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Higher order nonlinear evolution equations for a three-dimensional electromagnetic wave packet in a plasma

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Abstract. Starting from the full set of governing equations including the weak relativistic effect, three coupled nonlinear equations with higher order nonlinear and dispersive terms are derived, which describe the nonlinear evolution of a three-dimensional electromagnetic wave packet propagating in a plasma. Two particular cases are considered, where the space dependence of the dependent variables is either on the space coordinate along the direction of propagation of the wave or on that along an arbitrary fixed direction. In both cases the three coupled equations reduce to a single equation. This single equation is used to study the modulational instability of a uniform electromagnetic wave train. It is found that due to the inclusion of the weak relativistic effect there is always modulational instability, and a higher order nonlinear term in the evolution equation has a stabilizing influence.

1. Introduction

The nonlinear evolution of a one-dimensional transverse electromagnetic wave packet is governed by a nonlinear Schrödinger equation. However, if the intensity of the wave is sufficiently high and its temporal and spatial spectral width are not narrow enough, then higher order nonlinear and dispersive terms are to be included in the evolution equation. Such an evolution equation involving higher order nonlinear and dispersive terms for a one-dimensional electromagnetic wave packet has recently been obtained by Gromov and Talanov (1996). They have termed this equation as a third-order nonlinear Schrödinger equation. To derive this equation they have started from two coupled equations derived by Zakharov (1972) instead of starting from the full set of governing equations. In these two equations, the interaction of the wave with the wave induced second harmonic components has been neglected.

In the present investigation, we derive evolution equations for a three-dimensional electromagnetic wave packet starting from the full set of governing equations including the weak relativistic effect, which was shown to be important for electromagnetic waves by Sluijter and Montgomery (1965). It is found that the nonlinear evolution equations consist of three nonlinear coupled equations and they

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reduce to a single equation in the following two cases: (i) the space dependence of the amplitude of the wave is only on the space coordinate along the direction of propagation of the wave; and (ii) the space dependence of the amplitude of the wave is on the coordinate along an arbitrary fixed direction. The form of this single reduced equation is the same as that of the equation derived by Gromov and Talanov (1996) for one-dimensional electromagnetic wave packets in a plasma and is one order higher than the lowest order evolution equation, which is the nonlinear Schrödinger equation. Such higher order evolution equations for water waves have been obtained by many authors, for example Dysthe (1979), Hogan (1985), Janssen (1983), Stiassnie (1984) and Dhar and Das (1990). Modulational stability analysis of a uniform electromagnetic wave train is made from the single nonlinear evolution equation derived in the above-mentioned cases. The analysis reveals that there is modulational instability, and this appears due to the inclusion of the weak relativistic effect. Further a higher order nonlinear term in the evolution equation has a stabilizing influence.

2. Basic equations

The governing equations are the following equation of continuity and equation of motion for both electron and ion fluids together with Maxwell's equations. In the equation of motion for electron fluids, the weak relativistic effect has been included and nonlinear terms in the equation of continuity and equation of motion for ion fluid have been neglected (Dysthe and Pecseli 1977):

$$\frac{\partial n_{\mathrm{e}}}{\partial t} + \nabla \cdot \{(1+n_{\mathrm{e}})\mathbf{v}_{\mathrm{e}}\} = 0 \tag{2.1}$$

$$\left(\frac{\partial}{\partial t} + \mathbf{v}_{\mathrm{e}} \cdot \nabla\right) \frac{\mathbf{v}_{\mathrm{e}}}{\sqrt{1-\mathbf{v}_{\mathrm{e}}^2}} + \mathbf{E} + \frac{\beta_{\mathrm{e}}^2}{1+n_{\mathrm{e}}} \nabla n_{\mathrm{e}} + \mathbf{v}_{\mathrm{e}} \times \mathbf{B} = 0 \tag{2.2}$$

$$-\nabla \times \mathbf{B} + \frac{\partial \mathbf{E}}{\partial t} - (1+n_{\mathrm{e}})\mathbf{v}_{\mathrm{e}} + (1+n_{\mathrm{i}})\mathbf{v}_{\mathrm{i}} = 0 \tag{2.3}$$

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} = 0 \tag{2.4}$$

$$\frac{\partial n_{\mathrm{i}}}{\partial t} + \nabla \cdot \mathbf{v}_{\mathrm{i}} = 0 \tag{2.5}$$

$$\frac{\partial \mathbf{v}_{\mathrm{i}}}{\partial t} - \mathbf{E} + \beta_{\mathrm{i}}^2 \nabla n_{\mathrm{i}} = 0. \tag{2.6}$$

