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Weak stability conditions as limits of Bridgeland stability conditions

Tristan C. Collins, Jason Lo, Yun Shi, and Shing-Tung Yau

Abstract. In this article, we give a definition of weak stability condition on a triangulated category. The difference between our definition and existing definitions is that we allow objects in the kernel to have non-maximal phases. We then construct four types of weak stability conditions that naturally occur on Weierstraß ellitpic surfaces as limites of Bridgeland stability conditions.

1 Introduction

The notion of weak stability conditions was defined in [18] and has been studied by many authors (see, e.g., [3, 7, 18]). Recall that in [5], a Bridgeland stability condition on the derived category of coherent sheaves of a smooth projective variety can be equivalently defined as a pair $(Z, \mathcal{A})$, where Z is a group homomorphism from the Grothendieck group of the derived category to the complex numbers, called the central charge, and $\mathcal{A}$ is the heart of a bounded t-structure. Weak stability conditions are defined analogously, the main difference being weak stability conditions allow nonzero objects in the heart $\mathcal{A}$ to lie in the kernel of the central charge Z, i.e., there can be objects $0 \neq E \in \mathcal{A}$ such that $Z(E) = 0$.

In this article, we generalize the definition of weak stability conditions as defined in [18]. This is motivated by the observation that weak stability conditions also naturally occur as degenerations—or "limits"—of Bridgeland stability conditions. When the data $(Z, \mathcal{A})$ of a Bridgeland stability condition approaches a certain limit, it can happen that $\ker(Z)$ begins to have nonzero intersection with $\mathcal{A}$. One obvious issue is how to define the phases of the objects in $\ker(Z) \cap \mathcal{A}$. In the original definition of weak stability conditions in [18], the phase of any object in $\ker(Z) \cap \mathcal{A}$ is always set to be maximal. This is consistent with intuition, since $\ker(Z) \cap \mathcal{A}$ of weak stability conditions considered in [18] merely contains skyscraper sheaves. But if we consider weak stability conditions that occur as limits of Bridgeland stability conditions in general, it can happen that $\ker(Z) \cap \mathcal{A}$ contains objects not supported in dimension zero. In this article, we propose a natural generalization,we define the phases of the objects in $\ker(Z) \cap \mathcal{A}$ to be the limits of their Bridgeland phases as the data $(Z, \mathcal{A})$ of a Bridgeland stability condition approaches a limit.

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The first application of this generalized definition is in our followup paper [8], where we study the stability of certain objects with respect to Bridgeland stability conditions. It turns out that in some cases, it is easier to first show an object is stable with respect to weak stability conditions and then show it is stable with respect to Bridgeland stability conditions. Our new definition then allows us to use the stability of an object at a particular weak stability condition—considered as a limit of Bridgeland stability conditions—to conclude the stability of the object with respect to nearby Bridgeland stability conditions. More concretely, in the sequel [8] to this article, we use the above method to establish the stability of a class of line bundles on elliptic surfaces, which in turn allows us to compare Bridgeland stability of line bundles with the existence of solutions to their associated deformed Hermitian–Yang–Mills (dHYM) equation (also known as the Leung–Yau–Zaslow equation).

Besides the above application, we also hope that this generalization can give a better understanding of the Bridgeland stability manifold. In particular, we hope that the weak stability conditions defined in this article help give a better picture of the boundaries of certain open components of the Bridgeland stability manifold. We note that this direction has already been undertaken in [7], where the authors also take an approach towards weak stability conditions using slicings. Our definition of a weak stability condition, however, does not require the slicing to be locally finite, nor require the existence of a support property. We do, however, require the weak see-saw property. The main motivation behind our definition is that we imagine a weak stability condition to be a degeneration of Bridgeland stability conditions and, in our examples, we always define the phases of objects in the kernel to be limits of their phases with respect to the associated Bridgeland stability conditions. Our main application in this article is in proving the Bridgeland stability of specific objects, while that of [7] is in compactifying the Bridgeland stability manifold.

Note on a previous version. This manuscript is based on the portion of version 1 of the arxiv preprint [8] where we develop our theory of weak stability conditions. Version 2 of the arxiv preprint [8] no longer contains the theory of weak stability conditions; rather, it contains only the applications of weak stability conditions as discussed above.

1.1 Outline

In Section 2, we set up some notation for elliptic surfaces. In Section 3, we recall the definition of very weak stability conditions as defined in [18] and give a generalized definition of weak stability conditions. We also discuss several properties of weak stability conditions that are analogous to those of Bridgeland stability conditions, e.g., slicing, Harder–Narasimhan property etc. In Section 4, we show that a stronger version of Bogomolov–Gieseker inequality for slope stable sheaves holds for some nef diviors, too.

Then, in Section 5, we construct four types of weak stability conditions that satisfy our definition. These are the weak stability conditions we encounter in our followup paper, but they are of their own interests. In particular, they include the following types of limits of Bridgeland stability conditions: (i) when the ample class approaches a nef class; (ii) when the coefficient of ch_0 in the central charge Z of a Bridgeland

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stability condition approaches zero; (iii) when the limits of (i) and (ii) happen at the same time; and (iv) the image of the weak stability condition in (iii) under the relative Fourier Mukai transform. In each of these cases, it is easy to write down what the central charge of the weak stability condition "should" be; it is less clear, however, as to how to construct the right heart of t-structure so that both the positivity property and the Harder–Narasimhan property are satisfied. In addition, in order to construct a weak stability condition using our definition, we need to describe the objects in the intersection of the heart and the kernel of the central charge. Most of Section 5 are devoted to answering these technical questions.

2 Preliminaries on elliptic fibrations

2.1 Notation

Let X be a smooth projective variety X. We will write $D^b(X)$ to denote $D^b(\mathrm{Coh}(X))$, the bounded derived category of coherent sheaves on X.

In Section 3, we will give a definition of weak stability conditions on any triangulated category. In later sections, however, we will focus on weak stability conditions on the derived category of coherent sheaves on a Weierstraß elliptic surface, and so we also review the definition of such surfaces here.

2.2 Weierstraß elliptic surface

By a Weierstraß elliptic surface, we mean a flat morphism $p : X \to Y$ of smooth projective varieties of relative dimension 1, where X is a surface and

- the fibers of p are Gorenstein curves of arithmetic genus 1, and are geometrically integral;
- p has a section $s : Y \to X$ such that its image Θ does not intersect any singular point of any singular fiber of p.

The definition we use here follows that of [2, Definition 6.10]. Under our definition, the generic fiber of p is a smooth elliptic curve, and the singular fibers of p are either nodal or cuspidal curves. We usually write f to denote the class of a fiber for the fibration p, and write $e = -\Theta^2$. Often, we simply refer to X as a Weierstraß elliptic surface. Note that when $Y = \mathbb{P}^1$, X is K3 if and only if $e = 2$ [12, 2.3].

2.3 RDV coordinates for divisors

Suppose X is a Weierstraß elliptic surface. Given any divisor M on X of the form $M = a\Theta + bf$ where $a, b \in \mathbb{R}$ and $a \neq 0$, we can find real numbers $R_M \neq 0$ and D_M such that

$$M = R_M(\Theta + (D_M + e)f). \tag{2.3.1}$$

We also set

$$V_M = \frac{M^2}{2} = R_M^2\left(D_M + \tfrac{e}{2}\right).$$

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Note that when $D_M, V_M > 0$ (e.g., when M is ample), we can write

$$R_M = \sqrt{\frac{V_M}{D_M + \frac{e}{2}}}.$$

The coordinates R_M, D_M, V_M for divisors M are especially suited for computations on elliptic fibrations, and are inspired by symmetries first observed in [11]. For example, we have

$$\Theta M = R_M D_M \tag{2.3.2}$$

and, if W is a divisor written in the form (2.3.1), then

$$MW = R_M R_W(\Theta + (D_M + D_W + e)f)$$

which is reminiscent of multiplication for complex numbers in polar coordinates.

2.4 Bridgeland stability condition

Let X be a smooth projective variety over $\mathbb{C}$. In Section 3, we will give a generalized definition of weak stability condition, we first review the definition of Bridgeland stability condition.

Definition 2.5 A *slicing* $\mathcal{P}$ of $D^b(X)$ is a collection of subcategories $\mathcal{P}(\phi) \subset D^b(X)$ for all $\phi \in \mathbb{R}$ such that

(1) $\mathcal{P}(\phi)[1] = \mathcal{P}(\phi + 1)$,
(2) if $\phi_1 > \phi_2$ and $A \in \mathcal{P}(\phi_1)$, $B \in \mathcal{P}(\phi_2)$, then $\mathrm{Hom}(A, B) = 0$,
(3) every $E \in D^b(X)$ admits a Harder–Narasimhan (HN) filtration by objects in $\mathcal{P}(\phi_i)$ for some $1 \leq i \leq m$.

Definition 2.6 A Bridgeland stability condition on $D^b(X)$ with central charge Z is a slicing $\mathcal{P}$ satisfying the following properties

(1) For any non-zero $E \in \mathcal{P}(\phi)$ we have

$$Z(E) \in \mathbb{R}_{>0} e^{\sqrt{-1}\phi},$$

(2)

$$C := \inf\left\{\frac{|Z(E)|}{\|\mathrm{ch}(E)\|} : 0 \neq E \in \mathcal{P}(\phi), \phi \in \mathbb{R}\right\} > 0$$

where $\|\cdot\|$ is any norm on the finite dimensional vector space $H^{\mathrm{even}}(X, \mathbb{R})$.

Let $\mathbb{H}$ denote the upper-half complex plane together with the negative real axis

$$\{re^{i\pi\phi} : r \in \mathbb{R}_{>0}, \phi \in (0, 1]\}.$$

Proposition 2.7 (Bridgeland, [5]) *A Bridgeland stability condition on $D^b(X)$ is equivalent to the following data: the heart $\mathcal{A}$ of a bounded t-structure on $D^b(X)$, and a central charge $Z : K(\mathcal{A}) \to \mathbb{C}$ such that for every nonzero object $E \in \mathcal{A}$, one has*

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(i) $Z(E) \in \mathbb{H}$,
(ii) *E has a finite filtration*

$$0 = E_0 \subset E_1 \subset \cdots \subset E_{n-1} \subset E_n = E$$

such that $HN_i(E) = E_i/E_{i-1}$'s are semistable objects in $\mathcal{A}$ with decreasing phase ϕ. Furthermore, the central charge satisfies Definition 2.6 (2).

3 Definition of weak stability condition

We begin by recalling the following definition of a very weak pre-stability condition in [18]. Let us denote the strict upper-half complex plane by $\mathbb{H} = \{re^{i\pi\phi} : r \in \mathbb{R}_{>0}, \phi \in (0,1)\}$.

Definition 3.1 [18, Definition 2.1] A very weak pre-stability condition on a triangulated category $\mathcal{D}$ is a pair $(Z, \mathcal{A})$, where $\mathcal{A}$ is the heart of a bounded t-structure on $\mathcal{D}$, and $Z : K(\mathcal{D}) \to \mathbb{C}$ is a group homomorphism satisfying the following conditions:

(i) For any $E \in \mathcal{D}$, we have $Z(E) \in \mathbb{H} \cup \mathbb{R}_{\leq 0}$.

(ii) Let $\rho = -\frac{\Re(Z)}{\Im(Z)}$ be the associated slope function, where we set $\rho(E) = \infty$ if $\Im Z(E) = 0$. Then ρ satisfies the HN property.

In the next section, we will study weak stability conditions whose central charges are limits of central charges associated with Bridgeland stability conditions. Intuitively, we want the phases of the objects with respect to the limit stability conditions to be the limits of the phases with respect to the Bridgeland stability conditions. To achieve this goal, we modify Definition 3.1. In particular, we allow the objects in the kernel of the central charge to have phases other than 1, and define the phases of the objects in the kernel of the central charge to be the limits of the phases with respect to Bridgeland stability conditions.

Definition 3.2 A weak stability condition on $\mathcal{D}$ is a triple

$$\sigma = (Z, \mathcal{A}, \{\phi(K)\}_{K \in \ker(Z) \cap \mathcal{A}}),$$

where $\mathcal{A}$ is the heart of a bounded t-structure on $\mathcal{D}$, and $Z : K(\mathcal{D}) \to \mathbb{C}$ a group homomorphism satisfying:

(i) For any $E \in \mathcal{A}$, we have $Z(E) \in \mathbb{H} \cup \mathbb{R}_{\leq 0}$. For any $K \in \ker(Z) \cap \mathcal{A}$, we have $0 < \phi(K) \leq 1$.

(ii) (Weak see-saw property) For any short exact sequence

$$0 \to K_1 \to K \to K_2 \to 0$$

in $\ker(Z) \cap \mathcal{A}$, we have $\phi(K_1) \geq \phi(K) \geq \phi(K_2)$ or $\phi(K_1) \leq \phi(K) \leq \phi(K_2)$.

For any object $E \notin \ker(Z)$, define the phase of an object $E \in \mathcal{A}$ by

$$\phi(E) = (1/\pi)\arg Z(E) \in (0,1].$$

We further require

(iii) The phase function satisfies the HN property.

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Note that to talk about HN property in (iii) of the above definition, one will need the notion of the slope of a stability condition and the notion of semistability, which we define below.

3.2.1

Define the slope of an object with respect to a weak stability condition by $\rho(E) = -\cot(\pi\phi(E))$ Following [18], an object $E \in \mathcal{A}$ is (semi)-stable if for any nonzero subobject $F \subset E$ in $\mathcal{A}$, we have

$$\rho(F) < (\leq)\rho(E/F),$$

or equivalently,

$$\phi(F) < (\leq)\phi(E/F).$$

Remark 3.3 Note that the condition (ii) in Definition 3.2 (weak see-saw property) implies that for any short exact sequence in $\mathcal{A}$

$$0 \to F_1 \to F \to F_2 \to 0,$$

we have $\phi(F_1) \geq \phi(F_2)$ implies that $\phi(F_1) \geq \phi(F) \geq \phi(F_2)$, and $\phi(F_1) \leq \phi(F_2)$ implies that $\phi(F_1) \leq \phi(F) \leq \phi(F_2)$.

The weak seesaw property built into the definition of a weak stability condition also gives the following property, which comes naturally with Bridgeland stability conditions.

Lemma 3.3.1 *Let $(Z, \mathcal{A}, \{\phi(K)\}_{K \in ker(Z) \cap \mathcal{A}})$ be a weak stability condition. Then for any short exact sequence in $\mathcal{A}$*

$$0 \to E' \to E \to E'' \to 0$$

such that $\phi(E'), \phi(E'') > \phi_0$ for some constant ϕ_0, we also have $\phi(E) > \phi_0$.

Proof If at least one of $Z(E'), Z(E'')$ is nonzero, then the claim is clear. Suppose $Z(E'), Z(E'')$ are both zero. Then by the weak see-saw property, we have either $\phi(E') \geq \phi(E) \geq \phi(E'')$ or $\phi(E') \leq \phi(E) \leq \phi(E'')$. It follows that

$$\phi(E) \geq \min\{\phi(E'), \phi(E'')\} > \phi_0$$

and the claim follows. ∎

Let $(Z, \mathcal{A}, \{\phi(K)\}_{K \in \ker(Z) \cap \mathcal{A}})$ be a weak stability condition. Similar to Bridgeland stability conditions, for $0 < \phi \leq 1$, we define the full additive subcategories $\mathcal{P}(\phi)$ of $\mathcal{D}$ consisting of the objects in $\mathcal{A}$ which are semistable of phase ϕ together with the zero object in $\mathcal{D}$. For general ϕ define $\mathcal{P}(\phi + 1) = \mathcal{P}(\phi)[1]$.

Proposition 3.4 *$\mathcal{P}$ defines a slicing in the sense of Definition 2.5.*

Proof We only need to check the following axiom in Definition 3.3:

$$\text{If } \phi_1 > \phi_2, A_j \in \mathcal{P}(\phi_j), \text{ then } \mathrm{Hom}_{\mathcal{D}}(A_1, A_2) = 0.$$

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Without loss of generality, we assume that $A_2 \in \mathcal{A}$. Since $\mathcal{A}$ is the heart of a bounded t-structure, if $\phi_1 > 1$ we have $\mathrm{Hom}_{\mathcal{D}}(A_1, A_2) = 0$. So we may assume that $A_i \in \mathcal{A}$ for both i. Assuming there is a nonzero map $f : A_1 \to A_2$, we have the following short exact sequence in $\mathcal{A}$:

$$0 \to \ker(f) \to A_1 \to \mathrm{im}(f) \to 0,$$

$$0 \to \mathrm{im}(f) \to A_2 \to \mathrm{coker}(f) \to 0.$$

Since A_j's are semistable, we have $\phi(\ker(f)) \le \phi(\mathrm{im}(f)) \le \phi(\mathrm{coker}(f))$. Then by the weak see-saw property, we have

$$\phi(\ker(f)) \le \phi_1 \le \phi(\mathrm{im}(f)) \le \phi_2 \le \phi(\mathrm{coker}(f))$$

which contradicts $\phi_1 > \phi_2$. ∎

Let $\mathcal{A}$ be a category. Recall that the extension closure of a collection of subcategories of $\mathcal{A}$ is the smallest full subcategory of $\mathcal{A}$ which contains all the objects that can be formed by extentions of the objects in the collection of the subcategories. Let $\mathcal{P}((0,1])$ be the extension closure of all the $\mathcal{P}(\phi)$ where $\phi \in (0,1]$.

