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A Joint Framework for Consistently Pricing Interest Rates and Interest Rate Derivatives

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Abstract

Dynamic term structure models explain the yield curve variation well but perform poorly in pricing and hedging interest rate options. Most existing option pricing practices take the yield curve as given, thus having little to say about the fair valuation of the underlying interest rates. This paper proposes an $m + n$ model structure that bridges the gap in the literature by successfully pricing both interest rates and interest rate options. The first m factors capture the yield curve variation, whereas the latter n factors capture the interest rate options movements that cannot be effectively identified from the yield curve. We propose a sequential estimation procedure that identifies the m yield curve factors from the LIBOR and swap rates in the first step and the n options factors from interest rate caps in the second step. The three yield curve factors explain over 99% of the variation in the yield curve but account for less than 50% of the implied volatility variation for the caps. Incorporating three additional options factors improves the explained variation in implied volatilities to over 99%.

I. Introduction

In pricing interest rates and interest rate options, the literature takes two distinct approaches. The first approach, often referred to as a dynamic term structure model, captures the dynamics of the yield curve with a finite-dimensional state vector. Empirical studies show that well-designed dynamic term structure models can explain over 90% of the variation on the yield curve with as few as three factors (Dai and Singleton (2000)). However, these models perform poorly in pricing and hedging interest rate options (Li and Zhao (2006)).

In practice, the task of pricing interest rate options has mostly been handled by alternative approaches that take the yield curve as given and focus exclusively

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on the specification of the volatility structure.[1] Yet, by taking the yield curve as given, these models have little to say about the fair valuation of the underlying interest rates. Furthermore, accommodating the whole yield curve often necessitates accepting an infinite-dimensional state space and/or time-inhomogeneous model parametrization, both of which create difficulties for hedging practices.

In this paper, we propose an $m + n$ model structure that bridges the gap between the two existing approaches by successfully pricing both interest rates and interest rate options within a finite-dimensional framework. Under this framework, the first m factors capture the systematic variation of the yield curve and hence are referred to as the yield curve factors. The latter n factors capture the interest rate options movements that cannot be effectively identified from the yield curve. We label them as the interest rate options factors.

We elaborate on the model structure through a simple 3 + 3 Gaussian affine example. First, we use three factors to price the yield curve. Then, we capture the remaining interest rate options movements with another three factors. We estimate the model using eight years of data on U.S. dollar London Interbank Offered Rate (LIBOR), swap rates, and implied volatilities for the interest rate caps. The estimation is performed with a quasi-maximum likelihood method jointly with extended Kalman filter via a two-stage sequential procedure. In the first stage, we estimate the dynamics of the yield curve factors using the LIBOR and swap rates. In the second stage, we estimate the dynamics of the options factors based on the market quotes on the implied volatilities for the interest rate caps.

Despite its simple structure, our model performs well in pricing both interest rates and interest rate caps. The three yield curve factors explain over 99% of the variation in the yield curve but less than 50% of the variation in the implied volatility. By incorporating three additional options factors, the model also explains over 99% of the variation in the implied volatility.

We perform an out-of-sample analysis by reestimating the model using the first six years of data and then comparing the model's performance both in sample during the first six years and out of sample during the last two years. The subsample parameter estimates are similar to those from the whole sample, and the performance comparison shows no deterioration for the out-of-sample performance, suggesting that the model structure is stable.

For further robustness analysis, we also estimate an alternative specification for the three yield curve factors that allow for both stochastic central tendency and stochastic volatility. The estimation results show that incorporating stochastic volatility in the yield curve factors does not replace the need for the additional options factors in pricing the interest rate caps. Without the options factors, the three yield curve factors explain only 40% of the variation in the interest rate caps. Only after we incorporate the three options factors can the model explain over 99% of the interest rate cap variation.

[1]Examples include the forward rate models of Ho and Lee (1986), Hull and White (1993), and Heath, Jarrow, and Morton (1992); the market rate models of Brace, Gatarek, and Musiela (1997), Jamshidian (1997), Miltersen, Sandmann, and Sondermann (1997), and Musiela and Rutkowski (1997a); the string models of Goldstein (2000), Santa-Clara and Sornette (2001), and Longstaff, Santa-Clara, and Schwartz (2001b); and the nonparametric pricing approach of Aït-Sahalia (1996).

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The fact that interest rate factors identified from the yield curve cannot account for the movement of interest rate options has been documented by several recent studies. In a joint statistical analysis on LIBOR, swap rates, and implied volatilities for swaptions, Heidari and Wu (2003) find that three principal components extracted from the yield curve explain over 99% of the interest rate movements, but they only explain 60% of the variation in the implied volatility surface. They further find that three additional principal components extracted from the implied volatilities are needed to explain the movement in the implied volatility surface. Collin-Dufresne and Goldstein (2002) document similar evidence on implied volatilities for interest rate caps and refer to the evidence as unspanned stochastic volatility.

Several possible reasons contribute to the empirical findings. First, as suggested by Collin-Dufresne and Goldstein (2002), there are possibly interest rate stochastic volatility factors that are not spanned by the yield curve. Within the affine family of dynamic term structural models, they identify a set of parameter constraints so that the stochastic volatility of interest rates is not instantaneously correlated with the value of interest rates.[2] Nevertheless, Bikbov and Chernov (2009) show that these parametric constraints do not fully resolve the tension between the pricing of interest rate futures and options.

Generally speaking, interest rate volatility affects the curvature of the yield curve and hence is linked to the term structure, except under very special parametric restrictions as, for example, those identified in Collin-Dufresne and Goldstein (2002). Thus, their unspanned volatility specifications can only be regarded as special cases rather than the general rule. In particular, if a general dynamic term structure model has difficulty pricing both interest rates and interest rate options, it is unlikely that a constrained version of the model can do the job well.

A more plausible interpretation comes from misidentification and/or misspecification.[3] In practice, even if stochastic volatility factors affect the yield curve and are spanned by the yield curve in theory, we may still have difficulties in identifying these factors *statistically* from the yield curve if the dependence is weak. In this case, it is not a matter of identifying parametric constraints so that the volatility factors are indeed unspanned in theory, but a matter of designing an efficient factor structure and estimation procedure so that factors that affect the yield curve and the options can both be effectively identified.

Yet another possible source of failure for dynamic term structure models is that the models price interest rate options based on the model-implied fair values of the yield curve, thus ignoring any potential impacts of the residuals on the yield curve. A small misalignment between the observed interest rates and the model-implied values can lead to large variations in the implied volatilities on options. Such small and temporal misalignments occur naturally in a finite-dimensional framework, either due to model approximations or market imperfections. To avoid

[2]They also show that incorporating stochastic volatility under the Heath et al. (1992) framework automatically generates stochastic volatility unspanned by the term structure. However, as we have argued earlier, this class of models takes the yield curve as given and hence does not take a stance on the fair valuation of the underlying interest rates.

[3]We thank the referee for suggesting this interpretation.

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the bias introduced by such misalignments, most existing option pricing practices take the observed yield curve as given, either by accepting an infinite-dimensional state space or through time-inhomogeneous model parametrization.

Our $m + n$ model accommodates all these different scenarios. Under our model, we decompose the dynamics of each interest rate series into a systematic yield curve component and an orthogonal residual component that has little impact on the interest rate term structure but can have significant impacts on interest rate options. The residual component can proxy for i) the unspanned volatility of Collin-Dufresne and Goldstein (2002); ii) interest rate factors that cannot be effectively identified from the yield curve, even if they are spanned in theory by the yield curve; and iii) temporary misalignments between the observed interest rates and the fair value implied by a finite-dimensional dynamic term structure model. The orthogonal decomposition also allows us to design an efficient sequential estimation procedure, which estimates the yield curve component from the time series of interest rates in the first step and identifies the remaining options variations from the time series of options in a second step.

The remainder of the paper is structured as follows. Section II elaborates on the model structure through a 3 + 3 Gaussian affine example. Section III discusses the data and the estimation procedure. Section IV discusses the estimation results and model performance. Section V considers an alternative term structure specification. Section VI concludes.

II. The $m + n$ Model

We propose an $m + n$ model structure to price both interest rates and interest rate options. We use the first m factors to capture the systematic movement of the yield curve and the additional n factors to capture interest rate dynamics that cannot be effectively identified from the yield curve but nevertheless have important impacts on option pricing.

A. The Basic Model Structure

We fix a filtered complete probability space $\{\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{0\leq t\leq \mathcal{T}}\}$ satisfying the usual technical conditions with $\mathcal{T}$ being some finite, fixed time. Let F and E denote two vector Markov processes in some state space $\mathcal{D}_m \in \mathbb{R}^m$ and $\mathcal{D}_n \in \mathbb{R}^n$, respectively. We assume that the observed market quotes on interest rates, y_t, are subject to the following heuristic orthogonal decomposition:

$$y_t \quad = \quad Y(F_t) + \varepsilon(E_t),$$

where $Y(F_t)$ denotes the "fair value" of the interest rate, the dynamics of which are controlled by the state vector F_t, and $\varepsilon(E_t)$ denotes the remaining movements of the interest rate that have little impact on the yield curve variation but have significant impacts on interest rate options. We assume that this residual component is also governed by a finite-dimensional state vector E_t. We christen F_t as the *yield curve factors* and E_t as the *interest rate options factors*. We assume that the

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two state vectors are orthogonal to each other and satisfy the following stochastic differential equations under the statistical measure $\mathbb{P}$:

$$(1) \quad dF_t = \mu(F_t)dt + \Sigma(F_t)dW_t, \qquad dE_t = \mu(E_t)dt + \Sigma(E_t)dZ_t,$$

where $\mu(F_t)$ is an $m \times 1$ vector defining the drift and $\Sigma(F_t)$ is an $m \times m$ matrix defining the diffusion of the F process. Similarly, $\mu(E_t)$ is an $n \times 1$ vector and $\Sigma(E_t)$ is an $n \times n$ matrix defining the drift and diffusion of the E process. W_t and Z_t are independent Brownian motions with dimension m and n, respectively.

For any time $t \in [0, \mathcal{T}]$ and time of maturity $T \in [t, \mathcal{T}]$, let $P(F_t, T)$ denote the fair value at time t of a zero-coupon bond with maturity $\tau = T - t$, which is only a function of the yield curve factors but independent of the options factors. The fair values of the spot rates are defined as

$$(2) \qquad Y(F_t, T) \equiv \frac{1}{T-t} \ln P(F_t, T),$$

and the fair value of the instantaneous interest rate, or the short rate, r, is defined by continuity,

$$r(F_t) \equiv \lim_{T \downarrow t} \frac{-\ln P(F_t, T)}{T-t}.$$

We assume that there exists a risk-neutral measure $\mathbb{Q}$, which is absolutely continuous with respect to the statistical measure $\mathbb{P}$, such that the time-t fair value of a claim to a terminal payoff Π_T at time $T > t$ can be written as

$$(3) \qquad V(F_t, E_t, T) = \mathbb{E}_t^{\mathbb{Q}} \left[\exp\left(-\int_t^T r(F_s)ds \right) \Pi_T \right],$$

where $\mathbb{E}_t^{\mathbb{Q}}[\cdot]$ denotes the expectation operator conditional on filtration $\mathcal{F}_t$ and under measure $\mathbb{Q}$. Thus, the fair value of a zero-coupon bond can be computed from equation (3) by setting $\Pi_T = 1$ for all states. Since the payoff of a zero-coupon bond is a constant and hence state independent, the fair value of the zero-coupon bond is only a function of the fair value of the short rates during the life of the bond and thus independent of the options factors E, consistent with the original assumption.

Nevertheless, for state-contingent claims such as caps, the payoff at time T is determined by the market observed interest rates at that time, or one period earlier if paid in arrears. Thus, the payoff function Π_T depends on both the dynamics of the yield curve factors F and that of the residual options factors E. As a result, the value of the state-contingent claim will become a function of E as well, in addition to its dependence on the yield curve factors F. Therefore, E can be used to capture movements in interest rate options that cannot be identified statistically from the yield curve.

B. A 3 + 3 Gaussian Affine Example

To illustrate the idea of our modeling structure, we construct and estimate a concrete model within the analytically tractable Gaussian affine family. We assume that both the yield curve factors F and the options factors E have dimensions of three: $m = n = 3$. A yield curve dimension of three is consistent with the

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empirical evidence of Litterman and Scheinkman (1991), Knez, Litterman, and Scheinkman (1994), and Heidari and Wu (2003), as well as the current status quo in affine model design (Dai and Singleton (2000) and Duffee (2002)). The choice of three additional options factors is mainly motivated by the evidence in Heidari and Wu (2003) and Wadhwa (1999) on implied volatilities for swaptions.