In the above equations, the space coordinates (x, y, z) and time (t) have been made dimensionless by c/ω_{e} and ω_{e}^{-1}, respectively, where ω_{e} is the electron plasma frequency and c is the velocity of light; $\beta_{\mathrm{e}} = V_{\mathrm{e}}/c$ and $\beta_{\mathrm{i}} = V_{\mathrm{i}}/c$, where V_{e} and V_{i} are, respectively, the electron and ion thermal velocities. Also, in the above equations the electron (ion) number density n_{e} (n_i), electron (ion) fluid velocity $\mathbf{v}_{\mathrm{e}}$ $(\mathbf{v}_{\mathrm{i}})$, electric field intensity vector $\mathbf{E}$ and magnetic field intensity vector $\mathbf{B}$ have been made dimensionless by $N_0, c, c\omega_{\mathrm{e}}m_{\mathrm{e}}/e$ and $c\omega_{\mathrm{e}}m_{\mathrm{e}}/e$, respectively, where e is the charge (positive) of an electron, m_{e} is the mass of an electron and N_0 is the unperturbed number density of electrons and ions.

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3. Derivation of evolution equations

We assume that only electrons participate in the electromagnetic wave in the linear approximation, while ion motion comes through nonlinear interaction. That is, ions respond to the slowly varying part of the electric field (Dysthe and Pecseli 1977; Das and Sihi 1979). Keeping this in mind, we make the following perturbation expansion of the dependent variables:

$$\begin{aligned}
n_{\mathrm{e}} &= n_{\mathrm{e}0} + \sum_{m=1}^{\infty} (n_{\mathrm{em}} \exp i\psi + n_{\mathrm{em}}^{*} \exp -im\psi) \\
\mathbf{E} &= \mathbf{E}_0 + \sum_{m=1}^{\infty} (\mathbf{E}_{\mathrm{m}} \exp i\psi + \mathbf{E}_{\mathrm{m}}^{*} \exp -im\psi) \\
\mathbf{B} &= \mathbf{B}_0 + \sum_{m=1}^{\infty} (\mathbf{B}_{\mathrm{m}} \exp i\psi + \mathbf{B}_{\mathrm{m}}^{*} \exp -im\psi) \qquad (3.1) \\
\mathbf{v}_{\mathrm{e}} &= \mathbf{v}_{\mathrm{e}0} + \sum_{m=1}^{\infty} (\mathbf{v}_{\mathrm{em}} \exp i\psi + \mathbf{v}_{\mathrm{em}}^{*} \exp -im\psi) \\
\mathbf{v}_{\mathrm{i}} &= \mathbf{v}_{\mathrm{i}0}, \quad n_{\mathrm{i}} = n_{\mathrm{i}0}
\end{aligned}$$

where $n_{\mathrm{e}0}, n_{\mathrm{i}0}, \mathbf{v}_{\mathrm{e}0}, \mathbf{v}_{\mathrm{i}0}, \mathbf{E}_0, n_{\mathrm{em}}, \mathbf{v}_{\mathrm{em}}, \mathbf{E}_{\mathrm{m}}, \mathbf{B}_{\mathrm{m}}$ are slowly varying functions of x, y, z, t, i.e. they are functions of $x_1 = \epsilon x$, $y_1 = \epsilon y$, $z_1 = \epsilon z$, $t_1 = \epsilon t$, ϵ is a small parameter; $\psi = kx - \omega t$; ω, k satisfy the following linear dispersion relation in the dimensionless form:

$$\omega^2 = k^2 + 1. \qquad (3.2)$$

Substituting the expansions (3.1) in the governing equations (2.1)–(2.6) and then equating coefficients of $\exp im\psi$ for $m = 0, 1, 2$ on both sides of each equation, we get the following three sets of equations.

(i) For $m = 1$ we get the following equations from (2.1), (2.2), (2.4) and eliminant of $\mathbf{B}$ between (2.3) and (2.4),

$$-i\omega_1 n_{\mathrm{e}1} + ik_1 v_{\mathrm{e}x1} = a_{01}, \qquad (3.3)$$

$$\left(-i\omega_1 v_{\mathrm{e}x1} + E_{x1} + \beta_{\mathrm{e}}^2 ik_1 n_{\mathrm{e}1}, -i\omega_1 v_{\mathrm{e}y1} + E_{y1}, -i\omega_1 v_{\mathrm{e}z1} + E_{z1}\right) = (a_{11}, a_{21}, a_{31}), \qquad (3.4)$$

$$\begin{aligned}
&\left(-\omega_1^2 E_{x1} + i\omega_1 v_{\mathrm{e}x1}, \left\{k_1^2 - \omega_1^2\right\} E_{y1} + i\omega_1 v_{\mathrm{e}y1}, \left\{k_1^2 - \omega_1^2\right\} E_{z1} + i\omega_1 v_{\mathrm{e}z1}\right) \\
&\quad = (b_{11}, b_{21}, b_{31}), \qquad (3.5) \\
&(i\omega_1 B_{x1}, i\omega_1 B_{y1} + ik_1 E_{z1}, -i\omega_1 B_{z1} + ik_1 E_{y1}) = (c_{11}, c_{21}, c_{31}) \qquad (3.6)
\end{aligned}$$

where $\omega_1 = \omega + i\epsilon(\partial/\partial t_1)$, $k_1 = k - i\epsilon(\partial/\partial x_1)$; a_{i1}, b_{i1}, c_{i1} are contributions from nonlinear and higher order linear terms.