Proposition 3.5 *A weak stability condition can be equivalently defined by*

$$\sigma = (Z, \mathcal{P}, \{\phi(E)\}_{E \in \ker(Z) \cap \mathcal{P}((0,1])}),$$

where $\mathcal{P}$ is a slicing with properties as in [5, Definition 3.3], and the central charge $Z : K(\mathcal{D}) \to \mathbb{C}$ is a group homomorphism satisfying:

(i) *For $E \in \mathcal{P}(\phi)$,*
- *if $E \notin \mathcal{P}(\phi) \cap \ker(Z)$, then $Z(E) = m(E)e^{i\pi\phi}$ where $m(E) \in \mathbb{R}_{>0}$;*
- *if $E \in \mathcal{P}(\phi) \cap \ker(Z)$, then $\phi(E) = \phi$.*

(ii) *(weak see-saw property) For any short exact sequence*

$$0 \to K_1 \to K \to K_2 \to 0$$

in $\ker(Z) \cap \mathcal{P}((0,1])$, we have $\phi(K_1) \ge \phi(K) \ge \phi(K_2)$ or $\phi(K_1) \le \phi(K) \le \phi(K_2)$.

Proof Suppose we have a weak stability condition $(Z, \mathcal{A}, \{\phi(K)\}_{K \in \ker(Z) \cap \mathcal{A}})$ as in Definition 3.2. By Proposition 3.4, we have that the triple defines a slicing $\mathcal{P}$, where $\mathcal{A} = \mathcal{P}((0,1])$. Then the phase of the objects in $\ker(Z) \cap \mathcal{A}$ in Definition 3.2 defines the phases $\{\phi(E) : E \in \ker(Z) \cap \mathcal{P}((0,1])\}$ and the weak see-saw property implies (ii) in the Proposition.

Now assume that we have a triple $(Z, \mathcal{P}, \phi(E)|_{E \in \ker(Z) \cap \mathcal{P}((0,1])})$ as stated in the proposition. Since $\mathcal{P}$ is a slicing, the full subcategory $(\mathcal{P}(>0), \mathcal{P}(\le 1))$ defines a bounded t-structure on $\mathcal{D}$. Let $\mathcal{A} = \mathcal{P}((0,1])$ be the heart of this t-structure, then $\mathcal{A}$ is an abelian category. Then the definition of the triple $\sigma = (Z, \mathcal{P}, \phi(E)|_{E \in \ker(Z) \cap \mathcal{P}((0,1])})$ implies the weak see-saw property and defines the phases $\{\phi(E) : E \in \ker(Z) \cap \mathcal{A}\}$. ∎

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We also modify the criterion [5, Proposition 2.4] for the existence of HN filtrations, for use in later sections.

Proposition 3.6 *Let $\sigma = (Z, \mathcal{A}, \phi(E)|_{E\in \ker(Z)\cap\mathcal{A}})$ be a triple satisfying conditions (i) and (ii) in the Definition 3.2. Suppose Z satisfies the following chain conditions:*

(i) *There are no infinite sequences of subobjects in $\mathcal{A}$:*

$$\cdots \subset E_{i+1} \subset E_i \subset \cdots \subset E_1$$

with $\phi(E_{i+1}) > \phi(E_i/E_{i+1})$ for all i.

(ii) *There are no infinite sequences of quotients in $\mathcal{A}$:*

$$E_1 \twoheadrightarrow \cdots \twoheadrightarrow E_i \twoheadrightarrow E_{i+1} \twoheadrightarrow \cdots$$

with $\phi(K_i) > \phi(E_{i+1})$ for all i, where $K_i = \ker(E_i \to E_{i+1})$.

Then σ has the HN property.

Proof We follow Bridgeland's argument with certain modifications.

Claim 1 *For any $0 \neq E \in \mathcal{A}$ where E is not semistable, there exists a semistable subobject $A \subset E$ with $\phi(A) > \phi(E/A)$; similarly, there exists semistable quotient $E \twoheadrightarrow B$ with $\phi(K) > \phi(B)$ where $K = ker(E \twoheadrightarrow B)$.* ■

We prove the claim for semistable quotient, as the case of semistable subobject is similar. We have a sequence of short exact sequences

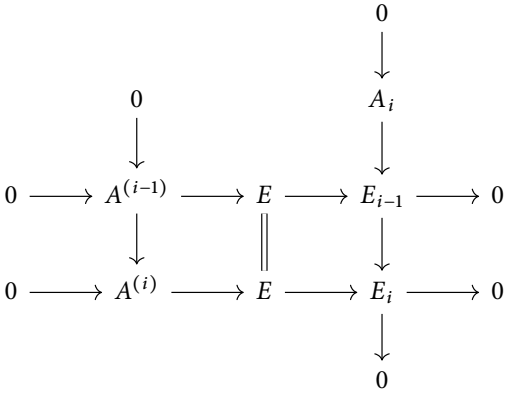
$$0 \to A_i \to E_{i-1} \to E_i \to 0,$$

with $E_0 = E$, and $\phi(A_i) > \phi(E_i)$ for all i. By the chain condition (ii), we have the sequence

$$E \twoheadrightarrow E_1 \twoheadrightarrow \cdots \twoheadrightarrow E_i \twoheadrightarrow \cdots$$

is stationary, hence there exists semistable quotient $E \twoheadrightarrow E_n$. Let $A^{(i)}$ be the kernel of $E \twoheadrightarrow E_i$. Next we show that $\phi(A^{(n)}) > \phi(E_n)$.

By the diagram



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we have short exact sequence

$$0 \to A^{(i-1)} \to A^{(i)} \to A_i \to 0.$$

We prove by induction that $\phi(A^{(i)}) > \phi(E_i)$. By induction hypothesis we have $\phi(A^{(i-1)}) > \phi(E_{i-1})$, which implies that

$$\phi(A^{(i-1)}) > \phi(E_{i-1}) \geq \phi(E_i).$$

We also know that $\phi(A_i) > \phi(E_i)$. By Lemma 3.3.1, we have $\phi(A^{(i)}) > \phi(E_i)$. This finishes the proof of Claim 1.

A maximally destabilising quotient (mdq) of an object $0 \neq E \in \mathcal{A}$ is defined to be a nonzero quotient $E \twoheadrightarrow B$ such that any nonzero semistable quotient $E \twoheadrightarrow B'$ satisfies $\phi(B') \geq \phi(B)$, with equality holding only if (i) $E \twoheadrightarrow B'$ factors via $E \twoheadrightarrow B$ and (ii) $\phi(K') \leq \phi(B')$ for $K' = \ker(B \to B')$. A routine argument shows B is semistable (the proof requires the use of condition (ii), which we did not need to impose in the case of Bridgeland stability conditions), and $\phi(E) \geq \phi(B)$.

We first check the mdq exists. If E is semistable, then itself is the mdq for E. If E is not semistable, then by Claim 1, there is a short exact sequence in $\mathcal{A}$:

$$0 \to A \to E \to E' \to 0.$$

with A semistable and $\phi(A) > \phi(E')$.

Claim 2 *If $E' \twoheadrightarrow B$ is a mdq for E', then the composition $E \twoheadrightarrow B$ is a mdq for E.*

Indeed, let $E \twoheadrightarrow B'$ be a quotient with B' semistable. If $\phi(B') \leq \phi(B)$, then $\phi(B') \leq \phi(E') < \phi(A)$. Then there is no map from A to B', and hence the map $E \to B'$ factors through E'. This implies that $\phi(B') \geq \phi(B)$, and hence $\phi(B') = \phi(B)$. Since $E' \twoheadrightarrow B$ is a mdq for E', the conclusion of the Claim follows.

We continue the process by replacing E by E'. By the chain condition (ii), this process terminates. So every non-zero object have a mdq.

Now we prove the existence of HN filtration under the assumption of the Proposition. Take $0 \neq E \in \mathcal{A}$, if E is semistable, we are done. If not, there is a short exact sequence in $\mathcal{A}$:

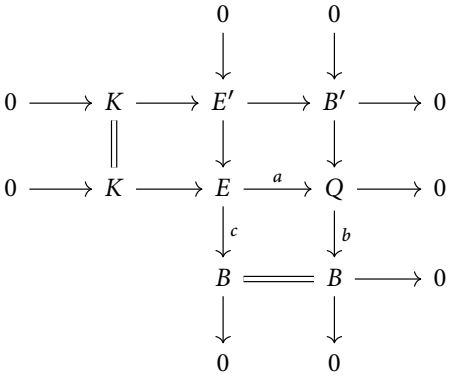
$$0 \to E' \to E \to B \to 0,$$

with $E \twoheadrightarrow B$ the mdq for E. By definition of mdq, B is semistable. Also by the construction in the previous paragraph, $\phi(E') > \phi(B)$. Let $E' \to B'$ be the mdq of E'. Consider the diagram (†) in the proof of [5, Proposition 2.4], in which K is taken to

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be the kernel of $E' \to B'$, and Q taken to be the cokernel of $K \to E$:

(3.6.1)



Claim 3 *We have* $\phi(B') > \phi(B)$.

First consider the case Q is semistable. Since $E \twoheadrightarrow B$ is mdq, we have $\phi(Q) \geq \phi(B)$. But since Q is semistable, we have $\phi(Q) \leq \phi(B)$. So $\phi(Q) = \phi(B)$. By the property of mdq, the map $E \to Q$ factors through $E \twoheadrightarrow B$, contradicting with $E' \to Q$ is nonzero.

From here on, assume Q is not semistable. Let $d : Q \twoheadrightarrow Q'$ be the mdq of Q. Since Q' is the mdq for Q, we have $\phi(B) \geq \phi(Q')$; since B is the mdq for E, we have $\phi(Q') \geq \phi(B)$. Hence $\phi(B) = \phi(Q')$. Now, that Q' is the mdq for Q implies b factors as $b = ed$ for some $e : Q' \twoheadrightarrow B$; on the other hand, that B is the mdq for E implies that da factors as fc for some $f : B \twoheadrightarrow Q'$. Overall, we have $ed = b$ and $da = fc$ which gives

$$(fe)da = fba = fc = da.$$

Since da is surjective, this means e is injective, and hence an isomorphism, i.e., $Q' \cong B$.

If $\phi(Q) > \phi(Q') = \phi(B)$. Applying the weak see-saw property to the short exact sequence $0 \to B' \to Q \to B \to 0$, we obtain $\phi(B') \geq \phi(Q) > \phi(B)$ and hence $\phi(B') > \phi(B)$.

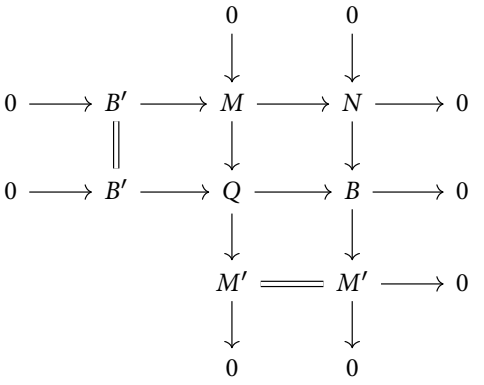
Assume $\phi(Q) = \phi(B)$. Since Q is not semistable, there is a short exact sequence in $\mathcal{A}$

$$0 \to M \to Q \to M' \to 0$$

such that M' is semistable and $\phi(M) > \phi(M')$. Since $\phi(Q) \geq \phi(M')$, $\phi(M') \geq \phi(B)$, we have $\phi(M') = \phi(B)$. Then the map $Q \to M'$ factors through $Q \to B$. Consider the following diagram:

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(3.6.2)



By the definition of mdq, we have $\phi(N) \leq \phi(M')$. Then $\phi(M) > \phi(M')$ implies that $\phi(B') > \phi(M') = \phi(B)$.

Repeating the process for E', by chain condition (i) the sequence terminates, and this is the HN filtration for E.

Definition 3.7 Let E be a σ-semistable object for some weak stability condition

$$\sigma = (Z_\sigma, \mathcal{P}, \{\phi_\sigma(K)\}|_{K \in \ker(Z_\sigma) \cap \mathcal{P}(0,1]}).$$

A Jordan–Hölder filtration of E is a filtration

$$0 = E_0 \subset E_1 \subset \ldots \subset E_n = E$$

such that the factors $gr_i(E) = E_i/E_{i-1}$ are stable, and for each i either $gr_i(E) \in \ker(Z_\sigma)$, or $\phi_\sigma(gr_i(E)) = \phi_\sigma(E)$.

Remark 3.8 We expect the weak stability conditions defined in this article give a compactification to the Bridgeland stability manifold. In particular, the weak stability conditions are the boundaries of certain open components. In the examples in Section 5, we see that with this "compatification", particular slices of Bridgeland stability manifold naively behave like blow up a point on $\mathbb{R}^2$. As a first step to make this precise, we need to generalize Bridgeland's topology to the set of weak stability conditions, we leave this exploration to a furture work.

Remark 3.9 Recall that in 3.2.1, we defined when an object is σ-stable in the abelian category $\mathcal{A} = \mathcal{P}((0,1])$. For a general t we say an object is stable in $\mathcal{P}(t)$ if it is a shift of a stable object in $\mathcal{P}(t')$ for some $0 < t' \leq 1$.

In Section 5, we will give four types of examples of weak stability conditions:

(i) when the ample class in $\sigma_{\omega,B}$ degenerates to a nef class;
(ii) when the coefficient of ch_0 in the central charge Z of a Bridgeland stability condition degenerates to zero;
(iii) when (i) and (ii) occur at the same time; and
(iv) when a relative Fourier-Mukai transform is applied to the weak stability condition in (iii).

These are natural classes of weak stability conditions we obtain when specific parameters of Bridgeland stability approach certain limits. One direct application of the these

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four weak stability conditions is in the sequel to this article, in which we obtain stability of line bundles at Bridgeland stability conditions by first studying their stability at these four types of weak stability conditions.

4 Bogomolov–Gieseker type inequalities

Since we will be dealing with weak stability conditions that arise from nef divisors in Section 5, we need to study Bogomolov–Gieseker-type inequalities for slope semistable sheaves with respect to nef divisors first.

Let X be a smooth projective surface, and ω an ample $\mathbb{R}$-divisor on X. The usual Bogomolov-Gieseker inequality for sheaves says the following: for any μ_ω-semistable torsion-free sheaf E on X, we have

$$\mathrm{ch}_2(E) \leq \frac{\mathrm{ch}_1(E)^2}{2\mathrm{ch}_0(E)}.$$

It is easily checked that this inequality is preserved under twisting the Chern character by a B-field.

When X is a K3 surface, we have the following, stronger Bogomolov–Gieseker inequality which is known to experts (e.g., see [1, Section 6]):

Proposition 4.1 *Let X be a K3 surface, and ω, B any $\mathbb{R}$-divisors on X where ω is ample. Then for any μ_ω-stable torsion-free sheaf E on X, we have*

$$\mathrm{ch}_2^B(E) \leq \frac{(\mathrm{ch}_1^B(E))^2}{2\mathrm{ch}_0(E)} - \mathrm{ch}_0(E) + \frac{1}{\mathrm{ch}_0(E)}. \tag{4.1.1}$$

Recall that on a smooth projective surface, a divisor class is nef if and only if it is movable. In [9, Section 2.6], Greb, Kebekus, and Peternell showed that given a nef divisor that is not necessarily ample, we can still define a notion of slope stability with the usual expected properties. More concretely, suppose ω is a nef divisor on a smooth projective surface X. Then for every nonzero coherent sheaf E on X, we can define the slope

$$\mu_\omega(E) = \begin{cases} \frac{\omega \mathrm{ch}_1(E)}{\mathrm{ch}_0(E)} & \text{if } \mathrm{ch}_0(E) \neq 0 \\ \infty & \text{if } \mathrm{ch}_0(E) = 0 \end{cases}.$$

We then declare a torsion-free sheaf E on X to be μ_ω-(semi)stable if, for every nonzero proper subsheaf A of E, we have $\mu_\omega(A) < (\leq) \mu_\omega(E)$. In particular, every torsion-free sheaf E on X admits a HN filtration

$$0 \neq E_0 \subsetneq E_1 \subsetneq \cdots \subsetneq E_m = E$$

where each subquotient E_i/E_{i-1} is μ_ω-semistable and

$$\mu_\omega(E_0) > \mu_\omega(E_1/E_0) > \cdots > \mu_\omega(E_m/E_{m-1}).$$

We write $\mu_{\omega,\max}(E)$ to denote $\mu_\omega(E_0)$, and $\mu_{\omega,\min}(E)$ to denote $\mu_\omega(E_m/E_{m-1})$. Then for every nonzero proper subsheaf A of E, we have $\mu_\omega(A) \leq \mu_{\omega,\max}(E)$.

In the rest of this section, we will show that on a Weierstraß elliptic K3 surface, $\mu_{\Theta+\epsilon f}$-stable torsion-free sheaves also satisfy the stronger Bogomolov–Gieseker

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inequality (4.1.1) even though $\Theta + ef$ is a nef divisor that is not ample (see Example 4.5). We begin with the following "openness of stability" result.

