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Mode conversion in plasmas with two-dimensional inhomogeneities

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Abstract. Most of the mode conversion theories considered so far assume only a plane-layered medium, i.e. a medium where the parameters depend on one spatial coordinate. We generalize the mode-conversion method of Cairns and Lashmore-Davies to plasmas with two-dimensional inhomogeneities. In the method presented here, the frequencies ω_1 and ω_2 of the uncoupled modes belonging to two different dispersion equations are considered as functions of the space variable $\mathbf{r}$ and the wave vector $\mathbf{k}$ and are coupled together via a small quantity η. We calculate the energy transmission and conversion coefficients analytically by solving two coupled wave amplitude equations in the electron cyclotron range of frequencies. The results are applicable to electron Bernstein wave heating of plasmas with two-dimensional inhomogeneity, e.g. spherical tokamaks.

1. Introduction

Supplementary heating of plasmas is a necessary condition to reach ignition temperatures. One essential method, which in addition controls the plasma properties, is radiofrequency heating (Cairns 1991). A high-power electromagnetic wave is launched into the plasma and will transfer its energy to the plasma particles. This process merely takes place indirectly, i.e. via mode conversion (Stix 1992). This process is based on the fact that, because of dispersion, at least two waves can propagate in the plasma. Hence, by tuning the incoming wave to some frequency of the plasma the wave can couple to one of the plasma oscillations and in an inhomogeneous medium, reaching the point where two roots of the local dispersion relation are close, can be converted into another mode, whilst some of the energy of the incoming mode is absorbed by the plasma.

Most of the mode conversion theories considered so far assume only a plane-layered medium, i.e. a medium whose parameters depend on one spatial coordinate. Such conditions are almost never encountered in practice and it is desirable to investigate mode conversion under more realistic assumptions. Mode conversion for four-dimensional space–time geometries was considered by Friedland et al. (1987) and Kaufman and Friedland (1987) generalizing previous work on one-dimensional mode conversion of the same authors and four-dimensional conversion coefficients

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were represented. In these papers the dispersion matrix D_{ij}, which fulfills the eigenvalue equation

$$\int d^4y D_{ij}(x,y)Z_j(y) = 0, \tag{1.1}$$

is used to calculate eigenvalues and then these eigenvalues are set to zero to obtain dispersion equations. The final conversion coefficients contain derivatives of these eigenvalues at points where two of the eigenvalues become equal. However, from the mathematical standpoint, this is obscure, because if we take as an example a cold plasma, then in the case of an isolated coupling point the derivatives of the eigenvalues become singular (see Budden (1972; 1985, ch. 16)) and hence do not exist in general. It is only in the case of the coalescence of two different coupling points where the derivative of the eigenvalues exist (see Budden (1974; 1985, ch. 17)), a condition which does not often occur. In general the problem is that the above cited papers of Kaufman, Friedland and coworkers consider the waves near and at the coupling point and circumvent the rapid variation of the polarization vector by introducing a substitute basis (Friedland et al. 1987) or use phase-space methods (Kaufman and Friedland 1987; Sabzevari 1992; Sabzevari 1993). However, the derivatives of the eigenvalues of the dispersion matrix remain in both cases, so the direct application of their final results to various cases could be misleading.

Another method to calculate mode conversion coefficients was proposed by Cairns and Lashmore-Davies (1983). In this method it is not necessary to consider polarization vectors and eigenvalues of dispersion matrices around coupling points. Instead, the frequencies ω_1 and ω_2 of the uncoupled modes belonging to two different dispersion equations are considered as functions of the space variable $\mathbf{x}$ and the wave vector $\mathbf{k}$ and are coupled together via a small quantity η which is only significant around the coupling point

$$[\omega - \omega_1(\mathbf{x},\mathbf{k})][\omega - \omega_2(\mathbf{x},\mathbf{k})] = \eta(\mathbf{x},\mathbf{k}). \tag{1.2}$$

We emphasize here that ω_1 and ω_2 belong to two different dispersion equations which do not result from a dispersion matrix. Expanding these two different modes ω_1 and ω_2 around a point $(\mathbf{x}_0,\mathbf{k}_0)$ where they become equal, constructing a coupled system of ordinary differential equations and finding asymptotic solutions conversion coefficients are derived (Cairns and Lashmore-Davies 1983). The essential point in the method of Cairns and Lashmore-Davies is that this method is equivalent to S-matrix methods, where the singularity in the interaction region is circumvented by asymptotic methods. Hence, concerning singularities, no mathematical obscurity does appear in this method. It is this method that we generalize here to higher dimensions. We restrict ourselves to two-dimensional geometries, because:

(a) two-dimensional inhomogeneity covers a large number of plasma configurations, including the main structure of the toroidal system of a tokamak, which is the main device where radiofrequency heating is applied to; and,

(b) second-order partial differential equations with two variables and varying coefficients are well treated in the mathematical literature.

In Sec. 2, the two-dimensional equations for the mode amplitudes are constructed from the local dispersion relation near the coupling point. Equations which conserve energy can be solved analytically to calculate the transmission and conversion

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coefficients by using the Wentrel–Kramers–Brillouin (WKB) approximation. For a finite temperature plasma, electron cyclotron resonances at the fundamental and the second harmonic become the dominant resonances for both the ordinary and extraordinary waves (Nishikawa and Wakatani 2000, ch. 12). In Secs 3 and 4 we analyse the mode conversion between the X-mode and O-mode near harmonic resonance, respectively, to obtain the appropriate dispersion relations in order to compute the transmission/conversion coefficients, i.e. the fraction of wave energy transmitted/converted in the mode conversion process.

