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On the Rees–Suschkewitsch structure theorem

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It was remarked in (1) that many of the results valid for compact (this term is taken to imply Hausdorff) semigroups with jointly continuous multiplication also hold if the multiplication is only separately continuous. It is the purpose of this note to show that this is true of the structure theorem for compact simple semigroups. Any unsubstantiated facts about semigroups which we use may be found in the second section of (1) (the results given there do not depend on the assumption that a semigroup should contain an identity).

By a simple semigroup, we mean one which has no proper ideals; so S is simple if and only if $SxS = S$ for each $x \in S$. A semigroup will be called left (resp. right) simple if it has no proper left (resp. right) ideals, i.e. $Sx = S$ (resp. $xS = S$) for all $x \in S$. Both left simple and right simple semigroups are clearly simple.

Completely simple semigroups may be defined to be simple semigroups which possess both minimal left and minimal right ideals (Clifford (7)). In particular, a simple semigroup which can be given a compact topology in which its multiplication is separately continuous is completely simple (from Lemma 2·2 of (1)).

If S is a completely simple semigroup, and g is any idempotent in S, Sg (resp. gS) is a minimal left (resp. right) ideal in S. The set E (resp. F) of idempotents in Sg (resp. gS) forms a semigroup in which multiplication is given by $e_1e_2 = e_1$ (resp. $f_1f_2 = f_2$); in particular, $eg = e$ for all $e \in E$ (and $gf = f$ for all $f \in F$). The subset $G = gSg$ of S is a group.

For each $x \in S$, there are precisely one $e \in E$ and one $f \in F$ for which $ex = x$ and $xf = x$. Put $\bar{x} = gxg$. Then the map

$$x \to (e, \bar{x}, f) \tag{1}$$

of $S \to E \times G \times F$ is well-defined; moreover, it has an inverse, for

$$e\bar{x}f = e(gxg)f = (eg)\,x(gf) = exf = x;$$

therefore it is a bijection. We find

Theorem 1. *A completely simple semigroup S is isomorphic with $E \times G \times F$ when multiplication is defined in the latter by*

$$(e_1, x_1, f_1)(e_2, x_2, f_2) = (e_1, x_1[f_1e_2]x_2, f_2), \tag{2}$$

where $[fe] = gfeg$ for all $f \in F$, $e \in E$; in particular, $[fg] = [ge] = g$ for all f, e.

This result is easily obtained by translating the structure of S to $E \times G \times F$ by means of the bijection (1) (see (3)). It is due to Rees (4) who generalized the result for the case of a finite semigroup given by Suschkewitsch (5). ⊙

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(The sign ⊙ indicates the end of a proof; it follows the statement of the result whenever the proof is omitted.)

A converse result is also due to Rees (4):

THEOREM 2. *Let E and F be any two sets, G any group, and $(f, e) \to [fe]$ any map of $F \times E$ into G. Then $E \times G \times F$ with multiplication defined by* (2) *is a completely simple semigroup.* ⊙

A corollary of Theorem 1 and Theorem 2 is

THEOREM 3. *A semigroup S is left simple and completely simple if and only if it is isomorphic to a direct product $E \times G$, where multiplication in E is defined by $e_1 e_2 = e_1$, and G is a group.*

For in this case, $gS = gSg$, being a group, contains only one idempotent, viz. g. ⊙

Wallace (6) announced that if S has a compact topology in which multiplication is jointly continuous, if E, G and F are given the topologies induced on them by that of S, and if $E \times G \times F$ is given the product topology, the isomorphism (1) becomes a homeomorphism as well. We shall show that the weaker hypothesis of separate continuity is sufficient.

From now on, S will denote a separately continuous, compact, simple semigroup. If $x \in S$ is mapped into $(e, \bar{x}, f)$ by the isomorphism (1), we write $\mathrm{pr}_1(x) = e$, $\mathrm{pr}_2(x) = \bar{x}$ and $\mathrm{pr}_3(x) = f$; (these maps are in a sense the projections of S onto E, G and F respectively).

LEMMA. pr_2 *is continuous.*

Proof. Since $\mathrm{pr}_2(x) = gxg$, this follows from separate continuity of multiplication. ⊙

PROPOSITION 1. *E, G and F are compact. G is a topological group.*

Proof. $G = \mathrm{pr}_2(S)$ is compact since pr_2 is continuous, and so is a topological group (Theorem 2 of (2)).