Proposition 4.2 *Let X be a smooth projective surface. Suppose H, H' are nef $\mathbb{R}$-divisors on X and the value*

$$c := \min\{Hc_1(E) : Hc_1(E) > 0, E \in \mathrm{Coh}(X)\}$$

exists. Then for any torsion-free sheaf E on X, there exists $\epsilon_0 > 0$ depending only on E, H, and H' such that, for any $0 < \epsilon < \epsilon_0$, we have

$$E \text{ is } \mu_H\text{-stable} \Rightarrow E \text{ is } \mu_{H+\epsilon H'}\text{-stable}.$$

Proof Suppose E is a μ_H-stable torsion-free sheaf. Since H' is nef, and hence movable, for any nonzero proper subsheaf A of E, we have $\mu_{H'}(A) \leq \mu_{H',\max}(E)$ [9, Corollary 2.24], where $\mu_{H',\max}(E)$ depends only on H' and E.

By our assumption on H, for any nonzero proper subsheaf A of E, the intersection number $H\mathrm{ch}_1(A)$ must be an integer multiple of c. As a result, there exists $\delta > 0$ depending only on E and H such that

$$\max\left\{\mu_H(A) = \frac{H\mathrm{ch}_1(A)}{\mathrm{ch}_0(A)} : A \text{ is a nonzero proper subsheaf of } E\right\} \leq \mu_H(E) - \delta.$$

Now fix an $\epsilon_0 > 0$ such that

$$\epsilon_0\,|\mu_{H',\max}(E)| < \frac{\delta}{2} \quad \text{and} \quad \epsilon_0|\mu_{H'}(E)| < \frac{\delta}{2}.$$

Note that ϵ_0 depends only on E, H' and δ. Then for any $0 < \epsilon < \epsilon_0$ and any nonzero proper subsheaf A of E, we have

$$\begin{aligned}
\mu_{H+\epsilon H'}(A) &= \mu_H(A) + \epsilon\mu_{H'}(A) \\
&\leq (\mu_H(E) - \delta) + \epsilon\mu_{H',\max}(E) \\
&< \mu_H(E) - \delta + \frac{\delta}{2} \\
&= \mu_H(E) - \frac{\delta}{2} \\
&< \mu_H(E) + \epsilon\mu_{H'}(E) \\
&= \mu_{H+\epsilon H'}(E).
\end{aligned}$$

This means that, for any $0 < \epsilon < \epsilon_0$, the sheaf E is also $\mu_{H+\epsilon H'}$-stable. ∎

Proposition 4.2 is similar to the result [9, Theorem 3.4] by Greb, Kebekus, and Peternell, but with slightly different assumptions.

Corollary 4.3 *Let X be a K3 surface. Suppose H, H' are nef $\mathbb{R}$-divisors on X satisfying the hypotheses in Proposition 4.2 as well as*

- *There exists $\epsilon_1 > 0$ such that $H + \epsilon H'$ is ample for all $\epsilon \in (0, \epsilon_1)$.*

Then any μ_H-stable torsion-free sheaf E on X satisfies the inequality (4.1.1).

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Proof Let ϵ_0 be as in Proposition 4.2. Then for any $0 < \epsilon < \epsilon_0$, the sheaf E is also $\mu_{H+\epsilon H'}$-stable. Since $H + \epsilon H'$ is an ample divisor, Proposition 4.1 applies to E and the claim follows. ∎

Here is a lemma we need for the example below:

Lemma 4.4 *Let $p : X \to Y$ be an elliptic surface with a section Θ. Then a divisor of the form $\Theta + af$ where $a \in \mathbb{R}_{>0}$ is ample if and only if $a > -\Theta^2$.*

Proof By the Nakai–Moishsezon criterion, $\Theta + af$ is ample if and only if $(\Theta + af)^2 = 2a - e > 0$ and $(\Theta + af)C > 0$ for every irreducible curve $C \subset X$.

Suppose C is an irreducible curve on X that is distinct from Θ. Then $\Theta C \geq 0$ and $fC \geq 0$. If $fC = 0$, then C is a vertical divisor, in which case $\Theta C > 0$ and $(\Theta + af)C > 0$. If $fC > 0$, then $(\Theta + af)C > 0$ as well.

Now suppose $C = \Theta$. Then $(\Theta + af)C = a + \Theta^2$. Since $a > 0$, it now follows that $\Theta + af$ is ample if and only if $a > -\Theta^2$. ∎

Example 4.5 Let X be a Weierstraß elliptic K3 surface (i.e., $e = 2$). If we take

$$H = \Theta + ef, \quad H' = f,$$

then these nef divisors satisfy the conditions in Proposition 4.2 since H is integral, while the extra condition in in Corollary 4.3 is satisfied by Lemma 4.4. Therefore, by Corollary 4.3, the sharper Bogomolov–Gieseker inequality (4.1.1) holds for $\mu_{\Theta+ef}$-stable torsion-free sheaves on X even though $\Theta + ef$ is a nef divisor.

5 Four types of weak stability conditions

In this section, we introduce four natural classes of weak stability conditions. Let $p : X \to Y = \mathbb{P}^1$ be a Weierstraß elliptic K3 surface throughout this section, so $e = -\Theta^2 = 2$. Let $i : \mathbb{P}^1 \hookrightarrow X$ denote the canonical section.

5.1 Weak stability condition at the origin

We consider the usual central charge (with zero B-field) as

$$Z_\omega = -\mathrm{ch}_2 + V_\omega \mathrm{ch}_0 + i\omega \mathrm{ch}_1$$

where $\omega = R_\omega(\Theta + (D_\omega + e)f)$. Note that according to our definitions of $R_\omega, D_\omega, V_\omega$ for ω ample, we have

$$V_\omega = \frac{\omega^2}{2}, \quad R_\omega = \sqrt{\frac{V_\omega}{D_\omega + \frac{e}{2}}}.$$

Rescaling the imaginary part of Z_ω, we obtain the central charge

$$Z_{V_\omega, D_\omega}(E) = -\mathrm{ch}_2(E) + V_\omega \mathrm{ch}_0(E) + i(\Theta + (D_\omega + e)f) \cdot \mathrm{ch}_1(E). \tag{5.1.1}$$

We consider the limit of Z_{V_ω, D_ω} when $D_\omega, V_\omega \to 0$

$$Z_H = -\mathrm{ch}_2 + iH\mathrm{ch}_1$$

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where $H = \Theta + ef$. We will now construct a heart $\mathcal{B}^0_{H,k}$ on which Z_H is a weak stability function. (We defer the verification of the HN property to the end of this section.)

Recall that for a proper morphism $f : X \to Y$ on smooth varieties, the Grothendieck–Riemann–Roch (GRR) theorem says

$$\mathrm{ch}(f_*\alpha).\mathrm{td}(T_Y) = f_*(\mathrm{ch}(\alpha).\mathrm{td}(T_X))$$

for any $\alpha \in K(X)$. Note that

$$\mathrm{td}(T_{\mathbb{P}^1}) = 1 + \tfrac{1}{2}c_1(T_{\mathbb{P}^1}) = 1 + [pt]$$

from [10, Example II8.20.1], while $\mathrm{td}(T_X) = (1, 0, 2[pt])$ since X is a K3 surface. Therefore, GRR gives

$$[X] \cap (\mathrm{ch}(i_*\mathcal{O}_{\mathbb{P}^1}(m)).(1, 0, 2[pt])) = i_*\left((\mathrm{ch}(\mathcal{O}_{\mathbb{P}^1}(m)).(1, [pt])) \cap [Y]\right)$$

from which we obtain

$$\begin{aligned}\mathrm{ch}_0(i_*\mathcal{O}_{\mathbb{P}^1}(m)) &= 0\\ \mathrm{ch}_1(i_*\mathcal{O}_{\mathbb{P}^1}(m)) &= i_*[Y] = \Theta\\ \mathrm{ch}_2(i_*\mathcal{O}_{\mathbb{P}^1}(m)) &= (m+1)[pt]\end{aligned}$$

Recall the definition of $\mu_{H,\max}$ and $\mu_{H,\min}$ before Proposition 4.2. Proceeding as in Tramel–Xia [19], we set

$$\begin{aligned}\mathcal{T}^a_H &= \langle E \in \mathrm{Coh}(X) : E \text{ is } \mu_H\text{-semistable}, \mu_{H,\min}(E) > a\rangle\\ \mathcal{F}^a_H &= \langle E \in \mathrm{Coh}(X) : E \text{ is } \mu_H\text{-semistable}, \mu_{H,\max}(E) \le a\rangle.\end{aligned}$$

Even though H is a nef divisor that is not ample, it is a movable class on X, and so the notions $\mu_{H,\min}$ and $\mu_{H,\max}$ make sense [9, Corollary 2.26]. Let

$$\mathcal{A}^a_H = \langle \mathcal{F}^a_H[1], \mathcal{T}^a_H\rangle.$$

Next, for any integer k we set

$$\mathcal{F}^a_{H,k} = \langle \mathcal{O}_\Theta(i) : i \le k\rangle$$

where $\mathcal{O}_\Theta(i)$ denotes $i_*\mathcal{O}_{\mathbb{P}^1}(i)$, and set

$$\mathcal{T}^a_{H,k} = \{E \in \mathcal{A}^a_H : \mathrm{Hom}(E, \mathcal{O}_\Theta(i)) = 0 \text{ for all } i \le k\}.$$

From [19, Lemma 3.2], we know $(\mathcal{T}^a_{H,k}, \mathcal{F}^a_{H,k})$ is a torsion pair in $\mathcal{A}^a_H$, allowing us to perform a tilt to obtain the heart

$$\mathcal{B}^a_{H,k} = \langle \mathcal{F}^a_{H,k}[1], \mathcal{T}^a_{H,k}\rangle.$$

Lemma 5.2 *Let X be a Weierstraß elliptic surface with canonical section Θ and $e = -\Theta^2 > 0$. Suppose $C \subseteq X$ is an irreducible curve not contained in Θ. Then $C.(\Theta + ef) > 0$.*

Proof If C is a vertical divisor, then $C.f = 0$ and $C.\Theta > 0$, in which case the lemma follows.

Suppose C is a horizontal divisor not contained in Θ. Then $C.\Theta \ge 0$, while $C.f > 0$ by [13, Lemma 3.15], in which case the lemma also follows. ∎

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We also know that the Bogomolov–Gieseker inequality holds for slope semistable sheaves with respect to a movable class on a surface: When X is a smooth projective surface and α is a nonzero movable class on X (e.g., when α is nef), for any μ_α-semistable torsion-free sheaf E we have the usual Bogomolov–Gieseker inequality

$$\mathrm{ch}_2(E) \leq \frac{\mathrm{ch}_1(E)^2}{2\mathrm{ch}_0(E)}, \tag{5.2.1}$$

where the inequality is strict if E is not locally free [9, Theorem 5.1].

Proposition 5.3 *Let X be a Weierstraß elliptic K3 surface. Then Z_H is a weak stability function on the heart $\mathcal{B}^0_{H,k}$ for $k = -1, -2$.*

Proof Take any $E \in \mathcal{B}^0_{H,k}$. It suffices to show that $Z_H(E) \in \mathbb{H}_0$ for either $E \in \mathcal{T}^0_{H,k}$ or $E \in \mathcal{F}^0_{H,k}$, when $k = -1, -2$.

Suppose $E \in \mathcal{T}^0_{H,k}$. Then $E \in \mathcal{A}^0_H$, and it is easy to see that $Z_H(E) \in \mathbb{H}$ when we are in one of the following cases:

- $E \in \mathcal{T}^0_H$ and $\mathrm{ch}_0(E) > 0$.
- $E \in \mathcal{T}^0_H$ is supported in dimension 1, and its support is not contained in Θ (use Lemma 5.2).
- $E \in \mathcal{T}^0_H$ is supported in dimension 0.
- $E = F[1]$ where F is a μ_H-semistable torsion-free sheaf with $\mu_H(F) < 0$.

When $E = F[1]$ where F is a μ_H-semistable torsion-free sheaf with $\mu_H(F) = 0$, we have

$$\begin{aligned} Z_H(E) &= Z_H(F[1]) \\ &= \mathrm{ch}_2(F) \\ &\leq \frac{\mathrm{ch}_1(F)^2}{2\mathrm{ch}_0(F)} \text{ by the Bogomolov–Gieseker inequality for} \\ &\qquad \mu_H\text{-semistability [9, Theorem 5.1]} \\ &\leq 0 \text{ by the Hodge Index [17, Theorem 7.14].} \end{aligned}$$

Note that the Hodge Index Theorem applies in the last step because $H^2 > 0$. Also, if E is a 1-dimensional sheaf supported on Θ, then because $\mathrm{Hom}(E, \mathcal{O}_\Theta(i)) = 0$ for all $i \leq k$, it follows that $E \in \langle \mathcal{O}_\Theta(i) : i > k \rangle$, which gives $Z_H(E) \in \mathbb{R}_{\leq 0}$. Since every object in $\mathcal{T}^0_{H,k}$ is an extension of objects of the forms above, we have $Z_H(\mathcal{T}^0_{H,K}) \subseteq \mathbb{H}_0$.

Lastly, if $E \in \mathcal{F}^0_{H,k}$, then $E \in \langle \mathcal{O}_\Theta(i) : i \leq k \rangle[1]$, in which case we also have $Z_H(E) \in \mathbb{H}_0$. ∎

Proposition 5.4 *Let X be a Weierstraß elliptic K3 surface. Then*

$$\mathcal{B}^0_{H,k} \cap ker Z_H = \begin{cases} \langle \mathcal{O}_\Theta(-1)[1], \mathcal{O}_X[1] \rangle & \text{if } k = -1 \\ \langle \mathcal{O}_X[1], \mathcal{O}_\Theta(-1) \rangle & \text{if } k = -2 \end{cases}.$$

Proof For convenience, we will write $\mathcal{A}$ and $\mathcal{B}$ to denote the hearts $\mathcal{A}^0_H$ and $\mathcal{B}^0_{H,k}$, respectively, in this proof.

Take any nonzero object $E \in \mathcal{B}^0_{H,k} \cap \ker Z_H$. Then $Z_H(\mathcal{H}^{-1}_{\mathcal{A}}(E)) = 0 = Z_H(\mathcal{H}^0_{\mathcal{A}}(E))$. In particular, we have $\mathrm{ch}_2(\mathcal{H}^{-1}_{\mathcal{A}}(E)) = 0$. Since

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$\mathcal{H}^{-1}_{\mathcal{A}}(E) \in \langle \mathcal{O}_\Theta(i) : i \leq k \rangle$, this means that

$$\mathcal{H}^{-1}_{\mathcal{A}}(E) \in \begin{cases} \langle \mathcal{O}_\Theta(-1) \rangle & \text{if } k = -1 \\ \{0\} & \text{if } k = -2 \end{cases}.$$

Also, since $\Im Z_H(\mathcal{H}^0_{\mathcal{A}}(E)) = 0$ and $\Im Z_H$ is nonnegative on $\mathcal{A}$, it follows that $\Im Z_H(H^0(\mathcal{H}^0_{\mathcal{A}}(E)))$ and $\Im Z_H(H^{-1}(\mathcal{H}^0_{\mathcal{A}}(E)))$ are both zero. As a result, the sheaf $H^0(\mathcal{H}^0_{\mathcal{A}}(E))$ must be a torsion sheaf supported on Θ by Lemma 5.2, while $M := H^{-1}(\mathcal{H}^0_{\mathcal{A}}(E))$ is a μ_H-semistable torsion-free sheaf with $\mu_H = 0$.

