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Three-wave coupling and weak turbulence of kinetic Alfvén waves

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The nonlinear dynamics of kinetic-Alfvén–wave (KAW) turbulence is studied. Weak KAW turbulence induced by three-wave interaction among parallel-propagating KAWs has a direct energy cascade in the wavenumber domain $k_{s\perp} > \rho_i^{-1}$ and an inverse cascade in the domain $k_{s\perp} < \rho_i^{-1}$, resulting in Kolmogorov-type spectra, $W_k \sim (k_z)^{-1/2}(k_\perp)^{-p}$, with exponents $p = 4$ and $p = 3.5$ respectively. The interaction including antiparallel-propagating KAWs, usually most effective, results in an inverse energy cascade over the whole $k_\perp$ range and $p = 2$ (at $k_\perp < \rho_i^{-1}$) and $p = 3.5$ (for $k_\perp > \rho_i^{-1}$) spectra. Three applications concerning KAW turbulence in flaring loops, in the Earth's magnetosphere and in tokamaks are considered. It is suggested that turbulent KAW spectra are common in space plasmas.

1. Introduction

A kinetic Alfvén wave (KAW) is an Alfvén wave with a large transverse wave-vector component, $k_\perp \gg k_z$ ($\mathbf{k}_\perp \perp \mathbf{B}_0$, $\mathbf{B}_0$ is the ambient magnetic field), so that the perpendicular wavelength becomes comparable to the ion gyroradius, $\rho_i = V_{Ti}/\Omega_i$ (Hasegawa and Chen 1976). The linear dispersion relation for KAWs may be written as

$$\omega_k = |k_z|\, V_A K_k, \tag{1.1}$$

where $\kappa = k_\perp \rho_i$, and the dispersion function $K_k = K_k(\kappa)$ has the following asymptotics:

$$K_k \sim \begin{cases} 1 + \dfrac{1}{2}\left(\dfrac{3}{4} + \dfrac{T_e}{T_i}\right)\kappa^2 & (\kappa^2 \ll 1), \\ \left(1 + \dfrac{T_e}{T_i}\right)^{1/2} \kappa & (\kappa^2 \gg 1). \end{cases} \tag{1.2}$$

KAWs have become known mainly because of the extensive investigations of their kinetic properties, which are important for processes of plasma heating and particle acceleration (see e.g. Hasegawa (1993) and references therein). It has been shown explicitly that finite-ion-gyroradius and electron-inertia effects produce E_z in Alfvén waves, which brings about collisionless wave-particle resonant interaction, resulting in enhanced plasma heating and anomalous transport in the presence of kinetic Alfvén waves. It has also been recognized that, owing to the unique combination of their MHD scales along $\mathbf{B}_0$ and their kinetic properties, KAWs play an important – often major – role in the dynamics of the both laboratory and space plasmas, and excitation of KAWs by various linear (Hasegawa and Chen 1976; Shukla *et al.* 1989; Voitenko 1989, 1995, 1996; Voitenko *et al.* 1990; Berk

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et al. 1993) and nonlinear (Shukla and Mamedow 1978; Sharma and Tripathi 1988; Hasan and Guha 1990; Stenflo and Shukla 1995; Yukhimuk *et al.* 1998) mechanisms have been widely discussed.

However, noting the strong tendency of KAWs towards nonlinear interaction among themselves at very low amplitudes (Voitenko 1998), we suggest that, despite various excitation mechanisms, the resulting KAW turbulence should possess some common characteristics, determined by the properties of the matrix elements (coupling coefficients) of the three-wave interaction among KAWs.

In order to take into account important kinetic and dispersive effects of finite ion Larmor radius, in the present paper the dynamics of the KAW turbulence caused by three-wave interaction among KAWs is studied using a dynamic equation and coupling coefficients (Sec. 2), found in a plasma kinetic model (Voitenko 1998). The nonlinear saturation level of KAW turbulence is given in Sec. 3. Dynamic properties of spectral energy and enstrophy transfer are investigated in Sec. 4, and the resulting turbulent spectra in the inertial range are found in Sec. 5. Discussion, applications and a summary follow in Secs 6–8.

2. Basic equations

Accounting for the linear and nonlinear finite-Larmor-radius effects, we study nonlinear saturation and properties of weak KAW turbulence in a low-β fully ionized uniform hydrogen plasma (β is the plasma/magnetic pressure ratio, lying in the range $m_e/m_i < \beta < 1$).

