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ON THE APPROXIMATION OF STABILITY FACTORS FOR GENERAL PARAMETRIZED PARTIAL DIFFERENTIAL EQUATIONS WITH A TWO-LEVEL AFFINE DECOMPOSITION

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Abstract. A new approach for computationally efficient estimation of stability factors for parametric partial differential equations is presented. The general parametric bilinear form of the problem is approximated by two affinely parametrized bilinear forms at different levels of accuracy (after an empirical interpolation procedure). The successive constraint method is applied on the coarse level to obtain a lower bound for the stability factors, and this bound is extended to the fine level by adding a proper correction term. Because the approximate problems are affine, an efficient offline/online computational scheme can be developed for the certified solution (error bounds and stability factors) of the parametric equations considered. We experiment with different correction terms suited for *a posteriori* error estimation of the reduced basis solution of elliptic coercive and noncoercive problems.

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1. Introduction

Stability factors of linear partial-differential operators control the well-posedness of the equations and the magnitude of many *a posteriori* error estimates for the finite element method [3,10]. In applications related to optimal design and/or control of PDEs, the state equations usually depend on a vector μ of real parameters – which can describe either physical or geometrical properties – and varying the parameters can cause large qualitative changes, both in terms of solutions and approximation stability. As a motivation, we mention the Helmholtz problem, parametrized with respect to the wave frequency: in this case the problem can become ill-posed at the resonance frequencies [11]. In this paper we consider and refer to the specific area of model reduction of parametric PDEs through reduced basis (RB) methods with certified *a posteriori* error bounds, where the "truth" finite element solution $u_h(\mu)$ is approximated by a solution $u_h^N(\mu)$ obtained through Galerkin projection in a low-dimensional subspace [30–32,34]. To certify the reduced solution we need to be able to give

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a computable *a posteriori* error estimator of the form

$$||u_h(\mu) - u_h^N(\mu)|| \leq \frac{||R(u_h^N(\mu))||_{X_h'}}{\beta_h(\mu)},$$

where $R(u_h^N(\mu))$ is a discrete residual computed in its dual norm and $\beta_h(\mu)$ a stability factor to be bounded from below [34]. *a priori* there is no guarantee that the reduced problem is stable (*i.e.* coercive or inf-sup stable in the more general noncoercive case) at a given parameter value μ, and so a computable lower bound $\beta_h^{\mathrm{LB}}(\mu) > 0$ (where it exists) is needed both to certify that there exists in fact a stable solution, as well as to quantify the approximation error to guarantee the accuracy and the reliability of the methodology.

Let X_h be a finite-dimensional subspace $H_0^1(\Omega) \subset X_h \subset H^1(\Omega)$ of the usual Sobolev space with inner product $(v,w)_1 := \int_\Omega (\nabla v \cdot \nabla w + vw)\, \mathrm{d}\Omega$ and norm $||v||_X := \sqrt{(v,v)_1}$, and $\Omega \subset \mathbb{R}^d$ a bounded polyhedral domain. Our problem of interest is a general (scalar, elliptic) parametric linear partial differential equation, written in the discretized weak form as follows: for given P-dimensional parameter vector $\mu \in \mathcal{P}$, find $u_h(\mu) \in X_h$ s.t.

$$a(u_h(\mu), v_h; \mu) = F(v_h; \mu) \qquad \forall v_h \in X_h. \tag{P}$$

The following assumptions are made in order to have a well-posed problem: (i) the parametric bilinear form $a(\cdot,\cdot;\mu)$ is continuous for all μ, and (ii) satisfies an inf-sup condition

$$\exists\, \beta_0 > 0 : \inf_{v \in X_h} \sup_{w \in X_h} \frac{a(v,w;\mu)}{||v||_X ||w||_X} = \beta_h(\mu) \geq \beta_0,$$