Since H is a nef divisor, the usual Bogomolov–Gieseker inequality holds for μ_H-semistable sheaves [9, Theorem 5.1]

$$\mathrm{ch}_2(M) \leq \frac{c_1(M)^2}{2\mathrm{ch}_0(M)}.$$

Also, since H is nef with $H^2 = e > 0$, the Hodge Index Theorem gives $c_1(M)^2 \leq 0$, and so we have $\mathrm{ch}_2(M) \leq 0$. On the other hand, by the construction of $\mathcal{B}$, we have

$$\mathrm{Hom}(H^0(\mathcal{H}^0_{\mathcal{A}}(E)), \mathcal{O}_\Theta(i)) = 0 \text{ for any } i \leq k$$

which means that $H^0(\mathcal{H}^0_{\mathcal{A}}(E))$ is an extension of a sheaf in $\langle \mathcal{O}_\Theta(i) : i > k \rangle$ by a 0-dimensional sheaf, giving us $\mathrm{ch}_2(H^0(\mathcal{H}^0_{\mathcal{A}}(E))) > k + 1 \geq 0$ (recall k is either -1 or -2). Since $\Re Z_H(\mathcal{H}^0_{\mathcal{A}}(E)) = 0$, this forces

$$\mathrm{ch}_2(M) = \mathrm{ch}_2(H^0(\mathcal{H}^0_{\mathcal{A}}(E))) = 0.$$

In particular, $H^0(\mathcal{H}^0(\mathcal{A}(E)))$ is forced to be a pure 1-dimensional sheaf, and

$$H^0(\mathcal{H}^0_{\mathcal{A}}(E)) \in \begin{cases} \{0\} & \text{if } k = -1 \\ \langle \mathcal{O}_\Theta(-1) \rangle & \text{if } k = -2 \end{cases}.$$

We now show that, in fact, M lies in $\langle \mathcal{O}_X \rangle$. Let M_i be the μ_H-Jordan Hölder factors of M (which exist by [9, Corollary 2.27]). By Example 4.5, each M_i satisfies the sharper Bogomolov–Giesker inequality (4.1). Together with the Hodge Index Theorem, for each i we have

$$\mathrm{ch}_2(M_i) \leq -\mathrm{ch}_0(M_i) + \frac{1}{\mathrm{ch}_0(M_i)}$$

where the right-hand side is strictly less than 0 if $\mathrm{ch}_0(M_i) > 1$, while equal to 0 if $\mathrm{ch}_0(M_i) = 0$. Since $0 = \mathrm{ch}_2(M) = \sum_i \mathrm{ch}_2(M_i)$, it follows that each M_i is a rank-one torsion-free sheaf with $\mathrm{ch}_2(M_i) = 0$. In fact, each M_i must be a line bundle, for if M_j is not locally free for some j, then

$$\mathrm{ch}_2(M_j) = \mathrm{ch}_2(M_j^{**}) - \mathrm{ch}_2(M_j^{**}/M_j) < \mathrm{ch}_2(M_j^{**}) \leq \frac{c_1(M_j^{**})^2}{2\mathrm{ch}_0(M_j^{**})} \leq 0 \tag{5.4.1}$$

where the second last inequality follows from the usual Bogomolov–Gieseker inequality, and the last inequality follows from the Hodge Index Theorem for H. This contradicts $\mathrm{ch}_2(M_j) = 0$, and so each M_i is a line bundle. Moreover, equality holds

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in the Hodge Index Theorem in (5.4.1), and so $c_1(M_i) \equiv 0$; since X is a K3 surface, this implies $c_1(M_i) = 0$ and hence $M_i \cong \mathcal{O}_X$. That is, $M \in \langle \mathcal{O}_X \rangle$.

So far, we have shown that any object E in $\mathcal{B}^0_{H,k} \cap \ker Z_H$ fits in an exact triangle in $D^b(X)$

$$E' \to E \to E'' \to E'[1]$$

where

- $E' \in \langle \mathcal{O}_\Theta(-1)[1] \rangle$ and $E'' \in \langle \mathcal{O}_X[1] \rangle$ if $k = -1$;
- $E' \in \langle \mathcal{O}_X[1] \rangle$ and $E'' \in \langle \mathcal{O}_\Theta(-1) \rangle$ if $k = -2$.

This proves the proposition. ∎

We actually have that the objects in $\mathcal{B}^0_{H,-1} \cap \ker(Z_H)$ are direct sums of $\mathcal{O}_\Theta(-1)[1]$ and $\mathcal{O}_X[1]$, and objects in $\mathcal{B}^0_{H,-2} \cap \ker(Z_H)$ are direct sums of $\mathcal{O}_\Theta(-1)$ and $\mathcal{O}_X[1]$, as shown in the following lemma.

Lemma 5.5 *Let X be a Weierstraß elliptic K3 surface. Then*

$$Ext^1(\mathcal{O}_\Theta(-1), \mathcal{O}_X[1]) = 0 = Ext^1(\mathcal{O}_X[1], \mathcal{O}_\Theta(-1)[1]),$$

$$Ext^1(\mathcal{O}_\Theta(-1), \mathcal{O}_\Theta(-1)) = 0,$$

and every object in $\mathcal{B}^0_{H,k} \cap ker Z_H$ is a direct sum of $\mathcal{O}_\Theta(-1)[1]$ and $\mathcal{O}_X[1]$ (resp. $\mathcal{O}_X[1]$ and $\mathcal{O}_\Theta(-1)$) if $k = -1$ (resp. $k = -2$).

Proof By Serre Duality,

$$\mathrm{Ext}^1(\mathcal{O}_\Theta(-1), \mathcal{O}_X[1]) \cong \mathrm{Hom}(\mathcal{O}_X, \mathcal{O}_\Theta(-1))$$

which must vanish, because every morphism $\mathcal{O}_X \to \mathcal{O}_\Theta(-1)$ of sheaves factors as $\mathcal{O}_X \to \mathcal{O}_\Theta \to \mathcal{O}_\Theta(-1)$, but the only morphism $\mathcal{O}_\Theta \to \mathcal{O}_\Theta(-1)$ is the zero map (e.g., use slope stability for torsion-free sheaves on Θ).

With the above vanishing and $\mathrm{Ext}^2(\mathcal{O}_X, \mathcal{O}_\Theta(-1)) \cong \mathrm{Hom}(\mathcal{O}_\Theta(-1), \mathcal{O}_X) = 0$, it follows that

$$\begin{aligned} \mathrm{dim}\mathrm{Ext}^1(\mathcal{O}_X, \mathcal{O}_\Theta(-1)) &= -\chi(\mathcal{O}_X, \mathcal{O}_\Theta(-1)) \\ &= (v(\mathcal{O}_X), v(\mathcal{O}_\Theta(-1))) \text{ by Riemann–Roch} \\ &= ((1,0,1),(0,\Theta,0)) \\ &= 0. \end{aligned}$$

Also by Serre duality, we have

$$\mathrm{Ext}^2(\mathcal{O}_\Theta(i), \mathcal{O}_\Theta(i)) \simeq \mathrm{Hom}(\mathcal{O}_\Theta(i), \mathcal{O}_\Theta(i)) \simeq \mathbb{C}.$$

By Riemann–Roch, we have

$$\chi(\mathcal{O}_\Theta(i), \mathcal{O}_\Theta(i)) = -(v(\mathcal{O}_\Theta(i)), v(\mathcal{O}_\Theta(i))) = 2.$$

Hence $2 - \mathrm{ext}^1(\mathcal{O}_\Theta(i), \mathcal{O}_\Theta(i)) = 2$, which implies that

$$\mathrm{Ext}^1(\mathcal{O}_\Theta(i), \mathcal{O}_\Theta(i)) = 0.$$

The second part of the proposition follows easily from Proposition 5.4. ∎

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5.6

To define a weak stability function, we still need to define the phases of objects in the subcategory $\mathcal{B}^0_{H,k} \cap \ker(Z_H)$ in such a way that the weak see-saw property is satisfied. We define the phases $\{\phi_{H,k}(K)|_{K\in\mathcal{B}^0_{H,k}\cap\ker(Z_H)}\}$ by taking the limit of the phase of $Z_{V_\omega,D_\omega}(K)$ as $V_\omega, D_\omega \to 0$.

We have

$$\lim_{V_\omega,D_\omega\to 0} \phi_{V_\omega,D_\omega}(\mathcal{O}_X[1]) = 1$$

and

$$\lim_{V_\omega,D_\omega\to 0} \phi_{V_\omega,D_\omega}(\mathcal{O}_\Theta(-1)) = \frac{1}{2}.$$

As a result, it is only when $k = -2$ that we have $0 < \phi_{H,k}(K) \le 1$ for any $K \in \mathcal{B}^0_{H,k} \cap \ker(Z_H)$. Given a point $b \in \mathbb{P}^1$, whose coordinates are given by $[V_\omega : D_\omega]$, we define

$$\phi^L_b(K) = \lim_{V_\omega,D_\omega\to 0,[V_\omega:D_\omega]=b} \phi_{V_\omega,D_\omega}(K).$$

We check the weak see-saw property in Definition 3.2.

Proposition 5.7 *Let*

$$0 \to K_1 \to K \to K_2 \to 0$$

be a short exact sequence in $\mathcal{B}^0_{H,-2} \cap \ker(Z_H)$. *Then we have*

(i) $\phi^L_b(K_1) \le \phi^L_b(K_2)$ *implies that* $\phi^L_b(K_1) \le \phi^L_b(K) \le \phi^L_b(K_2)$.
(ii) $\phi^L_b(K_1) \ge \phi^L_b(K_2)$ *implies that* $\phi^L_b(K_1) \ge \phi^L_b(K) \ge \phi^L_b(K_2)$.

Proof By lemma 5.5, we have for any $K \in \mathcal{B}_{H,-2} \cap \ker(Z_H)$, K is isomorphic to a direct sum of $\mathcal{O}_X[1]$ and $\mathcal{O}_\Theta(-1)$. Assume that

$$K_0 \simeq (\mathcal{O}_\Theta(-1))^{l_0} \oplus (\mathcal{O}_X[1])^{l_1},$$
$$K \simeq (\mathcal{O}_\Theta(-1))^{m_0} \oplus (\mathcal{O}_X[1])^{m_1},$$
$$K_1 \simeq (\mathcal{O}_\Theta(-1))^{n_0} \oplus (\mathcal{O}_X[1])^{n_1}.$$

To prove the proposition, it is enough to show the same relation for the Bridgeland slope. We define $\rho_{V_\omega,D_\omega} = -\frac{\Re(Z_{V_\omega,D_\omega})}{\Im(Z_{V_\omega,D_\omega})}$, and $\rho^L_b = -\cot(\pi\phi^L_b(_))$. We have

$$\begin{aligned}\rho^L_b(K) &= \lim_{V_\omega,D_\omega\to 0,[V_\omega:D_\omega]=b} \rho_{V_\omega,D_\omega}(K)\\ &= \lim_{V_\omega,D_\omega\to 0,[V_\omega:D_\omega]=b} -\frac{\Re(Z_{V_\omega,D_\omega}(K))}{\Im(Z_{V_\omega,D_\omega}(K))}\\ &= \lim_{V_\omega,D_\omega\to 0,[V_\omega:D_\omega]=b} \frac{V_\omega\cdot m_1}{D_\omega\cdot m_0}\end{aligned}$$

We write $b = V_\omega/D_\omega$, and $b \in [0,\infty]$. Then $\phi^L_b(K) = \frac{bm_1}{m_0}$. Similarly, we have $\phi^L_b(K_0) = \frac{bl_1}{l_0}$ and $\phi^L_b(K_1) = \frac{bn_1}{n_0}$. Since $m_0 = l_0 + n_0$ and $m_1 = l_1 + n_1$, we obtain the weak see-saw property in Definition 3.2. ∎

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For each $b \in \mathbb{P}^1$, we have a triple $\sigma_b^L = (Z_H, \mathcal{B}^0_{H,-2}, \{\phi_b(E)\}|_{E\in\ker(Z_H)\cap\mathcal{B}^0_{H,-2}})$. Finally we check the HN property in Definition 3.2.

Proposition 5.8 *The triple σ_b^L satisfies the HN property, hence defines a weak stability condition.*

Proof We follow the argument of [6].

We first check condition (i) in Proposition 3.6. Assume we have such a chain of subobjects in $\mathcal{B}^0_{H,-2}$ as in condition (i). Consider the short exact sequence in $\mathcal{B}^0_{H,-2}$:

$$0 \to E_{i+1} \to E_i \to F_i \to 0. \tag{5.8.1}$$

Since $\Im(Z_H(F_i)) \geq 0$, we have $\Im(Z_H(E_i)) \geq \Im(Z_H(E_{i+1}))$. Since $\Im(Z_H)$ has coefficients in $\mathbb{Z}$, we know that $\Im(Z_H)$ is discrete. Hence $\Im(Z_H(E_i))$ is constant for $i \gg 0$. Then $\Im(Z_H(F_i)) = 0$ for $i > n$ for some n.

Let $i > n$. If $F_i \notin \mathcal{B}^0_{H,-2} \cap \ker(Z_H)$, then $\phi_b^L(F_i) = 1$, contradicting $\phi_b^L(E_{i+1}) > \phi_b^L(F_i)$.

So we must have $F_i \in \ker(Z_H) \cap \mathcal{B}^0_{H,-2} = \langle \mathcal{O}_X[1], \mathcal{O}_\Theta(-1)\rangle$ for $i > n$. By Lemma 5.5, we have

$$\mathrm{Ext}^1(F_i, \mathcal{O}_\Theta(-1)) = \mathrm{Ext}^1(F_i, \mathcal{O}_X[1]) = 0.$$

Now, applying the functors $\mathrm{Hom}(-, \mathcal{O}_X[1])$ and $\mathrm{Hom}(-, \mathcal{O}_\Theta(-1))$ to (5.8.1), we see that the dimensions of $\mathrm{Hom}(E_i, \mathcal{O}_\Theta(-1))$ and $\mathrm{Hom}(E_i, \mathcal{O}_X[1])$ are both non-increasing as i increases, and for $i \gg 0$, we have

$$\mathrm{Hom}(F_i, \mathcal{O}_X[1]) = \mathrm{Hom}(F_i, \mathcal{O}_\Theta(-1)) = 0$$

which implies $F_i = 0$ by Proposition 5.5.

Next we consider condition (ii) in Proposition 3.6. Assuming we have a sequence

$$E = E_1 \twoheadrightarrow E_2 \twoheadrightarrow E_3 \twoheadrightarrow \dots$$

Consider the short exact sequence

$$0 \to K_i \to E_i \to E_{i+1} \to 0. \tag{5.8.2}$$

Similar to the argument above, we have $\Im(Z_H(E_i))$ is constant for $i \gg 0$. Omitting a finite number of terms, we can assume there are short exact sequences

$$0 \to L_i \to E \to E_i \to 0 \tag{5.8.3}$$

with $\Im(Z_H(L_i)) = 0$ for all $i \geq 0$. Also by the long exact sequence of cohomology, we have the surjective chain in $\mathrm{Coh}(X)$:

$$H^0(E_1) \twoheadrightarrow H^0(E_2) \twoheadrightarrow H^0(E_3) \twoheadrightarrow \dots$$

Since $\mathrm{Coh}(X)$ is noetherian, this sequence stabilizes. Hence we can assume that $H^0(E) \simeq H^0(E_i)$ for all i. Taking cohomology sheaves of (5.8.3), we have

$$0 \to H^{-1}(L_i) \to H^{-1}(E) \xrightarrow{f} H^{-1}(E_i) \to H^0(L_i) \to 0. \tag{5.8.4}$$

Consider the short exact sequence

$$0 \to L_{i-1} \to L_i \to B_i \to 0 \tag{5.8.5}$$

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where $B_i \cong K_{i-1}$ by the octahedral axiom. Taking cohomology sheaves shows that we have a chain

$$H^{-1}(L_1) \subset H^{-1}(L_2) \subset H^{-1}(L_3) \subset \cdots \subset H^{-1}(E)$$

which stabilizes after a finite number of terms. Hence by omitting finite number of terms, we can also assume that $H^{-1}(L_i) \simeq H^{-1}(L_{i+1})$ for all i. Then the exact sequence 5.8.4 becomes

$$0 \to Q \to H^{-1}(E_i) \to H^0(L_i) \to 0, \tag{5.8.6}$$

where Q is the image of f and is independent of i. We also have the following exact sequence of cohomology sheaves from (5.8.5).

$$0 \to H^{-1}(B_i) \to H^0(L_{i-1}) \to H^0(L_i) \to H^0(B_i) \to 0.$$

Since $\mathfrak{I}(Z_H(L_i)) = 0$, we have $\mathfrak{I}(Z_H(H^0(L_i))) = 0$, hence $H^0(L_i)$ lies in the extension closure of zero dimensional sheaves and sheaves in $\langle \mathcal{O}_\Theta(j) | j > -2 \rangle$. Since L_{i-1}, L_i, and B_i are all in $\mathcal{B}^0_{H,-2}$, it follows that $\mathfrak{I}(Z_H(B_i)) = 0$ for all $i \geq 0$ as well, and we have

(a) $H^{-1}(B_i) = 0$ or $\mathrm{ch}_2(H^{-1}(B_i)) < 0$,
(b) $\mathrm{ch}_2(H^0(L_{i-1})) \geq 0$,
(c) $\mathrm{ch}_2(H^0(L_i)) \geq 0$,
(d) $\mathrm{ch}_2(H^0(B_i)) \geq 0$.

Hence $\mathrm{ch}_2(H^0(L_i)) \geq \mathrm{ch}_2(H^0(L_{i-1}))$. Then we must have $\mathrm{ch}_2(H^0(L_i)) = \mathrm{ch}_2(H^0(L_{i-1}))$ for $i \gg 0$. If not, by the short exact sequence (5.8.6) we have $\mathrm{ch}_2(H^{-1}(E_i))$ has no upper bound. Then $-\mathrm{ch}_2(E_i) > 0$ for $i \gg 0$, which contradicts Z_H being a weak stability function on $\mathcal{B}^0_{H,-2}$ (Proposition 5.3).

