of learnable Borel equivalence relations on Cantor space is $\mathbf{\Pi}^1_1$-complete in the codes. Let us quickly recall the definitions. Let X and Y be Polish spaces, and $A \subseteq X$, $B \subseteq Y$. Then A is *Wadge reducible* to B, $A \leq_W B$ if there exists a continuous function $f : X \to Y$ such that for all $x \in X$, $x \in A$ if and only $f(x) \in B$. For a pointclass class $\mathbf{\Gamma}$, and $A \in \mathbf{\Gamma}$, A is $\mathbf{\Gamma}$-*complete* if for every $B \in \mathbf{\Gamma}$, $B \leq_W A$. We are now ready to pin down the complexity of the learnable Borel equivalence relations.

THEOREM 17. *The set of learnable $\mathbf{\Pi}^0_2$ equivalence relations on Cantor space is $\mathbf{\Pi}^1_1$-complete in the codes.*

PROOF. Recall that the set of well-founded trees in $\omega^{<\omega}$ is $\mathbf{\Pi}^1_1$-complete. We reduce it to the set of codes of learnable $\mathbf{\Pi}^0_2$ equivalence relations as follows. Given $x \in 2^\omega$ we might view it as a disjoint union $x = x_1 \oplus x_2$ where $x_1(n) = x(2n)$ and $x_2(n) = x(2n+1)$. We furthermore write p_x for the principal function of x, that is the function enumerating the set of natural numbers defined by x in order. Notice that $p_x \in \omega^\omega$ if and only if x is infinite, and thus this is already a Π^0_2 statement. Also recall that the set $\mathrm{INF} = \{\chi_X : X \subseteq \omega, |X| = \infty\}$ is $\mathbf{\Pi}^0_2$-complete. We are now ready to define our reduction. Given a tree $T \subseteq \omega^{<\omega}$ define a Π^0_2 equivalence relation E_T by

$$x \, E_T \, y \iff x = y \vee \left(p_{x_1}, p_{y_1} \in [T] \wedge x_2, y_2 \in \mathrm{INF}\right).$$

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Notice that if T is well-founded, then $E_T = id_{2^\omega}$. On the other hand, if T is ill-founded, then we claim that E_T is not learnable. Fix $x \in [T]$, and consider $[\langle x, 1^\infty\rangle]_{E_T}$, the equivalence class of $\langle x, 1^\infty\rangle$. If E_T were learnable, then by Theorem 13, $[\langle x, 1^\infty\rangle]_{E_T}$ would be $\Sigma^0_2(x)$. But then as

$$y \in \mathrm{INF} \iff \langle x, y\rangle \in [\langle x, 1^\infty\rangle]_{E_T}$$

we would get that INF is $\Sigma^0_2(x)$, a contradiction. Thus, the function $T \mapsto E_T$ is a continuous reduction from the set of well-founded trees to the set of codes of $\mathbf{\Pi}^0_2$ equivalence relations that are learnable. ⊣

Since the equivalence relation E_T produced in the above proof is id_{2^ω} if T is well-founded, and this equivalence relation is learnable by a computable learner, we obtain the following effective analog to Theorem 17.

COROLLARY 18. *The set of $\mathbf{\Pi}^0_2$ equivalence relations that are learnable by a computable learner is $\mathbf{\Pi}^1_1$-complete in the codes.*

Theorems 16 and 17 directly imply the following result.

COROLLARY 19. *The set of learnable equivalence relations is $\mathbf{\Pi}^1_1$-complete in the codes.*

Using the recursion theorem to identify computable codes for Borel equivalence relations with their indices and noticing that given a computable tree T we produce a computable Borel code for E_T we get similar results for the index sets of codes for learnable equivalence relations.

COROLLARY 20.

(1) *The set of indices of codes for learnable equivalence relations is Π^1_1-complete.*
(2) *The set of indices of codes for equivalence relations learnable by a computable learner is Π^1_1-complete.*

§4. Case study. In this section, we study whether various equivalence relations arising in logic are learnable, and, since learnability is closely tied to the Borel hierarchy, place these equivalence relations in this hierarchy.