We assume that under the statistical measure $\mathbb{P}$, the yield curve factors and the options factors are both governed by Ornstein-Uhlenbeck (OU) processes,

$$dF_t = -\kappa_F F_t dt + dW_t, \qquad dE_t = -\kappa_E E_t dt + dZ_t, \tag{4}$$

where $\kappa_F \in \mathbb{R}^{3\times 3}$ and $\kappa_E \in \mathbb{R}^{3\times 3}$ control the mean-reverting property of the two vector processes. For identification reasons, we normalize both state vectors to have zero long-run means and identity diffusion matrices. We further assume that the fair value of the short rate r is affine in the yield curve factor,

$$r(F_t) = a_r + b_r^\top F_t, \tag{5}$$

where $a_r \in \mathbb{R}$ is a scalar and $b_r \in \mathbb{R}^{3+}$ is a vector. Thus, the fair value of the yield curve only depends on the yield curve factors.

Finally, we close the model by assuming a flexible affine market price of risk for both sets of factors,

$$\gamma(F_t) = b_\gamma + \kappa_\gamma F_t, \qquad \lambda(E_t) = b_\lambda + \kappa_\lambda E_t, \tag{6}$$

with $b_\gamma, b_\lambda \in \mathbb{R}^3$ and $\kappa_\gamma, \kappa_\lambda \in \mathbb{R}^{3\times 3}$. Given these market price of risk specifications, the two types of factors remain OU under the risk-neutral measure $\mathbb{Q}$, but with adjustments to the drift terms:

$$dF_t = \left(-b_\gamma - \kappa_F^{\mathbb{Q}} F_t\right) dt + dW_t^{\mathbb{Q}}, \qquad dE_t = \left(-b_\lambda - \kappa_E^{\mathbb{Q}} E_t\right) dt + dZ_t^{\mathbb{Q}}, \tag{7}$$

with $\kappa_F^{\mathbb{Q}} = \kappa_F + \kappa_\gamma$ and $\kappa_E^{\mathbb{Q}} = \kappa_E + \kappa_\lambda$.

Our yield curve factor specification belongs to the affine class of Duffie and Kan (1996). The fair value of the zero-coupon bond with maturity τ is exponentially affine in the yield curve factor,

$$P(F_t, t+\tau) = \exp\left(-a(\tau) - b(\tau)^\top F_t\right), \tag{8}$$

where the coefficients $a(\tau)$ and $b(\tau)$ are determined by the following ordinary differential equations:

$$\begin{aligned} a'(\tau) &= a_r - b(\tau)^\top b_\gamma - b(\tau)^\top b(\tau)/2, \\ b'(\tau) &= b_r - \left(\kappa_F^{\mathbb{Q}}\right)^\top b(\tau), \end{aligned} \tag{9}$$

subject to the boundary conditions $a(0) = 0$ and $b(0) = 0$. The ordinary differential equations can be solved analytically in terms of the eigenvalues and eigenvectors of the $\kappa_F^{\mathbb{Q}}$ matrix. The fair value of the spot rate is affine in the yield curve factors,

$$Y(F_t, t+\tau) = \frac{1}{\tau}\left(a(\tau) + b(\tau)^\top F_t\right). \tag{10}$$

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The differences between the observed interest rates and the fair values are what we call residuals. They arise either due to the approximate (misspecified) nature of a model or due to temporal market imperfections. Regardless of the source, given well-specified yield curve factor dynamics, the residual on the interest rate term structure should be small. Thus, we can specify the E factor dynamics to capture movements in interest rate options while ignoring their impacts on the yield curve. In our estimation exercise, we price U.S. dollar caps, which are portfolios of options written on three-month LIBOR. Thus, we need to model the dynamics of the residuals on the three-month LIBOR. For this purpose, we assume that the residuals on the three-month LIBOR are a function of three remaining options factors. In particular, we assume the following functional form for the loading of the three options factors on the observed three-month rate:

$$y(t,t+h) = \frac{1}{h}\left(a(h)+b(h)^{\top}F_t+c_h^{\top}E_t\right), \tag{11}$$

where h denotes the exact maturity for the three-month LIBOR, $c_h \in \mathbb{R}^3$ is a vector of constant parameters that determine the loading of the options factors, and $y(t,t+h)$ is the continuously compounded interest rate derived from the observed LIBOR quote, based on the following relation:

$$\text{LIBOR}(h)_t = \frac{100}{h}\left(e^{hy(t,t+h)}-1\right).$$

The linear loading specification in equation (11) of the factors E on the spot rate, instead of on the LIBOR itself, is motivated partly by an analogy to the linear loading of the yield curve factors and partly by analytical tractability in pricing caps.

C. Caplet Pricing

We illustrate the impact of the options factor dynamics on interest rate options by pricing a caplet. Each cap contract consists of a series of caplets. The payoff of the ith caplet with unit dollar notional amount can be written as

$$\Pi_T^i = h\left(\text{LIBOR}(h)_T - K\right)^+,$$

where h denotes the maturity of the LIBOR and also the payment interval (tenor) of the cap contract, $T = t + ih$ is the settlement time (maturity) of the caplet, and K is the strike rate. For U.S. dollar caps, the payment is made in arrears, i.e., the payment of the ith caplet is determined at time T but paid one period later at $T+h$.

Based on equation (3), the time-t fair value of the ith caplet is given by

$$\text{CAPLET}_t^i = \mathbb{E}_t^{\mathbb{Q}}\left[\exp\left(-\int_t^{T+h} r\left(F_s\right)ds\right) h\left(\text{LIBOR}_T(h)-K\right)^+\right],$$

where the observed LIBOR rate depends on both the yield curve factors F and the remaining options factors E. Writing the simply compounded LIBOR rate in

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terms of the continuously compounded spot rate as in equation (11), and by the rule of iterated expectations, we have

$$\text{CAPLET}_t^i = \mathbb{E}_t^{\mathbb{Q}}\left[\exp\left(-\int_t^T r(F_s)\,ds\right) \times \left(e^{c_h^\top E_T} - (1+hK)\,e^{-a(h)-b(h)^\top F_T}\right)^+\right]. \tag{12}$$

Absent from the options factor ($E_t = 0$ for all t), the caplet is equivalent to a put option on a zero-coupon bond. In the presence of independent options movements E, the caplet can be regarded as an exchange option: The option holder has the right to exchange the fair value of a zero-coupon bond for a payoff that is a function of the options factor E.

The approach of dynamic term structure models in pricing options is akin to setting $E_t = 0$ for all t, even if they are present. On the other hand, the option pricing literature often incorporates time- and maturity-dependent parameters in pricing interest rates so that the observed yield curve is forced to match the fair value. If the observed yield curve differs from the fair value implied by a dynamic term structure model, the difference will be accommodated by time-inhomogeneous parameters and will be carried over permanently into the future. This practice amounts to the following modification of equation (12):

$$\text{CAPLET}_t^i = \mathbb{E}_t^{\mathbb{Q}}\left[\exp\left(-\int_t^T [r(F_s)+\mu(s)]\,ds\right) \times \left(1-(1+hK)\,e^{-a(t,h)-b(h)^\top F_T}\right)^+\right], \tag{13}$$

where $\mu(s)$ denotes a time-inhomogeneous parameter that accommodates the observed yield curve. This adjustment not only affects the discounting but also influences the payoff function, because now $a(t,h)$ becomes time-inhomogeneous. The adjustment leads to a dramatic increase in dimensionality, since we need a new parameter $\mu(t)$ for all t. Furthermore, although $\mu(t)$ is time varying and re-calibrated frequently, it is treated as a constant in pricing and hedging derivatives. The uncertainty, and hence risk, associated with this adjustment over time is ignored.

By comparison, under our specification in (12), we recognize the existence of potential model errors, market imperfections, or residual dynamics such as unspanned volatility that are difficult to identify from the yield curve but nevertheless have important impacts on option pricing. We explicitly account for the impacts of these residual dynamics on future terminal payoffs. Meanwhile, the discounting is still based on the fair value of the yield curve as in the practice of dynamic term structure models. This treatment makes the valuation of interest rate options consistent with the valuation of the underlying interest rates.

Carrying out the expectation operation in equation (12) leads to the following caplet pricing formula:

$$\text{CAPLET}_t^i = P(F_t, T+h)\left[(1+h\mathcal{R}_t)\,\mathcal{N}(d_1) - (1+hK)\,\mathcal{N}(d_2)\right], \tag{14}$$

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where $\mathcal{N}(\cdot)$ denotes the cumulative density of a standard normal variable and

$$d_1 = \frac{\ln\left(1+h\mathcal{R}_t\right)/\left(1+hK\right)+\frac{1}{2}\Sigma_t}{\sqrt{\Sigma_t}}, \quad d_2 = d_1-\sqrt{\Sigma_t} \tag{15}$$

are standardized variables. In equation (15), $\mathcal{R}_t$ denotes the residual-adjusted value of the forward three-month LIBOR, defined by

$$\left(1+h\mathcal{R}_t\right) = \frac{P\left(F_t,T\right)}{P\left(F_t,T+h\right)}\exp\left(c_h^{\top}\mathbb{E}_t^{\mathbb{T}}\left[E_T\right]+\frac{1}{2}c_h^{\top}\mathrm{Var}_t^{\mathbb{T}}\left[E_T\right]c_h\right),$$

Σ_t is the time-t conditional variance of $hy_T = a\left(h\right)+\left(h\right)^{\top}F_T+c_h^{\top}E_T$ under a forward measure $\mathbb{T}$, and y_T is the future observed value of the three-month continuously compounded spot rate. The conditional variance Σ_t can be evaluated as

$$\Sigma_t = b\left(h\right)^{\top}\mathrm{Var}_t^{\mathbb{T}}\left[F_T\right]b\left(h\right)+c_h^{\top}\mathrm{Var}_t^{\mathbb{T}}\left[E_T\right]c_h. \tag{16}$$

Appendix A provides details on the derivation of the option pricing formula. Appendix A also derives the conditional mean and variance of F_T and E_T under measures $\mathbb{Q}$ and $\mathbb{T}$.

Equation (14) shows how the dynamics of the yield curve residuals influence the pricing of an interest rate caplet. The impacts come from two sources, both due to the fact that the terminal payoff of the caplet is computed based on the observed market rate, not on some model-implied fair value. First, the forward rate ($\mathcal{R}_t$) is adjusted for the expected impact of the residual dynamics. The adjustment shows up both proportionally to the caplet price and also nonlinearly in the definition of the standardized variables d_1 and d_2. Second, the conditional variance of the underlying three-month LIBOR rate in the option pricing formula Σ_t is the conditional variance of the observed market rate (hy_T), not that of the fair value. Thus, Σ_t captures the aggregate contribution from both the yield curve factors F and the independent options factors E, as illustrated in (16).

III. Data and Estimation

To investigate the performance of our $m+n$ model, we propose a sequential estimation procedure and estimate the $3+3$ affine example using data on LIBOR, swap rates, and implied volatilities for interest rate caps.

A. Data Description

The data, obtained from Lehman Brothers, include weekly (Wednesday) closing mid quotes on i) U.S. dollar LIBOR rates at maturities of one, two, three, six, and 12 months; ii) swap rates at maturities of two, three, five, seven, 10, 15, and 30 years; and iii) at-the-money implied volatilities for interest rate caps at option maturities of one, two, three, four, five, seven, and 10 years. The sample spans eight years from April 6, 1994 to April 17, 2002 (420 weekly observations for each series).

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The U.S. dollar LIBOR rates are simply compounded interest rates. The maturities are computed following actual/360 day-count convention, starting two business days forward. The U.S. dollar swap rates have payment intervals of half years and are related to the zero-coupon bond prices (discount factors) by

$$\text{SWAP}(t, Nh) = 200 \times \frac{1 - p(t, t+nh)}{\sum_{i=1}^{N} p(t, t+ih)},$$

where $h = 0.5$ is the payment interval of the swap contract, and N is the number of payments over the maturity of the swap contract.

The at-the-money cap contracts are on three-month LIBOR rates, with a payment interval of three months. The payment is made in arrears. The strike price is set to the swap rate of the corresponding maturity. The implied volatility quotes are obtained under the framework of the Black (1976) model, where the LIBOR rate is assumed to follow a geometric Brownian motion. Given an implied volatility (IV) quote, the invoice price of the cap contract is computed according to the Black formula:

$$\begin{aligned} \text{CAP}(t, Nh) &= Lh \sum_{i=1}^{N-1} p(t, t+(i+1)h) \\ &\quad \times \left(R(t, t+ih, t+(i+1)h)\mathcal{N}(d_{1i}) - K\mathcal{N}(d_{2i})\right), \end{aligned} \tag{17}$$

where L denotes the notional value of the cap contract, $h = 0.25$ denotes the three-month payment interval, $R(t, t+ih, t+(i+1)h)$ denotes the forward LIBOR rate, K denotes the strike rate, and $\mathcal{N}(\cdot)$ denotes the cumulative normal function with

$$\begin{aligned} d_{1i} &= \frac{\ln(R(t, t+ih, t+(i+1)h)/K) + \text{IV}^2 ih/2}{\text{IV}\sqrt{ih}}, \\ d_{2i} &= d_{1i} - \text{IV}\sqrt{ih}. \end{aligned} \tag{18}$$

Table 1 reports the summary statistics on LIBOR, swap rates, and implied volatilities. The average interest rates exhibit an upward sloping term structure. The standard deviation of interest rates shows a hump-shaped term structure that peaks around six months. The interest rates are all highly persistent, with the weekly autocorrelation estimates ranging from 0.981 to 0.987. LIBOR rates also show some moderate excess kurtosis. The mean implied volatility exhibits a hump-shaped term structure that peaks at three-year maturity. The standard deviation of the implied volatility declines with increasing maturity. The implied volatilities are also persistent, with the weekly autocorrelation estimates ranging from 0.961 to 0.972.