Now that we know $\mathrm{ch}_2(H^0(L_i))$ is constant for $i \gg 0$, by (a) above we have $H^{-1}(B_i) = 0$ for $i \gg 0$ and we have

$$0 \to H^0(L_{i-1}) \to H^0(L_i) \to H^0(B_i) \to 0.$$

Then either $H^0(B_i) \in \langle \mathcal{O}_\Theta(-1) \rangle$ or $H^0(B_i) = 0$. Recall that we have $B_{i+1} \simeq K_i$. Thus we have for $i \gg 0$, $K_i \in \langle \mathcal{O}_\Theta(-1) \rangle$ or $K_i = 0$. Applying $\mathrm{Hom}(\mathcal{O}_\Theta(-1), _)$ to the short exact sequence (5.8.2) and recalling $\mathrm{Ext}^1(\mathcal{O}_\Theta(-1), \mathcal{O}_\Theta(-1)) = 0$ from Lemma 5.5, if $K_i \neq 0$ then we have

$$\dim \mathrm{Hom}(\mathcal{O}_\Theta(-1), E_i) > \dim \mathrm{Hom}(\mathcal{O}_\Theta(-1), E_{i+1}).$$

Hence K_i must vanish for $i \gg 0$, i.e., $E_i \simeq E_{i+1}$ for $i \gg 0$. This proves the HN property for σ^L_b. ∎

Remark 5.9 Since $\mathcal{B}^0_{H,-2} \cap \ker(Z_H)$ is a Serre subcategory of $\mathcal{B}^0_{H,-2}$, and every object of $\mathcal{B}^0_{H,-2} \cap \ker(Z_H)$ is a direct sum of $\mathcal{O}_X[1]$ and $\mathcal{O}_\Theta(-1)$ which have different phases with respect to σ^L_b, the only σ^L_b-semistable objects in $\mathcal{B}^0_{H,-2} \cap \ker(Z_H)$ are direct sums of $\mathcal{O}_X[1]$ itself and direct sums of $\mathcal{O}_\Theta(-1)$ itself.

We denote the slicing of σ^L_b by $\mathcal{P}_b$. The next two results shows that given $\phi \in \mathbb{R}$, every object $E \in \mathcal{P}_b(\phi)$ has a Jordan–Hölder filtration.

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Lemma 5.10 [6, Lemma 4.4] *Let α, β be real numbers such that $0 < \beta - \alpha < 1 - 2\epsilon$ for some $0 < \epsilon < \frac{1}{8}$. The thin subcategory $\mathcal{P}_b((\alpha, \beta))$ is of finite length.*

Proof We follow the argument of [6, Lemma 4.4]. Let $\phi = \frac{\alpha+\beta}{2}$. Then $\mathcal{P}_b((\alpha, \beta)) \subset \mathcal{P}_b((\phi - \frac{1}{2}, \phi + \frac{1}{2}))$. Define a function

$$f(E) = \Re(\exp(-i\pi\phi)Z_H(E)).$$

Then for any $E \in \mathcal{P}_b((\alpha, \beta))$, we have $f(E) > 0$ or $E \in \ker(Z_H)$. Given a strict short exact sequence

$$0 \to G \to E \to F \to 0$$

in $\mathcal{P}_b((\alpha, \beta))$, we have $f(E) = f(G) + f(F)$. We have the following two cases:

(i) the value of f decreases when taking a subobject,

(ii) the value of f does not change when taking a subobject and the corresponding quotient object is in $\ker(Z_H)$.

Assume that $\mathcal{O}_X[1] \in \mathcal{P}_b((\alpha, \beta))$ and/or $\mathcal{O}_\Theta(-1) \in \mathcal{P}_b((\alpha, \beta))$, then in case (ii) either the dimension of $\mathrm{Hom}(_, \mathcal{O}_X[1])$ or the dimension of $\mathrm{Hom}(_, \mathcal{O}_\Theta(-1))$ decreases when taking a subobject. Since Z_H is discrete, any chain in $\mathcal{P}_b((\alpha, \beta))$ of the form

$$E_0 \subset E_1 \subset \cdots \subset E_i \subset \cdots \subset E,$$

or of the form

$$\ldots \subset E_{j+1} \subset \cdots \subset E_1 \subset E$$

must terminate. ∎

Proposition 5.11 *Consider the weak stability condition σ_b^L. Any $E \in \mathcal{P}_b(\phi)$ has a finite Jordan–Hölder filtration into stable factors.*

Proof WLOG, we assume that $0 < \phi \leq 1$. If E is σ_b^L- stable, then we are done. From now on we assume that E is not σ_b^L-stable.

Step 1. We show that there exists a subobject A_1 of E in $\mathcal{B}^0_{H,-2}$, such that A_1 is σ_b^L-stable.

Since E is not stable, there exists a short exact sequence

$$0 \to E_1 \to E \to F_1 \to 0$$

in $\mathcal{B}^0_{H,-2}$ with $\phi_b(E_1) = \phi_b(E) = \phi_b(F_1) = \phi$. If E_1 is stable, then the claim is true. If not, we first prove the following Claim.

Claim: for any short exact sequence in $\mathcal{B}^0_{H,-2}$:

$$0 \to E_1^{(1)} \to E_1 \to E_1' \to 0 \tag{5.11.1}$$

with $\phi_b(E_1^{(1)}) \geq \phi_b(E_1')$, we have $\phi_b(E_1^{(1)}) = \phi$.

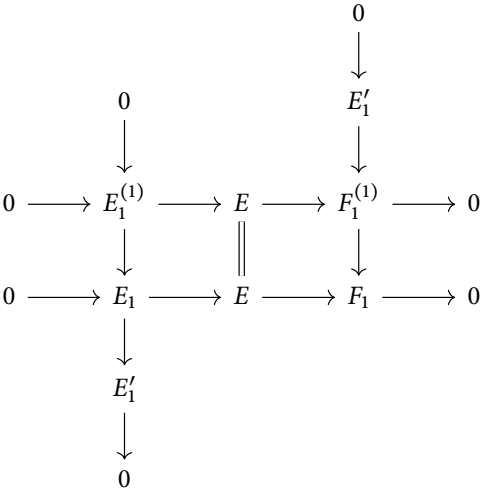
Note that by assumption we have $\phi_b(E_1^{(1)}) \geq \phi$. Composing with the map $E_1 \to E$, we form a short exact sequence in $\mathcal{B}^0_{H,-2}$:

$$0 \to E_1^{(1)} \to E \to F_1^{(1)} \to 0.$$

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By the diagram

(5.11.2)



we have $\phi_b(E_1^{(1)}) \geq \phi_b(F_1^{(1)})$. Since E is σ_b^L-semistable, we have

$$\phi_b(E_1^{(1)}) = \phi_b(F_1^{(1)}) = \phi$$

which proves the claim.

We have either $E_1 \in \mathcal{P}_b(\phi)$, or E_1 is not semistable and in which case there exists a short exact sequence in the form of (5.11.1) with

$$\phi = \phi_b(E_1^{(1)}) = \phi_b(E_1) > \phi_b(E_1').$$

This implies that $E_1' \in \ker(Z_H) \cap \mathcal{B}^0_{H,-2}$. Then if $E_1 \notin \mathcal{P}_b(\phi)$, we have either

$$\dim(\mathrm{Hom}(E_1^{(1)}, \mathcal{O}_X[1])) < \dim(\mathrm{Hom}(E_1, \mathcal{O}_X[1])),$$

or

$$\dim(\mathrm{Hom}(E_1^{(1)}, \mathcal{O}_\Theta(-1))) < \dim(\mathrm{Hom}(E_1, \mathcal{O}_\Theta(-1))).$$

Replace E_1 by $E_1^{(1)}$ and continue this process, we see that there exists m_1 such that $E_1^{(m_1)} \in \mathcal{P}_b(\phi)$.

If $E_1^{(m_1)}$ is stable, then we are done. If not, we replace E by $E_1^{(m_1)}$ and repeat the whole process for $E_1^{(m_1)}$. Then there exists a subobject of $E_1^{(m_1)}$ which is in $\mathcal{P}_b(\phi)$, we denote this object by $E_1^{(m_2)}$.

By Lemma 5.10, The chain

$$\ldots \subset E_1^{(m_i)} \subset \cdots E_1^{(m_2)} \subset E_1^{(m_1)} \subset E$$

in $\mathcal{P}_b(\phi)$ must terminate. Then there exists n_1 such that $E_1^{(n_1)}$ is stable. Denote this object by A_1.

Step 2. We show that E has a JH filtration.

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Consider the short exact sequence in $\mathcal{B}^0_{H,-2}$:

$$0 \to A_1 \to E \to B_1 \to 0 \tag{5.11.3}$$

If $B_1 \in \ker(Z_H) \cap \mathcal{B}^0_{H,-2}$, then by further filtering B_1 into stable factors, we have a JH filtration. If not, we have $\phi_b(B_1) = \phi_b(E)$. Then a similar diagram as (5.11.2) shows that for any short exact sequence

$$0 \to B_1' \to B_1 \to B_1^{(1)} \to 0 \tag{5.11.4}$$

with $\phi_b(B_1') \geq \phi_b(B_1^{(1)})$, we have $\phi_b(B_1^{(1)}) = \phi$. We have either $B_1 \in \mathcal{P}_b(\phi)$, or there exists a short exact sequence in the form of (5.11.4) such that $B_1' \in \ker(Z_H) \cap \mathcal{B}^0_{H,-2}$ and $\phi_b(B_1') > \phi_b(B_1^{(1)})$. Similar to the case of subobject, by comparing the dimension of $\mathrm{Hom}(\mathcal{O}_X[1], _)$ and $\mathrm{Hom}(\mathcal{O}_\Theta(-1), _)$, we also have $B_1^{(n_1)} \in \mathcal{P}_b(\phi)$ for some n_1.

We abuse notation and denote the kernel of $B_1 \to B_1^{(n_1)}$ by B_1'. Then B_1' admits a filtration

$$0 = B_{1,0}' \subset B_{1,1}' \subset B_{1,2}' \subset \cdots \subset B_{1,l_1-1}' \subset B_{1,l_1}' = B_1'$$

such that each of the quotient factors is stable in $\ker(Z_H) \cap \mathcal{B}^0_{H,-2}$. Denote the cokernel of $B_{1,i}' \to B_1$ by $B_{1,i}^{(n_1)}$, the kernel of $E \to B_{1,i}^{(n_1)}$ by $A_{1,i}^{(n_1)}$, and the kernel of $E \to B_1^{(n_1)}$ by $A_1^{(n_1)}$. Combining the short exact sequence 5.11.3, we have a set of short exact sequences

$$0 \to A_1 \to A_{1,i}^{(n_1)} \to B_{1,i}' \to 0$$

Hence we may further refine the map $A_1 \to E$ to a filtration

$$A_1 \subset A_{1,1}^{(n_1)} \subset A_{1,2}^{(n_1)} \subset \cdots A_{1,l_1}^{(n_1)} = A_1^{(n_1)} \subset E$$

with the last quotient factor isomorphic to $B_1^{(n_1)}$, and each of the previous quotient factors in $\ker(Z_H) \cap \mathcal{B}^0_{H,-2}$ and stable.

If $B_1^{(n_1)}$ is stable, then we have a JH filtration. If not, we replace E by $B_1^{(n_1)}$. Then there exists a subobject of $B_1^{(n_1)}$ which is stable. We denote the cokernel by B_2. Then there exist

$$0 \to B_2' \to B_2 \to B_2^{(n_2)} \to 0$$

where $B_2' \in \ker(Z_H) \cap B^0_{H,-2}$, and $B_2^{(n_2)} \in \mathcal{P}_b(\phi)$. We can further refine the filtration such that B_2' are stable objects in $\ker(Z_H) \cap B^0_{H,-2}$. Continue this process and we denote the kernel of $E \to B_i^{(n_i)}$ by $A_i^{(n_i)}$.

Note that there exists α, β such that the interval (α, β) of length < 1, and ϕ, $\frac{1}{2}$ and 1 are all in (α, β). Then

$$A_1 \subset A_{1,1}^{(n_1)} \subset \cdots \subset A_1^{(n_1)} \subset A_2 \subset A_{2,1}^{(n_2)} \subset \cdots \subset A_2^{(n_2)} \subset A_3 \subset \cdots \subset E$$

is a chain in $\mathcal{P}_b((\alpha, \beta))$. By Lemma 5.10, this chain must terminate. This is a Jordan–Hölder filtration of E. ∎

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Remark 5.12 Note that Proposition 5.11 shows that any object $E \in \mathcal{P}_b(\phi)$ has a JH filtration whose JH factors are in $\mathcal{P}((0,1])[k]$ for some $k \in \mathbb{Z}$. It is also worth pointing out that the σ_b^L-stable objects may not be stable in a general heart $\mathcal{P}_b((a, a+1])$ for $a \notin \mathbb{Z}$.

5.13 Weak stability conditions on the V-axis

Taking $D_\omega \to 0$ in the central charge formula (5.1.1), we obtain a central charge of the form

$$Z_{V_\omega,H}(E) = -\mathrm{ch}_2(E) + V_\omega \mathrm{ch}_0(E) + iH\mathrm{ch}_1(E),$$

where $H = \Theta + ef$. Since ω only appears through the volume V_ω, we omit ω in the notation.

Proposition 5.14 *We have $Z_{V,H}(\mathcal{B}^0_{H,k}) \in \mathbb{H}_0$ for $k = -1, -2$. Furthermore, we have when $k = -1$,*

$$ker(Z_{V,H}) \cap \mathcal{B}^0_{H,-1} = \langle \mathcal{O}_\Theta(-1)[1] \rangle,$$

when $k = -2$,

$$ker(Z_{V,H}) \cap \mathcal{B}^0_{H,-2} = \langle \mathcal{O}_\Theta(-1) \rangle.$$

Proof For any object $E \in \mathcal{B}^0_{H,k}$, there is a short exact sequence

$$0 \to F \to E \to T \to 0,$$

where $F \in \mathcal{F}^0_{H,k}[1]$ and $T \in \mathcal{T}^0_{H,k}$. For an object $T \in \mathcal{T}^0_{H,k}$, T fit into a short exact sequence

$$0 \to T_1 \to T \to T_2 \to 0$$

where $T_1 \in \mathcal{F}^0_H[1]$ and $T_2 \in \mathcal{T}^0_H$.

Claim 1 $Z_{V,H}(T_1) \in \mathbb{H}$. ■

Since $T_1 \in \mathcal{F}^0_H[1]$, we have $T_1[-1]$ is a torsion free sheaf with $\mu_H(\mathrm{HN}_i(T_1[-1])) \le 0$ for all i. If J is a μ_H-semistable coherent sheaf with $\mu_H(J) < 0$, then $\Im Z_{V,H}(J[1]) > 0$. If J is a μ_H-stable torsion-free sheaf with $\mu_H(J) = 0$, then by Corollary 4.3, we have

$$\begin{aligned}\Re(Z_{V,H}(J[1])) &= \mathrm{ch}_2(J) - V\mathrm{ch}_0(J) \\ &\le \frac{\mathrm{ch}_1(J)^2}{2\mathrm{ch}_0(J)} - V\mathrm{ch}_0(J) \\ &< 0\end{aligned}$$

Hence $Z_{V,H}(T_1) \in \mathbb{H}$.

Claim 2 *$Z_{V,H}(T_2) \in \mathbb{H}_0$. If J is a μ_H-semistable coherent sheaf with $\mu_H(J) > 0$, then $Z_{V,H}(J) \in \mathbb{H}$. It is enough to consider if T_2 is torsion and $\mu_H(T_2) = 0$. It is enough to assume that T_2 is a pure sheaf supported on dimension* 1. *Then by Lemma 5.2, we have $\mathrm{ch}_1(T_2) = n\Theta$, hence $T \in \langle \mathcal{O}_\Theta(m) \rangle$. Since $(\mathcal{T}_{H,k}, \mathcal{F}_{H,k})$ defines a torsion pair on $\mathcal{A}^0_H$ and T_2 is an $\mathcal{A}^0_H$-quotient of T which lies in $\mathcal{T}_{H,k}$, we have $T_2 \in \mathcal{T}_{H,k}$ as well and hence $T \in \langle \mathcal{O}_\Theta(m) | m > k \rangle$.*

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Since

$$\mathrm{ch}_2(\mathcal{O}_\Theta(m)) = m + 1,$$

we have for $m > k$,

$$Z(\mathcal{O}_\Theta(m)) = -\mathrm{ch}_2(\mathcal{O}_\Theta(m)) < -k - 1 \in \mathbb{R}_{\leq 0}.$$

This implies that $Z_{V,H}(T_2) \in \mathbb{H}_0$.

We are left to consider F. For $m \leq k$

$$Z(\mathcal{O}_\Theta(m)[1]) = \mathrm{ch}_2(\mathcal{O}_\Theta(m)) = m + 1 \leq k + 1 \in \mathbb{R}_{\leq 0}.$$

Hence $Z_{V,H}(F) \in \mathbb{H}_0$.

From the computation, we also have for $k = -1$

$$\ker(Z_{V,H}) \cap \mathcal{B}^0_{H,k} = \langle \mathcal{O}_\Theta(-1)[1] \rangle,$$

and for $k = -2$,

$$\ker(Z_{V,H}) \cap \mathcal{B}^0_{H,k} = \langle \mathcal{O}_\Theta(-1) \rangle.$$

We define the phases $\{\phi_{V,H}(K)\}_{K \in \ker(Z_{V,H}) \cap \mathcal{B}^0_{H,k}}$ by taking the limit of the phases of $\sigma_{V_\omega, D_\omega}(K)$ as $D_\omega \to 0$.

Since

$$Z_{V_\omega, D_\omega}(\mathcal{O}_\Theta(-1)) = iD_\omega,$$

we have

$$\lim_{D_\omega \to 0} \phi_{V_\omega, D_\omega}(\mathcal{O}_\Theta(-1)) = \frac{1}{2}.$$

Hence for $V \in \mathbb{R}_{>0}$, we define $\phi_{V,H}(\mathcal{O}_\Theta(-1)) = \frac{1}{2}$, which implies $\phi_{V,H}(\mathcal{O}_\Theta(-1)[1]) = \frac{3}{2}$.