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(ii) For $m = 2$ we get the following equations from (2.1), (2.2), (2.4) and eliminant of **B** between (2.3) and (2.4),

$$-i\omega_2 n_{e2} + ik_2 v_{ex2} = a_{02}, \tag{3.7}$$

$$\left(-i\omega_2 v_{ex2} + E_{x2} + \beta_e^2 ik_2 n_{e2},\ -i\omega_2 v_{ey2} + E_{y2},\ -i\omega_2 v_{ez2} + E_{z2}\right) = (a_{12}, a_{22}, a_{32}), \tag{3.8}$$

$$\left(-\omega_2^2 E_{x2} + i\omega_2 v_{ex2},\ \left\{k_2^2 - \omega_2^2\right\}E_{y2} + i\omega_2 v_{ey2},\ \left\{k_2^2 - \omega_2^2\right\}E_{z2} + i\omega_2 v_{ez2}\right) = (b_{12}, b_{22}, b_{32}) \tag{3.9}$$

where $k_2 = 2k - i\epsilon(\partial/\partial x_1)$, $\omega_2 = 2\omega + i\epsilon(\partial/\partial t_1)$; a_{j2}, b_{j2}, are contributions from nonlinear and higher order dispersive terms.

(iii) For $m = 0$ we get the following equations from (2.1), (2.2), (2.5) and (2.6),

$$\epsilon\frac{\partial n_{e0}}{\partial t_1} + \epsilon\nabla_1 \cdot \mathbf{v}_{e0} = a_{00} \tag{3.10}$$

$$\epsilon\frac{\partial \mathbf{v}_{e0}}{\partial t_1} + \mathbf{E}_0 + \beta_e^2\epsilon\nabla_1 n_{e0} = \hat{x}a_{10} + \hat{y}a_{20} + \hat{z}a_{30} \tag{3.11}$$

$$\epsilon\frac{\partial n_{i0}}{\partial t_1} + \epsilon\nabla_1 \cdot \mathbf{v}_{i0} = 0 \tag{3.12}$$

$$\epsilon\frac{\partial \mathbf{v}_{i0}}{\partial t_1} - \mathbf{E}_0 + \beta_i^2\epsilon\nabla_1 n_{i0} = 0 \tag{3.13}$$

where a_{j0} are contributions from nonlinear terms and $\hat{x}, \hat{y}, \hat{z}$ are unit vectors along the three coordinate axes. Here we need one more equation, which is the following charge neutrality condition:

$$n_{e0} = n_{i0}. \tag{3.14}$$

To solve the above three sets of equations we make the following perturbation expansion of the Fourier coefficients (Davey and Stewartson 1974):

$$F_1 = \sum_{n=1}^{\infty}\epsilon^n F_{1n}, \quad F_m = \sum_{n=2}^{\infty}\epsilon^n F_{mn} \quad (m = 0, 2) \tag{3.15}$$

where F_j stands for $n_{ej}, n_{ij}, v_{exj}, v_{eyj}, v_{ezj}, v_{ixj}, v_{iyj}, v_{izj}, E_{xj}, E_{yj}, E_{zj}, B_{xj}, B_{yj}, B_{zj}$, $j = 0, 1, 2$.

Substituting the expansions (3.15) in the above three sets of equations and then equating coefficients of like powers of ϵ on both sides, we get a sequence of equations for the determination of the quantities F_{mn}, $m = 0, 1, 2$.

From the first-order (i.e. lowest-order) and second-order equations of (3.3), (3.4), and x and z components of (3.5), (3.6), we get solutions for $E_{x11}, E_{z11}, \mathbf{v}_{e11}, \mathbf{B}_{11}$, $n_{e11}, E_{x12}, E_{z12}, \mathbf{v}_{e12}, \mathbf{B}_{12}, n_{e12}$ all expressed in terms of E_{y11} and E_{y12}. Next, from the first-order (i.e. lowest-order) and second-order equations of (3.7)–(3.9), we get solutions for $n_{e22}, \mathbf{v}_{e22}, \mathbf{E}_{22}$ and $n_{e23}, \mathbf{v}_{e23}, \mathbf{E}_{23}$, respectively, all expressed in terms of E_{y11} and E_{y12}. All of these perturbation solutions are given in Appendix A.