2. Two-dimensional wave amplitude equations

We consider an inhomogeneous plasma where both the density and magnetic field are functions of position. If a wave of given frequency ω_0 propagates into the location $(\mathbf{r}_0, \mathbf{k}_0)$ the incident wave and the fundamental or second cyclotron harmonic wave will be in resonance, that is, $\omega_0 = \omega_1(\mathbf{r}_0, \mathbf{k}_0) = \omega_2(\mathbf{r}_0, \mathbf{k}_0)$. Expanding about the resonance point gives

$$\begin{aligned} \omega_1 &= \omega_0 + \mathbf{a}\cdot\boldsymbol{\delta} + \mathbf{b}\cdot\boldsymbol{\xi} \\ \omega_2 &= \omega_0 + \mathbf{f}\cdot\boldsymbol{\delta} + \mathbf{g}\cdot\boldsymbol{\xi}, \end{aligned} \tag{2.1}$$

where $\boldsymbol{\delta} = \mathbf{k} - \mathbf{k}_0$, $\boldsymbol{\xi} = \mathbf{r} - \mathbf{r}_0$ and the vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{f}$ and $\mathbf{g}$ are

$$\begin{aligned} \mathbf{a} &= \frac{\partial\omega_1}{\partial\mathbf{k}}, \quad \mathbf{b} = \frac{\partial\omega_1}{\partial\mathbf{r}}, \\ \mathbf{f} &= \frac{\partial\omega_2}{\partial\mathbf{k}}, \quad \mathbf{g} = \frac{\partial\omega_2}{\partial\mathbf{r}}. \end{aligned} \tag{2.2}$$

Then around $\mathbf{r}_0$, (1.2) becomes

$$[\mathbf{a}\cdot\boldsymbol{\delta} + \mathbf{b}\cdot\boldsymbol{\xi}][\mathbf{f}\cdot\boldsymbol{\delta} + \mathbf{g}\cdot\boldsymbol{\xi}] = \eta_0, \tag{2.3}$$

where η_0 is the value of $\eta(\mathbf{r}, \mathbf{k})$ at $(\mathbf{r}_0, \mathbf{k}_0)$. By Fourier transformation $\boldsymbol{\delta} = -i\partial/\partial\boldsymbol{\xi}$ the behavior of the waves is described by two coupled partial differential equations. This is then the two-dimensional generalization of the coupled wave equation in Cairns and Lashmore-Davies (1983) for the two modes u_1 and u_2.

$$\begin{aligned} \mathbf{a}\cdot\frac{\partial u_1}{\partial\boldsymbol{\xi}} - i(\mathbf{a}\cdot\mathbf{k}_0 - \mathbf{b}\cdot\boldsymbol{\xi})u_1 &= i\sqrt{\eta_0}u_2 \\ \mathbf{f}\cdot\frac{\partial u_2}{\partial\boldsymbol{\xi}} - i(\mathbf{f}\cdot\mathbf{k}_0 - \mathbf{g}\cdot\boldsymbol{\xi})u_2 &= i\sqrt{\eta_0}u_1. \end{aligned} \tag{2.4}$$

It can be shown by direct calculations that from (2.4) and the WKB approximation $\nabla\omega = -\partial\mathbf{k}/\partial t$ follows

$$\frac{\partial}{\partial t}(|u_1|^2 + |u_2|^2) = 0. \tag{2.5}$$

Since $|u_1|^2$ and $|u_2|^2$ represent the energy fluxes, (2.5) is the energy conservation equation.

By eliminating u_2 from (2.4) we arrive at a second-order partial differential equation with two variables which is of the hyperbolic type (Smirnov 1964,

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Sec. 130)

$$a_1 f_1 \frac{\partial^2 u_1}{\partial \xi_1^2} + a_2 f_2 \frac{\partial^2 u_1}{\partial \xi_2^2} + (a_1 f_2 + a_2 f_1) \frac{\partial^2 u_1}{\partial \xi_1 \partial \xi_2}$$
$$- i[a_1 \mathbf{f} \cdot \mathbf{k}_0 + f_1 \mathbf{a} \cdot \mathbf{k}_0 - (a_1 g_1 + b_1 f_1)\xi_1 - (a_1 g_2 + b_2 f_1)\xi_2] \frac{\partial u_1}{\partial \xi_1}$$
$$- i[a_2 \mathbf{f} \cdot \mathbf{k}_0 + f_2 \mathbf{a} \cdot \mathbf{k}_0 - (a_2 g_1 + b_1 f_2)\xi_1 - (a_2 g_2 + b_2 f_2)\xi_2] \frac{\partial u_1}{\partial \xi_2}$$
$$+ [i\mathbf{f} \cdot \mathbf{b} + \eta_0 - (\mathbf{a} \cdot \mathbf{k}_0 - \mathbf{b} \cdot \boldsymbol{\xi})(\mathbf{f} \cdot \mathbf{k}_0 - \mathbf{g} \cdot \boldsymbol{\xi})] u_1 = 0. \tag{2.6}$$

By introducing the transformations

$$\begin{aligned} x &= \xi_2 - \frac{f_2}{f_1}\xi_1 \\ y &= \xi_2 - \frac{a_2}{a_1}\xi_1, \end{aligned} \tag{2.7}$$

which are the characteristic curves (Smirnov 1964, Sec. 134), we can transform (2.6) into the canonical form

$$\mathscr{L}(u_1) = \frac{\partial^2 u_1}{\partial x \partial y} + D(x,y)\frac{\partial u_1}{\partial x} + E(x,y)\frac{\partial u_1}{\partial y} + F(x,y)u_1 = 0, \tag{2.8}$$

with

$$\begin{aligned} D(x,y) &= -i\,\frac{a_1}{\alpha}\left[\mathbf{f} \cdot \mathbf{k}_0 + \frac{f_1}{\alpha}(\mathbf{a} \cdot \mathbf{g})x - \frac{a_1}{\alpha}(\mathbf{f} \cdot \mathbf{g})y\right] \\ E(x,y) &= i\,\frac{f_1}{\alpha}\left[\mathbf{a} \cdot \mathbf{k}_0 + \frac{f_1}{\alpha}(\mathbf{a} \cdot \mathbf{b})x - \frac{a_1}{\alpha}(\mathbf{f} \cdot \mathbf{b})y\right] \\ F(x,y) &= -\frac{a_1 f_1}{\alpha^2}(i\mathbf{f} \cdot \mathbf{b} + \eta_0) + D(x,y)E(x,y) \\ \alpha &= a_1 f_2 - a_2 f_1. \end{aligned} \tag{2.9}$$

Equation (2.8) can be solved analytically by using Riemann's method (Smirnov 1964, Sec. 143; Sommerfeld 1964, Sec. 11). In addition to the operator $\mathscr{L}(u_1)$, we consider the conjugate operator $\mathscr{M}$ as

$$\mathscr{M}(v) = v_{xy} - (Dv)_x - (Ev)_y + Fv = 0. \tag{2.10}$$

We wish to seek a solution of (2.10) satisfying the following conditions on the characteristic curves $x = x_0$ and $y = y_0$,

$$\begin{aligned} v_x - E(x,y_0)v &= 0 \\ v_y - D(x_0,y)v &= 0 \\ v(x_0,y_0) &= 1. \end{aligned} \tag{2.11}$$