4.1. Equivalence relations arising in computability theory.

THEOREM 21. *Turing equivalence is not learnable. In fact, it is not $\mathbf{\Pi}^0_3$.*[1]

PROOF. Assume towards a contradiction that $\equiv_T$ is $\Pi^0_3(a)$ for $a \in 2^\omega$. That is, suppose that there is a Π^0_3 formula

$$\psi(x, y, z) = \forall k \exists m \forall n R(z, x, y, k, m, n)$$

where R is a recursive relation and $\psi(x, y, a)$ holds if and only if $x \equiv_T y$. It is not hard to see that it is dense in the product forcing $\mathbb{P}^2$ that $a \oplus \dot{g}_i \not\geq_T a \oplus \dot{g}_{1-i}$ for $i < 2$. Thus, if we consider a sufficiently mutual generic pair g_1, g_2 relative to a, then $a \oplus g_1, a \oplus g_2$ are not Turing equivalent and so there is (p_1, p_2) such that

[1] We thank Uri Andrews for pointing out that our proof that Turing equivalence is not learnable can be modified to show that it is not $\mathbf{\Pi}^0_3$.

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$(p_1, p_2) \Vdash \exists k \forall m \exists n \neg R(a \oplus \dot{g}_1, a \oplus \dot{g}_2, a, k, m, n)$. In particular, there is k_1 such that the set

$$Q = \{(q_1, q_2) : (q_1, q_2) \Vdash \forall m \exists n \neg R(a \oplus \dot{g}_1, a \oplus \dot{g}_2, a, k_1, m, n)\}$$

is dense below (p_1, p_2). Using this fact we can build two a-recursive reals $(h_1, h_2) \succ (p_1, p_2)$ such that $\neg\psi(a \oplus h_1, a \oplus h_2, a)$. However, since they are a-recursive, clearly $h_1 \oplus a \equiv_T h_2 \oplus a \equiv_T a$, contradicting that $\psi(a)$ defines $\equiv_T$. Thus, Turing equivalence cannot be $\mathbf{\Pi}^0_3$-definable and so, by Theorem 6, it cannot be learnable.

It remains to build h_1 and h_2. We do this in stages; say at stage s we have built the finite strings h_1^s and h_2^s. Let h_1^{s+1} and h_2^{s+1} be the lexicographical minimal pair (q_1, q_2) that extends (h_1^s, h_2^s) and so that $(q_1, q_2) \Vdash \exists n \neg R(a \oplus \dot{g}_1, a \oplus \dot{g}_2, a, s, n)$. We can always find such a pair as Q is dense below (p_1, p_2) and the search is a-recursive since by the definition of the strong forcing relation the existential quantifier must be witnessed by $n < \min\{|q_1|, |q_2|\}$. ⊣

By Wadge's lemma [19] any set that is not $\mathbf{\Pi}^0_3$ is $\mathbf{\Sigma}^0_3$-hard. Hence, we obtain the following corollary.

COROLLARY 22. *Turing equivalence is $\mathbf{\Sigma}^0_3$-complete.*

Turing equivalence was a good candidate for being learnable, as it is naturally Σ^0_3. Coarser equivalence relations such as arithmetic equivalence or hyperarithmetic equivalence are quite far from being learnable. The former is not arithmetical and the latter is not even Borel. We assume that these results are well-known but let us quickly summarize the overall ideas needed to show this, which are quite similar to the ones used in Theorem 21.

For arithmetic equivalence, assume it was $\Sigma^0_n(x)$ for some $n \in \omega$ and a real x. Then, in the product forcing, there is (p_1, p_2) such that every n-generic pair extending (p_1, p_2) is not arithmetic equivalent. However, one can show that there are arithmetically equivalent n-generics, contradicting the above. The proof for hyperarithmetic equivalence is similar.

Let us now consider the equivalence relations of m and 1-equivalence. Recall that two reals x and y are m-equivalent, $x \equiv_m y$, if there exists a total computable function φ_e such that $x(n) = 1$ if and only if $y(\varphi_e(n)) = 1$. The two reals x, y are 1-equivalent, $x \equiv_1 y$, if $x \equiv_m y$ via a 1-1 function. Both of these relations are naturally Σ^0_3, m-equivalence being defined by

$$x \equiv_m y \iff \exists e\ \varphi_e \text{ total} \wedge \forall n (x(n) \leftrightarrow y(\varphi_e(n))) \qquad (*)$$

and 1-equivalence being defined analogously.