The y component of (3.5), which has not been used in obtaining the above-mentioned perturbation solutions, can be written in the following form after

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eliminating v_{ey1} by the use of the y-component of (3.4):

$$\begin{aligned}\left(k_1^2-\omega_1^2+1\right)E_{y1} = &-\epsilon\beta_{\mathrm{e}}^2\frac{\partial n_{\mathrm{e}1}}{\partial y_1}-[(\mathbf{v}_{\mathrm{e}}\cdot\nabla)v_{ey}]_1-[(\mathbf{v}_{\mathrm{e}}\times\mathbf{B})_y]_1\\ &-\frac{1}{2}\left[\frac{\partial}{\partial t}\left(v_{\mathrm{e}}^2 v_{ey}\right)\right]_1+\beta_{\mathrm{e}}^2\left[\left(-n_{\mathrm{e}}+n_{\mathrm{e}}^2\right)\frac{\partial n_{\mathrm{e}}}{\partial y}\right]_1-i\epsilon k_1\frac{\partial E_{x1}}{\partial y_1}\\ &+\epsilon^2\left(\frac{\partial^2 E_{y1}}{\partial z_1^2}-\frac{\partial^2 E_{z1}}{\partial y_1\partial z_1}\right)+\left[\frac{\partial}{\partial t}(n_{\mathrm{e}}v_{ey})\right]_1. \qquad (3.16)\end{aligned}$$

On the right-hand side of this equation we have only kept quadratic and cubic nonlinear terms as these are sufficient to derive the desired evolution equation involving terms up to order ϵ^4. By the notation $[\ \cdot\ ]_1$ we imply the coefficient of $\exp i\psi$ in Fourier expansion of the expression inside the bracket, which is obtained after substituting the expansions (3.1).

Keeping terms up to order ϵ^4 in (3.16), then substituting the solutions for various perturbed quantities and finally using the transformations,

$$\xi = x_1 - V_{\mathrm{g}}t_1 = \epsilon(x-V_{\mathrm{g}}t),\quad \eta = y_1 = \epsilon y,\quad \zeta = z_1 = \epsilon z,\quad \tau = \epsilon t_1 = \epsilon^2 t, \quad (3.17)$$

we arrive at the following evolution equation for $\alpha = \epsilon E_{y11} + \epsilon^2 E_{y12}$, where $N = \epsilon^2 n_{\mathrm{e}02} + \epsilon^3 n_{\mathrm{e}03}, V = \epsilon^2 v_{ey02} + \epsilon^3 v_{ey03}$ and $V_{\mathrm{g}} = d\omega/dk$ is the group velocity of the electromagnetic wave:

$$\begin{aligned}&i\frac{\partial\alpha}{\partial\tau}+\frac{1}{2\omega}\left(1-\frac{k^2}{\omega^2}\right)\frac{\partial^2\alpha}{\partial\xi^2}+\frac{1}{2\omega}\left(\frac{\partial^2\alpha}{\partial\eta^2}+\frac{\partial^2\alpha}{\partial\zeta^2}\right)\\ &\quad+\frac{i\epsilon k}{2\omega^3}\left(1-\frac{k^2}{\omega^2}\right)\frac{\partial^3\alpha}{\partial\xi^3}+\frac{i\epsilon k}{2\omega^3}\left(\frac{\partial^3\alpha}{\partial\xi\partial\eta^2}+\frac{\partial^3\alpha}{\partial\xi\partial\zeta^2}\right)\\ &\quad=-\frac{\mu_1}{2\omega}\alpha^2\alpha^*+\frac{1}{2\omega}N\alpha+\frac{i\epsilon}{2\omega^2}\frac{\partial}{\partial\eta}(V\alpha)+\frac{i\epsilon k}{2\omega^3}N\frac{\partial\alpha}{\partial\xi}\\ &\quad\quad-i\epsilon\left(\frac{\mu_2}{2\omega}+\frac{\mu_1 k}{\omega^3}\right)\alpha\alpha^*\frac{\partial\alpha}{\partial\xi}+\frac{i\epsilon k\mu_1}{2\omega^3}\alpha^2\frac{\partial\alpha^*}{\partial\xi} \qquad (3.18)\end{aligned}$$

where

$$\begin{aligned}\mu_1 &= \frac{9+8k^2-12k^2\beta_{\mathrm{e}}^2}{2\omega^2\left(3+4k^2-4k^2\beta_{\mathrm{e}}^2\right)},\\ \mu_2 &= \frac{12k}{\omega^2\left(3+4k^2-4k^2\beta_{\mathrm{e}}^2\right)^2}.\end{aligned} \qquad (3.19)$$

As the evolution equation (3.18) for α involves the quantities N and V, we need two more equations for these two quantities. These two equations are obtained from the zeroth harmonic equations (3.10)–(3.14).