B. A Sequential Estimation Procedure for the $m + n$ Model

We propose a sequential two-stage procedure for estimating the $m+n$ model. The first-stage estimates the yield curve factors using the LIBOR and swap rates. The second-stage estimates the options factors using cap option prices. At each

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TABLE 1

Summary Statistics of Interest Rates and Implied Volatilities

Entries are summary statistics of interest rates and implied volatilities. Mean, Std Dev, Skewness, Kurtosis, and Auto denote the sample estimates of the mean, standard deviation, skewness, excess kurtosis, and first-order autocorrelation, respectively. In the maturity column, LIBOR maturities are in months (m), swap and cap maturities are in years (y). The data are weekly closing mid quotes from Lehman Brothers from April 6, 1994 to April 17, 2002 (420 observations per series).

Maturity	Mean	Std Dev	Skewness	Kurtosis	Auto
Panel A. LIBOR and Swap Rates (%)					
1 m	5.259	1.113	−1.736	2.870	0.981
2 m	5.296	1.123	−1.754	2.892	0.983
3 m	5.341	1.137	−1.750	2.854	0.984
6 m	5.431	1.151	−1.667	2.631	0.986
12 m	5.638	1.148	−1.349	1.923	0.987
2 y	5.908	1.012	−0.805	0.837	0.986
3 y	6.101	0.909	−0.454	0.244	0.985
5 y	6.333	0.806	−0.100	−0.319	0.983
7 y	6.480	0.762	0.048	−0.516	0.982
10 y	6.633	0.731	0.130	−0.607	0.981
15 y	6.797	0.703	0.236	−0.604	0.981
30 y	6.893	0.698	0.393	−0.480	0.984
Panel B. Cap Implied Volatility (%)					
1 y	16.930	7.560	1.893	4.064	0.972
2 y	18.935	5.378	1.119	1.450	0.974
3 y	19.290	4.138	0.735	0.434	0.971
4 y	19.162	3.461	0.533	0.079	0.970
5 y	18.973	3.058	0.416	−0.094	0.968
7 y	18.237	2.490	0.362	0.024	0.967
10 y	17.339	2.071	0.415	0.193	0.961

stage, we cast the model into a state-space form, obtain efficient forecasts on the conditional mean and variance of the observed series using an extended Kalman filter, and build the likelihood function on the forecasting errors of the observed series, assuming that the forecasting errors are normally distributed. The model parameters are estimated by maximizing the likelihood function.

To estimate the 3 + 3 Gaussian affine model, in the first stage, we regard the yield curve factors as the unobservable states and specify the state propagation equation using an Euler approximation of the yield curve factor dynamics in equation (1),

$$F_t = \Phi_F F_{t-1} + \sqrt{\mathcal{Q}}\varepsilon_t, \tag{19}$$

where $\Phi_F = \exp(-\kappa_F \Delta t)$ denotes the autocorrelation matrix with $\Delta t = 1/52$ being the length of the weekly discrete-time interval, $\mathcal{Q} = I\Delta t$ denotes the instantaneous covariance matrix with I being an identity matrix of dimension three, and ε_t denotes an independent and identically distributed trivariate standard normal innovation vector. For notational clarity, we normalize the discrete-time interval to one.

The measurement equations for the first-stage estimation are constructed based on the observed LIBOR and swap rates by assuming additive, normally distributed pricing errors,

$$m_t = \begin{bmatrix} \text{LIBOR}(F_t, i) \\ \text{SWAP}(F_t, j) \end{bmatrix} + \sqrt{\mathcal{V}}e_t, \quad \begin{matrix} i = 1, 2, 3, 6, 12 \text{ months} \\ j = 2, 3, 5, 7, 10, 15, 30 \text{ years}, \end{matrix} \tag{20}$$

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where m_t denotes the measurement series generically, $\mathcal{V}$ denotes the covariance matrix of the measurement errors, and e_t denotes the standardized error vector, which has a standard normal distribution.

For the second stage, we regard the options factors as the unobservable states and specify the state propagation equation based on a discrete-time version of the options factor dynamics:

$$E_t = \Phi_E E_{t-1} + \sqrt{\mathcal{Q}}\varepsilon_t, \tag{21}$$

where $\Phi_E = \exp(-\kappa_E \Delta t)$ and $\mathcal{Q} = I\Delta t$. We construct the measurement equations based on the first-stage pricing error on the three-month LIBOR and the seven cap series, again assuming additive, normally distributed pricing errors,

$$m_t = \begin{bmatrix} \text{LIBOR}_t(i) - \text{LIBOR}(F_t, i) \\ \text{CAP}(F_t, E_t, j) \end{bmatrix} + \sqrt{\mathcal{V}}e_t, \tag{22}$$

$$i = 3 \text{ months}$$
$$j = 1, 2, 3, 4, 5, 7, 10 \text{ years},$$

where we convert the implied volatility quotes into dollar prices based on the Black formula in equation (17) and \$100 notational value.

For both stages of estimation, given initial guess of model parameters, we use the extended Kalman filter to update the conditional mean and conditional covariance matrix of the states and measurement variables. We further assume that the forecasting errors on the measurement series are normally distributed and define the weekly log likelihood function as

$$l_t(\Theta_i, m_t) = -\frac{1}{2}\left[\ln|\overline{A}_t| + (m_t - \overline{m}_t)^\top (\overline{A}_t)^{-1}(m_t - \overline{m}_t)\right], \quad i = 1, 2, \tag{23}$$

where Θ_i denotes the parameter set for stage-i estimation, and $\overline{m}_t$ and $\overline{A}_t$ denote the conditional mean and covariance matrix of the forecasts of the measurement series, respectively. The model parameters are estimated by maximizing the sum of the weekly log likelihood defined in (23).

Many econometric studies of affine models follow some variation of maximum likelihood estimation. While the state variables are in general not observable, they are often directly inverted from the observed discount bonds by assuming that m of these bonds are priced perfectly by the m factors. The other bonds are then assumed to be priced with errors, and the likelihood function can be constructed based on the conditional density of the latent variables and the pricing errors.[4]

However, in practice, it is arbitrary to decide which rates are priced exactly and which are priced with errors. A more reasonable assumption is that all interest rates or bond prices contain errors. An efficient estimation strategy should demand that the model values go through all the observed data points in a least

[4]This "exact pricing" assumption, or some variant of it, is maintained in, among others, Chen and Scott (1993), Duffee (1999), (2002), Duffie and Singleton (1997), Longstaff and Schwartz (1992), and Pearson and Sun (1994).

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square sense, rather than forcing the model to match an arbitrary set of points and ignoring the others.

A convenient approach to dealing with observation errors is to cast the model in state-space form augmented by measurement equations that relate the observed interest rates or bond prices to the underlying state variables. When the state variables are Gaussian and the measurement equations are linear, the Kalman filter yields the efficient state updates in the least square sense (Pennacchi (1991)). In our application, the state variables are Gaussian, but the measurement equations are nonlinear in the state variables. The extended Kalman filter deals with the nonlinearity in the measurement equation via a Taylor expansion. Examples of extended Kalman filtering in term structure model estimations include Duffee and Stanton (2003) and Leippold and Wu (2007).

The filtering technique fits naturally in our modeling framework. We first apply Kalman filtering to LIBOR and swap rates. The measurement errors on these interest rates are essentially what we call residuals on the yield curve. We make use of the residuals on the three-month LIBOR and the interest rate cap data in identifying the options factor dynamics in a second-stage estimation. We assume that the measurement errors on each series are independent but exhibit distinct variance σ_e^2. Thus, we have 31 parameters for the first-stage estimation: $\Theta_1 \equiv [\kappa_F \in \mathbb{R}^6, \kappa_F^{\mathbb{Q}} \in \mathbb{R}^6, b_r \in \mathbb{R}^{3+}, b_\gamma \in \mathbb{R}^3, a_r \in \mathbb{R}, \sigma_e^2 \in \mathbb{R}^{12+}]$. At this stage, we identify the yield curve factor dynamics from the 12 interest rate series. For the second-stage estimation, we have 26 parameters: $\Theta_2 \equiv [\kappa_E \in \mathbb{R}^6, \kappa_E^{\mathbb{Q}} \in \mathbb{R}^6, c_h \in \mathbb{R}^3, b_\lambda \in \mathbb{R}^3, \sigma \in \mathbb{R}^{8+}]$. At this stage, we take the yield curve factors extracted from the first stage as given and identify the options factor dynamics mainly from the interest rate options.

IV. Empirical Performance of the 3 + 3 Gaussian Affine Model

Based on the estimation results on the 3+3 Gaussian affine model, we discuss the different roles played by the yield curve factors and the options factors in pricing the interest rate term structure and interest rate options.

A. Pricing the Yield Curve with Three Yield Curve Factors

Panel A of Table 2 reports the summary properties of pricing errors on the yield curve obtained from the first-stage estimation. The pricing errors are defined as the difference in basis points (bps) between the market quotes on LIBOR and swap rates and the model-implied fair values as a function of the three yield curve factors. The last row reports the sample averages of the statistics over the 12 interest rate series. Overall, the pricing errors are very small. The average pricing error is less than 1 bp. The average standard deviation and mean absolute pricing errors are both less than 5 bps. The last column reports the percentage explained variation, defined as one minus the ratio of the pricing error variance to the variance of the original series, represented in percentage points. The explained variation estimates are over 99% for all but one series. Hence, the three yield curve factors capture the term structure of interest rates well.

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TABLE 2

Summary Statistics of Pricing Errors from the 3 + 3 Gaussian Affine Model

Entries report the summary statistics of the pricing errors on LIBOR and swap rates (Panel A of Table 2), obtained from the first-stage estimation, and on cap implied volatilities (Panel B), obtained from the second-stage estimation on the 3 + 3 Gaussian affine model. The pricing error is defined as the difference between the observed market quotes and the model-implied fair values. The columns titled Mean, Median, Std Dev, MAE, Auto, Max, and Min denote the mean, median, standard deviation, mean absolute error, first order autocorrelation, maximum, and minimum of the measurement errors at each maturity, respectively. The last column (VR) reports the percentage variance explained for each series by the three yield curve factors in Panel A and by the 3 + 3 Gaussian affine model in Panel B, defined as one minus the ratio of pricing error variance to the variance of the original series, in percentage points. The last row of each panel reports average statistics.

Maturity	Mean	Median	Std Dev	MAE	Auto	Max	Min	VR
Panel A. Errors on the Yield Curve (basis points)								
1 m	1.21	2.00	8.72	5.63	0.71	40.18	−53.46	99.39
2 m	0.60	0.84	3.94	2.39	0.57	28.80	−20.44	99.88
3 m	0.78	−0.16	5.77	2.83	0.77	41.34	−13.28	99.74
6 m	−3.20	−3.67	7.02	6.09	0.81	24.20	−26.50	99.63
12 m	−8.24	−8.12	6.27	8.88	0.57	13.07	−35.18	99.70
2 y	0.02	0.04	1.03	0.76	0.78	3.17	−4.21	99.99
3 y	0.48	0.44	1.21	0.96	0.74	6.61	−3.20	99.98
5 y	−0.02	0.01	1.45	1.14	0.75	4.07	−4.74	99.97
7 y	0.00	0.00	0.01	0.01	0.14	0.04	−0.07	100.00
10 y	0.53	0.50	3.02	2.43	0.88	12.19	−7.67	99.83
15 y	1.96	1.43	6.89	5.61	0.95	18.28	−17.22	99.04
30 y	−0.45	−1.19	12.90	10.31	0.95	32.37	−40.97	96.58
Average	−0.53	−0.66	4.85	3.92	0.72	18.69	−18.91	99.48
Panel B. Errors on Cap Implied Volatilities (%)								
1 y	−0.05	−0.03	0.38	0.26	0.33	2.65	−2.58	99.75
2 y	0.02	0.00	0.28	0.21	0.75	1.36	−0.69	99.72
3 y	0.03	0.03	0.09	0.07	0.41	0.30	−0.43	99.95
4 y	−0.06	−0.03	0.16	0.12	0.68	0.43	−0.79	99.79
5 y	0.07	0.06	0.15	0.12	0.59	0.80	−0.26	99.76
7 y	−0.02	−0.01	0.10	0.07	0.51	0.23	−0.54	99.85
10 y	0.02	0.01	0.16	0.12	0.63	0.89	−0.37	99.39
Average	−0.00	0.00	0.19	0.14	0.56	0.95	−0.81	99.75

Further inspection shows that the measurement errors are very small for swap rates at moderate maturities (two to 10 years), but are larger for LIBOR and long-maturity swap rates. The mean absolute errors are only about 1 bp for two-, three-, and five-year swap rates, and the model prices the seven-year swap rate almost perfectly. On the other hand, the mean absolute errors for the 12-month LIBOR and the 30-year swap rate are about 10 bps. The difference between short- and long-term swap rates may represent liquidity differences. The overall larger measurement errors on the LIBOR market may indicate some structural differences between the LIBOR and swap markets that cannot be accommodated by our model. The segmentation between the LIBOR and swap markets is well known in the industry (James and Webber (2000)). Longstaff, Santa-Clara, and Schwartz (2001a) find similar market segmentations between the cap market, which is based on LIBOR, and the swaption market, which is based on swap contracts. Our results indicate that such inconsistencies in the options market may actually start in the underlying interest rate market.