THEOREM 23. *If E is an equivalence relation so that $\equiv_1 \subseteq E \subseteq \equiv_T$, then E is not Π^0_3.*

PROOF. Suppose toward a contradiction that E is given by the Π^0_3 formula

$$\varphi(x, y) \iff \exists l \forall m \exists n R(x, y, l, m, n)$$

where R is a recursive relation. Then $\varphi(\dot{g}_1, \dot{g}_2)$ cannot be forced by any (p_1, p_2) because any two mutually generic $g_1 \succ p_1$, $g_2 \succ p_2$ are Turing incomparable and thus also $g_1 \not\equiv g_2$. Similar to the proof of Theorem 21, we can find (p_1, p_2) such

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that $(p_1, p_2) \Vdash \neg\varphi(\dot{g}_1, \dot{g}_2)$. We again build recursive $h_1 \succ p_1$ and $h_2 \succ p_2$ in stages with the added condition that h_1 and h_2 are infinite, coinfinite. Toward this let $h_1^0 = p_1$ and $h_2^0 = p_2$. Now suppose that we have defined h_1^s and h_2^s. If s is even, let $h_i^{s+1} = h_i^s{}^\frown 01$. If s is odd, let h_i^{s+1} be the lexicographical minimal extension q_i of h_i^s so that $(q_1, q_2) \Vdash \exists n \neg R(\dot{g}_1, \dot{g}_2, s, n)$. By the same argument as in the proof of Theorem 21 we can find such (q_1, q_2) recursively.

Odd stages ensure that the h_i are characteristic functions of infinite, coinfinite sets and even stages ensure that $\neg\varphi(h_1, h_2)$. However, as our construction is recursive, so are h_1 and h_2 and as two recursive infinite coinfinite sets are 1-equivalent, $h_1 \; E \; h_2$, contradicting that φ defines E. ⊣

Theorem 23 immediately implies that m-equivalence is not Π^0_3 as it is a subequivalence relation of Turing equivalence that contains 1-equivalence.

The fact that we could assume that the relation R in the proof of Theorem 23 was recursive, allowed us to build recursive and infinite coinfinite reals, ensuring that they are 1-equivalent. Without this assumption, we cannot guarantee anymore that the elements built are both recursive, and thus our proof fails. In fact, if we allowed $\mathrm{TOT} = \{e : \varphi_e \text{ total}\}$ as an oracle, then the definition of m-equivalence in Eq. $(*)$ becomes Σ^0_2.

THEOREM 24. *Both $\equiv_m$ and $\equiv_1$ are $\Sigma^0_2(\mathrm{TOT})$. Hence, they are learnable by a* TOT*-computable learner.*

4.2. Jump hierarchies. Let us recall the notion of a jump hierarchy. Roughly speaking, the Turing jump assigns to a set X its corresponding halting set X'. This operation can be iterated to obtain "halting sets relative to halting sets" $X^{(n)}$ and so on—first, via finite iterations and then, via transfinite recursion. A jump hierarchy is a set that encodes in its columns a countable number of iterations of the Turing jump. In this way, any computable well-order gives rise to a jump hierarchy.

DEFINITION 7. Given an infinite computable linear order $\prec$ on natural numbers let $[\prec n] = \{m : m \prec n\}$. A set $H \subseteq \omega$ is a *jump hierarchy supported on* $\prec$ if

$$\forall n \in \omega \; H^{[n]} = (H^{[\prec n]})'$$

where $H^{[n]} = \{m : \langle m, n\rangle \in H\}$.

We can immediately observe that this definition is Π^0_2. If $\alpha = (\omega, \prec)$ is isomorphic to a computable ordinal, then the corresponding jump hierarchy is uniquely defined. In such cases, we abuse the notation and use $\emptyset^{(\alpha)}$ to denote the unique jump hierarchy supported on $\prec$. It is a well-known fact that if α is an infinite computable ordinal, then the corresponding jump hierarchy $\emptyset^{(\alpha+1)}$ is a Σ^0_α-complete set (see for example [17, Theorem V.16] or [1, p. 75]).

Jump hierarchies allow us to exemplify relatively simple, i.e., defined by a Π^0_2 formula, equivalence relations, which are uniformly learnable but which require learners with computational power at arbitrarily high levels of the hyperarithmetic hierarchy. To this end, given a computable linear order $\prec$ and the corresponding Π^0_2 class $\mathbf{H}$ consisting of jump hierarchies supported on $\prec$ we define $\equiv_\prec$ as follows.

$$x \equiv_\prec y \iff x = y \text{ or } x^{[1]} = y^{[1]} = z \in \mathbf{H}.$$

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PROPOSITION 25. *Given a presentation $(\omega, \prec)$ of an ordinal α, the equivalence relation $\equiv_\prec$ is uniformly learnable and for every z, if $\equiv_\prec$ is z-learnable, then $z \geq_T 0^{(\alpha)}$.*