We see that it is only when $k = -2$ that we have a chance of having $0 < \phi_{V,H}(K) \leq 1$ for all $K \in \ker(Z_{V,H}) \cap \mathcal{B}^0_{H,k}$. Since $\ker(Z_{V,H}) \cap \mathcal{B}^0_{H,-2} = \langle \mathcal{O}_\Theta(-1) \rangle$, for any $K \in \ker(Z_{V,H}) \cap \mathcal{B}^0_{H,-2}$, we have

$$\lim_{D_\omega \to 0} \phi_{V_\omega, D_\omega}(K) = \frac{1}{2}.$$

As a result, for $V \in \mathbb{R}_{>0}$, $K \in \ker(Z_{V,H}) \cap \mathcal{B}^0_{H,-2}$ we define $\phi_{V,H}(K) = \frac{1}{2}$. Then the weak see-saw property follows automatically. To summarize, the triple

$$\sigma_{V,H} = (Z_{V,H}, \mathcal{B}^0_{H,-2}, \{\phi_{V,H}(K)\}_{K \in \ker(Z_{V,H}) \cap \mathcal{B}^0_{H,-2}})$$

defines a weak stability function.

The HN property of $\sigma_{V,H}$ can be shown exactly the same way as Proposition 5.8, and so $\sigma_{V,H}$ is a weak stability condition in the sense of Definition 3.2. We denote the slicing of $\sigma_{V,H}$ by $\mathcal{P}_{V,H}$.

Remark 5.15 From the computation in Lemma 5.5, we know that $\mathrm{Ext}^1(\mathcal{O}_\Theta(-1), \mathcal{O}_\Theta(-1)) = 0$. Hence following the same proof of Propostiion 5.11,

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we have any object $E \in \mathcal{P}_{V,H}(\phi)$ has a JH filtration whose stable factors are either $\sigma_{V,H}$-stable objects in $\mathcal{P}_{V,H}(\phi)$ or stable objects in $\ker(Z_{V,H}) \cap \mathcal{B}^0_{H,-2}$.

5.16 Weak stability conditions on the *D*-axis

Taking $V_\omega \to 0$ in the central charge formula (5.1.1), we obtain a central charge of the form

$$Z_{D_\omega}(E) = -\mathrm{ch}_2(E) + i\omega \mathrm{ch}_1(E)$$

where $\omega = \Theta + (D_\omega + e)f$. If ω is clear from the context, we omit ω in the notation.

Proposition 5.17 *Suppose ω is ample. We have $Z_D(\mathrm{Coh}^{\omega,0}) \in \mathbb{H}_0$. Furthermore, $ker(Z_D) \cap \mathrm{Coh}^{\omega,0} = \langle \mathcal{O}_X[1] \rangle$.*

Proof We only need to show $Z_D(E[1]) \in \mathbb{H}_0$ for E a μ_ω-semistable torsion-free sheaf with $\mu_\omega(E) = 0$. In this case,

$$Z_D(E[1]) = \mathrm{ch}_2(E) \leq \frac{\mathrm{ch}_1(E)^2}{2\mathrm{ch}_0(E)} \in \mathbb{R}_{\leq 0}.$$

Hence $Z_D(E[1]) \in \mathbb{H}_0$.

Furthermore, if $E \in \ker(Z_D) \cap \mathrm{Coh}^{\omega,0}$, then we have $E = E'[1]$ for $E' \in \mathrm{Coh}(X)$. By the generalized Bogomolov–Gieseker inequality on K3 surfaces, (e.g., see [1, Section 6] or (4.1.1)), if F is a μ_ω-stable object, then

$$\mathrm{ch}_2(F) \leq \frac{\mathrm{ch}_1(F)^2}{2\mathrm{ch}_0(F)} - \mathrm{ch}_0(F) + \frac{1}{\mathrm{ch}_0(F)}. \tag{5.17.1}$$

If furthermore $F[1] \in \ker(Z_D) \cap \mathrm{Coh}^{\omega,0}$, then $\mathrm{ch}_0(F) = 1$. This implies that F is an ideal sheaf. Since $\mathrm{ch}_1(F) \cdot \omega = 0$ and $\mathrm{ch}_2(F) = 0$, we have $F \simeq \mathcal{O}_X$. Hence $E' \in \langle \mathcal{O}_X \rangle$. ∎

We define the phases $\{\phi_D(K)\}_{K \in \ker(Z_D) \cap \mathrm{Coh}^{\omega,0}}$ by taking the limit of the phases of $Z_{V_\omega, D_\omega}(K)$ as $V_\omega \to 0$. We have

$$Z_{V_\omega, D_\omega}(\mathcal{O}_X[1]) = -V_\omega.$$

Hence

$$\lim_{V_\omega \to 0} \phi_{V_\omega, D_\omega}(\mathcal{O}_X[1]) = 1.$$

For $D \in \mathbb{R}_{>0}$, we define $\phi_D(K) = 1$ for any $K \in \ker(Z_D) \cap \mathrm{Coh}^{\omega,0} = \langle \mathcal{O}_X[1] \rangle$.

Proposition 5.18 *Given $D \in \mathbb{Q}_{>0}$, the triple*

$$\sigma_D = (Z_D, \mathrm{Coh}^{\omega,0}, \{\phi_D(K)\}_{K \in ker(Z_D) \cap \mathrm{Coh}^{\omega,0}})$$

satisfies the HN property, hence defines a weak stability condition.

Proof The argument follows the proof of Lemma 2.18 in [18].

Since ω is ample, it is well known that the abelian category $\mathrm{Coh}^{\omega,0}$ is noetherian (e.g., see [16, Lemma 6.17]. As a result, we only need to check condition (i) in

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Proposition 3.6. Assume we have such a chain of subobjects in $\mathrm{Coh}^{\omega,0}$ as in condition (i). Consider the short exact sequence in $\mathrm{Coh}^{\omega,0}$:

$$0 \to E_{i+1} \to E_i \to F_i \to 0.$$

Since $\Im(Z_D(F_i)) \geq 0$, we have $\Im(Z_D(E_i)) \geq \Im(Z_D(E_{i+1}))$. The assumption $D_\omega \in \mathbb{Q}_{>0}$ implies that $\Im(Z_D)$ is discrete. Hence $\Im(Z_D(E_i))$ is constant for $i \gg 0$. Then $\Im(Z_D(F_i)) = 0$ for $i > n$ for some n.

Let $i > n$. If $F_i \notin \ker(Z_D) \cap \mathrm{Coh}^{\omega,0}$, then $\phi_D(F_i) = 1$, contradicting $\phi_D(E_{i+1}) > \phi_D(F_i)$.

So we must have $F_i \in \ker(Z_D) \cap \mathrm{Coh}^{\omega,0} = \langle \mathcal{O}_X[1] \rangle$ for $i > n$. Since $\mathrm{Ext}^1(\mathcal{O}_X[1], \mathcal{O}_X[1]) = 0$, the dimension of $\mathrm{Hom}(E_i, \mathcal{O}_X[1])$ decreases as i increases. This implies that $F_i = 0$ for i large enough. This proves the HN property for σ_D when $D \in \mathbb{Q}_{>0}$. ∎

5.19 Weak stability condition after the relative Fourier-Mukai transform

Recall that on a smooth projective surface X, we usually denote the central charge of a standard Bridgeland stability condition as

$$\begin{aligned} Z_{\omega,B} &= -\int_X e^{-(B+i\omega)}\mathrm{ch}(E) = -\int_X e^{-i\omega}\mathrm{ch}^B(E) \\ &= -\mathrm{ch}_2^B(E) + V_\omega \mathrm{ch}_0^B(E) + iR_\omega(\Theta + (D_\omega + e)f)\mathrm{ch}_1^B(E) \end{aligned} \tag{5.19.1}$$

for $\mathbb{R}$-divisors ω and B. To construct a heart that pairs with $Z_{\omega,B}$ to form a stability condition on $D^b(X)$, we usually consider the torsion pair $(\mathcal{T}_{\omega,B}, \mathcal{F}_{\omega,B})$ where

$$\begin{aligned} \mathcal{T}_{\omega,B} &= \langle E \in \mathrm{Coh}(X) : \mu_{\omega,B,\min}(E) > 0 \rangle \\ \mathcal{F}_{\omega,B} &= \langle E \in \mathrm{Coh}(X) : \mu_{\omega,B,\max}(E) \leq 0 \rangle. \end{aligned}$$

The heart $\mathcal{A}_{\omega,B} = \langle \mathcal{F}_{\omega,B}[1], \mathcal{T}_{\omega,B} \rangle$ then pairs with $Z_{\omega,B}$ to form a Bridgeland stability condition on $D^b(X)$.

Now let us return to the case of $p : X \to B$ being a Weierstraß elliptic surface.Denoting the projections from $X \times_B X$ to each factors by π_i, and the projection to B by ρ. The relative Fourier-Mukai transform is an autoequivalence

$$\Phi : D^b(X) \to D^b(X),$$

which is defined by $\Phi = \mathbf{R}\pi_{2,*}(\pi_1^*(_) \otimes \mathcal{P})$, where $\mathcal{P}$ is the relative Poincaré sheaf. Seting $v = \mathbf{R}^1 p_* \mathcal{O}_X$, then

$$\mathcal{P} := \mathcal{I}_\Delta \otimes \pi_1^* \mathcal{O}(\Theta) \otimes \pi_2^* \mathcal{O}(\Theta) \otimes \rho^* v^{-1}.$$

The definition of $\Phi, \mathcal{P}$ and their properties can be found in [2, Section 6.2.3].

Given ω and B, when attempting to solve the central charge equation

$$Z_{\omega',B'}(\Phi(E)) = TZ_{\omega,B}(E) \quad \text{for all } E \in D^b(X)$$

for some $T \in \mathrm{GL}^+(2,\mathbb{R})$ and ω', B', it is easy to see from the solution in [14, 8.5], that when ω^2 is small, the above equation admits a solution where $V_{\omega'}$, i.e., the coefficient of ch_0 in $Z_{\omega',B'}$, is forced to be negative. This prompts us to consider the central charge

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with Todd class on a Weierstraß elliptic K3 surface X:

$$\begin{aligned} Z^{td}_{\omega',B'}(E) &= (e^{\omega'+B'}, v(E)) = -\int e^{-i\omega'}\mathrm{ch}^{B'}(E)\sqrt{\mathrm{td}(X)} \\ &= -\mathrm{ch}_2^{B'}(E) + (\tfrac{(\omega')^2}{2} - 1)\mathrm{ch}_0^{B'}(E) + i\omega'\mathrm{ch}_1^{B}(E) \\ &= -\mathrm{ch}_2^{B'}(E) + (V_{\omega'} - 1)\mathrm{ch}_0^{B'}(E) + i\omega'\mathrm{ch}_1^{B'}(E). \end{aligned} \tag{5.19.2}$$

In Appendix A, we show that when we impose the relations

$$\begin{aligned} D_{\omega'} &= \frac{1}{2}(R_\omega^2 + R_B^2)(2D_\omega + e) \\ &= V_\omega + \frac{1}{2}R_B^2(2D_\omega + e). \\ R_{B'} &= -\frac{R_B(2D_\omega + e)}{2D_{\omega'} + e}. \\ V_{\omega'} &= D_\omega - \frac{1}{2}\frac{R_B^2(2D_\omega + e)^2}{2D_{\omega'} + e} + 1 \\ R_{B'}D_{B'} &= -R_{B'} + R_B D_B + R_B - 1 \end{aligned} \tag{5.19.3}$$

there is an appropriate $T \in \mathrm{GL}^+(2,\mathbb{R})$ such that the slightly different central charge equation

$$Z^{td}_{\omega',B'}(\Phi(E)) = TZ_{\omega,B}(E) \quad \text{for all } E \in D^b(X) \tag{5.19.4}$$

holds.

5.20

We are interested in solving the equation (5.19.4) on an elliptic K3 surface for the case $B = -\alpha$ where $\alpha := c_1(L) = \Theta + (D_\alpha + e)f$. In this case, we have $R_B = -R_\alpha = -1$ and $D_B = D_\alpha$, and (5.19.3) simplifies to

$$\begin{aligned} D_{\omega'} &= V_\omega + D_\omega + 1 \\ R_{B'} &= \frac{D_\omega + 1}{D_\omega + V_\omega + 2} \\ V_{\omega'} &= \frac{D_\omega V_\omega - 1}{D_\omega + V_\omega + 2} + 1 \\ R_{B'}D_{B'} &= -\frac{D_\omega + 1}{V_\omega + D_\omega + 2} - (D_\alpha + 2). \end{aligned} \tag{5.20.1}$$

Then equations in (5.20.1) define a map from $\mathbb{R}^4$ to $\mathbb{R}^4$. We denote this map by Φ_Z. In formula (5.19.1) for the central charge $Z_{\omega,B}$, if we choose the parameters

$$D_\omega = 0, \quad V_\omega = 0, \quad R_\omega = 1, \quad B = -\alpha$$

then $Z_{\omega,B}$ reduces to $Z_H((-)\otimes \mathcal{O}_X(-B))$ where Z_H is as defined in 5.1. With these choices, the relations (5.20.1) give

$$D_{\omega'} = 1, \quad R_{B'} = \tfrac{1}{2}, \quad V_{\omega'} = \tfrac{1}{2}, \quad R_{B'}D_{B'} = -D_\alpha - \tfrac{5}{2}$$

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which in turn give

$$\omega' = R_{\omega'}(\Theta + (D_{\omega'} + e)f) = \tfrac{1}{2}(\Theta + 3f) \text{ where } R_{\omega'} = \sqrt{\frac{V_{\omega'}}{D_{\omega'} + \frac{e}{2}}}$$

$$B' = \tfrac{1}{2}(\Theta + (-2D_\alpha - 3)f).$$

In summary, with the RDV coordinates for ω, B, ω', B' chosen as above, the equation

$$Z^{td}_{\omega',B'}(\Phi(E)) = TZ_H(E \otimes \mathcal{O}_X(-B)) \quad \text{for all } E \in D^b(X) \tag{5.20.2}$$

holds for the appropriate T.

In what follows, we will write ω'_0, B'_0 to denote the specific ω', B' above, i.e.,

$$\begin{aligned} \omega'_0 &= \tfrac{1}{2}(\Theta + 3f) \\ B'_0 &= \tfrac{1}{2}(\Theta + (-2D_\alpha - 3)f). \end{aligned} \tag{5.20.3}$$

We will informally think of the left-hand side of (5.20.2) as "$\Phi_Z(Z_H)$" and denote it by $Z'_0 := Z^{td}_{\omega'_0,B'_0}$. Also note that ω'_0 is ample.

Proposition 5.21 *Using the notations above, we have* $Z'_0(\mathrm{Coh}^{\omega'_0,B'_0}) \in \mathbb{H}_0$.

Proof Since ω'_0 is ample, it is easy to see that if $E \in \mathrm{Coh}(X)$ is a torsion sheaf, or is a slope semistable torsion-free sheaf with $\mu_{\omega'_0,B'_0}(E) > 0$, then $Z'_0(E) \in \mathbb{H}$. Also if $E \in \mathrm{Coh}(X)$ is slope semistable with $\mu_{\omega'_0,B'_0}(E) < 0$, then $Z'_0(E[1]) \in \mathbb{H}$. It remains to check that when $E \in \mathrm{Coh}(X)$ is a slope stable torsion-free sheaf with $\mu_{\omega'_0,B'_0}(E) = 0$, we have $Z'_0(E[1]) \in \mathbb{H}_0$, i.e., $\Re(Z'_0(E[1])) \le 0$. We have

$$\begin{aligned} \Re(Z'_0(E[1])) &= \mathrm{ch}_2^{B'_0}(E) - (V_{\omega'_0} - 1)\mathrm{ch}_0^{B'_0}(E) \\ &= \mathrm{ch}_2^{B'_0}(E) + \frac{1}{2}\mathrm{ch}_0^{B'_0}(E). \end{aligned}$$

By the generalized BG inequality on K3 surfaces in Proposition 4.1, we have

$$\Re(Z'_0(E[1])) \le \frac{(\mathrm{ch}_1^{B'_0}(E))^2}{2\mathrm{ch}_0(E)} - \frac{1}{2}\mathrm{ch}_0(E) + \frac{1}{\mathrm{ch}_0(E)}.$$

By Hodge index theorem, we have $(\mathrm{ch}_1^{B'_0}(E))^2 \le 0$, hence $\Re(Z'_0(E[1])) < 0$ when $\mathrm{ch}_0(E) \ge 2$. We only need to consider the case when $\mathrm{ch}_0(E) = 1$. In this case we have

$$\Re(Z'_0(E[1])) \le \frac{1}{2}(\mathrm{ch}_1^{B'_0}(E))^2 + \frac{1}{2}.$$

We have

$$\begin{aligned} (\mathrm{ch}_1^{B'_0}(E))^2 &= \mathrm{ch}_1(E)^2 - 2B'_0\mathrm{ch}_1(E) + (B'_0)^2 \\ &= \mathrm{ch}_1(E)^2 - (\Theta + (-2D_\alpha - 3)f)\mathrm{ch}_1(E) + (-D_\alpha - 2). \end{aligned}$$

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Hence $(\mathrm{ch}_1^{B_0'}(E))^2$ is an integer. If $(\mathrm{ch}_1^{B_0'}(E))^2 = 0$, Hodge index theorem implies that $\mathrm{ch}_1^{B_0'}(E)$ is numerically trivial. This implies that

$$\mathrm{ch}_1(E) = B_0' = \frac{1}{2}(\Theta + (-2D_\alpha - 3)f),$$

contradicting $E \in \mathrm{Coh}(X)$. Hence $(\mathrm{ch}_1^{B_0'}(E))^2 \leq -1$, which implies that $\Re(Z_0'(E[1])) \leq 0$. ∎

We analyze the objects in the kernel of the central charge. Let $K \in \ker(Z_0') \cap \mathrm{Coh}^{\omega_0', B_0'}$, then $K = E[1]$ for some sheaf $E \in \mathcal{F}_{\omega', B'}$.