B. Pricing Caps With and Without Additional Options Factors

Given the estimated yield curve factors, we first price interest rate caps based on the model-implied yield curve, ignoring the potential impact of the yield curve

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residuals and unspanned volatility. The pricing results show that although the three yield curve factors can explain over 99% of variation on the yield curve, they are far from sufficient in capturing the movement of implied volatilities. The three yield curve factors only explain 48.35% of the aggregate variation in implied volatilities. Li and Zhao (2006) report similar poor option pricing and hedging performance for three-factor quadratic term structure models. Bikbov and Chernov (2009) show that most three-factor dynamic term structure models perform well in explaining the yield curve, but poorly in explaining the eurodollar futures options, even if they apply the parametric specification proposed by Collin-Dufresne and Goldstein (2002) to account for unspanned stochastic volatility. Thus, the poor performance of the yield curve factors in option pricing is not specific to our model design, but generic to dynamic term structure models that ignore the potential impacts of the residuals and unspanned or unidentified volatility movements.

The performance in pricing interest rate caps improves dramatically when we include the three additional options factors. Panel B of Table 2 reports the summary statistics of the pricing errors on implied volatilities based on the estimated 3 + 3 model. Here, the pricing errors are defined as the differences in percentage points between market quotes and model-implied values on the implied volatility. The statistics show negligible mean pricing errors. The standard deviations of the pricing errors are all within 40 bps, and the mean absolute pricing errors are all within 30 bps. Both numbers are below the half-a-percentage-point benchmark for average bid-ask spreads on implied volatilities for interest rate caps. Finally, the last column shows that the 3 + 3 model explains all cap series over 99%.

The four graphs in Figure 1 contrast the pricing of the implied volatilities with and without the options factors at four sample dates. The circles denote the market quotes on implied volatility. The solid lines represent the fair value computed from the estimated 3 + 3 model, which almost always go through the data circles. In contrast, the dashed lines, which represent the pricing from only the three yield curve factors while ignoring the options factors, deviate significantly from the data points. Without the help of the options factors, the yield curve factors alone either underprice or overprice the interest rate caps.

C. Yield Curve Factor Dynamics and the Term Structure of Interest Rates

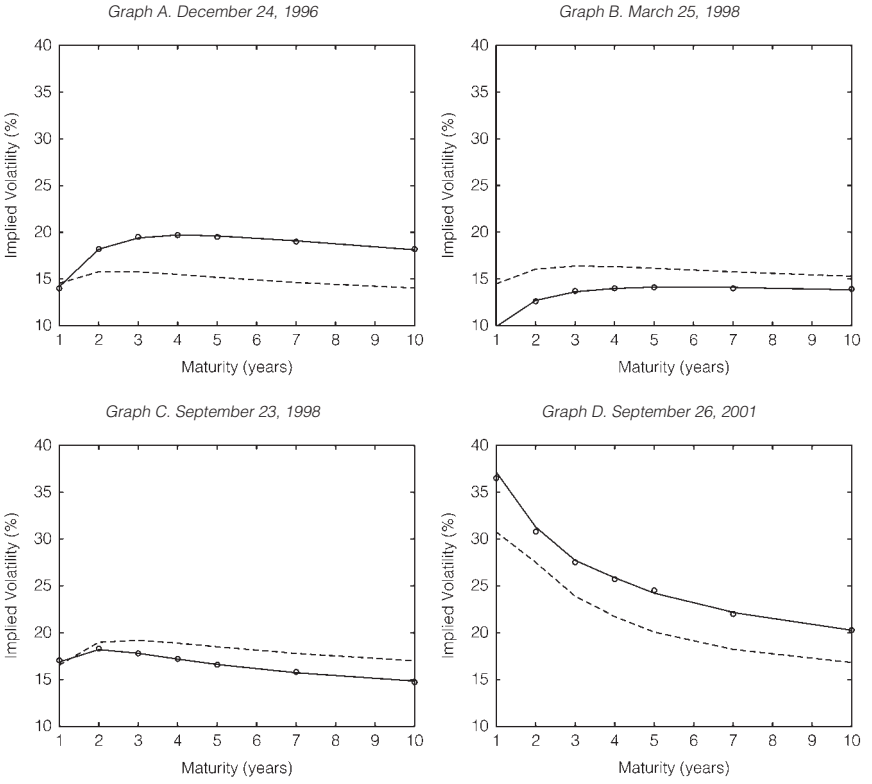
Panel A of Table 3 reports the parameter estimates and the absolute magnitudes of the t-statistics (in parentheses) from the first-stage estimation on the yield curve factor dynamics. The small diagonal elements for κ_F and $\kappa_F^{\mathbb{Q}}$ show that the yield curve factors are highly persistent under both the statistical measure $\mathbb{P}$ and the risk-neutral measure $\mathbb{Q}$. Furthermore, the significant and large estimates of the off-diagonal elements suggest that the three yield curve factors have strong dynamic interactions with one another. The estimates for b_r show that the loadings of the first two factors on the short rate are small, but the loading of the third factor is large and statistically significant. The estimates on b_γ reveal negative market price for yield curve factors, significantly so for the first factor. The estimate for a_r represents the long-run mean for the instantaneous short rate.

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FIGURE 1

Pricing Cap Implied Volatilities With and Without Additional Options Factors

Circles denote market quotes on cap implied volatilities. Solid lines denote the fair value computed from the estimated 3 + 3 Gaussian affine model. Dashed lines represent the pricing from only the three Gaussian affine yield curve factors while ignoring the additional options factors.



According to equation (8), the coefficients $[a(\tau), b(\tau)]$ determine the term structure of interest rates. The fair values of continuously compounded spot rates are linked to the yield curve factors F_t by

$$Y(F_t, t+\tau) = \left[\frac{a(\tau)}{\tau}\right] + \left[\frac{b(\tau)}{\tau}\right]^{\top} F_t. \tag{24}$$

Since we normalize the long-run mean of the state vector to zero, the intercept $a(\tau)/\tau$ defines the mean term structure of the spot rate. The slope coefficient $b(\tau)/\tau$ captures the instantaneous response of the spot rate to unit shocks in the yield curve factors and can thus be regarded as the loading of the yield curve factors on the term structure. The ordinary differential equations in (9) show that the risk-neutral dynamics of the yield curve factors ($\kappa_F^{\mathbb{Q}}$ and b_γ) interact with the short rate loading function (a_r and b_r) to determine the coefficients $[a(\tau), b(\tau)]$ and hence whole term structure of interest rates.

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TABLE 3

Full Sample Parameter Estimates of the 3 + 3 Gaussian Affine Model

Entries report the maximum likelihood estimates of the 3+3 Gaussian affine model parameters and the absolute magnitudes of the *t*-statistics (in parentheses). The estimation uses the full sample of over eight years of weekly data from April 6, 1994 to April 17, 2002 (420 weekly observations for each series). Panel A of Table 3 reports parameters (Θ_1) that are related to the three yield curve factors and are estimated using 12 LIBOR and swap rates. Panel B reports the parameters (Θ_2) that are related to the three options factors and are estimated using seven cap series.

Panel A. Θ_1

κ_F			$\kappa_F^{\mathbb{Q}}$			b_r	b_γ	a_r
0.0184	0	0	0.0018	0	0	0.0004	–0.1376	
(0.07)	—	—	(114.9)	—	—	(1.09)	(105.3)	
–0.1545	0.0516	0	0.6550	1.3857	0	0.0000	–1.8440	0.0364
(0.51)	(0.19)	—	(4.56)	(9.18)	—	(0.08)	(9.41)	(4.54)
–3.3403	–2.1957	0.1280	–3.6299	–3.9651	1.2092	0.0056	–2.4902	
(5.84)	(2.75)	(0.41)	(13.7)	(20.7)	(10.0)	(32.2)	(1.28)	

Panel B. Θ_2

κ_E			$\kappa_E^{\mathbb{Q}}$			c_h	b_λ
0.0189	0	0	0.5776	0	0	0.0000	–0.2472
(0.02)	—	—	(0.68)	—	—	(0.24)	(0.57)
0.6195	0.0496	0	0.0080	0.0013	0	0.0001	–0.0904
(0.60)	(0.06)	—	(0.02)	(0.13)	—	(1.60)	(0.50)
0.1364	0.5125	0.1079	–3.0837	1.4572	0.5385	0.0004	–0.9471
(0.12)	(0.91)	(0.15)	(2.68)	(1.88)	(0.66)	(11.6)	(0.40)

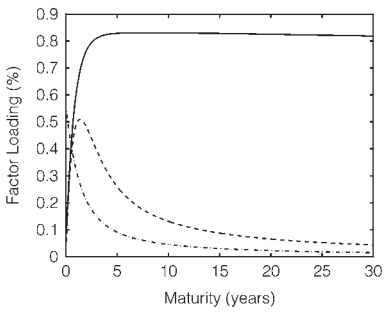
The three lines in Graph A of Figure 2 plot the coefficients $b(\tau)/\tau$, which measure the contemporaneous response of the spot rate curve to unit shocks from each of the three yield curve factors. The coefficients are computed based on the parameter estimates in Panel A of Table 3. The three lines illustrate how the three factors control the variation of interest rates at different maturities. The first factor (the solid line) loads up most heavily on long-term interest rates, the loading of the second factor (the dashed line) peaks around three-year maturity, and the last factor (the dashed-dotted line) mainly loads up on the short end of the yield curve.

FIGURE 2

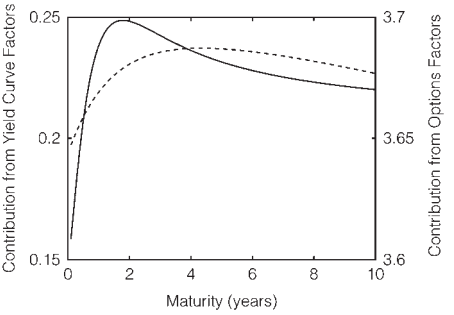
Factor Contributions to Interest Rates and Conditional Volatilities

The three lines in Graph A of Figure 2 represent the contemporaneous response of the spot rate curve to unit shocks on the three yield curve factors under the estimated 3 + 3 Gaussian affine model. The two lines in Graph B plot the contribution of the yield curve factors (solid line, with scale on the left *y*-axis) and options factors (dashed line, with scale on the right *y*-axis) to the conditional volatility of the three-month LIBOR at different conditioning horizons, in annualized percentages.

Graph A. Interest Rate Term Structure



Graph B. Conditional Volatility Term Structure



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D. Options Factor Dynamics and the Term Structure of Conditional Variance

Since the three yield curve factors explain over 99% of the variation in interest rates, the remaining options factors have minimal impacts on the term structure of interest rate levels. Nevertheless, they can have important impacts on cap pricing, mainly through their contribution to the conditional variance of the observed three-month interest rates. As shown in equation (14), caplet pricing depends crucially on the conditional variance dynamics of the observed three-month interest rate. According to equation (16), this conditional variance can be decomposed into two components: the contribution from the yield curve factors (F) and the contribution from the options factors (E):

$$\Sigma_t = b(h)^\top \mathrm{Var}_t^{\mathbb{T}}[F_T]\, b(h) + c_h^\top \mathrm{Var}_t^{\mathbb{T}}[E_T]\, c_h. \tag{25}$$

Appendix A shows that the conditional variance matrix of F_T and E_T are controlled by their respective risk-neutral dynamics $\kappa_F^{\mathbb{Q}}$ and $\kappa_E^{\mathbb{Q}}$:

$$\mathrm{Var}_t^{\mathbb{T}}[F_T] = \int_t^T e^{-\kappa_F^{\mathbb{Q}} s} e^{-\kappa_F^{\mathbb{Q}\top} s} ds, \qquad \mathrm{Var}_t^{\mathbb{T}}[E_T] = \int_t^T e^{-\kappa_E^{\mathbb{Q}} s} e^{-\kappa_E^{\mathbb{Q}\top} s} ds. \tag{26}$$

We have discussed the estimates on $\kappa_F^{\mathbb{Q}}$ and their impacts on the term structure of interest rate levels in the previous subsection. In equation (25), $b(h)$ denotes the loading coefficient of the three-month spot rate on the three yield curve factors, which are computed at $[0.006, 0.006, 0.0012]^\top$. Hence, the conditional variance of the third yield curve factor contributes more significantly to the conditional variance of the three-month rate. Nevertheless, the large and significant off-diagonal elements of $\kappa_F^{\mathbb{Q}}$ suggest that the first- and second-yield curve factors also have strong impacts on the conditional variance of the third factor.