except possibly at finitely many $\mu \in \mathcal{P}$. We call $\beta_h(\mu)$ the (discrete) *parametric stability factor*; (iii) the parametric linear form $F(\cdot;\mu)$ is bounded for all μ; (iv) the parameter domain $\mathcal{P} \subset \mathbb{R}^P$ is a bounded subset of a low-dimensional parameter space. Assumptions (i)–(iii) guarantee the existence and uniqueness of solutions. Assumption (iv) limits the parameter domain to one that we can explore via an efficient discrete sampling algorithm[2]. The most challenging one of our four assumptions is (ii) because it is usually not possible to derive analytical expressions for the parametric stability factor $\beta_h(\mu)$. Even if theoretical lower bounds can be obtained, they can be very pessimistic and not useful for practical computational applications, such as *a posteriori* error estimation. Therefore our interest is in providing numerical methods for obtaining efficient lower bounds for $\beta_h(\mu)$. Some algorithms for deriving lower bounds for parametric stability factors have been proposed in literature, among them we mention the successive constraint method (SCM) [5,17,18,29,34] with recent applications in viscous flows characterized by an increasing complexity in terms of geometrical parametrization [22,25]. This algorithm (briefly detailed in Appendix A) is based on an offline-online computational approach, where the offline stage consists of the solution of a large number of eigenproblems of the underlying PDE at different parametric points. Using this information, a parametric lower bound for the stability factor can be constructed in a way that permits its efficient evaluation in the online stage with a computational complexity independent of the dimension of X_h. To realize such an offline/online splitting we need to have a problem that is suited to allow the assumption of *affine decomposition*, *i.e.*

$$a(u,v;\mu) = \sum_{q=1}^{Q} \Theta_q^a(\mu) a_q(u,v), \quad F(v;\mu) = \sum_{q'=1}^{Q'} \Theta_{q'}^F(\mu) F_q(v), \tag{AD}$$

for given Q, $Q' \geq 1$, where the parametric bilinear forms $a_q(\cdot,\cdot)$ are continuous, and the parametric linear forms $F_q(\cdot)$ are bounded. If assumption (AD) does not hold, we cannot *a priori* decouple the parametric dependence of the solutions from the computational effort of solving them online. A way to resolve this issue is to exploit

[2]We have experimented with test cases where P is at least 4. For higher-dimensional parametrizations the *curse of dimensionality* may greatly reduce the effectivity of methods based on discrete sampling of the parameter space.

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an approximate affine decomposition of the form

$$a(u,v;\mu)=\sum_{q=1}^{Q}\Theta_q^a(\mu)a_q(u,v)+e^a(u,v;\mu,Q),\quad F(v;\mu)=\sum_{q=1}^{Q'}\Theta_q^F(\mu)F_q(v)+e^F(v;\mu,Q'),\tag{AAD}$$

where the error terms $e^a(u,v;\mu,Q)$ and $e^F(v;\mu,Q')$ are within a prescribed tolerance. For example, the empirical interpolation method (EIM) can be used to give such approximations, see [2, 23], but with the following complication: Q may become quite big also in problems with a reasonable parametrization [36].

Remark 1.1. The computational complexity of the offline stage of the SCM for the parametric stability factor $\beta_h(\mu)$ depends inherently on $Q\mathcal{N}$, where $\mathcal{N}:=\dim(X_h)$. The dependence on $\mathcal{N}$ is due to eigenvalues calculation.

Thus we observe that already for rather small problems the size of the approximate affine expansion may cause the offline stage to become potentially very expensive if rigorous stability factors are desired. In our experience, for the SCM proposed in previous works, convergence may turn to be quite slow, even for rather modest parametric complexity Q, and a relevant range of variation of parameters, and thus improvements need to be made. Our proposed approach is to limit the number of affine terms Q used in the SCM without sacrificing the accuracy of the lower bound $\beta_h^{\mathrm{LB}}(\mu)$. This improvement will allow to treat problems with a more complex nonaffine parametrization, like the ones arising with geometrical parametrization of more complex shapes [22, 25].

The structure of the paper is as follows. In Section 2 we discuss our proposed two-level algorithm for the approximation of lower bounds of stability factors for complex nonaffine operators, based on two nested approximated affine decompositions (obtained through empirical interpolation): the finer approximated form containing the coarser one, a suitable correction factor bounding the coarse and the fine approximations levels and the usual successive constraint method for lower bounds computations. We address an exhaustive description of SCM and EIM methods in Appendices A and B, respectively. We show how to obtain three possible approximations of the correction factor in the elliptic coercive and noncoercive cases in Sections 3 and 4, respectively. Then, we apply the two-level procedure to a nonaffinely parametrized Poisson problem in 5 and to a nonaffinely parametrized Helmholtz problem in Section 6, showing some numerical results and the computational performances and speedups – both form a qualitative and quantitative point of view – obtained by means of the two-level (coarse and fine) algorithm. In the last Section 7 we provide some conclusions and perspectives.