PROOF. First, since the jump hierarchy supported on $\prec$ is unique, it is straightforward to see that $\equiv_\prec$ is learnable. Now, using the definition of $\equiv_\prec$ and our characterization of learnability (Theorem 6), we notice that $x = \emptyset^{(\alpha)}$ if and only if $\emptyset \oplus x \equiv_\prec \omega \oplus x$ if and only if $\exists n \forall m \varphi(\emptyset \oplus x, \omega \oplus x, n, m)$ where φ is $\Delta^0_1(z)$ and z computes a learner for $\equiv_\prec$. Fixing a witness n for φ we obtain a $\Pi^0_1(z)$ definition of $\emptyset^{(\alpha)}$ and therefore, $\emptyset^{(\alpha)}$ is the unique path in a z-computable binary tree. Each isolated path in a z-computable binary tree is z-computable, hence z computes $\emptyset^{(\alpha)}$. ⊣

While the notion of jump hierarchy was originally defined with well-founded linear orders in mind—we start with the degree of computable functions and iterate the jump from there—a surprising consequence of the Gandy basis theorem is that there exist nonstandard jump hierarchies, supported on ill-founded linear orders. However, these are no longer unique. Indeed, Harrison [9] noted that when an ill-founded linear order supports a jump hierarchy, then it supports a continuum many of them. Moreover, none of these non-standard jump hierarchies can be hyperarithmetic. Indeed, one can show that if a jump hierarchy contains an infinite descending sequence, then it must compute all hyperarithmetic reals. In fact, one can go further, and observe that if $\prec$ is ill-founded and supports a jump hierarchy, then no infinite $\prec$-descending sequence is hyperarithmetic. Thus, we can make the following observation.

PROPOSITION 26. *Given an ill-founded computable order $\prec$, which supports a jump hierarchy, the equivalence relation $\equiv_\prec$ is not uniformly learnable.*

PROOF. As in the proof of Proposition 25 we assume that z learns $\equiv_\prec$ and apply Theorem 6 to argue that there is a $\Pi^0_1(z)$ definition of some non-empty subset of the class of jump hierarchies supported on $\prec$. Therefore, there exists a member of this class which is arithmetic in z—in fact, even z'-computable. However, by Louveau's separation theorem, z may be assumed to be hyperarithmetic, as $\equiv_\prec$ is hyperarithmetic, in fact even Π^0_2. This contradicts the fact that no jump hierarchy supported on an ill-founded order is hyperarithmetic. ⊣

One consequence of Proposition 26 is that for equivalence relations of the form $\equiv_\prec$ learnability identifies exactly the well-founded part of $\prec$. More formally, $\equiv_{\prec n}$ is learnable if and only if $\prec n$ is well-founded. This observation is closely connected to the following result, which is interesting in its own right.

THEOREM 27. *Let $\prec$ be an ill-founded computable order that supports a jump hierarchy. The well-founded part $\mathcal{O}_\prec$ of $\prec$ is Π^1_1 but not Σ^1_1 and not Π^1_1-complete.*

PROOF. The well-founded part of any computable linear order $\prec$ is clearly Π^1_1 as it is defined by

$$n \in \mathcal{O}_\prec \iff (\forall \bar{a} \in [\prec n]^\omega) \exists i \; a_i \prec a_{i+1}$$

and $[\prec n] = \{m : m \prec n\}$ is uniformly computable. Since $\prec$ supports a jump hierarchy, it contains no infinite descending hyperarithmetic sequence. In particular, $\overline{\mathcal{O}_\prec}$ cannot be hyperarithmetic, i.e., Δ^1_1, and hence, $\mathcal{O}_\prec$ is not Δ^1_1, which together with it being Π^1_1 implies that it is not Σ^1_1.

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To see that $\mathcal{O}_\prec$ is not Π^1_1-complete suppose toward a contradiction that the Π^1_1-complete set $\mathrm{WO} = \{e : \varphi_e$ computes a diagram of a well-ordered set$\}$ reduces to $\mathcal{O}_\prec$ via a computable function computed by g. For simplicity, we assume that $\prec$ is a strict ordering. We use g to define a computable linear ordering $\leq_{a,b}$ uniformly for each pair of natural numbers a and b. The ordering is defined by induction—at each stage s, $\leq_{a,b}$ is a finite linear ordering of size s, starting from an empty ordering at $s = 0$. Suppose we have defined the ordering on the first s natural numbers and we are at stage $s + 1$ of the construction. If $g_s(a) \prec g_s(b)$, then we add $s + 1$ as the smallest element so far in $\leq_{a,b}$. Otherwise, i.e., if $g_s(b) \prec g_s(a)$ or one of the two computations did not halt after s steps, then we add s as the largest element so far in $\leq_{a,b}$. This finishes the construction.