Panel B of Table 3 reports the second-stage estimates and absolute magnitudes of the t-statistics (in parentheses) on the options factor dynamics. The two off-diagonal elements in the last row of $\kappa_E^{\mathbb{Q}}$ are large and significant. Thus, the first and second options factors strongly impact the dynamics of the third options factor. The estimates on the loading coefficients c_h are close to zero for the first two elements but significantly positive for the third element, suggesting that only the conditional variance of the third options factor contributes significantly to Σ_t. Nevertheless, the first two options factors influence the conditional variance dynamics of the third options factor through the two off-diagonal elements in the last row of $\kappa_E^{\mathbb{Q}}$.

Based on the estimated model parameters in Table 3, we compute the contribution to the conditional variance Σ_t from both the yield curve factors and the options factors according to equations (25) and (26). Graph B of Figure 2 plots the contributions in annualized percentage volatility at different option maturities. The solid line, with scale on the left side, denotes the contribution from the yield curve factors, $\sqrt{b(h)^\top \mathrm{Var}_t^{\mathbb{T}}[F_T] b(h)/(T-t)}$. The contribution generates a hump shape that peaks around one-and-a-half-year maturity. The dashed line, with scale on the right side, denotes the contribution from the options factors,

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$\sqrt{c_h^\top \mathrm{Var}_t^{\mathbb{T}}[E_T]c_h/(T-t)}$, which also generates a hump-shaped term structure but peaks at four-year maturity. Not only does the hump shape from the options factors better match the sample mean term structure of the implied volatility (Table 1), but the different scales of the two lines also show that the conditional volatility contribution from the options factors is several times larger than the conditional volatility contribution from the yield curve factors. Therefore, although the options factors have minimal impact on the interest rate term structure, they have large impacts on the term structure of conditional variance and hence on option pricing.

E. Out-of-Sample Performance

To study the out-of-sample performance, we reestimate the model using the first six years of data from April 6, 1994 to April 5, 2000 (314 weekly observations for each series). Then, we use these estimated model parameters to compare the model performance both in sample during the first six years and out of sample during the last two years from April 12, 2000 to April 17, 2002 (106 weekly observations for each series). If the model is well specified and our estimation generates stable model parameters, we would expect the model's out-of-sample performance to be similar to its in-sample performance.

Table 4 reports the model parameter estimates and absolute magnitudes of the *t*-statistics (in parentheses) using the subsample of six years of data. The estimates are close to what we have obtained from the full sample in Table 3, indicating that the estimation generates stable model parameters and that the interest rate behavior has not experienced dramatic changes over the last two years of our data sample. Table 5 compares the summary statistics of the in-sample pricing errors during the first six years (on the left side) and out-of-sample pricing errors during the last two years (on the right side). The statistics show that the model performs well both in sample and out of sample. There is no visible deterioration for the out-of-sample performance, showing that our 3 + 3 model is well specified and robust.

V. Alternative Term Structure Specifications

Within the generic $m + n$ factor structure, the choice of the number of factors and the specifications of the factor dynamics are governed by balanced considerations for parsimony, analytical tractability, and empirical performance. In previous sections, we have chosen a 3 + 3 Gaussian affine specification to illustrate the theoretical idea and its empirical performance. The choice of the 3 + 3 factor structure is motivated by previous factor analysis on the interest rate and implied volatility term structures. The choice of the Gaussian affine specification is mainly motivated by analytical tractability. Under this specification, we can derive both the interest rates and cap prices in analytical forms. The analytical forms facilitate our illustration on the different roles played by the yield curve factors and the options factors. Our empirical analysis further shows that the Gaussian affine specification generates satisfactory performance for both yield curve and cap pricing.

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TABLE 4

Subsample Parameter Estimates of the 3 + 3 Gaussian Affine Model

Entries report the maximum likelihood estimates of the 3+3 Gaussian affine model parameters and the absolute magnitudes of the *t*-statistics (in parentheses). The estimation uses the first six years of weekly data from April 6, 1994 to April 5, 2000 (314 weekly observations for each series). Panel A of Table 4 reports parameters (Θ_1) that are related to the three yield curve factors and are estimated using 12 LIBOR and swap rates. Panel B reports the parameters (Θ_2) that are related to the three options factors and are estimated using seven cap series.

Panel A. Θ_1

κ_F			$\kappa_F^{\mathbb{Q}}$			b_r	b_γ	a_r
0.0185	0	0	0.0018	0	0	0.0003	–0.1494	
(0.03)	—	—	(58.8)	—	—	(0.55)	(3.42)	
–0.0199	0.0513	0	1.0989	1.5224	0	0.0000	–0.2228	0.0350
(0.03)	(0.06)	—	(7.66)	(49.2)	—	(0.08)	(0.00)	(0.00)
–2.9868	–2.5614	0.1142	–3.8703	–4.2298	0.4027	0.0043	–2.5060	
(3.77)	(2.55)	(0.27)	(7.79)	(10.4)	(43.9)	(16.7)	(0.01)	

Panel B. Θ_2

κ_E			$\kappa_E^{\mathbb{Q}}$			c_h	b_λ
0.0192	0	0	0.7355	0	0	0.0000	–0.2460
(0.02)	—	—	(1.54)	—	—	(0.39)	(0.63)
0.9714	0.0505	0	–0.1493	0.0013	0	0.0000	–0.0305
(1.02)	(0.07)	—	(0.84)	(0.15)	—	(0.57)	(0.22)
0.3690	–0.0466	0.1184	–4.6000	2.7318	0.8449	0.0003	–1.1346
(0.24)	(0.05)	(0.07)	(3.93)	(1.61)	(1.63)	(6.41)	(0.27)

TABLE 5

In-Sample and Out-of-Sample Performance of the 3 + 3 Gaussian Affine Model

Entries report the summary statistics of both in-sample and out-of-sample pricing errors on LIBOR and swap rates in Panel A of Table 5, and on cap implied volatilities in Panel B under the 3 + 3 Gaussian affine model. The pricing error is defined as the difference between the observed market quotes and the model-implied fair values. The columns titled Mean, Median, Std Dev, MAE, Auto, Max, and Min denote, respectively, the mean, median, standard deviation, mean absolute error, first-order autocorrelation, maximum, and minimum of the measurement errors at each maturity. Under VR, we report the percentage variance explained for each series by the three yield curve factors in Panel A and by the 3 + 3 Gaussian affine model in Panel B. The estimation is based on the first six years of data.

	In-Sample Errors (1994–2000)								Out-of-Sample Errors (2000–2002)							
Maturity	Mean	Median	Std Dev	MAE	Auto	Max	Min	VR	Mean	Median	Std Dev	MAE	Auto	Max	Min	VR
Panel A. Errors on the Yield Curve (basis points)																
1 m	2.27	2.75	9.61	6.15	0.68	45.50	–48.26	95.57	14.13	15.21	12.13	15.18	0.91	38.29	–9.38	99.60
2 m	1.41	1.84	5.25	3.44	0.61	33.64	–25.36	98.57	7.44	7.98	6.91	8.22	0.80	21.88	–8.92	99.87
3 m	1.58	0.53	6.02	3.14	0.77	39.57	–14.54	98.07	1.98	1.88	4.13	3.34	0.47	11.93	–15.71	99.95
6 m	–3.37	–4.05	6.01	5.46	0.83	22.55	–24.06	98.14	–10.72	–11.84	10.07	11.84	0.84	7.25	–33.90	99.73
12 m	–9.22	–9.03	4.65	9.37	0.46	5.01	–25.09	99.25	–21.04	–22.19	14.89	21.12	0.86	3.10	–57.40	99.34
2 y	–0.44	–0.37	1.96	1.38	0.82	9.28	–8.53	99.90	–7.28	–7.00	6.50	7.77	0.96	3.25	–21.13	99.80
3 y	0.46	0.39	1.02	0.84	0.71	4.34	–2.10	99.98	–0.18	–0.45	1.16	0.97	0.62	2.87	–2.59	99.99
5 y	0.26	0.18	1.17	0.86	0.73	4.47	–3.90	99.97	2.36	2.13	2.50	2.75	0.92	7.53	–2.36	99.93
7 y	0.00	0.00	0.01	0.01	0.43	0.04	–0.09	100.00	0.02	0.02	0.02	0.02	0.16	0.07	–0.07	100.00
10 y	0.16	–0.04	1.63	1.25	0.71	5.53	–4.14	99.95	–4.90	–4.26	3.70	5.01	0.93	1.87	–12.19	99.73
15 y	1.02	0.97	3.89	2.98	0.90	11.57	–10.70	99.69	–9.87	–8.59	5.27	9.87	0.92	–2.62	–22.00	99.24
30 y	–0.81	–1.41	8.91	6.89	0.93	22.71	–30.56	98.40	–21.40	–17.79	9.85	21.40	0.90	–6.46	–46.09	96.39
Avg	–0.56	–0.69	4.18	3.48	0.71	17.02	–16.44	98.96	–4.12	–3.74	6.43	8.96	0.77	7.41	–19.31	99.46
Panel B. Errors on Cap Implied Volatilities (%)																
1 y	–0.25	–0.24	0.54	0.47	0.63	1.01	–2.04	98.27	–0.35	–0.23	1.32	0.86	0.54	5.87	–5.04	98.79
2 y	0.03	0.03	0.18	0.14	0.56	0.66	–0.59	99.76	0.47	0.41	0.48	0.51	0.88	1.69	–0.20	99.64
3 y	0.03	0.03	0.11	0.09	0.35	0.33	–0.49	99.89	–0.03	–0.01	0.13	0.09	0.58	0.24	–0.81	99.95
4 y	–0.05	–0.04	0.11	0.09	0.40	0.35	–0.40	99.85	–0.28	–0.25	0.26	0.29	0.93	0.11	–0.90	99.67
5 y	0.07	0.04	0.16	0.13	0.64	0.87	–0.26	99.63	–0.11	–0.09	0.13	0.13	0.67	0.10	–0.48	99.89
7 y	–0.04	–0.03	0.12	0.09	0.51	0.32	–0.60	99.70	0.11	0.10	0.10	0.12	0.64	0.41	–0.06	99.90
10 y	0.03	0.03	0.18	0.14	0.60	0.67	–0.45	98.99	0.30	0.31	0.31	0.34	0.92	1.00	–0.24	98.64
Avg	–0.03	–0.02	0.20	0.16	0.53	0.60	–0.69	99.44	0.01	0.04	0.39	0.34	0.74	1.35	–1.10	99.50

One limitation of the Gaussian affine specification is that the model generates constant volatilities for the continuously compounded interest rates. A model with constant volatility cannot be expected to capture the strong variation in implied volatilities. In this section, we consider an alternative specification for the three yield curve factors, in which we explicitly incorporate both stochastic central tendency and stochastic volatility for the interest rate dynamics. We derive bond and option pricing equations under this specification, and we empirically investigate whether the explicit inclusion of stochastic volatility in the interest rate term structure alleviates the need for additional options factors in pricing interest rate caps.