Low-frequency ($\omega < \Omega_i$), magnetically incompressible ($\tilde{B}_z = 0$) KAWs are described by two fluctuating electromagnetic wave potentials, a scalar $\tilde{\phi}$ and a (z-component) vector $\tilde{A}_z$ (Hasegawa and Chen 1976). The charge and current-density perturbations $\tilde{q}_\alpha$ and $\tilde{\jmath}_\alpha$, induced in the αth plasma species by these potentials, obey the quasineutrality equation, $\sum_\alpha \tilde{q}_\alpha = 0$, and Ampere's law in the z direction, $\sum_\alpha \tilde{\jmath}_\alpha = -(c/4\pi)\nabla^2 \tilde{A}_z$. The difference between MHD and kinetic theories here is in the calculation of $\tilde{q}_\alpha$ and $\tilde{\jmath}_\alpha$. To take into account finite-Larmor-radius effects correctly, we have solved the Vlasov kinetic equations up to second-order (in the wave potentials) perturbations, and calculated $\tilde{q}_\alpha$ and $\tilde{\jmath}_\alpha$ using the perturbed parts $\tilde{f}_\alpha$ of the velocity distribution functions $F_\alpha = F_\alpha^0 + \tilde{f}_\alpha$. This allowed us to derive the dynamic equation for the Fourier amplitudes of the KAW magnetic potential A_k

$$\left(\frac{\partial}{\partial t} - \gamma_k\right) A_k^\sigma = \sum_{\mathbf{k}_1,\mathbf{k}_2} \sum_{\sigma_1,\sigma_2} \delta(\sigma\mathbf{k} - \sigma_1\mathbf{k}_1 - \sigma_2\mathbf{k}_2)\delta\left(\sigma\omega_k - \sigma_1\omega_1 - \sigma_2\omega_2\right) \times U(k,k_1,k_2) A_1^{\sigma_1} A_2^{\sigma_2} \qquad (2.1)$$

and the matrix element

$$U(\mathbf{k},\mathbf{k}_1,\mathbf{k}_2) = \frac{V_A}{4B_0}(s_1K_1 - s_2K_2)\frac{K_k}{\kappa^2}\left(s_1\frac{\kappa_1^2}{K_1} + s_2\frac{\kappa_2^2}{K_2} + \frac{\kappa^2}{K_k}\right)\sigma_1\sigma_2\left(\mathbf{k}_1 \times \mathbf{k}_2\right)_z, \quad (2.2)$$

valid for an arbitrary value of the 'kinetic variable' $\kappa = k_\perp \rho_i$ (Voitenko 1998). Here the subscripts 2 and 1 indicate $\mathbf{k}_2$ and $\mathbf{k}_1$ correspondingly, and the kinetic increment (decrement) due to wave–particle interaction at Čerenkov resonance is

$$\gamma_k = \frac{\pi}{2}\omega_k \frac{\kappa^2}{K_k^2}\frac{T_e}{T_i} V_{Te}^2 \sum_\alpha \frac{n_\alpha}{n_0}\frac{m_e}{m_\alpha}\left.\frac{\partial \overline{F}_{0\alpha}}{\partial V_z}\right|_{V_z = V_k}. \qquad (2.3)$$

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The function

$$K_k = K(\kappa) = \kappa \left(\frac{1}{1-\Lambda_0} + \frac{T_e}{T_i} \right)^{1/2}$$

determines the dispersive properties of KAWs, $\Lambda_0 = I_0(\kappa^2)\exp(-\kappa^2)$, the normalized distribution function

$$\overline{F}^0_\alpha = 2\pi \int dV_\perp \, V_\perp J_0^2 \left(\frac{k_\perp V_\perp}{\Omega_\alpha} \right) F^0_\alpha(V_z, V_\perp),$$

$\kappa = k_\perp \rho_i$, and $I_0(\kappa^2)$ is the modified Bessel function.

We estimate the rate of three-wave interaction by maximizing the product $|U(\mathbf{k}_1, \mathbf{k}, -\mathbf{k}_2) U(-\mathbf{k}_2, -\mathbf{k}, \mathbf{k}_1)|$ in the limits of strong and weak wave dispersion.

(i) In the long-wavelength domain, $\kappa^2 \ll 1$, the wavenumbers of dominant triads

$$\frac{k_{1\perp}}{k_\perp} = 0.717, \qquad \frac{k_{2\perp}}{k_\perp} = 1.49, \tag{2.4}$$

and the rate of nonlinear interaction

$$\gamma_k^{NL} = 0.375 \Omega_i \frac{V_A}{V_{Ti}} \kappa^3 \left| \frac{B_k}{B_0} \right| \tag{2.5}$$

for parallel-propagating waves ($s_1 = s_2 = 1$). For counterstreaming waves ($s_1 = 1, s_2 = -1$),

$$\frac{k_{1\perp}}{k_\perp} = 0.776, \qquad \frac{k_{2\perp}}{k_\perp} = 0.442, \tag{2.6}$$

and

$$\gamma_k^{NL} = 0.28 \Omega_i \frac{V_A}{V_{Ti}} \kappa^2 \left| \frac{B_k}{B_0} \right|. \tag{2.7}$$

(ii) In the long-wavelength domain, $\kappa^2 \gg 1$, the wavenumbers of dominant triads

$$\frac{k_{1\perp}}{k_\perp} = 0.726, \qquad \frac{k_{2\perp}}{k_\perp} = 1.465, \tag{2.8}$$

and the interaction rate

$$\gamma_k^{NL} = 0.25 \Omega_i \frac{V_A}{V_{Ti}} \kappa^2 \left| \frac{B_k}{B_0} \right| \tag{2.9}$$

for parallel-propagating waves. For counterstreaming waves,

$$\frac{k_{1\perp}}{k_\perp} = 0.707, \qquad \frac{k_{2\perp}}{k_\perp} = 0.704, \tag{2.10}$$

and

$$\gamma_k^{NL} = 0.18 \Omega_i \frac{V_A}{V_{Ti}} \kappa^2 \left| \frac{B_k}{B_0} \right|. \tag{2.11}$$

Here we have put $T_e = T_i$ for simplicity.