Eliminating E_{y0} between the y-component of (3.11) and (3.13), we get the following equation from the lowest and next order:

$$-\frac{k}{\omega}\frac{\partial V}{\partial\xi}+c_{\mathrm{s}}^2\frac{\partial N}{\partial\eta}+\epsilon\frac{\partial V}{\partial\eta}=-\frac{\mu}{\omega^2(1+\mu)}\frac{\partial}{\partial\eta}(\alpha\alpha^*)+\frac{i\epsilon k}{\omega^4}\frac{\partial}{\partial\eta}\left(\alpha\frac{\partial\alpha^*}{\partial\xi}-\alpha^*\frac{\partial\alpha}{\partial\xi}\right), \qquad (3.20)$$

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where $\mu = m_e/m_i$ and

$$c_s^2 = \frac{m_e V_e^2 + m_i V_i^2}{c^2(m_e + m_i)} \tag{3.21}$$

is the dimensionless ion-acoustic speed.

Now, to find an equation for N we proceed as follows. Taking the dot product of (3.11) and (3.13) with the vector ∇_1 and then eliminating $\nabla_1 \cdot \mathbf{v}_{e0}$ and $\nabla_1 \cdot \mathbf{v}_{i0}$ by the use of (3.10) and (3.12), we get two equations. Finally, eliminating $\nabla_1 \cdot \mathbf{E}_0$ from these two equations we arrive at the following equation for N where we keep terms up to order ϵ^5 and the charge neutrality condition (3.14):

$$\left(\frac{k^2}{\omega^2} - c_s^2\right)\frac{\partial^2 N}{\partial \xi^2} - c_s^2\left(\frac{\partial^2 N}{\partial \eta^2} + \frac{\partial^2 N}{\partial \zeta^2}\right) - \frac{2\epsilon k}{\omega}\frac{\partial^2 N}{\partial \xi \partial \tau}$$
$$= \frac{\mu}{\omega^2(1+\mu)}\nabla_\xi^2(\alpha\alpha^*) - \frac{i\epsilon\mu k}{\omega^4(1+\mu)}\nabla_\xi^2\left(\alpha\frac{\partial \alpha^*}{\partial \xi} - \alpha^*\frac{\partial \alpha}{\partial \xi}\right)$$
$$- \frac{i\epsilon\mu}{\omega^2 k(1+\mu)}\frac{\partial^2}{\partial \xi \partial \eta}\left(\alpha\frac{\partial \alpha^*}{\partial \eta} - \alpha^*\frac{\partial \alpha}{\partial \eta}\right) \tag{3.22}$$

where

$$\nabla_\xi^2 = \frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2} + \frac{\partial^2}{\partial \zeta^2}.$$

The nonlinear evolution of a three-dimensional electromagnetic wave packet in a plasma is governed by the three coupled equations (3.18), (3.20) and (3.22).

We now consider the following cases in which the three coupled equations mentioned above reduce to a single equation.

3.1. The space dependence of the dependent variables is only on the space coordinate along the direction of propagation of the wave

In this case α, N and V depend on ξ and τ only. If we set

$$\alpha = \epsilon\alpha^{(0)} + \epsilon^2\alpha^{(1)}, \quad N = \epsilon^2 N^{(0)} + \epsilon^3 N^{(1)}, \quad V = \epsilon^2 V^{(0)} + \epsilon^3 V^{(1)} \tag{3.23}$$

then we get the following solutions for $V^{(0)}, N^{(0)}, \partial\alpha^{(0)}/\partial\tau$ and $N^{(1)}$ from (3.18), (3.20) and (3.22):

$$V^{(0)} = 0, \quad N^{(0)} = \frac{\mu}{(1+\mu)\left(k^2 - \omega^2 c_s^2\right)}\alpha^{(0)}\alpha^{(0)*},$$

$$\frac{\partial \alpha^{(0)}}{\partial \tau} = \frac{i}{\omega}\left(1 - \frac{k^2}{\omega^2}\right)\frac{\partial^2 \alpha^{(0)}}{\partial \xi^2} + i\left\{\frac{\mu_1}{2\omega} - \frac{\mu}{2\omega(1+\mu)\left(k^2 - c_s^2\omega^2\right)}\right\}\alpha^{(0)^2}\alpha^{(0)*},$$

$$N^{(1)} = \frac{\mu}{(1+\mu)\left(k^2 - c_s^2\omega^2\right)}\left(\alpha^{(0)}\alpha^{(1)*} + \alpha^{(1)}\alpha^{(0)*}\right)$$
$$- \frac{i\mu k}{\omega^2(1+\mu)\left(k^2 - c_s^2\omega^2\right)}\left\{1 + \frac{\omega^2 - k^2}{k^2 - c_s^2\omega^2}\right\}\left(\alpha^{(0)}\frac{\partial \alpha^{(0)*}}{\partial \xi} - \alpha^{(0)*}\frac{\partial \alpha^{(0)}}{\partial \xi}\right). \tag{3.24}$$