A. Stochastic Central Tendency and Stochastic Volatility

In the alternative specification, we use the three yield curve factors to capture the level, the stochastic central tendency, and the stochastic volatility of the fair value of the instantaneous interest rate. Their $\mathbb{P}$-dynamics are

$$\begin{aligned} dr_t &= \kappa_r(\theta_t - r_t)\,dt + \sqrt{v_t}\,dW_t^r, \\ d\theta_t &= \kappa_\theta(\theta_r - \theta_t)\,dt + \sigma_\theta\,dW_t^\theta, \quad \text{and} \\ dv_t &= \kappa_v(\theta_v - v_t)\,dt + \sigma_v\sqrt{v_t}\,dW_t^v, \end{aligned} \tag{27}$$

with $\rho = E[dW^r dW^v]/dt$ capturing the correlation between innovations in the short rate and its volatility. We assume that other pairs of Brownian motions are independent of each other. In vector forms, we can write the dynamics of the three yield curve factors $F_t \equiv [r_t, \theta_t, v_t]^\top$ as

$$dF_t = \kappa_F(\theta_F - F_t)\,dt + \sqrt{V(F_t)}\,dW_t, \tag{28}$$

with

$$\kappa_F\theta_F = \begin{bmatrix} 0 \\ \kappa_\theta\theta_r \\ \kappa_v\theta_v \end{bmatrix}, \qquad \kappa_F = \begin{bmatrix} \kappa_r & -\kappa_r & 0 \\ 0 & \kappa_\theta & 0 \\ 0 & 0 & \kappa_v \end{bmatrix}, \tag{29}$$

$$V(F_t) = \begin{bmatrix} v_t & 0 & v_t\rho\sigma_v \\ 0 & \sigma_\theta^2 & 0 \\ v_t\sigma_v\rho & 0 & \sigma_v^2 v_t \end{bmatrix}.$$

We further assume flexible and yet tractable forms on the market prices of the three sources of risks, $\gamma(F_t) = [(\gamma_0^r + \gamma_1^r r_t)/\sqrt{v_t} + \gamma_1^{rv}\sqrt{v_t}, \gamma_0^\theta + \gamma_1^\theta\theta_t, (\gamma_0^v + \gamma_1^{vr} r_t)/\sqrt{v_t} + \gamma_1^v\sqrt{v_t}]^\top$, with which we can derive the factor dynamics under measure $\mathbb{Q}$ as

$$dF_t = \kappa_F^{\mathbb{Q}}\left(\theta_F^{\mathbb{Q}} - F_t\right)dt + \sqrt{V(F_t)}\,dW_t^{\mathbb{Q}}, \tag{30}$$

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with

$$\kappa_F^{\mathbb{Q}}\theta_F^{\mathbb{Q}} = \begin{bmatrix} -\gamma_0^r - \gamma_0^v\rho \\ \kappa_\theta\theta_r - \gamma_0^\theta\sigma_\theta \\ \kappa_v\theta_v - (\gamma_0^v + \gamma_0^r\rho)\sigma_v \end{bmatrix}, \tag{31}$$

$$\kappa_F^{\mathbb{Q}} = \begin{bmatrix} \kappa_r + \gamma_1^r + \gamma_1^{vr}\rho & -\kappa_r & \gamma_1^{rv} + \gamma_1^v\rho \\ 0 & \kappa_\theta + \gamma_1^\theta\sigma_\theta & 0 \\ (\gamma_1^{vr} + \gamma_1^r\rho)\sigma_v & 0 & \kappa_v + (\gamma_1^{rv}\rho + \gamma_1^v)\sigma_v \end{bmatrix}.$$

Our yield curve factor specification belongs to the $A_2(3)$ classification of affine models in Dai and Singleton (2000), and our market price specification belongs to the "essentially affine" specification of Duffee (2002). Compared to the generic $A_2(3)$ specification, our specification is more parsimonious, and it assigns a specific meaning to each factor, similar to Balduzzi, Das, Foresi, and Sundaram (1996). A more general specification can in principle generate better performance; yet, identification concerns motivate us to choose a more parsimonious specification. The explicit meanings of the three factors in our specification also enable us to better understand the contribution of each factor to interest rate and cap pricing. We label this new specification the LCV model, with the three letters capturing the initials of the three factors (level, central tendency, and volatility).

As in equation (11), we add residuals to the continuously compounded three-month spot rate and maintain the Gaussian affine assumption on the three options factors.

B. Bond and Option Pricing under the 3 + 3 LCV Model

Given the affine structure on the LCV specification, the fair values of zero-coupon bonds remain exponentially affine in the yield curve factors as in (8), where the coefficients $a(\tau)$ and $b(\tau)$ are determined by the following ordinary differential equations:

$$a'(\tau) = b(\tau)^\top \kappa_F^{\mathbb{Q}}\theta_F^{\mathbb{Q}} - \frac{1}{2}b(\tau)_2^2(\sigma_\theta)^2, \tag{32}$$

$$b'(\tau) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \left(\kappa^{\mathbb{Q}}\right)^\top b(\tau) - \frac{1}{2}\begin{bmatrix} 0 \\ 0 \\ b(\tau)_1^2 + 2b(\tau)_1 b(\tau)_3\rho\sigma_v + b(\tau)_3^2(\sigma_v)^2 \end{bmatrix},$$

starting at $a(0)=0$ and $b(0)=0$. The ordinary differential equations can be readily solved with standard numerical procedures such as the fourth-order Runge-Kutta method.

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To price interest rate caps, we follow Appendix A and write the value of the ith caplet as

$$\text{CAPLET}^i = \mathbb{E}_t^{\mathbb{Q}}\left[\exp\left(-\int_t^T r(s)\,ds\right) \times \left(e^{c_h^\top E_T} - (1+hK)\,e^{-a(h)-b(h)^\top F_T}\right)^+\right]. \tag{33}$$

Since the yield curve factors F_T are no longer normally distributed, we cannot directly solve the expectation in analytical forms. Nevertheless, we can solve the Fourier transform of the caplet value in analytical form. Given the Fourier transform, we apply fast Fourier inversion to obtain the caplet value.

Specifically, we perform the caplet valuation in several steps. First, we define the following valuation function:

$$\phi(a_0 + b_0^\top F_t + c_0^\top E_T) \equiv \mathbb{E}_t^{\mathbb{Q}}\left[\exp\left(-\int_t^T r(F_s)ds\right) \times \exp(a_0 + b_0^\top F_T + c_0^\top E_T)\right], \tag{34}$$

where $\exp(a_0+b_0^\top F_T+c_0^\top E_T)$ denotes a terminal payoff function. Given the affine factor dynamics and the exponential affine payoff structure, we can solve the value function ϕ as an exponential affine function of the yield curve and options factors,

$$\phi(a_0 + b_0^\top F_t + c_0^\top E_T) = \exp\left(-a(\tau) - b(\tau)^\top F_t\right) \times \exp\left(c_0^\top \mathbb{E}_t^{\mathbb{Q}}[E_T] + \frac{1}{2}c_0^\top \text{Var}_t^{\mathbb{Q}}[E_T]\,c_0\right), \tag{35}$$

where the coefficients $(a(\tau), b(\tau))$ satisfy the same ordinary differential equations as in equation (32), but with different initial conditions to accommodate the different terminal payoffs: $a(0) = -a_0$ and $b(0) = -b_0$. In (35), we explicitly separate the contribution of the yield curve factors from the contribution of the options factors as two exponential affine forms. This separation is important numerically, as we can precalculate and store the contribution from the yield curve factors when we perform the second-stage estimation on the options factors. The risk-neutral conditional mean and variance of the options factors E_T are given in equation (A-8) in Appendix A.

Second, we let $G_{a,b}(k)$ denote the value of a contingent claim, defined as

$$G_{a,b}(k) \equiv \mathbb{E}_t^{\mathbb{Q}}\left[\exp\left(-\int_t^T r_s ds\right) e^{-a_T} 1_{b_T \le k}\right].$$

Then, the caplet value in (33) can be written as

$$\text{CAPLET}^i = G_{-c_h^\top E_T, -c_h^\top E_T - a(h) - b(h)^\top F_T}(k) - (1+hK)\,G_{a(h)+b(h)^\top F_T, -c_h^\top E_T - a(h) - b(h)^\top F_T}(k), \tag{36}$$

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with $k = -\ln(1 + hK)$ because we can rewrite the exercise condition as

$$-c_h^\top E_T - a(h) - b(h)^\top F_T \leq -\ln(1 + hK) = k. \tag{37}$$

Third, we define the Fourier transform of the caplet value as

$$\begin{aligned}\varphi(u) \equiv \int_{-\infty}^{\infty} e^{iuk} \Big[& G_{-c_h^\top E_T, -c_h^\top E_T - a(h) - b(h)^\top F_T}(k) \\ & - e^{-k} G_{a(h)+b(h)^\top F_T, -c_h^\top E_T - a(h) - b(h)^\top F_T}(k) \Big] dk,\end{aligned} \tag{38}$$

which we can solve analytically in terms of the value function defined in (34) and solved in (35),

$$\varphi(u) = \frac{1}{iu(iu-1)} \phi\left(-iua(h) - iub(h)^\top F_T - (iu-1) c_h^\top E_T\right). \tag{39}$$

We leave the details of the derivation in Appendix B. The Fourier transform is generalized, because u needs to take complex values: $u = u_r + iu_i$. For the Fourier transform of the caplet to be well-defined, the imaginary part of u needs to be positive: $u_i > 0$.

Finally, given the generalized Fourier transform $\varphi(u)$, we compute the caplet value via the inversion formula,

$$\text{CAPLET}_t^i = \frac{1}{2\pi} \int_{iu_i - \infty}^{iu_i + \infty} e^{-iuk} \varphi(u) du, \tag{40}$$

which involves an integral along a straight line in the complex u-plane parallel to the real axis. The integral can also be written as

$$\text{CAPLET}_t^i = \frac{e^{u_i k}}{\pi} \int_0^{\infty} e^{-iu_r k} \varphi(u_r + iu_i) du_r, \tag{41}$$

which can be approximated on a finite interval by

$$\text{CAPLET}_t^i \approx \frac{e^{u_i k}}{\pi} \sum_{j=0}^{N-1} e^{-iu_r(j)k} \varphi(u_r(j) + iu_i) \Delta u_r, \tag{42}$$

where $\{u_r(j)\}_{j=0}^{N-1}$ are the nodes of u_r and Δu_r is the grid of the nodes.

We apply the fast Fourier transform (FFT) to evaluate the approximation in equation (42). The FFT is a mapping of $\mathbf{f} = (f_0, \ldots, f_{N-1})^\top$ on the vector of Fourier coefficients $\mathbf{d} = (d_0, \ldots, d_{N-1})^\top$, such that

$$d_j = \frac{1}{N} \sum_{j=0}^{N-1} f_j e^{jm\frac{2\pi}{N} i}, \qquad m = 0, 1, \ldots, N-1. \tag{43}$$

FFT allows the efficient calculation of $\mathbf{d}$ if N is an even number, say $N = 2^n$, $n \in \mathbb{N}$. The algorithm reduces the number of multiplications in the required N summations from an order of 2^{2n} to that of $n2^{n-1}$, a very considerable reduction.

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By a suitable choice of Δu_r and a discretization scheme of k, we can cast the approximation in equation (42) in the form of equation (43) to take advantage of the computing efficiency of the FFT. For our application, we set $u_r(j) = \eta j$, where $\eta = \Delta u_r$ is used as a control parameter for the summation grid. We set $k_m = -b + \lambda m$, with $\lambda = 2\pi/(\eta N)$ being the strike grid and b being a parameter that controls the strike range. Then, we can cast the approximation in equation (42) in the form of equation (43),

$$\text{CAPLET}(k_m) \approx \frac{1}{N}\sum_{j=0}^{N-1} f_j e^{jm\frac{2\pi}{N}i}, \qquad m = 0, 1, \ldots, N-1, \tag{44}$$

with

$$f_j = \frac{N}{\pi} e^{u_i k_m + ibu_r(j)} \eta \varphi(u_r(j) + iu_i). \tag{45}$$

This Fourier transform pricing method for interest rate caps is applicable to any specifications where the value function ϕ can be solved analytically. Therefore, we can price interest rate caps efficiently under the $m + n$ model structure for, among others, all affine (Duffie, Pan, and Singleton (2000)) and quadratic (Leippold and Wu (2002)) factor dynamics specifications.

C. Pricing Performance of the 3 + 3 LCV Model

We estimate the 3 + 3 LCV model the same way as we have estimated the 3 + 3 Gaussian affine model. Table 6 reports the summary statistics of the pricing errors on the yield curve (Panel A) and on the implied volatilities (Panel B). The statistics in Panel A show that overall, the performance of the LCV model is similar to that of the Gaussian affine model in pricing interest rates. Our finding is in line with the finding in Bikbov and Chernov (2009): Different specifications of three-factor affine models generate similar performance in pricing the interest rate term structure.

Although the LCV model contains a stochastic volatility factor, its performance in pricing interest rate caps remains poor. In the absence of the options factors, the estimated interest rate factors alone can explain only about 40% of the variation in implied volatilities for the interest rate caps. With the options factors, as shown in Panel B of Table 6, the 3+3 LCV model can explain over 99% of the variation in implied volatilities. Therefore, the explicit inclusion of a stochastic volatility factor in the interest rate dynamics does not alleviate the need for the additional options factors.