The general form of this solution is

$$\begin{aligned} &v(x,y;x_0,y_0) \\ &= \exp\left[\frac{if_1\mathbf{a} \cdot \mathbf{k}_0}{\alpha}(x - x_0) - \frac{ia_1 f_1(\mathbf{f} \cdot \mathbf{b})}{\alpha^2}\,y(x - x_0) + \frac{if_1^2(\mathbf{a} \cdot \mathbf{b})}{2\alpha^2}\left(x^2 - x_0^2\right)\right] \\ &\quad \times \exp\left[-\frac{ia_1\mathbf{f} \cdot \mathbf{k}_0}{\alpha}(y - y_0) - \frac{ia_1 f_1(\mathbf{a} \cdot \mathbf{g})}{\alpha^2}x(y - y_0) + \frac{ia_1^2(\mathbf{f} \cdot \mathbf{g})}{2\alpha^2}\left(y^2 - y_0^2\right)\right] \\ &\quad \times W(x,y;x_0,y_0). \end{aligned} \tag{2.12}$$

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Substituting (2.12) into (2.10) yields

$$\frac{\partial^2 W}{\partial x \partial y} - \frac{ia_1 f_1(\mathbf{f}\cdot\mathbf{b})}{\alpha^2}(x-x_0)\frac{\partial W}{\partial x} - \frac{ia_1 f_1(\mathbf{a}\cdot\mathbf{g})}{\alpha^2}(y-y_0)\frac{\partial W}{\partial y} - \left[\frac{a_1 f_1}{\alpha^2}(i\mathbf{f}\cdot\mathbf{b}+\eta_0) + \left(\frac{a_1 f_1}{\alpha^2}\right)^2 (\mathbf{a}\cdot\mathbf{g})(\mathbf{f}\cdot\mathbf{b})(x-x_0)(y-y_0)\right] W = 0. \quad (2.13)$$

Fortunately, (2.13) can be modified to a differential equation with one variable by an appropriate change of variable (see, for example, Sommerfeld (1964, Sec. 11)). By introducing the following transformations

$$z = (x-x_0)(y-y_0)$$
$$W = \left[\frac{ia_1 f_1}{\alpha^2}(\mathbf{a}\cdot\mathbf{g}-\mathbf{f}\cdot\mathbf{b})z\right]^{-1/2} \exp\left[\frac{ia_1 f_1}{2\alpha^2}(\mathbf{a}\cdot\mathbf{g}+\mathbf{f}\cdot\mathbf{b})z\right] G(z) \quad (2.14)$$

(2.13) reduces to the standard form given by Whittaker and Watson (1927) for the confluent hypergeometric function

$$\frac{d^2 G}{dz^2} + \left\{\frac{1}{4z^2} + \frac{a_1 f_1}{2\alpha^2}\left[i(\mathbf{a}\cdot\mathbf{g}-\mathbf{f}\cdot\mathbf{b}) - 2\eta_0\right]\frac{1}{z} + \frac{1}{4}\left(\frac{a_1 f_1}{\alpha^2}\right)^2 (\mathbf{a}\cdot\mathbf{g}-\mathbf{f}\cdot\mathbf{b})^2\right\} G = 0, \quad (2.15)$$

for which the solution is (Gradshteyn and Ryzhik 1965, pp. 1059, (9.220))

$$G(z) = \left[\frac{ia_1 f_1}{\alpha^2}(\mathbf{a}\cdot\mathbf{g}-\mathbf{f}\cdot\mathbf{b})z\right]^{1/2} \exp\left[-\frac{ia_1 f_1}{2\alpha^2}(\mathbf{a}\cdot\mathbf{g}-\mathbf{f}\cdot\mathbf{b})z\right] \times {}_1F_1\left(-\frac{i\eta_0}{\mathbf{a}\cdot\mathbf{g}-\mathbf{f}\cdot\mathbf{b}}, 1, \frac{ia_1 f_1}{\alpha^2}(\mathbf{a}\cdot\mathbf{g}-\mathbf{f}\cdot\mathbf{b})z\right). \quad (2.16)$$

By substituting (2.16) into (2.14) and inserting the result into (2.12) we have

$$\begin{aligned} v(x,y;x_0,y_0) &= \exp\left[\frac{if_1\mathbf{a}\cdot\mathbf{k}_0}{\alpha}(x-x_0) - \frac{ia_1 f_1\mathbf{f}\cdot\mathbf{b}}{\alpha^2}y_0(x-x_0) + \frac{if_1^2\mathbf{a}\cdot\mathbf{b}}{2\alpha^2}(x^2-x_0^2)\right] \\ &\quad \times \exp\left[-\frac{ia_1\mathbf{f}\cdot\mathbf{k}_0}{\alpha}(y-y_0) - \frac{ia_1 f_1\mathbf{a}\cdot\mathbf{g}}{\alpha^2}x(y-y_0) + \frac{ia_1^2\mathbf{f}\cdot\mathbf{g}}{2\alpha^2}(y^2-y_0^2)\right] \\ &\quad \times {}_1F_1\left(-\frac{i\eta_0}{\mathbf{a}\cdot\mathbf{g}-\mathbf{f}\cdot\mathbf{b}}, 1, \frac{ia_1 f_1}{\alpha^2}(\mathbf{a}\cdot\mathbf{g}-\mathbf{f}\cdot\mathbf{b})(x-x_0)(y-y_0)\right). \end{aligned} \quad (2.17)$$

Equation (2.17) is the Riemann function of $\mathscr{L}(u_1)$ which satisfies the condition $v(x_0,y_0)=1$. If we interchange the points (x,y) and (x_0,y_0) in $v(x,y;x_0,y_0)$, the solution u_1 of the operator $\mathscr{L}(u_1)$ can be obtained (Smirnov 1964, Sec. 144)

$$u_1(x_0,y_0;x,y) = v(x,y;x_0,y_0).$$

For large $|z|$ the asymptotic expansion of ${}_1F_1$ is (Farina 1973)

$${}_1F_1(-i\gamma,1,ikz) \overset{|z|\to\infty}{\sim} \frac{e^{\pm\pi\gamma}}{\Gamma(1+i\gamma)}(ikz)^{i\gamma} + \frac{e^{ikz}}{\Gamma(-i\gamma)}(ikz)^{-1-i\gamma}, \quad (2.18)$$

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where the plus (minus) sign is related to $\mathrm{Im}\, z > 0 (\mathrm{Im}\, z < 0)$. Using (2.18) with

$$\gamma = \frac{\eta_0}{\mathbf{a}\cdot\mathbf{g} - \mathbf{f}\cdot\mathbf{b}}, \quad k = \frac{ia_1 f_1}{\alpha^2}(\mathbf{a}\cdot\mathbf{g} - \mathbf{f}\cdot\mathbf{b}) \quad \text{and} \quad z = (x - x_0)(y - y_0)$$

the asymptotic form of u_1 at large distances may be obtained.