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By use of these solutions, the three coupled equations (3.18), (3.20) and (3.22) can be reduced to the following single evolution equation in this case:

$$i\frac{\partial\alpha}{\partial\tau}+\beta_1\frac{\partial^2\alpha}{\partial\xi^2}+i\beta_2\frac{\partial^3\alpha}{\partial\xi^3}=\lambda_1\alpha^2\alpha^*+i\lambda_2\alpha\alpha^*\frac{\partial\alpha}{\partial\xi}+i\lambda_3\alpha^2\frac{\partial\alpha^*}{\partial\xi}, \tag{3.25}$$

where

$$\begin{aligned}
\beta_1&=\frac{\omega^2-k^2}{2\omega^3},\quad \beta_2=\frac{k(\omega^2-k^2)}{2\omega^5},\\
\lambda_1&=-\frac{\mu_1}{2\omega}+\frac{\mu}{2\omega(1+\mu)\big(\omega^2-k^2c_{\mathrm{s}}^2\big)},\\
\lambda_2&=-\frac{k\mu_1}{\omega^3}-\frac{\mu_2}{2\omega}+\frac{k\mu}{\omega^3(1+\mu)\big(\omega^2-k^2c_{\mathrm{s}}^2\big)}-\frac{k\mu(\omega^2-k^2)}{2\omega^3(1+\mu)\big(\omega^2-k^2c_{\mathrm{s}}^2\big)^2},\\
\lambda_3&=\frac{k\mu_1}{2\omega^3}-\frac{k\mu}{2\omega^3(1+\mu)\big(\omega^2-k^2c_{\mathrm{s}}^2\big)}-\frac{k\mu(\omega^2-k^2)}{2\omega^3(1+\mu)\big(\omega^2-k^2c_{\mathrm{s}}^2\big)^2}.
\end{aligned} \tag{3.26}$$

3.2. The space dependence of all the dependent variables is only on the space coordinate along an arbitrary fixed direction

Let the direction cosines of this fixed line be l, m, n and let X be the coordinate along this line. Then we have

$$\frac{\partial}{\partial\xi}=l\frac{\partial}{\partial X},\quad \frac{\partial}{\partial\eta}=m\frac{\partial}{\partial X},\quad \frac{\partial}{\partial\zeta}=n\frac{\partial}{\partial X}. \tag{3.27}$$

By use of these relations and by the same setting for α, N, V as given by (3.23), we get the following solutions for $V^{(0)}, N^{(0)}, \partial\alpha^{(0)}/\partial\tau$ and $N^{(1)}$ from (3.18), (3.20) and (3.22),

$$\begin{aligned}
V^{(0)}&=\frac{\mu klm}{\omega(1+\mu)\big(k^2l^2-\omega^2c_{\mathrm{s}}^2\big)}\alpha^{(0)}\alpha^{(0)*},\\
N^{(0)}&=\frac{\mu}{(1+\mu)\big(k^2l^2-\omega^2c_{\mathrm{s}}^2\big)}\alpha^{(0)}\alpha^{(0)*},\\
\frac{\partial\alpha^{(0)}}{\partial\tau}&=\frac{i(\omega^2-k^2l^2)}{2\omega^3}\frac{\partial^2\alpha^{(0)}}{\partial X^2}+i\left\{\frac{\mu_1}{2\omega}-\frac{\mu}{2\omega(1+\mu)\big(k^2l^2-c_{\mathrm{s}}^2\omega^2\big)}\right\}\alpha^{(0)^2}\alpha^{(0)*},\\
N^{(1)}&=\frac{\mu}{(1+\mu)\big(k^2l^2-c_{\mathrm{s}}^2\omega^2\big)}\big(\alpha^{(0)}\alpha^{(1)*}+\alpha^{(1)}\alpha^{(0)*}\big)\\
&\quad-\frac{i\mu kl}{\omega^2(1+\mu)\big(k^2l^2-c_{\mathrm{s}}^2\omega^2\big)}\left\{1+\frac{\omega^2-k^2l^2}{k^2l^2-c_{\mathrm{s}}^2\omega^2}+\frac{m^2\omega^2}{k^2}\right\}\\
&\quad\times\left(\alpha^{(0)}\frac{\partial\alpha^{(0)*}}{\partial\xi}-\alpha^{(0)*}\frac{\partial\alpha^{(0)}}{\partial\xi}\right).
\end{aligned} \tag{3.28}$$

By use of these solutions, we arrive at the following single evolution equation from the three coupled equations (3.18), (3.20) and (3.22) (this equation is the same as (3.25) obtained in the previous case; the only difference is that the coefficients