The matrix element $U(k, k_1, k_2)$ is a key mathematical object of the nonlinear wave theory. With (2.1)–(2.11), we proceed to find the nonlinear saturation level, the spectral energy transfer, and the turbulent spectra of weak KAW turbulence.

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3. Nonlinear saturation level

The nonlinear saturation of KAW amplitudes in many cases is initiated by the parametric decay instability of the modes initially excited. Let some source give rise to KAW excitation with growth rate γ_k^s; it may be a plasma kinetic instability, in which case $\gamma_k^s = \gamma_k^L$. Then the KAW amplitude grows in accordance with (2.3) up to the nonlinear saturation value B_k^s/B_0, at which excitation of the initial wave is balanced by its decay and consequent energy transfer into secondary waves:

$$\gamma_k^s = \gamma_k^{NL}.$$

Taking for γ_k^{NL} the maximum values (2.7) and (2.9), we get an estimate of the nonlinear saturation level:

$$\left(\frac{B_k}{B_0}\right)_{\rm sat} \approx 4\left(\frac{\gamma_k^s}{\Omega_i}\right)\frac{V_{Ti}}{V_A}\kappa^{-2}. \tag{3.1}$$

If decay into counterstreaming KAWs is forbidden, the saturation amplitude in the domain $k_\perp < \rho_i$ is

$$\left(\frac{B_k}{B_0}\right)_{\rm sat} \approx 3\left(\frac{\gamma_k^s}{\Omega_i}\right)\frac{V_{Ti}}{V_A}\kappa^{-3}. \tag{3.2}$$

In (3.1) and (3.2) we observe the tendency for excited turbulence at $\kappa \approx 1$ to saturate at a very low level. For example, with $\gamma_k^s/\Omega_i = 0.01$ and $\beta_i = 0.01$, one can see that this mechanism provides a limit on KAW amplitudes in the source region at a sufficiently low level, $B_k^s/B_0 = 0.001\kappa^{-\nu}(\ll 1$ for $\kappa \approx 1)$.

4. Cascade processes and spectral energy transfer in KAW turbulence

Let us consider the excitation of KAW turbulence by a source, pumping energy into the mode $\mathbf{k}_s$ (wave vector) with rate $2\gamma_k^s$. When the amplitude of the growing mode reaches the saturation value (3.1) or (3.2), the parametric decay of the pump $\mathbf{k}_s$ stops its growth, spreading out the wave spectrum. That is, the decay into parallel-propagating KAWs initially spreads the spectrum into both higher- and lower-wavenumber domains, $k_{1\perp} < k_{s\perp} < k_{2\perp}$, whereas decay into counterstreaming KAWs spreads the spectrum into the lower-wavenumber domain, $k_{1\perp} < k_{2\perp} < k_{s\perp}$. But the common dynamics and direction of the energy transfer in $\mathbf{k}$-space are not clear at this stage.

To examine the direction of spectral energy transfer in KAW turbulence, we use the model of dominant triads in the cascade process, successfully applied to different kinds of turbulence (see e.g. Hasegawa 1985; Pakyari and Pavlenko 1996). In this model the spectral energy cascade is considered as a series of consequent decays beginning with the first step of the source-wave decay into a two secondary waves (decay products), having the largest growth rate. Any decay product of the nth step in turn excites two decay products of the $(n+1)$th step, and so on, eventually resulting in energy transfer in $\mathbf{k}$ space, dynamical properties of which are determined by the form of the matrix element (2.2). The wavenumbers of the dominant decay products at any step are determined by maximizing $|U(\mathbf{k}_1, \mathbf{k}, -\mathbf{k}_2)U(-\mathbf{k}_2, -\mathbf{k}, \mathbf{k}_1)|$, and are given by (2.4) and (2.8).