Now, we use Smullyan's double recursion theorem to argue that there exists e_0 and e_1 such that φ_{e_0} computes the diagram of $\leq_{e_0,e_1}$ and φ_{e_1} computes the diagram of $\leq_{e_1,e_0}$. Note that $\leq_{e_0,e_1}$ is ill-founded if and only if $\leq_{e_1,e_0}$ is well-founded. Suppose that $g(e_0) \prec g(e_1)$. This means that $\leq_{e_0,e_1}$ is ill-founded. However, by our assumption g is an appropriate reduction, and hence, g maps e_0 outside of $\mathcal{O}_\prec$. But since $\leq_{e_1,e_0}$ is well-founded, $g(e_1) \in \mathcal{O}_\prec$. This is a contradiction, as the well-founded part of $\prec$ is closed downwards. Similarly, we obtain a contradiction for the case when $g(e_1) \prec g(e_0)$. ⊣

4.3. Model theory. As mentioned in the introduction, the work in this paper was inspired by the work of Fokina, Kötzing, and San Mauro [7] and others on learning the isomorphism relation on countably many isomorphism types. One of the first questions that arises is whether it is possible to learn the isomorphism relation on the whole space of structures. This should of course be impossible, as, in general, the isomorphism relation is not even Borel. The following proposition makes this observation precise in terms of non-uniform learnability.

PROPOSITION 28. *Let τ be a countable vocabulary containing at least one relation symbol. Then $Mod(\tau)$ is not non-uniformly learnable.*

PROOF. We will assume without loss of generality that τ contains exactly one unary relation symbol R.

Consider structures $\mathcal{A}_n$ such that $R^{\mathcal{A}_n}(m)$ if and only if $m \leq n$, and a structure $\mathcal{A}_\infty$ where $\{m : R^{\mathcal{A}_\infty}(m)\}$ is an infinite, coinfinite subset of A. Given $x \in 2^\omega$, let $\mathcal{B}_x$ be such that $R^{\mathcal{B}_x}(2n) \iff x(n) = 1$ and $\neg R^{\mathcal{B}_x}(2n + 1)$ for all n. Clearly, the map $x \mapsto \mathcal{B}_x$ is continuous and $\mathcal{B}_x \cong \mathcal{A}_\infty$ if and only if $x \in \mathrm{INF}$.

Now, assume there was a learner L for the sequence $(\mathcal{A}_\infty, \mathcal{A}_1, \dots)$, i.e., there are $\mathbf{\Sigma}^0_2$ sets separating the isomorphism classes. Then, we would get that $x \in \mathrm{INF}$ if and only if $\mathcal{B}_x \in S$ where S is the $\mathbf{\Sigma}^0_2$ set containing $Iso(\mathcal{A}_\infty)$. But then INF would be $\mathbf{\Sigma}^0_2$, contradicting that INF is $\mathbf{\Pi}^0_2$-complete. ⊣

REMARK. Various examples of countable classes of structures C that are explanatory learnable are known [3]. Thus, Proposition 28 fails if we restrict our attention to τ-structures satisfying a fixed $L_{\omega_1\omega}$ sentence φ.

The following is an immediate corollary of Propositions 28 and 1. It generalizes the result by Miller that no structure in relational language has a Σ^{in}_2 Scott sentence [16].

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COROLLARY 29. *There is no vocabulary τ such that for all $\mathcal{A}_0, \cdots \in Mod(\tau)$ there are $\mathbf{\Sigma}^0_2$ sets $(S_i)_{i\in\omega}$ such that $\mathcal{A}_i \subseteq S_i$ and $S_i \cap S_j = \emptyset$ if and only if $\mathcal{A}_i \not\cong \mathcal{A}_j$.*

If we consider Borel learnability, then the picture becomes a bit more interesting. Hjorth and Kechris [10] showed that an equivalence relation induced by the Borel action of a closed subgroup of S_∞, the permutation group of the natural numbers, is potentially $\mathbf{\Sigma}^0_2$ if and only if it is *essentially countable*, i.e., Borel reducible to a countable equivalence relation. Since the isomorphism relation on any class of structures is an orbit equivalence relation induced by S_∞ we get the following as a corollary of Theorem 13.

COROLLARY 30. *Given an $L_{\omega_1\omega}$ sentence φ we have that $\cong_\varphi$ is Borel learnable if and only if $\cong_\varphi$ is essentially countable.*

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