Proposition 5.22 *Let E be a $\mu_{\omega', B'}$-stable sheaf with $E[1] \in \ker(Z_0') \cap \mathrm{Coh}^{\omega_0', B_0'}$. Then either $E \simeq \mathcal{O}(-(D_\alpha + 1)f)$ or $E \simeq \mathcal{O}(\Theta - (D_\alpha + 2)f)$.*

Proof From the proof of Proposition 5.21, we know that $\mathrm{ch}_0(E) = 1$, and that equality holds for E in the generalized BG inequality, giving us

$$\Re(Z_0'(E[1])) = \tfrac{1}{2}(\mathrm{ch}_1^{B_0'}(E))^2 + \tfrac{1}{2},$$

hence $\mathrm{ch}_2(E) = \frac{1}{2}\mathrm{ch}_1(E)^2$. Consider the short exact sequence

$$0 \to E \to E^{**} \to Q \to 0.$$

Since E is $\mu_{\omega_0', B_0'}$-stable of rank 1, we have Q is supported on dimension 0. Then $\mathrm{ch}_2(E) = \frac{1}{2}\mathrm{ch}_1(E)^2$ implies that $E \simeq E^{**}$, and hence E is a line bundle.

Since $\Im(Z_0'(E)) = 0$ and $\Re(Z_0'(E)) = 0$, we have

$$\begin{aligned} &\mathrm{ch}_1(E) \cdot (\Theta + 3f) - \frac{1}{2}(1 + (-2D_\alpha - 5) + 2) = 0 \\ &\frac{1}{2}\mathrm{ch}_1(E)^2 - \mathrm{ch}_1(E) \cdot \frac{1}{2}(\Theta + (-2D_\alpha - 3)f) + \frac{1}{4}(-2D_\alpha - 4) + \frac{1}{2} = 0. \end{aligned} \tag{5.22.1}$$

Denote $\mathrm{ch}_1(E) \cdot \Theta$ by x, and $\mathrm{ch}_1(E) \cdot f$ by y. Then equation (5.22.1) becomes

$$\begin{aligned} x + 3y &= -D_\alpha - 1 \\ \mathrm{ch}_1(E)^2 &= -(2D_\alpha + 6)y. \end{aligned} \tag{5.22.2}$$

For any $a, b \in \mathbb{R}$ such that $a + b = D_\alpha + 1$, we have $(\mathrm{ch}_1(E) + (a\Theta + bf)) \cdot \omega_0' = 0$. Then the Hodge index theorem implies that

$$(\mathrm{ch}_1(E) + (a\Theta + bf))^2 \leq 0. \tag{5.22.3}$$

Substituting $b = D_\alpha - a + 1$, equation (5.22.3) implies that

$$(2a + 1)y \geq -a^2$$

for all $a \in \mathbb{R}$. Taking $a = 0$, we have $y \geq 0$. Taking $a = -1$, we have $y \leq 1$. Since $y \in \mathbb{Z}$, we have $y = 0$ or $y = 1$. If $y = 0$, then HIT taking equality forces $\mathrm{ch}_1(E)$ to be numerically equivalent to $-(D_\alpha + 1)f$. Similarly, if $y = 1$, then $\mathrm{ch}_1(E)$ is numerically equivalent to $\Theta - (D_\alpha + 2)f$. ∎

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5.23

Define

$$L_0 := \mathcal{O}(-(D_\alpha + 1)f),$$

and

$$L_1 := \mathcal{O}(\Theta - (D_\alpha + 2)f).$$

Let E, y be the same as in Proposition 5.22. We denote the phase of E in $\sigma' = (Z^{td}_{\omega',B'}, \mathrm{Coh}^{\omega',B'})$ by $\phi_{\sigma'}(E)$.

Precomposing with the map Φ_Z, we have

$$\begin{aligned} \Im(Z_{\omega',B'}(E)) &= \frac{\sqrt{(D_\omega+1)(V_\omega+1)}}{D_\omega + V_\omega + 2}(y(D_\omega + V_\omega) - D_\omega). \\ \Re(Z_{\omega',B'}(E)) &= \frac{V_\omega - D_\omega}{D_\omega + V_\omega + 2}y + \frac{D_\omega V_\omega + D_\omega}{D_\omega + V_\omega + 2}. \end{aligned} \tag{5.23.1}$$

For the rest of the section, we always consider σ' as a function of V_ω and D_ω.

If $E \simeq L_0$, i.e., $y = 0$, we have

$$\begin{aligned} \Im(Z_{\omega',B'}(L_0)) &= -D_\omega \frac{\sqrt{(D_\omega+1)(V_\omega+1)}}{D_\omega + V_\omega + 2} \\ \Re(Z_{\omega',B'}(L_0)) &= \frac{D_\omega V_\omega + D_\omega}{D_\omega + V_\omega + 2}. \end{aligned} \tag{5.23.2}$$

Since $D_\omega > 0$, $V_\omega > 0$, we have $\Im(Z_{\omega',B'}(L_0[1])) > 0$, and since

$$-\frac{\Re(Z_{\omega',B'}(L_0[1]))}{\Im(Z_{\omega',B'}(L_0[1]))} = \frac{V_\omega + 1}{\sqrt{(D_\omega+1)(V_\omega+1)}}$$

it is easy to see that

$$\lim_{D_\omega, V_\omega \to 0^+} \phi_{\sigma'}(L_0[1]) = \frac{3}{4}.$$

If $E \simeq L_1$, i.e., $y = 1$, we have

$$\begin{aligned} \Im(Z_{\omega',B'}(L_1)) &= V_\omega \frac{\sqrt{(D_\omega+1)(V_\omega+1)}}{D_\omega + V_\omega + 2} \\ \Re(Z_{\omega',B'}(L_1)) &= \frac{D_\omega V_\omega + V_\omega}{D_\omega + V_\omega + 2}. \end{aligned} \tag{5.23.3}$$

In this case we have $\Im(Z_{\omega',B'}(L_1)) > 0$, and

$$-\frac{\Re(Z_{\omega',B'}(L_1))}{\Im(Z_{\omega',B'}(L_1))} = -\frac{D_\omega + 1}{\sqrt{(D_\omega+1)(V_\omega+1)}}$$

and so

$$\lim_{D_\omega, V_\omega \to 0^+} \phi_{\sigma'}(L_1) = \frac{1}{4}.$$

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From the computation, we see that if $K \in \ker(Z_0') \cap \mathrm{Coh}^{\omega',B'}$ is taken to be $L_1[1]$, then

$$\lim_{D_\omega, V_\omega \to 0^+} \phi_{\sigma'}(K) = \frac{5}{4}.$$

If we define

$$\phi_{\sigma_0'}(K) = \lim_{D_\omega, V_\omega \to 0^+} \phi_{\sigma'}(K), \tag{5.23.4}$$

the triple

$$(Z_0', \mathrm{Coh}^{\omega_0', B_0'}, \{\phi_{\sigma_0'}(K)\}|_{K \in Ker(Z_0') \cap \mathrm{Coh}^{\omega_0', B_0'}})$$

does not satisfy condition (i) in Definition 3.2. To define a weak stability condition, we construct a heart $\mathcal{B}$ such that $L_1 \in \mathcal{B}$ and $Z_0'(\mathcal{B}) \in \mathbb{H}_0$. To simplify notation, for the rest of this section, we denote $\mathrm{Coh}^{\omega_0', B_0'}$ by $\mathcal{A}$. Recall that $\mathcal{A}$ is constructed by tilting $\mathrm{Coh}(X)$ at a torsion pair $(\mathcal{T}_{\omega_0', B_0'}, \mathcal{F}_{\omega_0', B_0'})$. For the rest of this section, we denote this torsion pair by $(\mathcal{T}, \mathcal{F})$. We construct $\mathcal{B}$ by tilting $\mathcal{A}$ at a torsion pair.

It is noted in [18] that the full subcategory $\mathcal{A}_{\ker(Z_0')} := \ker(Z_0') \cap \mathcal{A}$ is an abelian subcategory of $\mathcal{A}$.

Lemma 5.24 *Any object in $\mathcal{A}_{\ker(Z_0')}$ is a direct sum of copies of $L_0[1]$ and $L_1[1]$.*

Proof Let $K \in \mathcal{A}_{\ker(Z_0')}$, we know that $K \simeq E[1]$ for some coherent sheaf $E \in \mathcal{F}_{\omega', B'}$. By Proposition 5.22, we have E has a filtration by copies of L_0 and L_1.

We have $\mathrm{Hom}(L_1, L_0) = H^0(O(-\Theta + f)) = 0$. Also, Serre duality implies that

$$\mathrm{Hom}(L_0, L_1) \cong H^2(O(-\Theta + f)) \simeq H^0(O(\Theta - f))^* = 0.$$

By Riemann–Roch, we have

$$\mathrm{Ext}^1(L_1, L_0) = H^1(O(-\Theta + f)) = 0$$

and

$$\mathrm{Ext}^1(L_0, L_1) = H^1(O(\Theta - f)) = 0.$$

■

Define the subcategory $\mathcal{T}_{\mathcal{A}} \subset \mathcal{A}$ by

$$\mathcal{T}_{\mathcal{A}} = \langle L_1[1] \rangle.$$

Define its right orthogonal in $\mathcal{A}$ by

$$\mathcal{F}_{\mathcal{A}} = \{F \in \mathcal{A} | \mathrm{Hom}(L_1[1], F) = 0\}.$$

Proposition 5.25 *The pair $(\mathcal{T}_{\mathcal{A}}, \mathcal{F}_{\mathcal{A}})$ defines a torsion pair on $\mathcal{A}$.*

Proof The argument is similar to the proof of Lemma 3.2 in [19].

We need to show that for any $E \in \mathcal{A}$, there exists an exact triangle

$$0 \to T \to E \to F \to 0$$

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with $T \in \mathcal{T}_{\mathcal{A}}$ and $F \in \mathcal{F}_{\mathcal{A}}$. Assume that $E \notin \mathcal{F}_{\mathcal{A}}$, then there exists $T \in \mathcal{T}_{\mathcal{A}}$ with a nonzero map to E. Completing the triangle, we have

$$T \to E \to F \to T[1].$$

Since $H^i(T) = 0$ for all $i \neq -1$, by long exact sequence of cohomology, we have

$$0 \to H^{-2}(F) \to H^{-1}(T) \to H^{-1}(E) \to H^{-1}(F) \to 0.$$

Consider the map $H^{-1}(T)/H^{-2}(F)[1] \to E$ via the composition

$$H^{-1}(T)/H^{-2}(F)[1] \to H^{-1}(E)[1] \to E.$$

Completing the triangle, we obtain the following exact triangle

$$H^{-1}(T)/H^{-2}(F)[1] \to E \to F_1 \to H^{-1}(T)/H^{-2}(F)[2]. \tag{5.25.1}$$

Claim 1 *$H^{-1}(T)/H^{-2}(F)[1] \in \mathcal{T}_{\mathcal{A}}$. We have the following short exact sequence of sheaves:*

$$0 \to H^{-2}(F) \to H^{-1}(T) \to H^{-1}(T)/H^{-2}(F) \to 0. \tag{5.25.2}$$

Since $H^{-1}(T) \in \mathcal{F}$, we have $H^{-2}(F) \in \mathcal{F}$. Also since $H^{-1}(T)/H^{-2}(F)$ is a subsheaf of $H^{-1}(E)$, we have $H^{-1}(T)/H^{-2}(F) \in \mathcal{F}$. Then the short exact sequence (5.25.2) is a short exact sequence in $\mathcal{A}[-1]$. Then $H^{-1}(T)[1] \in \mathcal{A}_{\ker(Z'_0)}$ implies that $H^{-2}(F)[1] \in \mathcal{A}_{\ker(Z'_0)}$ and $H^{-1}(T)/H^{-2}(F)[1] \in \mathcal{A}_{\ker(Z'_0)}$. Hence we have equation (5.25.2) is a short exact sequence in $\mathcal{A}_{\ker(Z'_0)}[-1]$. From the proof of Lemma 5.24, $H^{-1}(T)[1] \in \mathcal{T}_{\mathcal{A}}$ implies that $H^{-1}(T)/H^{-2}(F)[1] \in \mathcal{T}_{\mathcal{A}}$. ■

Claim 2 *$F_1 \in \mathcal{A}$. Indeed by long exact sequence, we have $H^0(F_1) \simeq H^0(E) \in \mathcal{T}$, and*

$$0 \to H^{-1}(T)/H^{-2}(F) \to H^{-1}(E) \to H^{-1}(F_1) \to 0.$$

Let G be any subsheaf of $H^{-1}(F_1)$, and let R be the cokernel. Let K be the kernel of the composition $H^{-1}(E) \to H^{-1}(F_1) \to R$. Then we have a short exact sequence of coherent sheaves:

$$0 \to H^{-1}(T)/H^{-2}(F) \to K \to G \to 0.$$

$H^{-1}(T)/H^{-2}(F)[1] \in \mathcal{A}_{\ker(Z'_0)}$ implies that $\mu_{\omega',B'}(K) = \mu_{\omega',B'}(G)$. Since K is a subsheaf of $H^{-1}(E)$, $\mu_{\omega',B'}(K) \leq 0$. Hence we have $\mu_{\omega_1,B_1}(G) \leq 0$, which implies that $H^{-1}(F_1) \in \mathcal{F}$.

Hence the exact triangle (5.25.1) is a short exact sequence in $\mathcal{A}$, and $H^{-1}(T)/H^{-2}(F)[1] \in \mathcal{T}_{\mathcal{A}}$. If $F_1 \in \mathcal{F}_{\mathcal{A}}$, then we are done. If not, we continue the process for F_1. Thus we obtain a sequence of quotients

$$F_1 \twoheadrightarrow F_2 \twoheadrightarrow F_3 \twoheadrightarrow \dots \tag{5.25.3}$$

in $\mathcal{A}$, where the kernels K_i's lie in $\mathcal{T}_{\mathcal{A}}$. Applying $\mathrm{Hom}(L_1[1], -)$ to the short exact sequences

$$0 \to K_i \to F_i \to F_{i+1} \to 0,$$

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we have

$$0 \to \mathrm{Hom}(L_1[1], K_i) \to \mathrm{Hom}(L_1[1], F_i) \to \mathrm{Hom}(L_1[1], F_{i+1}) \to \mathrm{Ext}^1(L_1[1], K_i) = 0.$$

Hence the dimension of $\mathrm{Hom}(L_1[1], F_i)$ decreases as i increases. So we have the sequence (5.25.3) must stabilize after a finite number of steps.

Definition 5.26 We define an abelian category $\mathcal{B}$ by tilting $\mathcal{A}[-1]$ at the torion pair $(\mathcal{T}_{\mathcal{A}}[-1], \mathcal{F}_{\mathcal{A}}[-1])$. Equivalently, $\mathcal{B} = \langle \mathcal{F}_{\mathcal{A}}, \mathcal{T}_{\mathcal{A}}[-1] \rangle$.

5.26.1

Let us set $\mathcal{B}_{\ker(Z'_0)} = \{E \in \mathcal{B} : Z'_0(E) = 0\}$. For any $E \in \mathcal{B}_{\ker(Z'_0)}$, we can fit it in a short exact sequence $0 \to E' \to E \to E'' \to 0$ in $\mathcal{B}$ where $E' \in \mathcal{F}_{\mathcal{A}}$ and $E'' \in \mathcal{T}_{\mathcal{A}}[-1]$. Since $Z'_0(E'') = 0$, it follows that $Z'_0(E') = 0$ and hence $E' \in \mathcal{A}_{\ker(Z'_0)}$. Moreover, since $\mathrm{Hom}(\mathcal{T}_{\mathcal{A}}, E') = 0$, it follows from Lemma 5.24 that E' is a direct sum of copies of $L_0[1]$ while E'' is a direct sum of copies of L_1. Now, note that $\mathrm{Ext}^1(L_1, L_0[1]) \cong \mathrm{Hom}(L_0, L_1) \cong H^0(X, \mathcal{O}_X(\Theta - f))$, which vanishes because $\mathcal{O}_X(\Theta - f)$ is a line bundle of negative degree with respect to an ample divisor of the form $\Theta + (2+\epsilon)f$ for a small $\epsilon > 0$. Overall, we see that E itself is isomorphic to a direct sum of copies of $L_0[1]$ and L_1.