We show that E (and by symmetry, F) is compact by proving it is closed in S. Let $e \in \bar{E}$, the closure of E, and let $(e_\alpha)_{\alpha \in A}$ be some net in E with $e_\alpha \to e$. Then since multiplication is separately continuous,

$$eg = \lim_{\alpha \in A} e_\alpha g = \lim_{\alpha} e_\alpha = e \quad \text{and} \quad e^2 = \lim_{\alpha \in A} \lim_{\beta \in A} e_\alpha e_\beta = \lim_{\alpha} \lim_{\beta} e_\alpha = e.$$

So e is an idempotent in Sg, i.e. $e \in E$. ⊙

We can now give a quick proof of Wallace's result.

THEOREM 4. *If multiplication in the compact simple semigroup S is jointly continuous, the map* (1) *is a homeomorphism.*

Proof. The joint continuity ensures that $(e, x, f) \to exf$ is continuous. Since this map is biunique, and both $E \times G \times F$ and S are compact, it is a homeomorphism. ⊙

To prove that (1) is a homeomorphism in the separately continuous case does not seem so easy. We first give a situation in which, as in groups, compactness and separate continuity imply joint continuity.

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THEOREM 5. *Multiplication in a compact, left simple semigroup S is jointly continuous.*

Proof. Using Theorem 3, we identify S with $E \times G$.

Since multiplication in S is separately continuous, the map

$$(e, x) \to (e, xy) = (e, x)(g, y)$$

is continuous for each $y \in G$. Moreover, it has a continuous inverse $(e, x) \to (e, xy^{-1})$ and so is a homeomorphism. We may now consider on G, regarded as a subset of the space of continuous functions mapping S onto S, the topology of pointwise convergence, in which a sub-basic set of neighbourhoods of $y \in G$ is given by

$$V = \{z\colon (e, xz) \in N = N(e, xy)\},$$

where (e, x) runs through S and N runs through the set of neighbourhoods in S of (e, xy). But again by separate continuity, the map $y \to (e, xy) = (e, x)(g, y)$ of $G \to S$ is continuous for each $(e, x) \in S$, and therefore V is also a neighbourhood of y in the induced topology on G. The topology of pointwise convergence is therefore coarser than, and so coincides with, the induced, compact topology on G.

Under these conditions (i.e. having a group G of homeomorphisms of a compact space S, which is separately continuous and compact in the topology of pointwise convergence) we may apply Ellis's Theorem 1 (2), to conclude that the map

$$[(e, x), y] \to (e, xy)$$

of $S \times G \to S$ is continuous. Since the map $(e', y) \to y$ of $S \to G$ is continuous by the lemma above, the map $[(e, x), (e', y)] \to (e, x)(e', y) = (e, xy)$ of $S \times S \to S$ is continuous. ⊙

THEOREM 6. *The map* (1) *is a homeomorphism of S with $E \times G \times F$.*

Proof. We shall show that the maps pr_1, pr_2 and pr_3 are continuous, and it will then follow from the theory of topological products that the biunique map (1) $S \to E \times G \times F$ is continuous. Since both spaces are compact, this will establish the homeomorphism.

Sg is a compact subsemigroup of S which is homeomorphic with $E \times G \times \{g\}$ by Theorem 4 and Theorem 5. Therefore the map $(e, x, g) \to (e, g, g)$ of $Sg \to E$ is continuous. Since multiplication is separately continuous,

$$(e, x, f) \to (e, x, f)(g, g, g) = (e, x, g)$$

maps $S \to Sg$ continuously. pr_1 is just the composition of these two maps. Similarly, pr_3 is continuous. pr_2 is continuous by the lemma. ⊙

Since $(g, g, f)(e, g, g) = (g, [fe], g)$, we deduce

PROPOSITION 2. *If S is a separately (resp. jointly) continuous compact simple semigroup, the map $(f, e) \to [fe]$ of $F \times E \to G$ is separately continuous (resp. continuous).* ⊙

We now show that we have characterized compact simple semigroups.

THEOREM 7. *Let E and F be any two compact sets; let G be any compact (topological) group; let $(f, e) \to [fe]$ be any separately continuous (resp. continuous) map of $F \times E \to G$ (there are at least the trivial ones $[fe] = x$ for all (f, e) and some fixed $x \in G$); and let a*

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product be defined in $S = E \times G \times F$ *by* (2). *Then* S *is a compact simple semigroup* (*resp. jointly continuous compact simple semigroup*).

Proof. From Theorem 2, S is a simple semigroup; it is clearly compact, and multiplication is easily seen to have the desired continuity properties. ⊙

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