Let us turn back to (2.15). This equation can be written in the general form of the Schrödinger wave equation which we will examine using the WKB approximation (Bender and Orszag 1978, ch. 10)

$$\frac{d^2G(z)}{dz^2} = Q(z)G(z), \tag{2.19}$$

where

$$Q(z) = -\frac{1}{4z^2} - \frac{a_1 f_1}{2\alpha^2}[i(\mathbf{a}\cdot\mathbf{g} - \mathbf{f}\cdot\mathbf{b}) - 2\eta_0]\frac{1}{z} - \frac{1}{4}\left(\frac{a_1 f_1}{\alpha^2}\right)^2 (\mathbf{a}\cdot\mathbf{g} - \mathbf{f}\cdot\mathbf{b})^2. \tag{2.20}$$

For very large values of z, we have

$$Q_\infty \approx -\frac{1}{4}\left(\frac{a_1 f_1}{\alpha^2}\right)^2 (\mathbf{a}\cdot\mathbf{g} - \mathbf{f}\cdot\mathbf{b})^2. \tag{2.21}$$

Assuming that z_1 and z_2 with $z_1 < z_2$, are the points at which $Q(z) = 0$. The WKB approximation to $G(z)$ in the regions $z > z_2$ and $z < z_1$ approach plane waves as $z \to \infty$ and $z \to -\infty$ (Bender and Orszag 1978, ch. 10)

$$\begin{aligned} G(z) &= A_\pm(-Q(z))^{-1/4} \exp\left(\pm i \int_{z_2}^{z} dz' \sqrt{-Q(z')}\right) \\ &\sim A_\pm(-Q_\infty)^{-1/4} e^{\pm iI} \exp(\pm i\sqrt{-Q_\infty}\, z), \quad z \to \infty \end{aligned} \tag{2.22}$$

$$\begin{aligned} G(z) &= B_\pm(-Q(z))^{-1/4} \exp\left(\mp i \int_{z}^{z_1} dz' \sqrt{-Q(z')}\right) \\ &\sim B_\pm(-Q_\infty)^{-1/4} e^{\pm iJ} \exp(\mp i\sqrt{-Q_\infty}\, z), \quad z \to -\infty \end{aligned} \tag{2.23}$$

where $I = \int_{z_2}^{\infty} dz'[\sqrt{-Q(z')} - \sqrt{-Q_\infty}] - \sqrt{-Q_\infty} z_2$ and $J = \sqrt{-Q_\infty} z_1 + \int_{-\infty}^{z_1} dz'$ $[\sqrt{-Q(z')} - \sqrt{-Q_\infty}]$. We assume a unit incident right-moving plane wave at $z = -\infty$. This gives rise to a right-moving transmitted plane wave at $z = \infty$ and a left-moving reflected wave at $z = -\infty$. We formulate these boundary conditions as asymptotic relations

$$G(z) \sim \exp(-i\sqrt{-Q_\infty}\, z) + C\exp(i\sqrt{-Q_\infty}\, z), \quad z \to -\infty \tag{2.24}$$

$$G(z) \sim D\exp(-i\sqrt{-Q_\infty}\, z), \quad z \to \infty. \tag{2.25}$$

Replacing these equations into (2.14) and the resultants into (2.12), separately, gives the relations for $v(x, y; x_0, y_0)$ at large distances

$$\begin{aligned} &v(x, y; x_0, y_0) \\ &= \exp\left[\frac{if_1\mathbf{a}\cdot\mathbf{k}_0}{\alpha}(x - x_0) - \frac{ia_1 f_1 \mathbf{f}\cdot\mathbf{b}}{\alpha^2} y(x - x_0) + \frac{if_1^2 \mathbf{a}\cdot\mathbf{b}}{2\alpha^2}(x^2 - x_0^2)\right] \\ &\quad \times \exp\left[-\frac{ia_1\mathbf{f}\cdot\mathbf{k}_0}{\alpha}(y - y_0) - \frac{ia_1 f_1 \mathbf{a}\cdot\mathbf{g}}{\alpha^2} x(y - y_0) + \frac{ia_1^2 \mathbf{f}\cdot\mathbf{g}}{2\alpha^2}(y^2 - y_0^2)\right] \\ &\quad \times z^{-1/2}\left[\exp\left(\frac{ia_1 f_1 \mathbf{f}\cdot\mathbf{b}}{\alpha^2} z\right) + C\exp\left(\frac{ia_1 f_1 \mathbf{a}\cdot\mathbf{g}}{\alpha^2} z\right)\right] \quad z \to -\infty. \end{aligned} \tag{2.26}$$

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$$
\begin{aligned}
&v(x,y;x_0,y_0)\\
&\quad=\exp\left[\frac{if_1\mathbf{a}\cdot\mathbf{k}_0}{\alpha}(x-x_0)-\frac{ia_1f_1\mathbf{f}\cdot\mathbf{b}}{\alpha^2}y(x-x_0)+\frac{if_1^2\mathbf{a}\cdot\mathbf{b}}{2\alpha^2}(x^2-x_0^2)\right]\\
&\qquad\times\exp\left[-\frac{ia_1\mathbf{f}\cdot\mathbf{k}_0}{\alpha}(y-y_0)-\frac{ia_1f_1\mathbf{a}\cdot\mathbf{g}}{\alpha^2}x(y-y_0)+\frac{ia_1^2\mathbf{f}\cdot\mathbf{g}}{2\alpha^2}(y^2-y_0^2)\right]\\
&\qquad\times Dz^{-1/2}\exp\left(\frac{ia_1f_1\mathbf{f}\cdot\mathbf{b}}{\alpha^2}z\right)\quad z\to\infty
\end{aligned}
\tag{2.27}
$$

The objective is to compute the constants C and D using WKB theory. Where $z>z_2$ the WKB approximation to the solution $G(z)$ of (2.19) which satisfies the boundary conditions is (Bender and Orszag 1978, ch. 10)

$$
G(z)=De^{iI}\left[\frac{Q(z)}{Q_\infty}\right]^{-1/4}\exp\left(-i\int_{z_2}^{z}dz'\sqrt{-Q(z')}\right).
$$