Let us present total-energy W_k and enstrophy (dispersive part of the energy) $\mathscr{E}_k$ conservation laws at any decay step in the forms

$$W_s^{(n)} = W_1^{(n)} + W_2^{(n)}, \tag{4.1}$$

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$$W_s^{(n)} = \eta_1 W_1^{(n)} + \eta_2 W_2^{(n)}, \tag{4.2}$$

where the coefficients η_1 and η_2 are to be determined from the expression relating W_k and $\mathscr{E}_k$. Then the energy partitition at any step is

$$W_1^{(n)} = \xi_w W_s^{(n)}, \tag{4.3}$$

$$W_2^{(n)} = (1-\xi_w) W_s^{(n)}, \tag{4.4}$$

where the energy-partitition coefficient

$$\xi_w = \frac{\eta_2 - 1}{\eta_2 - \eta_1}.$$

Beginning with the first step, after n decays we shall observe binomial distribution for energy partitition among the $\mathbf{k}$ modes $k_\perp^{(r)}$, initially placed at wavenumber $k_{s\perp}$:

$$W(k_\perp = \sigma_1^{n-r}\sigma_2^r k_{s\perp}) = \binom{n}{r} \xi_w^{n-r}(1-\xi_w)^r W_s. \tag{4.5}$$

For $n \gg 1$, this distribution attains its maximum value at $r/n \approx 1-\xi_w$, i.e. energy is concentrated around the rth mode, $r \approx n(1-\xi_w)$, with wavenumber

$$k_{\perp w} = k_\perp(r/n = 1-\xi_w) = \sigma_2^{(1-\xi)n}\sigma_1^{\xi n} k_{s\perp} = q_w^n k_{s\perp}. \tag{4.6}$$

We call $k_{\perp w}$ the energy-containing wavenumber, and

$$q_w = \sigma_2^{1-\xi_w}\sigma_1^{\xi_w} \tag{4.7}$$

the energy-transfer coefficient.

It follows from (4.6), that, for $q_w > 1$, the energy, initially placed into a mode with wavenumber $k_{s\perp}$, after n decays is transferred into higher-wavenumber modes, $k_{\perp w} \approx q_w^n k_{s\perp} > k_{s\perp}$. When $n \to \infty$, the peak of the spectral energy distribution shifts upwards, and the energy-containing wavenumber $k_{\perp w} \to \infty$.

In contrast, for $q_w < 1$, the energy is transferred into lower-wavenumber modes, $k_{\perp w} = q_w^n k_{s\perp} < k_{s\perp}$. When $n \to \infty$, the energy-containing wavenumber $k_{\perp w}^2 \to 0$, and the peak of the spectral energy distribution shifts downwards towards $k_\perp^2 = 0$. Therefore, depending on the condition $q_w > (<)\, 1$, the energy will be transferred by three-wave interactions into higher- (lower-) wavenumber modes, forming a direct (inverse) turbulent cascade.

In terms of enstrophy, (4.1) and (4.2) read as

$$\mathscr{E}_s^{(n)} = \eta_1^{-1}\mathscr{E}_1^{(n)} + \eta_2^{-1}\mathscr{E}_2^{(n)}, \tag{4.8}$$

$$\mathscr{E}_s^{(n)} = \mathscr{E}_1^{(n)} + \mathscr{E}_2^{(n)}, \tag{4.9}$$

resulting in the enstrophy-partitition coefficient

$$\xi_\varepsilon = \eta_1 \xi_w,$$

a binomial enstrophy distribution among the $\mathbf{k}$ modes $k_\perp^{(r)}$ after n decays

$$\mathscr{E}(k_\perp = \sigma_1^{n-r}\sigma_2^r k_{s\perp}) = \binom{n}{r} \xi_\varepsilon^{n-r}(1-\xi_\varepsilon)^r \mathscr{E}_s, \tag{4.10}$$

an enstrophy-containing wavenumber

$$k_{\perp\varepsilon} = k_\perp(r/n = 1-\xi_\varepsilon) = \sigma_2^{(1-\xi_\varepsilon)n}\sigma_1^{\xi_\varepsilon n} k_{s\perp} = q_\varepsilon^n k_{s\perp}, \tag{4.11}$$

and an enstrophy-transfer coefficient

$$q_\varepsilon = \sigma_2^{1-\xi_\varepsilon}\sigma_1^{\xi_\varepsilon}. \tag{4.12}$$

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Introducing the number of wave quanta $N_k = W_k/\omega_k$ and using the conservation properties of quantum numbers, following from (2.1) (see e.g. Stenflo 1994), we can write the conservation laws for the z component of momentum,

$$k_z N_k = k_{1z} N_1 + k_{2z} N_2 \tag{4.13}$$

and for the total wave energy,

$$\omega_k N_k = \omega_1 N_1 + \omega_2 N_2, \tag{4.14}$$

in any decay step (the index n is dropped here). With KAW dispersion, (4.13) may be rewritten in the form

$$\frac{W_k}{K_k} = \frac{W_1}{K_1} + s_2 \frac{W_2}{K_2}, \tag{4.15}$$

from which follows the conservation law for enstrophy $\mathscr{E}_k$.

Let us first examine the turbulent cascades in the long- and short-wavelength domains, created by decays including parallel-propagating KAWs ($s_1 = s_2 = 1$).