Table 7 reports the maximum likelihood estimates of the model parameters for the LCV model. From the parameter estimates on the yield curve factors, we compute the $b(\tau)/\tau$ coefficients, which measure the contemporaneous response of the spot rate curve to unit shocks on the three yield curve factors. The three lines in Graph A of Figure 3 plot the three components of the coefficients, each multiplied by the sample standard deviation of the corresponding yield curve factor that we have extracted from the estimation. Hence, the three lines represent

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TABLE 6

Summary Statistics of Pricing Errors from the 3 + 3 LCV Model

Entries report the summary statistics of the pricing errors on LIBOR and swap rates (Panel A of Table 6), obtained from the first-stage estimation, and on cap implied volatilities (Panel B), obtained from the second-stage estimation under the 3 + 3 LCV model. The pricing error is defined as the difference between the observed market quotes and the model-implied fair values. The columns titled Mean, Median, Std Dev, MAE, Auto, Max, and Min denote the mean, median, standard deviation, mean absolute error, first order autocorrelation, maximum, and minimum of the measurement errors at each maturity, respectively. The last column (VR) reports the percentage variance explained for each series by the three yield curve factors in Panel A and by the 3 + 3 LCV model in Panel B, defined as one minus the ratio of pricing error variance to the variance of the original series, in percentage points. The last row of each panel reports average statistics.

Maturity	Mean	Median	Std Dev	MAE	Auto	Max	Min	VR
Panel A. Errors on the Yield Curve (basis points)								
1 m	1.16	1.75	8.46	5.30	0.71	38.79	−57.06	99.42
2 m	0.54	0.65	3.37	1.97	0.58	25.56	−17.53	99.91
3 m	0.72	−0.32	6.00	2.89	0.79	44.01	−11.36	99.72
6 m	−3.27	−3.67	7.17	6.27	0.83	24.55	−26.54	99.61
12 m	−8.29	−8.30	6.39	8.94	0.59	12.09	−36.44	99.69
2 y	−0.01	−0.04	1.27	0.93	0.82	3.80	−5.45	99.98
3 y	0.45	0.42	1.09	0.87	0.74	5.92	−2.96	99.99
5 y	−0.05	0.01	1.41	1.11	0.75	3.80	−4.49	99.97
7 y	0.00	0.00	0.01	0.01	0.10	0.04	−0.08	100.00
10 y	0.54	0.51	2.99	2.41	0.88	11.98	−7.48	99.83
15 y	1.81	1.28	6.84	5.52	0.95	18.09	−17.23	99.05
30 y	−0.88	−1.20	13.14	10.40	0.94	32.51	−43.16	96.45
Average	−0.61	−0.74	4.84	3.88	0.72	18.43	−19.15	99.47
Panel B. Errors on Cap Implied Volatilities (%)								
1 y	−0.03	−0.24	1.49	0.97	0.80	10.14	−2.45	96.12
2 y	0.11	0.07	0.39	0.29	0.75	2.19	−0.99	99.46
3 y	0.05	0.06	0.11	0.09	0.38	0.36	−0.51	99.93
4 y	0.02	0.02	0.10	0.08	0.53	0.47	−0.48	99.91
5 y	0.10	0.06	0.16	0.13	0.66	0.86	−0.27	99.74
7 y	0.02	0.03	0.10	0.07	0.62	0.31	−0.45	99.85
10 y	0.01	0.01	0.19	0.13	0.71	0.77	−0.61	99.19
Average	0.04	0.00	0.36	0.25	0.64	2.16	−0.82	99.17

the responses of the yield curve to one-standard-deviation shocks on each of the three yield curve factors.

The instantaneous short rate has a large impact at short maturities, but its effect declines rapidly as the yield curve maturity increases. On the other hand, the central tendency factor does not enter the instantaneous short rate, but it imposes an increasingly large and positive impact on the yield curve as the maturity increases. The magnitude of the impact flattens out after five years. Finally, positive shocks on the short rate variance v_t lower the long-term yields by virtue of the convexity effect.

Under the LCV specification, the contribution of the yield curve factors to cap pricing is not as clear as in the Gaussian affine case, given the lack of analytical solutions. Nevertheless, equation (33) shows that despite the distributional variations of the LCV specification, the yield curve factors contribute to interest rate cap pricing mainly through the conditional variance of $b(h)^\top F_T$, as in the Gaussian affine case. To compute the conditional variance, we first derive the cumulant exponent of $b(h)^\top F_T$ as

$$(46) \qquad k(u) \equiv \ln \mathbb{E}_t^{\mathbb{Q}} \left[\exp(ub(h)^\top F_T\right] = -a(\tau, u) - b(\tau, u)^\top F_t,$$

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TABLE 7

Full Sample Parameter Estimates of the 3 + 3 LCV Model

Entries report the maximum likelihood estimates of the 3 + 3 LCV model parameters and the absolute magnitudes of the t-statistics (in parentheses). The estimation uses the full sample of over eight years of weekly data from April 6, 1994 to April 17, 2002 (420 weekly observations for each series). Panel A of Table 7 reports parameters (Θ_1) that are related to the three yield curve factors and are estimated using 12 LIBOR and swap rates. Panel B reports the parameters (Θ_2) that are related to the three options factors and are estimated using seven cap series. The scale of v_t and ω_t are multiplied by 10^4.

Panel A. Θ_1

Θ_1	κF	$\kappa_F^{\mathbb{Q}}$	θ_F	$\kappa_F^{\mathbb{Q}}\theta_F^{\mathbb{Q}}$	σ_F	ρ
r_t	1.6309 (4.66)	−1.6309	0	1.1761 (5.47)	−1.1583	0.1172 (4.14)	0	0.0546 (2.32)	0	0.1339 (1.87)
θ_t	0	0.0345 (0.26)	0	0	1.2662 (6.95)	0	0.0368 (0.00)	0.0705 (9.08)	0.0165 (4.55)	
v_t	0	0	0.0002 (0.00)	1.2058 (3.33)	0	0.1292 (4.14)	0.5065 (0.00)	0.1419 (3.62)	0.1154 (5.22)	

Panel B. Θ_2

κE	$\kappa_E^{\mathbb{Q}}$	c_h	b_λ
867.5238 (0.01)	0 —	0 —	1.1086 (9.56)	0 —	0 —	0.0000 (0.00)	−0.0314 (0.65)
−39.5458 (0.33)	3.6469 (2.24)	0 —	0.0015 (0.00)	0.0658 (1.68)	0 —	0.0000 (0.07)	0.0689 (2.19)
−1.7530 (0.00)	−2.2360 (1.09)	21.6735 (8.07)	−19.811 (0.90)	3.9355 (0.89)	0.2325 (2.60)	0.0003 (0.89)	−0.8215 (0.83)

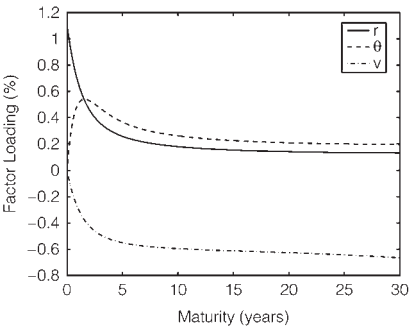
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FIGURE 3

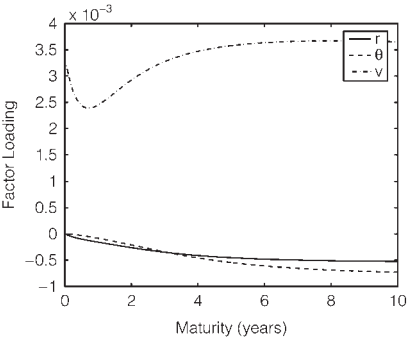
Yield Curve Factor Contributions to Interest Rates and Conditional Variances under the 3 + 3 LCV Model

Lines plot the impacts of one-standard-deviation shocks on the three LCV yield curve factors on the spot rate curve (Graph A of Figure 3) and the conditional variance of the three-month LIBOR at different horizons (Graph B).

Graph A. Interest Rate Term Structure



Graph B. Conditional Variance Term Structure



where the coefficients satisfy the following ordinary differential equations:

$$
\begin{aligned}
(47) \qquad a'(\tau) &= b(\tau)^{\top} \kappa_F^{\mathbb{Q}} \theta_F^{\mathbb{Q}} - \frac{1}{2} b(\tau)_2^2 \sigma_\theta^2, \\
b'(\tau) &= -\left(\kappa_F^{\mathbb{Q}}\right)^{\top} b(\tau) \\
&\quad - \frac{1}{2} \begin{bmatrix} 0 \\ 0 \\ b(\tau)_1^2 + 2b(\tau)_1 b(\tau)_3 \rho\sigma_v + b(\tau)_3^2 (\sigma_v)^2 \end{bmatrix},
\end{aligned}
$$

starting at $a(0)=0$, $b(0)=-ub(h)$. Then, we can compute the conditional variance of $b(h)^{\top} F_T$ numerically by taking second derivatives on the cumulant exponent $k(u)$ against the cumulant coefficient u and evaluating the derivative at $u = 0$:

$$
\begin{aligned}
(48) \qquad \frac{1}{\tau} \mathrm{Var}_t^{\mathbb{Q}} \left[b(h)^{\top} F_T \right] &= -\left[\frac{\partial^2 a(\tau, u)}{\tau \partial u^2} \right]_{u=0} - \left[\frac{\partial^2 b(\tau, u)}{\tau \partial u^2} \right]_{u=0}^{\top} F_t \\
&= v_0(\tau) + v_1(\tau)^{\top} F_t.
\end{aligned}
$$

In Graph B of Figure 3, we plot the coefficients $v_1(\tau)/\tau$ to show the relative contribution of the yield curve factors to the conditional variance at different maturities. Increasing the interest rate level and its long-run central tendency lowers the conditional variance slightly at longer maturities, but the main contribution to the conditional variance comes from the stochastic short rate variance factor v_t. As expected, the stochastic variance factor generates the most variation in the conditional variance.

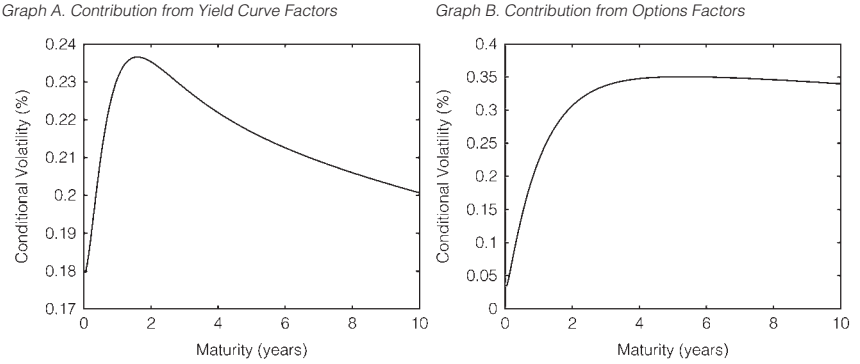
Replacing F_t with its sample averages in equation (48), we can compute the sample average of the conditional variance of $b(h)^{\top} F_T$. Graph A of Figure 4 plots the sample average of the conditional variance in volatility percentage points. The

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FIGURE 4

Conditional Volatility Contribution from Yield Curve Factors and Options Factors under the 3 + 3 LCV Model

Lines denote the conditional volatility at different horizons on the three-month LIBOR generated from the three yield curve factors (Graph A of Figure 4) and from the three options factors (Graph B) under the estimated 3 + 3 LCV model.



conditional volatility shows a hump-shaped term structure, consistent with the observed term structure on the implied volatilities.

For comparison, we also compute the conditional variance contributed by the options factors, $c_h^\top \mathrm{Var}_t\left[E_T\right] c_h$, which is given in equation (A-8) in Appendix A. Graph B of Figure 4 plots the conditional variance contribution from the options factors in volatility percentage points. The variance contribution from the yield curve factors is smaller than the contribution from the options factors, especially at long maturities. Therefore, introducing stochastic volatility in interest rates does not replace the need for additional options factors.

VI. Conclusion

Empirical analysis suggests that systematic factors identified from the yield curve cannot fully explain the movements in implied volatilities for interest rate options. Collin-Dufresne and Goldstein (2002) regard the findings as evidence for unspanned stochastic volatility and identify parametric constraints within the affine family of dynamic term structural models such that the stochastic volatility of interest rates is not instantaneously correlated with the value of interest rates. In practice, even if stochastic volatility factors affect the yield curve and are spanned by the yield curve in theory, one may still have difficulties in identifying these factors statistically from the observed yield curve if the dependence is weak. Furthermore, another possible source of the failure for dynamic term structure models in pricing interest rate options is that these models price interest rate options based on the model-implied fair values of the yield curve. A small misalignment between the underlying observed interest rates and the model-implied values can lead to large variations in the option implied volatilities.

In this paper, we propose a flexible $m + n$ model framework that can efficiently and successfully price both interest rates and interest rate options, regardless of the underlying rationale for the independent movements between the two

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markets. We use m yield curve factors to capture the systematic movement on the yield curve, and we use n additional options factors to capture the remaining interest rate movements that have little impact on the interest rate term structure but significant impacts on interest rate options. The orthogonal decomposition also allows us to design an efficient sequential estimation procedure, which estimates the yield curve component from the time series of interest rates in a first step and identifies the remaining options variations from the time series of options in a second step.