This expression asymptotically matches to

$$
G(z)=De^{iI}\exp\left(\int_{z_1}^{z_2}dz'\sqrt{Q(z')}+\frac{i\pi}{4}\right)\left[\frac{Q(z)}{-Q_\infty}\right]^{-1/4}\exp\left(-\int_{z_1}^{z}dz'\sqrt{Q(z')}\right),
$$

which decays exponentially as z increases toward z_2. We connect to the oscillatory solution which is valid for $z<z_1$ and obtain the WKB approximation (Bender and Orszag 1978, ch. 10)

$$
\begin{aligned}
G(z)&=De^{iI}\exp\left(\int_{z_1}^{z_2}dz'\sqrt{Q(z')}\right)\left[\frac{Q(z)}{Q_\infty}\right]^{-1/4}\\
&\quad\times\left[\exp\left(i\int_{z}^{z_1}dz'\sqrt{-Q(z')}\right)+i\exp\left(-i\int_{z}^{z_1}dz'\sqrt{-Q(z')}\right)\right].
\end{aligned}
\tag{2.28}
$$

We let $z\to-\infty$. So using the definition of J, (2.28) becomes

$$
\begin{aligned}
G(z)&=De^{iI}\exp\left(\int_{z_1}^{z_2}dz'\sqrt{Q(z')}\right)\\
&\quad\times[e^{iJ}\exp(-i\sqrt{-Q_\infty}z)+ie^{-iJ}\exp(i\sqrt{-Q_\infty}z)],\quad z\to-\infty.
\end{aligned}
\tag{2.29}
$$

Substituting (2.29) into (2.14) and then the result into (2.12), obtains the asymptotic solution to $v(x,y;x_0,y_0)$

$$
\begin{aligned}
&v(x,y;x_0,y_0)\\
&\quad=\exp\left[\frac{if_1\mathbf{a}\cdot\mathbf{k}_0}{\alpha}(x-x_0)-\frac{ia_1f_1\mathbf{f}\cdot\mathbf{b}}{\alpha^2}y(x-x_0)+\frac{if_1^2\mathbf{a}\cdot\mathbf{b}}{2\alpha^2}(x^2-x_0^2)\right]\\
&\qquad\times\exp\left[-\frac{ia_1\mathbf{f}\cdot\mathbf{k}_0}{\alpha}(y-y_0)-\frac{ia_1f_1\mathbf{a}\cdot\mathbf{g}}{\alpha^2}x(y-y_0)+\frac{ia_1^2\mathbf{f}\cdot\mathbf{g}}{2\alpha^2}(y^2-y_0^2)\right]\\
&\qquad\times De^{iI}z^{-1/2}\exp\left(\int_{z_1}^{z_2}dz'\sqrt{Q(z')}\right)\\
&\qquad\times[e^{iJ}\exp(-i\sqrt{-Q_\infty}z)+ie^{-iJ}\exp(i\sqrt{-Q_\infty}z)],\quad z\to-\infty.
\end{aligned}
\tag{2.30}
$$

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Comparing (2.30) with that in (2.26) gives expressions for the constants C and D

$$C = ie^{-i2J},$$
$$D = \exp\left(-\int_{z_1}^{z_2}\sqrt{Q(z')}\,dz'\right)e^{-i(I+J)}.$$

The transmission coefficient becomes

$$\mathscr{T} = |D|^2 \sim \exp\left(-2\int_{z_1}^{z_2}\sqrt{Q(z')}\,dz'\right). \tag{2.31}$$

On substituting (2.20) into (2.31), after some algebra, it can be found the transmission coefficient as

$$\mathscr{T} = \exp\left(-\frac{2\pi\eta_0}{\mathbf{a}\cdot\mathbf{g}-\mathbf{f}\cdot\mathbf{b}}\right). \tag{2.32}$$

In the one-dimensional case, (2.32) reduces to the relation found by Cairns and Lashmore-Davies (1983). Using the transmission coefficient, the conversion coefficient for the two-dimensional inhomogeneous plasma can be calculated by (Swanson 1989, ch. 6)

$$\mathscr{C} = \mathscr{T}(1-\mathscr{T}). \tag{2.33}$$

3. Propagation of the X-mode near the second harmonic

The dispersion relation for the electromagnetic waves in the cold plasma approximation is given by (Stix 1992, ch. 2)

$$\frac{k_\perp^2 c^2}{\omega^2}\epsilon_{xx}^{\text{cold}} = \epsilon_{xx}^{\text{cold}}\epsilon_{yy}^{\text{cold}} - \epsilon_{xy}^{\text{cold}}\epsilon_{yx}^{\text{cold}}, \tag{3.1}$$

where $\epsilon^{\text{cold}} = n_i\epsilon_{ij}n_j/n^2$ is the dielectric function, and its tensor elements ϵ_{ij} are in Stix's notation (Stix 1992, ch. 1)

$$\epsilon_{xx}^{\text{cold}} = \epsilon_{yy}^{\text{cold}} = 1 - \frac{\omega_{\text{pe}}^2}{\omega^2-\Omega_{\text{e}}^2}, \quad \epsilon_{xy}^{\text{cold}} = -\epsilon_{yx}^{\text{cold}} = -i\frac{\Omega_{\text{e}}\omega_{\text{pe}}^2}{\omega\left(\omega^2-\Omega_{\text{e}}^2\right)}. \tag{3.2}$$

For the full wave analysis we must consider the general form of the hot plasma dispersion relation. The dielectric tensor elements ϵ_{ij} are given by using the susceptibility tensor χ (Stix 1992, ch. 10), in which the finite temperature effects enter the function ϵ through the parameter $\lambda = k_\perp^2\kappa_B T/m_{\text{e}}\Omega_{\text{e}}^2$. Since now the parameter λ is small, we calculate the resonance terms of ϵ_{ij} near $\omega = 2\Omega_{\text{e}}$ up to first order of λ using the zero-order (cold plasma) approximation. In this approximation, the elements $\epsilon_{xx} = \epsilon_{yy}$ and $\epsilon_{xy} = -\epsilon_{yx}$ are

$$\begin{aligned}\epsilon_{xx} = \epsilon_{yy} &= \epsilon_{xx}^{\text{cold}} + \frac{\omega_{\text{pe}}^2\lambda}{2\omega k_\parallel W}Z_0(\zeta_2)\\ \epsilon_{xy} = -\epsilon_{yx} &= \epsilon_{xy}^{\text{cold}} + i\frac{\omega_{\text{pe}}^2\lambda}{2\omega k_\parallel W}Z_0(\zeta_2),\end{aligned} \tag{3.3}$$

and the other elements ϵ_{xz}, ϵ_{zx}, ϵ_{yz} and ϵ_{zy} vanish. Here $W = \sqrt{2\kappa_{\text{B}}T/m_{\text{e}}}$ and Z_0 is the plasma dispersion function with the argument $\zeta_2 = (\omega - 2\Omega_{\text{e}})/k_\parallel W$ (see, for example, Stix 1992, ch. 10). Introducing the variables $q = \omega/\Omega_e$ and $p = \omega_{\text{pe}}^2/\omega^2$,