(i) $\kappa_s^2 \ll 1$. In the case of weak dispersion, (4.13) and (4.15) represent conservation of the 'non-dispersive' (main) part of the wave energy. Extracting from the total energy (4.14) the non-dispersive part (4.15), after substitution of (1.2) for the Ks, we get the conservation law for enstrophy in the long-wavelength domain:

$$k_\perp^2 W_k = k_{1\perp}^2 W_1 + k_{2\perp}^2 W_2. \tag{4.16}$$

The relation between enstrophy and energy in this case is

$$\mathscr{E}_k = 0.5 \left(\frac{3}{4} + \frac{T_e}{T_i} \right) \rho_i^2 k_\perp^2 W_k,$$

and the wave vector components of the dominant decay products are $k_{1\perp} = 0.717 k_{s\perp}$ and $k_{2\perp} = 1.49 k_{s\perp}$. The resulting values $\sigma_1 = 0.717$, $\sigma_2 = 1.49$, $\eta_1 = \sigma_1^2$ and $\eta_2 = \sigma_2^2$ give us an energy-transfer coefficient $q_w = 0.883$ and an enstrophy-transfer coefficient $q_\varepsilon = 1.138$. Therefore inverse energy ($q_w < 1$) and direct enstrophy ($q_\varepsilon > 1$) cascades are formed in the long-wavelength domain.

(ii) $\kappa_s^2 \gg 1$. Alternatively, in the case of strong dispersion, (4.13) and (4.15) represent conservation of the 'dispersive' part of the wave energy; after substitution of (1.2) for the Ks we get the conservation law for enstrophy in the short-wavelength domain:

$$k_\perp^{-1} W_k = k_{1\perp}^{-1} W_1 + k_{2\perp}^{-1} W_2, \tag{4.17}$$

and

$$\mathscr{E}_k = (1 + T_e/T_i)^{-1} \rho_i^{-1} k_\perp^{-1} W_k.$$

Together with the wavenumbers of the dominant modes $k_{1\perp} = 0.726 k_{s\perp}$ and $k_{2\perp} = 1.465 k_{s\perp}$, this gives $\sigma_1 = 0.726$, $\sigma_2 = 1.465$, $\eta_1 = \sigma_1^{-1}$ and $\eta_2 = \sigma_2^{-1}$. The resulting energy-transfer coefficient $q_w = 1.063$ and enstrophy-transfer coefficient $q_\varepsilon = 0.942$ indicate a direct energy cascade and an inverse enstrophy cascade in the short-wavelength domain.

The above results show that three-wave resonant interaction among parallel-propagating KAWs results in an inverse energy cascade in the domain $\kappa_s^2 \ll 1$ and a direct energy cascade in the domain $\kappa_s^2 \gg 1$, or, in other words, escape of spectral energy from the region $\kappa^2 \approx 1$. For the other conserved quantity, enstrophy $\mathscr{E}_k$, in

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this case we observe enstrophy transport towards wavenumber modes $\sim \rho_i^{-1}$ by a direct turbulent cascade at $k_\perp < \rho_i^{-1}$, and an inverse cascade at $k_\perp > \rho_i^{-1}$.

The decays including antiparallel-propagating KAWs ($s_1 = 1, s_2 = -1$) can significantly modify the cascade process, especially in the domain $k_\perp > \rho_i^{-1}$, where the cascade direction may be reversed. To show this, we note that, under the action of two decay types with energy-transfer coefficients $q'_w < 1$ and $q''_w > 1$, the cumulative transfer coefficient will be $q_w = q'_w\,(q''_w)^{c_\gamma}$, where $c_\gamma = \gamma''_{NL}/\gamma'_{NL}$ is the ratio of the corresponding decay growth rates and $\left(\gamma'_{NL}\right)^{-1}$ is taken as the time step. With $q''_w = 1.063$ ($s_2 = 1$ decay), $q'_w = 0.707$ ($s_2 = -1$ decay) and $c_\gamma = 1.6$, we get $q_w = 0.78 < 1$, indicating an inverse energy cascade at $k_\perp > \rho_i^{-1}$.

In the domain $k_\perp < \rho_i^{-1}$, the cooperative action of the $s_2 = -1$ and $s_2 = 1$ decay processes results in a very strong inverse energy cascade with $c_\gamma \sim \kappa^{-1} \gg 1$, $q'_w = 0.78$

5. Kolmogorov-type spectra of KAW turbulence

Symmetry properties of the matrix elements make it possible to construct a simple algorithm for the calculation of stationary turbulent spectra in the inertial (no sinks, no sources) wavenumber range, provided that the wave dispersion and normalized matrix element are uniform functions of the wavenumbers (Zakharov 1984). Namely, let anisotropic waves with $k_\perp \gg k_z$ have the dispersion relation

$$\omega_k = \bar{\omega} k_z^a k_\perp^b, \tag{5.1}$$

and let the normalized matrix element of their interaction be a uniform function with exponents p and q:

$$U_N(\alpha k_z, \alpha k_{1z}, \alpha k_{2z}; \beta \mathbf{k}_\perp, \beta \mathbf{k}_{1\perp}, \beta \mathbf{k}_{2\perp}) = \alpha^p \beta^q U_N(k_z, k_{1z}, k_{2z}; \mathbf{k}_\perp, \mathbf{k}_{1\perp}, \mathbf{k}_{2\perp}). \tag{5.2}$$

Then the kinetic equation for waves has stationary solutions, determining the form of the inertial turbulent spectra:

$$N_k = \tilde{N} k_z^{-\mu} k_\perp^{-\nu}, \tag{5.3}$$

where N_k is the plasmon number density in k-space and for μ and ν there are two solutions:

$$\mu_1 = 1 + p, \qquad \nu_1 = 2 + q, \tag{5.4}$$

corresponding to the constant spectral flux of enstrophy in the long-wavelength domain (energy in the short-wavelength domain), and

$$\mu_2 = \tfrac{1}{2}(3 + 2p - a), \qquad \nu_2 = \tfrac{1}{2}(4 + 2q - b), \tag{5.5}$$

corresponding to the constant spectral flux of energy in the long-wavelength domain (enstrophy in the short-wavelength domain).