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of the equation are not the same):

$$i\frac{\partial\alpha}{\partial\tau}+\bar{\beta}_1\frac{\partial^2\alpha}{\partial X^2}+i\bar{\beta}_2\frac{\partial^3\alpha}{\partial X^3}=\bar{\lambda}_1\alpha^2\alpha^*+i\bar{\lambda}_2\alpha\alpha^*\frac{\partial\alpha}{\partial X}+i\bar{\lambda}_3\alpha^2\frac{\partial\alpha^*}{\partial X}, \tag{3.29}$$

where

$$\bar{\beta}_1=\frac{1}{2\omega}\left(1-\frac{k^2l^2}{\omega^2}\right),\quad \bar{\beta}_2=\frac{kl}{2\omega^3}\left(1-\frac{k^2l^2}{\omega^2}\right),$$

$$\bar{\lambda}_1=-\frac{\mu_1}{2\omega}+\frac{\mu}{2\omega(1+\mu)\left(k^2l^2-\omega^2c_{\rm s}^2\right)},$$

$$\bar{\lambda}_2=-\frac{\mu_1kl}{\omega^3}-\frac{\mu_2l}{2\omega}+\frac{\mu kl}{2\omega^3(1+\mu)\left(k^2l^2-\omega^2c_{\rm s}^2\right)}$$

$$\times\left\{\frac{(\omega^2-k^2l^2)}{\left(k^2l^2-\omega^2c_{\rm s}^2\right)}+2+m^2\left(\frac{\omega^2}{k^2}+2\right)\right\},$$

$$\bar{\lambda}_3=\frac{\mu_1kl}{2\omega^3}-\frac{\mu kl}{2\omega^3(1+\mu)\left(k^2l^2-\omega^2c_{\rm s}^2\right)}\left\{\frac{(\omega^2-k^2l^2)}{\left(k^2l^2-\omega^2c_{\rm s}^2\right)}+1+m^2\left(\frac{\omega^2}{k^2}-1\right)\right\}. \tag{3.30}$$

4. Modulational instability

As we recover the evolution equation (3.25) from the evolution equation (3.29) by setting $l=1,m=0,n=0,X=\xi$, we derive the modulational instability condition from the latter equation.

For a uniform wave train solution of (3.29) we take

$$\alpha=a_0\exp(-i\Delta\omega\tau)\equiv\alpha_0, \tag{4.1}$$

where a_0 and $\Delta\omega$ are two real constants. The latter is the nonlinear amplitude-dependent frequency shift and is given by

$$\Delta\omega=\bar{\lambda}_1a_0^2. \tag{4.2}$$

To investigate the modulational instability of the uniform wave train (4.1), we introduce the following perturbation in the uniform solution

$$\alpha=\alpha_0[1+\hat{\alpha}(X,\tau)] \tag{4.3}$$

where $\hat{\alpha}$ is complex.

Substituting (4.3) into (3.29) and then linearizing we get an equation for $\hat{\alpha}$. Next, writing $\hat{\alpha}=\hat{\alpha}_{\rm r}+i\hat{\alpha}_{\rm i}$ (where $\hat{\alpha}_{\rm r}$ and $\hat{\alpha}_{\rm i}$ are real and imaginary parts of $\hat{\alpha}$) in this equation and then separating into real and imaginary parts, we get two equations which produce the following nonlinear dispersion relation, when the space–time dependence of $\hat{\alpha}_{\rm r}$ and $\hat{\alpha}_{\rm i}$ is assumed to be of the form $\exp i(pX-\Omega\tau)$:

$$\Omega+\bar{\beta}_2p^3+\bar{\lambda}_2a_0^2p=\pm\left[\bar{\beta}_1p^2\left(\bar{\beta}_1p^2+2\bar{\lambda}_1a_0^2\right)+\bar{\lambda}_3^2p^2a_0^4\right]^{1/2}. \tag{4.4}$$

For sufficiently small p, i.e. for long wavelength plane wave perturbation, there is instability if

$$\bar{\lambda}_1<0. \tag{4.5}$$

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From the expression for $\bar{\lambda}_1$ given by (3.30) we find that $\bar{\lambda}_1$ is negative and this is due to the inclusion of the weak relativistic effect. The term $\bar{\lambda}_3^2 p^2 a_0^4$, appearing in (4.4), which comes from the higher order term $i\bar{\lambda}_3\alpha^2(\partial\alpha^*/\partial X)$ of the nonlinear evolution equation (3.29), is always positive. Hence, this higher order nonlinear term in the evolution equation has a stabilizing influence.