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(3.3) can be written as

$$\epsilon_{xx} = \epsilon_{yy} = 1 - \frac{pq^2}{q^2-1} + \frac{p\lambda\omega}{2k_\parallel W} Z_0(\zeta_2),$$
$$\epsilon_{xy} = -\epsilon_{yx} = -i\frac{pq}{q^2-1} + i\frac{p\lambda\omega}{2k_\parallel W} Z_0(\zeta_2). \tag{3.4}$$

Now, by using the dispersion relation of the cold plasma approximation i.e. (3.1), the following relation is obtained

$$\frac{k_\perp^2 c^2}{\omega^2}\epsilon_{xx} = \left(1 - \frac{p\,q^2}{q^2-1}\right)^2 - \frac{p^2q^2}{(q^2-1)^2} + \left(1 - \frac{pq^2}{q^2-1}\right)\frac{p\lambda\omega}{k_\parallel W} Z_0(\zeta_2)$$
$$+ \frac{p^2 q\lambda\omega}{k_\parallel W(q^2-1)} Z_0(\zeta_2). \tag{3.5}$$

If we assume $p \gg 1$, i.e. the high plasma density limit, the dispersion relation becomes

$$\frac{k_\perp^2 c^2}{\omega^2}\epsilon_{xx} - \frac{p^2q^2}{q^2-1} + \frac{p^2\lambda\omega q(q-1)}{k_\parallel W(q^2-1)} Z_0(\zeta_2) = 0. \tag{3.6}$$

Replacing the parameter λ into (3.6) yields

$$\frac{k_\perp^2 c^2}{\omega^2}\,\epsilon_{xx} - \frac{p^2q^2}{q^2-1}\left[1 - \frac{k_\perp^2 W(q-1)}{2\Omega_e k_\parallel} Z_0(\zeta_2)\right] = 0. \tag{3.7}$$

Near the $q = 2$ resonance (3.7) leads to the approximate dispersion relation

$$k_\perp^2 \approx \frac{2\Omega_e k_\parallel}{W Z_0(\zeta_2)}, \tag{3.8}$$

where

$$Z_0(\zeta_2) = -\frac{k_\parallel W}{\omega - 2\Omega_e} - \frac{k_\parallel^3 W^3}{2(\omega - 2\Omega_e)^3} - \cdots. \tag{3.9}$$

The parallel wave vector $k_\parallel = \mathbf{k}\cdot\mathbf{B}/|B|$, considering magnetic shear effect, can be written as $k_\parallel = k_x B_x/B + k_y B_y/B + k_z B_z/B$ where B_x is the radial magnetic field component and, B_y and B_z are the poloidal and toroidal magnetic field components respectively, in the toroidal geometry (Stix 1992, ch. 13). Replacing (3.9) into (3.8) results in the closed form

$$(\omega - 2\Omega_e)\left[\omega^2 - \omega\left(4\Omega_e - \frac{k_\perp^2 W^2}{2\Omega_e}\right) + \left(4\Omega_e^2 - k_\perp^2 W^2\right)\right] = -\frac{k_\parallel^2 k_\perp^2 W^4}{4\Omega_e}, \tag{3.10}$$

in which we keep the first two terms of $Z_0(\zeta_2)$ since we know that near the $\omega = 2\Omega_e$ resonance still $|\zeta_2| \gg 1$. Equation (3.10) is equivalent to

$$(\omega - 2\Omega_e)(\omega - \omega_X)(\omega - \omega') = -\frac{k_\parallel^2 k_\perp^2 W^4}{4\Omega_e} \tag{3.11}$$

where ω_X and ω' are the two branches of the X-mode, and we assume that the branch ω_X propagates near $2\Omega_e$. It can immediately be found from (3.10) and (3.11) that

$$\omega_X + \omega' = 4\Omega_e - \frac{k_\perp^2 W^2}{2\Omega_e}. \tag{3.12}$$

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Then dispersion relation takes the form

$$(\omega - 2\Omega_{\rm e})(\omega - \omega_X) = -\frac{k_\parallel^2 k_\perp^2 W^4}{4\Omega_{\rm e}(\omega - \omega')}.$$

Evaluating ω' from (3.12) gives

$$(\omega - 2\Omega_{\rm e})(\omega - \omega_X) = -\frac{k_\parallel^2 k_\perp^2 W^4}{4\Omega_{\rm e}(\omega - 2\Omega_{\rm e} + k_\perp^2 W^2/2\Omega_{\rm e})}. \tag{3.13}$$

Comparing (1.2) with (3.13), the coupling parameter η_0 at $\omega_X \approx 2\Omega_{\rm e}$ is found to be

$$|\eta_0| = \frac{k_\parallel^2 k_\perp^2 W^4}{4\Omega_{\rm e}\omega - 8\Omega_{\rm e}^2 + 2k_\perp^2 W^2}. \tag{3.14}$$

To estimate the transmission coefficient from (2.32) we need the vectors **a**, **b**, **f** and **g**, given by (2.2). Equation (3.14) reveals that the two modes decouple when $n_\parallel \to 0$. Hence away from the coupling point, where $n_\parallel = 0$, the dispersion relation is of the usual form for the incident X-mode (Stix 1992, ch. 2)

$$\omega_X^4 - \omega_X^2\left(2\omega_{\rm pe}^2 + \Omega_{\rm e}^2 + c^2k_\perp^2\right) + c^2k_\perp^2\left(\Omega_{\rm e}^2 + \omega_{\rm pe}^2\right) + \omega_{\rm pe}^4 = 0. \tag{3.15}$$