The normalized matrix element is the coupling coefficient entering the kinetic equation for the quantum-number density N_k:

$$N_k = \frac{W_k}{\omega_k}. \tag{5.6}$$

Using the norm

$$N_k = c_k^2 \left|A_k\right|^2, \tag{5.7}$$

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the expression for U_N follows from (2.1):

$$U_N(\mathbf{k},\mathbf{k}_1,\mathbf{k}_2) = \frac{c_k}{c_1 c_2} U(\mathbf{k},\mathbf{k}_1,\mathbf{k}_2). \tag{5.8}$$

The norm coefficients c are found from the energy density in KAWs,

$$W_k = \frac{k_\perp^2 A_k^2}{4\pi}, \tag{5.9}$$

and (5.6),

$$c_k^2 = \frac{k_\perp^2}{4\pi\omega_k}.$$

The normalized matrix elements of three-wave interaction (5.8) and wave dispersion (1.1) become uniform functions of the wavenumbers k_z and $k_\perp$ with uniformity indexes $a = 1$, $b = 1$, $p = \frac{1}{2}$ and $q = \frac{5}{2}$ in the limit of strong wave dispersion. Using these uniformity indexes in (5.4) and (5.5), we find two kinds of Kolmogorov-type power-law spectra of KAW turbulence in the domain $k_\perp > \rho_i^{-1}$: one formed by an enstrophy cascade,

$$W_k \sim (k_z)^{-1/2}(k_\perp)^{-3}, \tag{5.10}$$

and another formed by an energy cascade,

$$W_k \sim (k_z)^{-1/2}(k_\perp)^{-3.5}. \tag{5.11}$$

In the limit of weak wave dispersion, the wave dispersion (1.1) can be made uniform only if all the interacting KAWs propagate in the same direction ($s_1 = s_2 = 1$). Then, moving with the Alfvén-velocity reference system, we get the uniformity indexes $a = 1$, $b = 2$, $p = \frac{1}{2}$ and $q = 3$, and turbulent spectra

$$W_k \sim (k_z)^{-1/2}(k_\perp)^{-5}, \tag{5.12}$$

formed by an enstrophy cascade, and

$$W_k \sim (k_z)^{-1/2}(k_\perp)^{-4}, \tag{5.13}$$

formed by an energy cascade.

It is interesting to note that the same KAW spectra in the domain $k_\perp < \rho_i^{-1}$ have been reported by Mikhailovskii *et al.* (1989), using a Grad-type magnetohydrodynamic approach, and by Voitenko (1989), using a mixed kinetic/MHD approach. However, comparing (2.5) and (2.7), one should note that interactions including KAWs propagating in opposite directions ($s_1 = 1, s_2 = -1$), neglected in these papers, are much more effective than those among parallel-propagating KAWs ($s_1 = s_2 = 1$) in the domain $k_\perp < \rho_i^{-1}$. Therefore the spectra (5.12) and (5.13) may be formed under special conditions only, when $s_2 = -1$ interactions are reduced (Voitenko 1998). To find the spectrum formed by $s_2 = -1$ interactions, we take into account that secondary (decay) waves propagating in the opposite ($k_z < 0$) direction carry a small fraction of energy,

$$W_2 \approx 0.2\left(\frac{T_e}{T_i} + \frac{3}{4}\right)\rho_i^2 k_\perp^2 W_k \ll W_k, \tag{5.14}$$

and supply a sink, completing the enstrophy balance of interacting $k_z > 0$ and $k_{1z} > 0$ waves. In that case the energy will be redistributed along the spectrum of parallel-propagating KAWs, carrying the main part of the energy. Taking the uniformity indexes of parallel-propagating KAWs $a = 1$, $b = 2$, and the uniformity

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indexes of the matrix element $p = \frac{1}{2}$, $q = 1$, we get the spectrum formed by the enstrophy cascade,

$$W_k \sim (k_z)^{-1/2}(k_\perp)^{-3}, \tag{5.15}$$

and the spectrum formed by the energy cascade,

$$W_k \sim (k_z)^{-1/2}(k_\perp)^{-2}. \tag{5.16}$$

6. Discussion

Taking into account the directions of enstrophy and energy cascades, determined in the previous section, we can find the turbulent KAW spectra formed with different source positions in **k**-space. In the case of KAWs propagating in the same direction ($s_1 = s_2 = 1$), the following spectra are formed:

(i) if $k_{s\perp} < \rho_i^{-1}$,

$$W_k \sim \begin{cases} (k_z)^{-1/2}(k_\perp)^{-5} & \text{(direct enstrophy cascade for } k_{s\perp} < k_\perp < \rho_i^{-1}\text{)}, \\ (k_z)^{-1/2}(k_\perp)^{-4} & \text{(inverse energy cascade for } k_\perp < k_{s\perp}\text{)}; \end{cases} \tag{6.1}$$

(ii) if $k_{s\perp} = \rho_i^{-1}$,

$$W_k \sim \begin{cases} (k_z)^{-1/2}(k_\perp)^{-4} & \text{(inverse energy cascade for } k_\perp < k_{s\perp}\text{)}, \\ (k_z)^{-1/2}(k_\perp)^{-3.5} & \text{(direct energy cascade for } k_{s\perp} < k_\perp\text{)}; \end{cases} \tag{6.2}$$

(iii) if $k_{s\perp} > \rho_i^{-1}$,

$$W_k \sim \begin{cases} (k_z)^{-1/2}(k_\perp)^{-3} & \text{(inverse enstrophy cascade for } \rho_i^{-1} < k_\perp < k_{s\perp}\text{)}, \\ (k_z)^{-1/2}(k_\perp)^{-3.5} & \text{(direct energy cascade for } k_{s\perp} < k_\perp\text{)}. \end{cases} \tag{6.3}$$

Different spectra may be formed by $s_1 = -s_2 = 1$ interaction in the domain $k_{s\perp} < \rho_i^{-1}$:

$$W_k \sim \begin{cases} (k_z)^{-1/2}(k_\perp)^{-3} & \text{(direct enstrophy cascade for } k_{s\perp} < k_\perp < \rho_i^{-1}\text{)}, \\ (k_z)^{-1/2}(k_\perp)^{-2} & \text{(inverse energy cascade for } k_\perp < k_{s\perp}\text{)}. \end{cases} \tag{6.4}$$

In realistic situations at sufficiently high and low wavenumbers (say, at $k_{\perp\,\max}$ and $k_{\perp\,\min}$), different mechanisms of energy redistribution or dissipation come into play, balancing energy transfer induced by three-wave interaction, and placing upper and lower cut-offs on KAW spectra and spectral cascades.

7. Applications

(i) We apply the results presented here to KAW turbulence, observed in the auroral magnetosphere (Gurnett *et al.* 1984; Dubinin *et al.* 1988). In the companion paper (Voitenko 1998), we have shown that three-wave interactions are important in the evolution of KAWs propagating along auroral field lines from the distant magnetosphere towards the Earth. We proceed further here, considering spectral energy distributions in KAW turbulence.

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Let as assume some source in a distant magnetosphere, pumping the energy into modes $k_\perp \approx k_{s\perp}$; this may be current-driven KAW instability, excited at altitudes $h = (2-4) \times 10^4$ km (Voitenko *et al.* 1990). In accordance with the results presented in Sec. 5, the turbulent KAW spectrum formed by three-wave $s_1 = -s_2 = 1$ interaction in the domain $k_{s\perp} < \rho_i^{-1}$ varies as $(k_\perp)^{-2}$ in the vicinity of the source; the spectrum $\sim (k_\perp)^{-4}$, formed by $s_1 = s_2 = 1$ interaction, can appear at smaller wavenumbers, where $s_1 = -s_2 = 1$ interaction may be forbidden by the finite correlation length of waves. Then the tendency for turbulent KAW spectra is to become steeper, up to $\sim (k_\perp)^{-4}$, as a result of dominant role of $s_1 = s_2 = 1$ interaction at low wavenumbers. Also, since the growth rate of nonlinear interaction attains its maximum at $k_{2\perp} = 0.776 k_{s\perp}$, the energy transfer is most efficient into modes concentrated around $(k_\perp)^{(1)} \approx 0.776 k_{s\perp}$, $(k_\perp)^{(2)} \approx 0.776\,(k_\perp)^{(1)}$, and so on. As a result, the humps on the above-mentioned spectra appear at $k_\perp \approx (0.776)^n\, k_{s\perp}$, where $n = 1, 2, 3, \ldots$ is the decay (step) number.

All of these features of turbulent KAW spectra, following from our theoretical considerations, have been revealed by spacecraft observations (Gurnett *et al.* 1984; Dubinin *et al.* 1988).

The general tendency for the KAW turbulence is energy concentration in the spectrum $\sim (k_\perp)^{-4}$ at low wavenumbers. Spectra $\sim (k_\perp)^{-4}$ have been often observed in space plasmas with intense field-aligned currents (Gurnett *et al.* 1994; Labelle and Treumann 1988). These observations supply strong evidence for the nonlinear spreading of spectra generated by the current-driven KAW instability at $k_{s\perp} \lesssim \rho_i^{-1}$ (Voitenko *et al.* 1990).