Under this framework, we estimate a 3 + 3 Gaussian affine example using eight years of data on LIBOR, swap rates, and interest rate caps. The estimation results show that the three yield curve factors explain over 99% of the variation on the yield curve, but they only explain less than 50% of the variation on implied volatilities for the interest rate caps. Incorporating three additional options factors improves the explained percentage variation on implied volatilities to over 99%. We also estimate an alternative term structure specification that allows for both stochastic central tendency and stochastic volatility. Nevertheless, the estimation results show that incorporating stochastic volatility in the interest rate term structure does not replace the need for additional options factors in pricing interest rate caps.

Appendix A. Caplet Pricing

Given the terminal payoff of the ith caplet,

$$\Pi_T^i = h\left(\text{LIBOR}(h)_T - K\right)^+,$$

with $T = t + ih$, its fair value can be computed via the following expectation:

$$\begin{aligned}\text{CAPLET}_t^i &= \mathbb{E}_t^{\mathbb{Q}}\left[\exp\left(-\int_t^{T+h} r(s)\,ds\right) h\left(\text{LIBOR}(h)_T - K\right)^+\right] \\ &= \mathbb{E}_t^{\mathbb{Q}}\left[\exp\left(-\int_t^{T} r(s)\,ds\right)\left(e^{hy(T,T+h)} - 1 - hK\right)^+ e^{\left(-a(h)-b(h)F_T\right)}\right],\end{aligned}$$

where the second line is obtained by representing LIBOR in terms of the continuously compounded spot rate ($y(T, T + h)$) and then applying the rule of iterated conditional expectation.

Furthermore, since

$$hy(T, T+h) - a(h) - b(h)^\top F_T = c_h^\top E_T,$$

we have

$$\text{(A-1)}\quad \text{CAPLET}^i = \mathbb{E}_t^{\mathbb{Q}}\left[\exp\left(-\int_t^T r(s)\,ds\right)\left(e^{c_h^\top E_T} - (1+hK)\,e^{-a(h)-b(h)^\top F_T}\right)^+\right].$$

To facilitate the expectation, we perform the following measure change:

$$\text{(A-2)}\quad \text{CAPLET}_t^i = P(F_t, T)\,\mathbb{E}_t^{\mathbb{T}}\left[\left(e^{c_h^\top E_T} - (1+hK)\,e^{-a(h)-b(h)^\top F_T}\right)^+\right],$$

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where $\mathbb{E}_t^{\mathbb{T}}[\cdot]$ denotes expectation under the fair-value *forward measure* $\mathbb{T}$, defined by the following Radon-Nikodým derivative (see Musiela and Rutkowski (1997b), p. 316):

$$\text{(A-3)} \qquad \frac{d\mathbb{T}}{d\mathbb{Q}} \equiv \frac{\exp\left(-\int_0^T r(s)\,ds\right)}{P(F_0,T)}, \quad \mathbb{Q}-\text{a.s.}$$

Conditional on the filtration $\mathcal{F}_t$, the above Radon-Nikodým derivative satisfies, for every $t \in [0,T]$,

$$\text{(A-4)} \qquad \eta_t \equiv \left.\frac{d\mathbb{T}}{d\mathbb{Q}}\right|_{\mathcal{F}_t} = \frac{\exp\left(-\int_0^t r(s)\,ds\right)P(F_t,T)}{P(F_0,T)}.$$

By Itó's lemma, the dynamics for the bond price $P(F_t,T)$ under measure $\mathbb{Q}$ becomes

$$\frac{dP(F_t,T)}{P(F_t,T)} = r(t)dt - b(T-t)^\top dW_t^{\mathbb{Q}}.$$

Thus, the dynamics of η_t can be written as

$$\eta_t = \exp\left(-\int_0^t b(T-u)^\top dW_u^{\mathbb{Q}} - \frac{1}{2}\int_0^t b(T-u)^\top b(T-u)du\right),$$

and $W_t^{\mathbb{T}}$ defined by the formula

$$W_t^{\mathbb{T}} = W_t^{\mathbb{Q}} - \int_0^T b(T-u)du$$

follows a standard Brownian motion under the forward measure $\mathbb{T}$. The yield curve factor dynamics under $\mathbb{T}$ are then adjusted as follows:

$$dF_t = \left(-b_\gamma - b(T-t) - \kappa_F^* F_t\right)dt + dW_t^{\mathbb{T}}.$$

It can be shown that under measure $\mathbb{T}$, and conditional on filtration $\mathcal{F}_t$, F_T is Gaussian, with mean and variance given by

$$\begin{aligned}\text{(A-5)}\quad \mathbb{E}_t^{\mathbb{T}}[F_T] &= e^{-\kappa_F^{\mathbb{Q}}(T-t)}F_t - \left(I - e^{-\kappa_F^{\mathbb{Q}}(T-t)}\right)\left(\kappa_F^{\mathbb{Q}}\right)^{-1} b_\gamma \\ &\quad - \left(I - e^{-\kappa_F^{\mathbb{Q}}(T-t)}\right)\left(\kappa_F^{\mathbb{Q}\top}\kappa_F^{\mathbb{Q}}\right)^{-1} b_r + \text{Var}_t^{\mathbb{T}}[F_t]\left(\kappa_F^{\mathbb{Q}\top}\right)^{-1} b_r, \\ \text{Var}_t^{\mathbb{T}}[F_T] &= \int_t^T e^{-\kappa_F^{\mathbb{Q}}s} e^{-\kappa_F^{\mathbb{Q}\top}s} ds.\end{aligned}$$

The matrix integral in (A-5) can be solved based on the eigenvalues and eigenvectors of $\kappa_F^{\mathbb{Q}}$. Specifically, let U and D denote the matrices formed by the eigenvectors and eigenvalues of $\kappa_F^{\mathbb{Q}}$, such that $\kappa_F^{\mathbb{Q}} = UDU^{-1}$, and we have

$$\text{(A-6)} \qquad \int_t^T e^{-\kappa_F^{\mathbb{Q}}s} e^{-\kappa_F^{\mathbb{Q}\top}s} ds = U\left[\frac{\mathcal{U}(i,j)\left(1-\exp(-(D(i,i)+D(j,j))(T-t))\right)}{D(i,i)+D(j,j)}\right]U^\top,$$

with $\mathcal{U} = \left(U^\top U\right)^{-1}$.

Equation (A-2) can be regarded as the pricing equation for an exchange option. Directly taking expectations yields

$$\text{(A-7)}\quad \text{CAPLET}_t^i = P(F_t,T)\,\mathbb{E}_t^{\mathbb{T}}\left[e^{c_h^\top E_T}\right]\mathcal{N}(d_1) - (1+hK)\,\mathbb{E}_t^T\left[e^{-a(h)-b(h)^\top F_T}\right]\mathcal{N}(d_2),$$

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where $\mathcal{N}(\cdot)$ denotes the cumulative density of a standard normal variable, and

$$d_1 = \frac{\ln\left(\mathbb{E}_t^{\mathbb{T}}\left[e^{c_h^\top E_T}\right] / (1+hK)\,\mathbb{E}_t^{\mathbb{T}}\left[e^{-a(h)-b(h)^\top F_T}\right]\right) + \frac{1}{2}\Sigma_t}{\sqrt{\Sigma_t}}, \quad d_2 = d_1 - \sqrt{\Sigma_t},$$

with Σ_t being the time-t conditional variance of $hy_T = a(h) + (h)^\top F_T + c_h^\top E_T$ under measure $\mathbb{T}$, and y_T being the future observed value of the three-month continuously compounded spot rate. The conditional variance Σ_t can be evaluated as

$$\Sigma_t = b(h)^\top \mathrm{Var}_t^{\mathbb{T}}[F_T]\, b(h) + c_h^\top \mathrm{Var}_t^{\mathbb{T}}[E_T]\, c_h.$$

The conditional mean and variance of F_T under measure $\mathbb{T}$ are given in (A-5). The conditional mean and variance of E_T are given by

$$\text{(A-8)} \quad \mathbb{E}_t^{\mathbb{T}}[E_T] = \mathbb{E}_t^{\mathbb{Q}}[E_T] = e^{-\kappa_E^{\mathbb{Q}}(T-t)}E_t - \left(I - e^{-\kappa_E^{\mathbb{Q}}(T-t)}\right)\left(\kappa_E^{\mathbb{Q}}\right)^{-1} b_\lambda,$$

$$\mathrm{Var}_t^{\mathbb{T}}[E_T] = \mathrm{Var}_t^{\mathbb{Q}}[E_T] = \int_t^T e^{-\kappa_E^{\mathbb{Q}} s} e^{-\kappa_E^{\mathbb{Q}\top} s} ds,$$

where the integral in (A-8) can be evaluated as in (A-6). Furthermore, in (A-7), we have

$$\mathbb{E}_t^{\mathbb{Q}}\left[e^{c_h^\top E_T}\right] = \exp\left(c_h^\top \mathbb{E}_t^{\mathbb{T}}[E_T] + \frac{1}{2} c_h^\top \mathrm{Var}_t^{\mathbb{T}}[E_T]\, c_h\right),$$

$$\mathbb{E}_t^{\mathbb{Q}}\left[e^{-a(h)-b(h)^\top F_T}\right] = P(F_t, T+h)/P(F_t, T).$$

Rearrange, and we have

$$\mathrm{CAPLET}_t^i = P(F_t, T+h)\left[(1+h\mathcal{R}_t)\,\mathcal{N}(d_1) - (1+hK)\,\mathcal{N}(d_2)\right],$$

where $\mathcal{R}_t$ can be regarded as the residual-adjusted value of the forward three-month LIBOR, defined by

$$(1+h\mathcal{R}_t) = \frac{P(F_t, T)}{P(F_t, T+h)} \exp\left(c_h^\top \mathbb{E}_t[E_T] + \frac{1}{2} c_h^\top \mathrm{Var}_t[E_T]\, c_h\right),$$

and d_1 can be rewritten as

$$d_1 = \frac{\ln(1+h\mathcal{R}_t)/(1+hK) + \frac{1}{2}\Sigma_t}{\sqrt{\Sigma_t}}.$$

Appendix B. Generalized Fourier Transforms

For the value of a contingent claim $G_{a,b}(k)$, defined as

$$G_{a,b}(k) \equiv \mathbb{E}_t^{\mathbb{Q}}\left[\exp\left(-\int_t^T r_s ds\right) e^{-a_T} 1_{b_T \le k}\right],$$

we define its generalized Fourier transform as

$$\varphi_G(u) \equiv \int_{-\infty}^{\infty} e^{iuk} G_{a,b}(k)\, dk,$$

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where $u = u_r + iu_i$ with real numbers (u_r, u_i). To solve the transform, we first perform integration by parts,

$$\varphi(u) \equiv \int_{-\infty}^{\infty} e^{iuk} G_{a,b}(k)\, dk = G_{a,b}(k)\, \frac{e^{iuk}}{iu}\Big|_{-\infty}^{\infty} - \frac{1}{iu}\int_{-\infty}^{\infty} e^{iuk} dG_{a,b}(k).$$

Since $G_{a,b}(\infty) = \mathbb{E}_t^{\mathbb{Q}}[\exp(-\int_t^T r_s ds)e^{-a_T}] = \phi(-a_T) > 0$, the limit term is well defined and vanishes only when $u_i > 0$. Applying Fubini's theorem on the second term, we have

$$\begin{aligned}\varphi_G(u) &= -\frac{1}{iu}\int_{-\infty}^{\infty} e^{iuk} dG_{a,b}(k) = -\frac{1}{iu}\mathbb{E}_t^{\mathbb{Q}}\left[\exp\left(-\int_t^T r_s ds\right) e^{-a_T + iub_T}\right] \\ &= -\frac{1}{iu}\phi(-a_T + iub_T).\end{aligned}$$

Applying this general result to the specific case of the caplet, we have

$$\begin{aligned}\varphi(u) \equiv & \int_{-\infty}^{\infty} e^{iuk}\Big[G_{-e_T,-e_T-a(h)-b(h)^\top F_T}(k) - e^{-k} \\ & \times G_{a(h)+b(h)^\top F_T,-e-a(h)-b(h)^\top F_T}(k)\Big] dk \\ = & -\frac{1}{iu}\phi\left(e_T - iue_T - iua(h) - iub(h)^\top F_T\right) \\ & + \frac{1}{iu-1}\phi\left(-a(h) - b(h)^\top F_T - (iu-1)\left(e_T + a(h) + b(h)^\top F_T\right)\right) \\ = & \frac{1}{iu(iu-1)}\phi\left(-iua(h) - iub(h)^\top F_T + (1-iu)e_T\right),\end{aligned}$$

which can be solved as an exponential affine function of F_t and E_t as in (35).

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