Taking ω_1 and ω_2 corresponding to ω_X and $2\Omega_{\rm e}$ respectively, the vector **a** can be evaluated from (2.2)

$$a_x = \frac{c^2\left(3\Omega_{\rm e}^2 - \omega_{\rm pe}^2\right)k_x}{16\Omega_{\rm e}^3 - 2\Omega_{\rm e}\left(2\omega_{\rm pe}^2 + \Omega_{\rm e}^2 + c^2k_\perp^2\right)}, \quad a_y = \frac{c^2\left(3\Omega_{\rm e}^2 - \omega_{\rm pe}^2\right)k_y}{16\Omega_{\rm e}^3 - 2\Omega_{\rm e}\left(2\omega_{\rm pe}^2 + \Omega_{\rm e}^2 + c^2k_\perp^2\right)}$$

and we also have

$$g_x = \frac{\partial 2\Omega_{\rm e}}{\partial x} = \frac{2\Omega_{\rm e}}{L_x}, \quad g_y = \frac{\partial 2\Omega_{\rm e}}{\partial y} = \frac{2\Omega_{\rm e}}{L_y}$$

where

$$L_x = \frac{1}{\Omega_{\rm e}}\frac{\partial \ln \Omega_{\rm e}}{\partial x} \quad \text{and} \quad L_y = \frac{1}{\Omega_{\rm e}}\frac{\partial \ln \Omega_{\rm e}}{\partial y}.$$

With the assumption that the electron cyclotron frequency $\Omega_{\rm e}$ depends weakly on k^2, the vector **f** can be neglected, and we arrive at

$$\frac{|\eta_0|}{\mathbf{a}\cdot\mathbf{g}} = \frac{k_\parallel^2 k_\perp^2 W^4(9 - 5\alpha^2 + \alpha^4)}{2c^2(3-\alpha^2)^2\left(2\Omega_{\rm e}\omega - 4\Omega_{\rm e}^2 + k_\perp^2 W^2\right)\left(k_x/L_x + k_y/L_y\right)}, \tag{3.16}$$

where $\alpha = \omega_{\rm pe}/\Omega_{\rm e}$. The parameter $\eta_0/\mathbf{a}\cdot\mathbf{g}$ depends on the position of the resonance point through the dependency of $\Omega_{\rm e}$ and $\omega_{\rm pe}$ on position, i.e.

$$\omega_{\rm pe}^2 = \frac{4\pi e^2 n(x,y)}{m_{\rm e}}, \quad \Omega_{\rm e} = \frac{eB(x,y)}{m_{\rm e}c}.$$

The transmission coefficient can now be calculated by substituting (3.16) into (2.32)

$$\mathscr{T} = \exp\left\{-\frac{\pi k_\parallel^2 k_\perp^2 W^4(9 - 5\alpha^2 + \alpha^4)}{c^2(3-\alpha^2)^2\left(2\Omega_{\rm e}\omega - 4\Omega_{\rm e}^2 + k_\perp^2 W^2\right)\left(k_x/L_x + k_y/L_y\right)}\right\}. \tag{3.17}$$

As a check we consider the one dimensional case in which (3.16) reduces to

$$\frac{\eta_0}{ag} = \frac{R}{2c^2}\frac{k_x W^2}{(3-\alpha^2)^2}(9 - 5\alpha^2 + \alpha^4), \tag{3.18}$$

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where R is the scale length of magnetic field. Replacing the value of k_x at $\omega \approx 2\Omega_e$ into (3.18) gives

$$\frac{\eta_0}{ag} = \frac{R\omega}{2c}\frac{9-5\alpha^2+\alpha^4}{(3-\alpha^2)^2}\left[\frac{(4-\alpha^2)^2-4}{3-\alpha^2}\right]^{1/2}\frac{T_e}{m_e c^2}. \tag{3.19}$$

This is not exactly the same as the relation given by Cairns and Lashmore-Davies (1983), because of our different definition for α and different form of η_0, but both relations show the same behavior with respect to $T_e/m_e c^2$.

According to (2.33), the mode-conversion coefficient attains its maximum value at $\mathscr{T} = 0.5$. For this condition $\exp(-2\pi\eta_0/\mathbf{a}\cdot\mathbf{g}) = 0.5$, i.e. $\eta_0/\mathbf{a}\cdot\mathbf{g} \approx 0.11$ and we have from (3.16)

$$\frac{k_\parallel^2 k_\perp^2 W^4(9-5\alpha^2+\alpha^4)}{c^2(3-\alpha^2)^2\left(2\Omega_e\omega - 4\Omega_e^2 + k_\perp^2 W^2\right)(k_x/L_x + k_y/L_y)} \approx 0.22. \tag{3.20}$$

Equation (3.20) is the relation between the incident wave frequency ω and the components of the wave vector.

4. Propagation of the O-mode near the fundamental harmonic

For this case the dispersion relation in cold plasma approximation is (Stix 1992, ch. 2)

$$\frac{k_\perp^2 c^2}{\omega^2} = 1 - \frac{\omega_{pe}^2}{\omega^2}. \tag{4.1}$$

Including the finite temperature effect and keeping the terms of the first order of λ modifies the cold plasma dispersion relation to

$$\begin{aligned}\frac{k_\perp^2 c^2}{\omega^2} &= 1 - \frac{\omega_{pe}^2}{\omega^2} + \frac{\omega_{pe}^2}{\omega}\frac{\lambda(\omega-\Omega_e)}{k_\parallel^2 W^2}[1+\zeta_1 Z_0(\zeta_1)] \\ &= 1 - \frac{\omega_{pe}^2}{\omega^2} - \frac{\omega_{pe}^2\lambda}{2\omega(\omega-\Omega_e)},\end{aligned} \tag{4.2}$$

where $\zeta_1 = (\omega-\Omega_e)/k_\parallel W$ with the parameters λ, W and Z_0 akin to Sec. 3. By manipulating (4.2), it can be found the dispersion relation for the O-mode propagating near the fundamental harmonic

$$(\omega-\Omega_e)\left(\omega - \sqrt{k_\perp^2 c^2 + \omega_{pe}^2}\right) = \frac{\omega_{pe}^2 k_\perp^2 W^2 \omega}{4\Omega_e^2\left(\omega + \sqrt{k_\perp^2 c^2 + \omega_{pe}^2}\right)}. \tag{4.3}$$

Comparing (1.2) with (4.3) gives the coupling constant at the resonance point $\omega_0 \approx \Omega_e$

$$\eta_0 = \frac{\omega_{pe}^2 k_\perp^2 W^2 \omega}{4\Omega_e^2\left(\omega + \sqrt{k_\perp^2 c^2 + \omega_{pe}^2}\right)}. \tag{4.4}$$