The inverse turbulent cascade transports wave energy into the low-wavenumber domain, where dissipation is rather weak. In these cases KAWs with larger amplitudes and lower $k_\perp$ can excite vortical structures (Shukla and Stenflo 1995), simultaneously observed on the background smaller-scale turbulence (see e.g. Dubinin *et al.* 1988).

(ii) One of the most intriguing discoveries of the *Yohkoh* mission is the impulsive heating of collisionless coronal plasma in solar flares. A heating mechanism incorporating intermediate turbulence of KAWs has been proposed by Voitenko (1996). Here we present an estimation of the saturation level of the KAW turbulence excited by proton beams in flaring loops. Taking a typical value for the (linear) instability growth rate $\gamma_k^L/\Omega_i = 5 \times 10^{-2}$ in a flaring loop with magnetic field $B_0 = 100$ G, Alfvén velocity $V_A = 10^8$ cm s^{-1} and ion temperature $T_i = 5 \times 10^6$ K, we get the saturation level of the wave amplitude in the source ($k_\perp \rho_i \lesssim 1$) region: $B_k^{(s)}/B_0 = 5 \times 10^{-2}$. The (non-thermal) velocity of wave motions, calculated from the saturation level, $V_{\rm pl} = 50 \times 10^5$ cm s^{-1}, is three times smaller than the $V_{\rm pl} = 150$ km s^{-1} observed at flare onset (Antonucci and Dodero 1995). We consider this discrepancy between the calculated and observed non-thermal velocity to be a significant indication that a turbulent cascade and concentration of KAW energy at lower wavenumbers occur in flaring loops. With a spectrum $\sim (k_\perp)^{-2}$, we calculate the lower boundary of the inertial range, $k_{\perp\min} = k_{s\perp}/3 \lesssim 0.7 \times 10^{-2}$ cm^{-1}, where the energy of KAW turbulence is concentrated.

(iii) As linear kinetic theory shows, KAWs may easily be excited in tokamaks by various destabilizing factors, for example by fusion α-particles (see e.g. Gang 1992). Ion Compton scattering (Gang 1992; Wising and Gang 1994), transferring energy into modes $k_\perp > k_{s\perp}$, has been considered as the most important mechanism

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for the spectral dynamics in α-particle-driven KAW turbulence. To estimate the relative importance of ion Compton scattering and three-wave interaction in KAW spectral dynamics, we write down the condition that ion Compton scattering be more effective:

$$\gamma_k^{NL}(k_\perp) < \gamma_k^{I-C}(k_\perp),$$

resulting in the parametric relation

$$S < \left(\frac{R}{a}\right)^2 \beta \left(\frac{\rho_i}{a}\right)^{-1},$$

where $S\,(\approx 100)$ is a constant determined by the source of the turbulence, and R and a are the major and minor tokamak radii. With ITER-like parameters, this condition may be rewritten as $S < 10$–100, indicating that both ion Compton scattering and three-wave interaction are of the same order. When three-wave interaction is more efficient, the energy is ultimately being placed into the largest scales. Since the directions of energy transfer induced by ion Compton scattering and three-wave interaction are opposite to each other, a careful examination of their relative importance under concrete conditions is required.

8. Conclusions

The dynamic equation governing kinetic KAW instabilities and weak nonlinear interaction among KAWs in a low-β plasma has been used to study weak KAW turbulence. Kinetic and dispersive finite-Larmor-radius corrections retained in both the linear and nonlinear parts of the dispersion relation allow one to find the nonlinear saturation level of kinetic KAW instability, turbulent spectra, and the directions of the spectral energy and enstrophy fluxes.

The remarkable property of three-wave interaction among parallel-propagating KAWs is spectral energy escaping from the region $k_\perp \approx \rho_i^{-1}$. As a result, inverse and direct energy cascades and Kolmogorov-type spectra $W_k \sim (k_z)^{-1/2}(k_\perp)^{-4}$ and $W_k \sim (k_z)^{-1/2}(k_\perp)^{-3.5}$, may be formed in the domains $\kappa < 1$ and $\kappa > 1$ respectively. If the interaction including counterstreaming KAWs dominates, the turbulent energy always cascades into lower wavenumbers, forming $W_k \sim (k_z)^{-1/2}(k_\perp)^{-3.5}$ and $W_k \sim (k_z)^{-1/2}(k_\perp)^{-2}$ spectra in the limits $\kappa > 1$ and $\kappa < 1$. In contrast, enstrophy tends to concentrate at wavenumbers $k_\perp \approx \rho_i^{-1}$, resulting in $W_k \sim (k_z)^{-1/2}(k_\perp)^{-5}$ and $W_k \sim (k_z)^{-1/2}(k_\perp)^{-3}$ spectra.

Of critical significance for the energy and enstrophy cascades and the resulting turbulent spectra is the type of three-wave interaction, as well as the position of the source in $\mathbf{k}$-space with respect to $k_\perp = \rho_i^{-1}$.

Three examples have shown that three-wave couplings and resulting spectral dynamics in KAW turbulence may be of great importance in both space and laboratory plasmas.

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