We define $\mathcal{B}_{\ker(Z'_0)} := \mathcal{B} \cap \ker(Z'_0)$. Consider the $\mathbb{P}^1$ whose coordinate is given by $[D_\omega : V_\omega]$. Let $a \in \mathbb{P}^1$. For any $K \in \mathcal{B}_{\ker(Z'_0)}$, we define

$$\phi_a^R(K) = \lim_{V_\omega, D_\omega \to 0, [V_\omega : D_\omega] = a} \phi_{\sigma'}(K),$$

where $\sigma' = (Z^{td}_{\omega', B'}, \mathrm{Coh}^{\omega', B'})$, and $Z^{td}_{\omega', B'}$ is considered as a function of V_ω, D_ω via precomposing with Φ_Z.

Consider the triple

$$\sigma_a^R = (Z'_0, \mathcal{B}, \{\phi_a^R(K)\}_{K \in \mathcal{B}_{\ker(Z'_0)}}).$$

Lemma 5.27 *The triple σ_a^R satisfies condition (i) and (ii) in Definition 3.2.*

Proof Since for any $K \in \mathcal{B}_{\ker(Z'_0)}$ we have

$$\begin{aligned} \lim_{V_\omega, D_\omega \to 0, [V_\omega : D_\omega] = a} \phi_{\sigma'}(L_1) = \frac{1}{4} &\le \lim_{V_\omega, D_\omega \to 0, [V_\omega : D_\omega] = a} \phi_{\sigma'}(K) \\ &\le \lim_{V_\omega, D_\omega \to 0, [V_\omega : D_\omega] = a} \phi_{\sigma'}(L_0[1]) = \frac{3}{4}, \end{aligned}$$

we see that condition (i) is satisfied. The proof of condition (ii) is the same as Proposition 5.7. ∎

To show σ_a^R defines a weak stability condition, we are left with checking the HN property.

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Proposition 5.28 *The abelian category $\mathcal{B}$ is a noetherian abelian category.*

Proof Since ω' is ample, it is well known that $\mathcal{A}$ is noetherian, e.g., see [16, Lemma 6.17]. We use the criterion of [4, Lemma 5.5.2] to show $\mathcal{B}$ is noetherian. In particular, we show that for any accending chain

$$F_0 \subset F_1 \subset \cdots \tag{5.28.1}$$

with $F_i \in \mathcal{F}_{\mathcal{A}}$ and $\operatorname{coker}(F_0 \to F_i) \in \mathcal{T}_{\mathcal{A}}$ stabilizes after a finite number of steps.

Consider the short exact sequence in $\mathcal{A}$:

$$0 \to F_0 \to F_i \to G_i \to 0.$$

By the snake lemma, we have

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F_0 & \longrightarrow & F_i & \longrightarrow & G_i & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F_0 & \longrightarrow & F_{i+1} & \longrightarrow & G_{i+1} & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & C_i & \xrightarrow{\simeq} & K_i & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

where all the short exact sequences are in $\mathcal{A}$. Since G_i and G_{i+1} are in $\mathcal{T}_{\mathcal{A}}$, we have $C_i \simeq K_i \in \mathcal{T}_{\mathcal{A}}$.

Let us apply $\operatorname{Hom}(L_1, _)$ to the short exact sequence

$$0 \to F_i \to F_{i+1} \to C_i \to 0.$$

Since $L_1, F_{i+1} \in \mathcal{B}$, we have $0 = \operatorname{Ext}^{-1}(L_1, F_{i+1}) = \operatorname{Hom}(L_1[1], F_{i+1})$. Since $\operatorname{Hom}(L_1, L_1[1]) = H^1(X, \mathcal{O}_X) = 0$, we also have $\operatorname{Hom}(L_1, C_i) = 0$. Hence the dimension of $\operatorname{Hom}(L_1, F_i)$ decreases as i increases. At the point where the dimension of $\operatorname{Hom}(L_1, F_i)$ becomes stationary, we obtain $\operatorname{Ext}^{-1}(L_1, C_i) = 0$; since C_i is an extension of copies of $L_1[1]$ which has no self-extensions, it follows that $C_i = 0$. Thus the chain (5.28.1) stabilizes after a finite number of steps. ∎

Theorem 5.29 *Given $a \in \mathbb{P}^1$, the weak stability function σ_a^R satisfies the HN property.*

Proof The argument follows the proof of [18, Lemma 2.18].

Since $\mathcal{B}$ is noetherian, we only need to check (i) in the Proposition 3.6. Assuming we have such a chain of subobjects in $\mathcal{B}$, consider the short exact sequence in $\mathcal{B}$:

$$0 \to E_{i+1} \to E_i \to F_i \to 0.$$

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Since $\Im(Z_0'(E_{i+1})) \leq \Im(Z_0'(E_i))$. Since Z_0' has coefficients in $\mathbb{Q}$, we know that the image of Z_0' is discrete. Hence $\Im(Z_0'(E_i))$ is constant for $i \gg 0$. Then $\Im(Z_0'(F_i)) = 0$ for $i > n$ for some n.

Let $i > n$, if $F_i \notin \mathcal{B}_{\ker(Z_0')}$, then $\phi(F_i) = 1$, contradicting with $\phi(E_{i+1}) > \phi(F_i)$.

We must have $F_i \in \mathcal{B}_{\ker(Z_0')}$. From 5.26.1, we know F_i must be a direct sum of copies of $L_0[1]$ and L_1. From the Ext-group calculations in the proof of Lemma 5.24, we know that $\mathrm{Ext}^1(F_i, L_1) = 0 = \mathrm{Ext}^1(F_i, L_0[1])$. Hence either the dimension of $\mathrm{Hom}(E_i, L_1)$ or $\mathrm{Hom}(E_i, L_0[1])$ decreases as i increases, we much have $F_i = 0$ for i large enough. ∎

Intuitively, the local picture near the weak stability conditions σ_b^L's and σ_a^R's are $\mathcal{B}^l_{(0,0)}\mathbb{R}^2$, the blow up of $\mathbb{R}^2$ at the origin.

A Solving a central charge equation

In this section, for given $\mathbb{R}$-divisors ω, B on a Weirstraß elliptic K3 surface X, we solve the central charge equation

$$Z^{td}_{\omega',B'}(\Phi(-)) = TZ_{\omega,B}(-) \tag{A.0.1}$$

for some $T \in \mathrm{GL}^+(2,\mathbb{R})$ and ω', B'. Since $\sqrt{\mathrm{td}(X)} = (1,0,1)$ on a K3 surface,

$$\begin{aligned} Z^{td}_{\omega,B}(F) &= -\int_X e^{-(B+i\omega)}\mathrm{ch}(F)\sqrt{\mathrm{td}(X)} \\ &= -\mathrm{ch}_2^B(F) + (\tfrac{\omega^2}{2} - 1)\mathrm{ch}_0^B(F) + i\omega\mathrm{ch}_1^B(F) \\ &= -\mathrm{ch}_2^B(F) + (V_\omega - 1)\mathrm{ch}_0^B(F) + i\omega\mathrm{ch}_1^B(F) \end{aligned}$$

which differs from $Z_{\omega,B}(F)$ by a change a variable in the coefficient for ch_0^B. Therefore, to solve the equation (A.0.1), we first solve the equation

$$Z_{\omega',B'}(\Phi(-)) = TZ_{\omega,B}(-). \tag{A.0.2}$$

Even though (A.0.2) was solved in [14, Section 10] (see also ([15, Appendix A]), the solution there was written in terms of a coordinate system different from the RDV coordinates introduced in 2.3. Instead of applying a change of variables to the solution in [14] (which is cumbersome), we outline a direct solution which uses the RDV coordinates.

To begin with, note that

$$\begin{aligned} Z_{\omega,B} &= -\mathrm{ch}_2^B + \tfrac{\omega^2}{2}\mathrm{ch}_0^B + i\omega\mathrm{ch}_1^B \\ &= -\mathrm{ch}_2^B + V_\omega\mathrm{ch}_0^B + iR_\omega(\Theta + (D_\omega + e)f)\mathrm{ch}_1^B \\ &= -\mathrm{ch}_2 + R_B\Theta\mathrm{ch}_1 + R_B(D_B + e)f\mathrm{ch}_1 + (V_\omega - V_B)\mathrm{ch}_0 \\ &\quad + iR_\omega(\Theta\mathrm{ch}_1 + (D_\omega + e)f\mathrm{ch}_1 - R_B(D_B + D_\omega + e)\mathrm{ch}_0). \end{aligned}$$

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Now we set

$$\text{(A.0.3)}\quad Z'_{\omega,B} = \begin{pmatrix} 1 & -R_B/R_\omega \\ 0 & 1/R_\omega \end{pmatrix} Z_{\omega,B}$$
$$= -\mathrm{ch}_2 + L_{\omega,B} f\mathrm{ch}_1 + M_{\omega,B}\mathrm{ch}_0 + i\big(\Theta\mathrm{ch}_1 + (D_\omega + e) f\mathrm{ch}_1 + N_{\omega,B}\mathrm{ch}_0\big)$$

where the terms

$$\begin{aligned} L_{\omega,B} &= R_B(D_B - D_\omega) \\ M_{\omega,B} &= V_\omega - V_B + R_B^2(D_B + D_\omega + e) \\ N_{\omega,B} &= -R_B(D_B + D_\omega + e) \end{aligned}$$

depend only on ω and B.

Solving (A.0.2) is now in turn equivalent to solving

$$\text{(A.0.4)}\qquad Z'_{\omega',B'}(\Phi(E)) = TZ'_{\omega,B}(E) \quad \text{for all } E \in D^b(X).$$

For an object $E \in D^b(X)$, if we write

$$\mathrm{ch}_0(E) = n,\ \ f\mathrm{ch}_1(E) = d,\ \ \Theta\mathrm{ch}_1(E) = c,\ \ \mathrm{ch}_2(E) = s,$$

then from [2, Section 6.2.6] we have

$$\begin{aligned} &\mathrm{ch}_0(\Phi(E) = d,\ \ f\mathrm{ch}_1(\Phi E) = -n,\ \ \Theta\mathrm{ch}_1(\Phi E) = s - \tfrac{e}{2}d + en, \\ &\quad \mathrm{ch}_2(\Phi E) = -c - ed + \tfrac{e}{2}n. \end{aligned}$$

Using this notation for Chern classes, we now have

$$Z'_{\omega,B}(E) = -s + L_{\omega,B}d + M_{\omega,B}n + i(c + (D_\omega + e)d + N_{\omega,B}n)$$

while

$$Z'_{\omega',B'}(\Phi E) = -(\tfrac{e}{2} + L_{\omega',B'})n + (M_{\omega',B'} + e)d + c + i(-D_{\omega'}n + (N_{\omega',B'} - \tfrac{e}{2})d + s).$$

It is then clear that if the following relations are satisfied

$$\text{(A.0.5)}\quad -L_{\omega,B} = N_{\omega',B'} - \tfrac{e}{2},\ \ M_{\omega,B} = D_{\omega'},\ \ D_\omega + e = M_{\omega',B'} + e,\ \ N_{\omega,B} = -(\tfrac{e}{2} + L_{\omega',B'})$$

and we take $T = -i = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, then the equation (A.0.4) would hold.

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The four relations in (A.0.5), when written out in RDV coordinates, correspond to

$$R_B(D_B - D_\omega) = R_{B'}(D_{B'} + D_{\omega'} + e) + \tfrac{e}{2} \tag{A.0.6}$$

$$V_\omega - V_B + R_B^2(D_B + D_\omega + e) = D_{\omega'} \tag{A.0.7}$$

$$D_\omega + e = V_{\omega'} - V_{B'} + R_{B'}^2(D_{B'} + D_{\omega'} + e) + e \tag{A.0.8}$$

$$-R_B(D_B + D_\omega + e) = -\left(\tfrac{e}{2} + R_{B'}(D_{B'} - D_{\omega'})\right) \tag{A.0.9}$$

respectively. Noting that for any divisor W we have $V_W = \frac{1}{2}R_W^2(2D_W + e)$, relation (A.0.7) gives

$$D_{\omega'} = V_\omega + R_B^2(D_\omega + \tfrac{e}{2}). \tag{A.0.10}$$

Adding (A.0.6) and (A.0.9) together gives

$$R_{B'} = \frac{R_B(-2D_\omega - e)}{2D_{\omega'} + e}. \tag{A.0.11}$$

Now (A.0.8) yields

$$V_{\omega'} = D_\omega - \frac{R_B^2(2D_\omega + e)^2}{2(2D_{\omega'} + e)}. \tag{A.0.12}$$

Also, subtracting (A.0.9) from (A.0.6) (and noting $e = 2$) gives

$$R_{B'}D_{B'} = R_BD_B + R_B - R_{B'} - 1. \tag{A.0.13}$$

Finally, we see that when we the RDV coordinates of ω, B, ω', B' satisfy (A.0.10), (A.0.11), (A.0.13), and

$$V_{\omega'} = D_\omega - \frac{R_B^2(2D_\omega + e)^2}{2(2D_{\omega'} + e)} + 1 \tag{A.0.14}$$

(instead of (A.0.12)), and we choose

$$T = \begin{pmatrix} 1 & -R_{B'}/R_{\omega'} \\ 0 & 1/R_{\omega'} \end{pmatrix}^{-1} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -R_B/R_\omega \\ 0 & 1/R_\omega \end{pmatrix},$$

the parameters B, ω, B', ω' and T together solve the central charge equation (A.0.1).

The equations in (5.19.3) are precisely (A.0.10), (A.0.11), (A.0.13), and (A.0.14).

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References

[1] D. Arcara, A. Bertram, and M. Lieblich, *Bridgeland-stable moduli spaces for K-trivial surfaces*. J. Eur. Math. Soc. **15**(2013), no. 1, 1–38.
[2] C. Bartocci, U. Bruzzo, and D. Hernández-Ruipérez, *Fourier-Mukai and Nahm transforms in geometry and mathematical physics*, Progress in Mathematics, 276, Springer, Birkhäuser, 2009.
[3] A. Bayer, E. Macrì, and Y. Toda, *Bridgeland stability conditions on threefolds I: Bogomolov-Gieseker type inequalities*. J. Algebr. Geom. **23**(2014), 117–163.
[4] A. Bondal and M. Van den Bergh, *Generators and representability of functors in commutative and noncommutative geometry*. Mosc. Math. J. **3**(2003), no. 1, 1–36.

https://doi.org/10.4153/S0008414X25000100 Published online by Cambridge University Press

[5] T. Bridgeland, *Stability conditions on triangulated categories*. Ann. Math. **166**(2007), 317–345.
[6] T. Bridgeland, *Stability conditions on K3 surfaces*. Duke Math. J. **141**(2008), 241–291.
[7] N. Broomhead, D. Pauksztello, D. Ploog, and J. Woolf, *Partial compactification of stability manifolds by massless semistable objects*. Preprint, 2022. arXiv:2208.03173 [math.RT].
[8] T. Collins, J. Lo, Y. Shi, and S.-T. Yau, *Stability for line bundles and deformed Hermitian-Yang-Mills equation on some elliptic surfaces*. Preprint, 2023. arXiv:2306.05620 [math.AG].
[9] D. Greb, S. Kebekus, and T. Peternell, *Movable curves and semistable sheaves*. Int. Math. Res. Not. **2016**(2016), no. 2, 536–570.
[10] R. Hartshorne, *Algebraic geometry*, Graduate Texts in Mathematics, 52, Springer-Verlag, New York, NY, 1977.
[11] W. Liu and J. Lo, *Private communication*, 2020.
[12] W. Liu, J. Lo, and C. Martinez, *Fourier-Mukai transforms and stable sheaves on Weierstrass elliptic surfaces*. Bull. Braz. Math. Soc. **55**(2024), 47.
[13] J. Lo, *t-structures on elliptic fibrations*. Kyoto J. Math. **56**(2016), no. 4, 701–735.
[14] J. Lo, *Weight functions, tilts, and stability conditions*. Preprint, 2020. arXiv:2007.06857 [math.AG].
[15] J. Lo and C. Martinez, *Geometric stability conditions under autoequivalences and applications: Elliptic surfaces*. J. Geom. Phys. **194**(2023), 104994.
[16] E. Macrí and B. Schmidt, *Lectures on Bridgeland stability*. In: L. Brambila Paz, C. Ciliberto, E. Esteves, M. Melo, C. Voisin (eds.), *Moduli of curves*, volume 21 of Lecture Notes of the Unione Matematica Italiana, CIMAT Guanajuato, Mexico, 2016, pp. 139–211.
[17] J. McKernan, *Classification of surfaces*. https://math.mit.edu/~mckernan/Teaching/07-08/Autumn/18.735/l_7.pdf [math.mit.edu].
[18] D. Piyaratne and Y. Toda, *Moduli of Bridgeland semistable objects on 3-folds and Donaldson-Thomas invariants*. J. Reine Angew. Math. **2019**(2019), no. 747, 175–219.
[19] R. Tramel and B. Xia, *Bridgeland stability conditions on surfaces with curves of negative self-intersection*. Adv. Geom. **22**(2022), no. 3, 383–408.

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