Away from the coupling point the dispersion relation for the O-mode is given by

$$\omega_0^2 - k_\perp^2 c^2 - \omega_{pe}^2 = 0. \tag{4.5}$$

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Taking ω_1 and ω_2 as ω_0 and Ω_e, respectively, gives the components of the vectors **a** and **g** as

$$a_x = \frac{c^2 k_x}{\Omega_e}, \quad a_y = \frac{c^2 k_y}{\Omega_e}$$

$$g_x = \frac{\partial \Omega_e}{\partial x} = \frac{\Omega_e}{L_x}, \quad g_y = \frac{\partial \Omega_e}{\partial y} = \frac{\Omega_e}{L_y}$$

where again

$$L_x = \frac{1}{\Omega_e}\frac{\partial \ln \Omega_e}{\partial x} \quad \text{and} \quad L_y = \frac{1}{\Omega_e}\frac{\partial \ln \Omega_e}{\partial y}.$$

Assuming the weak dependence of Ω_e on k^2, results in

$$\frac{\eta_0}{\mathbf{a}\cdot\mathbf{g}} = \frac{\omega_{pe}^2 k_\perp^2 W^2 \omega}{4\Omega_e^2 c^2 \left(\omega + \sqrt{k_\perp^2 c^2 + \omega_{pe}^2}\right)(k_x/L_x + k_y/L_y)}. \tag{4.6}$$

Substituting (4.6) into (2.32) gives the transmission coefficient for O-mode

$$\mathscr{T} = \exp\left\{-\frac{\pi \omega_{pe}^2 k_\perp^2 W^2 \omega}{2\Omega_e^2 c^2 \left(\omega + \sqrt{k_\perp^2 c^2 + \omega_{pe}^2}\right)(k_x/L_x + k_y/L_y)}\right\}. \tag{4.7}$$

In the one-dimensional case (4.6) is transformed into

$$\frac{\eta_0}{ag} = \frac{R}{4\Omega_e^2 c^2}\frac{\omega_{pe}^2 k_x W^2 \omega}{\Omega_e + \sqrt{k_x c^2 + \omega_{pe}^2}}. \tag{4.8}$$

Substituting the value of k_x at $\omega_0 \approx \Omega_e$ from (4.5) into (4.8) leads to

$$\frac{\eta_0}{ag} = \frac{R}{4}\frac{\omega_{pe}^2}{\Omega_e^2}\left(1 - \frac{\omega_{pe}^2}{\Omega_e^2}\right)^{1/2}\frac{\Omega_e}{c}\frac{T_e}{m_e c^2}, \tag{4.9}$$

with $v_t^2 = T_e/m_e$, this is exactly the same as relation which is obtained by Cairns and Lashmore-Davies (1982).

Akin to Sec. 3, the condition for maximum mode-conversion ratio obtains

$$\frac{\omega_{pe}^2 k_\perp^2 W^2 \omega}{\Omega_e^2 c^2 \left(\omega + \sqrt{k_\perp^2 c^2 + \omega_{pe}^2}\right)(k_x/L_x + k_y/L_y)} \approx 0.44. \tag{4.10}$$

5. Conclusions

An analytical model has been developed to study the energy transfer in mode conversion theory with two-dimensional inhomogeneities. We have found the transmission and mode conversion coefficients for the excited secondary waves by using the local dispersion relation and asymptotic analysis of the differential equations governing the modes propagating through the plasma. We have also described ECRH by our two-dimensional model of mode conversion between the O-mode and X-mode and the electron Bernstein waves. The condition for which the power mode-conversion coefficient achieves the maximum value, i.e. $\mathscr{T} = 0.5$, gives the relation between the incident wave frequency and the wave vector (index of refraction) which can be solved numerically for the given magnetic field, density and

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temperature profiles. In O-B and X-B heating in spherical Torus type tokamaks, since the magnetic shear is strong and the poloidal field is comparable with the toroidal field, the model and the above results can be applied to verify the wave energy of the launched wave necessary to excite the electron Bernstein waves, which have no density cutoffs and propagate into the core of the plasma.

References

Bender, C. M. and Orszag, S. A. 1978 *Advanced Mathematical Methods for Scientists and Engineers*. New York: McGraw-Hill.

Budden, K. G. 1972 *J. Atmose. Terrest. Phys.* **34**, 1909.

Budden, K. G. and Smith, M. S. 1974 *Proc. R. Soc. London* A **341**, 1.

Budden, K. G. 1985 *The Propagation of Radio Waves*. Cambridge: Cambridge University Press.

Cairns, R. A. and Lashmore-Davies, C. N. 1982 *Phys. Fluids* **25**, 1605.

Cairns, R. A. and Lashmore-Davies, C. N. 1983 *Phys. Fluids* **26**, 1268.

Cairns, R. A. 1991 *Radiofrequency Heating of Plasmas*. Bristol: Adam Hilger.

Farina, J. E. G. 1973 *The International Encyclopedia of Physical Chemistry and Chemical Physics, Topic 2*, Vol. 4. Oxford: Pergamon Press, p. 42.

Friedland, L., Goldner, G. and Kaufman, A. N. 1987 *Phys. Rev. Lett.* **58**, 1392.

Gradshteyn, I. S. and Ryzhik, I. M. 1965 *Table of Integrals, Series and Products*, 4th edn. Reading, MA: Academic Press.

Kaufman, A. N. and Friedland, L. 1987 *Phys. Lett.* A **123**, 387.

Nishikawa, K. and Wakatani, M. 2000 *Plasma Physics*, 3rd edn. Berlin: Springer.

Sabzevari, B. 1992 *J. Plasma Phys.* **47**, 49–60.

Sabzevari, B. 1993 *J. Plasma Phys.* **49**, 413–427.

Smirnov, V. I. 1964 *A Course of Higher Mathematics*, Vol. IV. Oxford: Pergamon Press, pp. 387–426.

Sommerfeld, A. 1964 *Partial Differential Equations in Physics: Lectures on Theoretical Physics*. New York: Academic Press.

Stix, T. H. 1992 *Waves in Plasmas*. New York: AIP.

Swanson, D. G. 1989 *Plasma Waves*. Reading, MA: Academic Press.

Whittaker, E. T. and Watson, G. N. 1927 *A Course of Modern Analysis*, 4th edn. New York: Cambridge University Press.

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