5. Conclusions

The nonlinear evolution of the amplitude of a one-dimensional electromagnetic wave packet propagating in a plasma is governed by a nonlinear Schrödinger equation. However, if the intensity of the wave is sufficiently high and its temporal and spatial spectral width is not narrow enough, then higher order nonlinear and dispersive terms are to be included in the nonlinear Schrödinger equation. Such an equation for a one-dimensional electromagnetic wave packet has recently been derived by Gromov and Talanov (1996). As they have derived this equation from two coupled equations obtained by Zakharov (1972), which disregard the interaction of the wave with the wave induced second harmonic component, in the present paper nonlinear evolution equations are derived for a three-dimensional electromagnetic wave packet starting from the full set of governing equations including the weak relativistic effect. The evolution equations are now found to consist of three coupled equations, and they reduce to a single equation when the space dependence of the amplitude of the wave is either on the space coordinate along the direction of propagation of the wave or on the space coordinate along an arbitrary fixed direction. This single evolution equation is same as that derived by Gromov and Talanov (1996) with differences only in coefficients. This single equation has been used to obtain the modulational instability condition for a uniform electromagnetic wave train. It is found that there is always modulational instability and this is due to inclusion of the weak relativistic effect. A higher order nonlinear term in the evolution equation is found to have a stabilizing influence.

Appendix A

$$E_{x11} = 0, \quad E_{z11} = 0, \quad v_{ex11} = 0, \quad v_{ey11} = -\frac{i}{\omega}\alpha^{(0)},$$

$$v_{ez11} = 0, \quad B_{x11} = 0, \quad B_{y11} = 0, \quad B_{z11} = \frac{k}{\omega}\alpha^{(0)},$$

$$n_{e11} = 0, \quad E_{x12} = \frac{i}{k}\frac{\partial\alpha^{(0)}}{\partial\eta}, \quad E_{z12} = 0, \quad v_{ex12} = \frac{1}{k\omega}\frac{\partial\alpha^{(0)}}{\partial\eta},$$

$$v_{ey12} = -\frac{i}{\omega}\alpha^{(1)} + \frac{k}{\omega^3}\frac{\partial\alpha^{(0)}}{\partial\xi}, \quad v_{ez12} = 0,$$

$$B_{x12} = \frac{i}{\omega}\frac{\partial\alpha^{(0)}}{\partial\zeta}, \quad B_{y12} = 0, \quad B_{z12} = \frac{k}{\omega}\alpha^{(1)} - \frac{i}{\omega}\left(1 - \frac{k^2}{\omega^2}\right)\frac{\partial\alpha^{(0)}}{\partial\xi},$$

$$n_{e12} = 0, \quad E_{x22} = -\frac{ik\alpha^{(0)^2}}{\omega^2\{3 + 4k^2(1 - \beta_e^2)\}}, \quad E_{y22} = 0,$$

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$$E_{z22} = 0, \quad v_{ex22} = -\frac{k\alpha^{(0)^2}}{\omega\{3 + 4k^2(1-\beta_e^2)\}}, \quad v_{ey22} = 0,$$

$$v_{ez22} = 0, \quad n_{e22} = -\frac{2k^2\alpha^{(0)^2}}{\omega^2\{3 + 4k^2(1-\beta_e^2)\}},$$

$$E_{x23} = -\frac{2ik\alpha^{(0)}\alpha^{(1)}}{\omega^2\{3 + 4k^2(1-\beta_e^2)\}} + \frac{\{-3 + (7 - 4\beta_e^2)k^2 + 12(1-\beta_e^2)k^4\}}{\omega^4\{3 + 4(1-\beta_e^2)k^2\}^2}\alpha^{(0)}\frac{\partial\alpha^{(0)}}{\partial\xi},$$

$$E_{y23} = -\frac{1}{3\omega^2}\alpha^{(0)}\frac{\partial\alpha^{(0)}}{\partial\eta}, \quad E_{z23} = \frac{1}{3\omega^2}\alpha^{(0)}\frac{\partial\alpha^{(0)}}{\partial\zeta},$$

$$v_{ex23} = -\frac{4k\alpha^{(0)}\alpha^{(1)}}{\omega\{3 + 4k^2(1-\beta_e^2)\}} + \frac{2i}{\omega^3\{3 + 4k^2(1-\beta_e^2)\}}$$

$$\times \left[3 - 8k^2(2-\beta_e^2) + 4k^4(1+\beta_e^2)\right]\alpha^{(0)}\frac{\partial\alpha^{(0)}}{\partial\xi},$$

$$v_{ey23} = \frac{2i}{3\omega^3}\alpha^{(0)}\frac{\partial\alpha^{(0)}}{\partial\eta}, \quad v_{ez23} = \frac{2i}{3\omega^3}\alpha^{(0)}\frac{\partial\alpha^{(0)}}{\partial\zeta},$$

$$n_{e23} = -\frac{4k^2\alpha^{(0)}\alpha^{(1)}}{\omega^2\{3 + 4k^2(1-\beta_e^2)\}} + \frac{4ik\{3 - 4k^4(1-\beta_e^2)\}}{\omega^4\{3 + 4k^2(1-\beta_e^2)\}}\alpha^{(0)}\frac{\partial\alpha^{(0)}}{\partial\xi}.$$

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