2. Two-level affine decomposition in the approximation of stability factors

We introduce in this section a general procedure for the approximation of stability factors based on a two-level affine (coarse and fine) decomposition of the parametrized operators. Consider two different affine approximations to the general parametric bilinear form $a(\cdot,\cdot;\mu)$: a coarse-level affine approximation

$$a(v,w;\mu)=a^c(v,w;\mu)+e^c(v,w;\mu)$$

and a fine-level affine approximation

$$a(v,w;\mu)=a^f(v,w;\mu)+e^f(v,w;\mu)$$

being

$$a^c(v,w;\mu)=\sum_{q=1}^{Q_c}\Theta_q^c(\mu)a_q^c(v,w),\tag{2.1}$$

$$a^f(v,w;\mu)=a^c(v,w;\mu)+\sum_{q=Q_c+1}^{Q_f}\Theta_q^f(\mu)a_q^f(v,w),\tag{2.2}$$

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such that $a^c(\cdot,\cdot;\mu)$ and $a^f(\cdot,\cdot;\mu)$ fulfill a *hierarchical* property and $a^c(\cdot,\cdot;\mu)$ can be seen as a truncation of $a^f(\cdot,\cdot;\mu)$[3]. We assume that the error terms are bounded, for all $\mu\in\mathcal{P}$, by

$$|e^c(v,w;\mu)|\le\delta_c\|v\|\|w\|,\qquad |e^f(v,w;\mu)|\le\delta_f\|v\|\|w\|,$$

being $\delta_c>0$, $\delta_f>0$ the continuity constants for the coarse-level and the fine-level approximation, respectively.

Moreover, let us denote

$$\beta_c(\mu)=\inf_{v\in X_h}\sup_{w\in X_h}\frac{a^c(v,w;\mu)}{||v||_X||w||_X},\qquad \beta_f(\mu)=\inf_{v\in X_h}\sup_{w\in X_h}\frac{a^f(v,w;\mu)}{||v||_X||w||_X}.$$

The idea is to take the fine-level affinely parametrized problem as the exact problem, and to use the coarse-level as a surrogate for computing bounds for the stability factor bound. Thus, the fine-level affine approximation is assumed to be computed with a very small tolerance w.r.t to the nonaffine problem. A preliminary idea for this approach is in [21] (with a test case related with [20] dealing with potential flows) but without developing a fine correction on the coarse error bounds (the certification was exploited on the coarse level and the convergence of the approximation was demonstrated by a fixed point algorithm). We can prove the following general relationship between the stability factors of the coarse- and fine-level problems:

Theorem 2.1. *Define the coarse-level supremizer operator* $T_c^\mu: X_h\to X_h$ *as*

$$(T_c^\mu v,w)_X=a^c(v,w;\mu)\quad \text{for all } w\in X_h. \tag{2.3}$$

Then we have that

$$\beta_f(\mu)\ge\beta_c(\mu)+\varepsilon_{cf}(\mu),$$

where the correction term is

$$\varepsilon_{cf}(\mu):=\inf_{v\in X_h}\frac{\sum_{q=Q_c+1}^{Q_f}\Theta_q^f(\mu)a_q^f(v,T_c^\mu v)}{||v||_X||T_c^\mu v||_X}. \tag{2.4}$$

Proof. Using the property of the supremizer operator

$$\sup_{w\in X_h}\frac{a^c(v,w;\mu)}{||v||_X||w||_X}=\frac{a^c(v,T_c^\mu v;\mu)}{||v||_X||T_c^\mu v||_X}\quad \text{for all } v\in X_h$$

and the assumption that the coarse- and fine-level affine bilinear forms agree on the part a^c, we have that

$$\begin{aligned}\beta_f(\mu)&=\inf_{v\in X_h}\sup_{w\in X_h}\frac{a^f(v,w;\mu)}{||v||_X||w||_X}=\inf_{v\in X_h}\sup_{w\in X_h}\frac{a^c(v,w;\mu)+\sum_{q=Q_c+1}^{Q_f}\Theta_q^f(\mu)a_q^f(v,w)}{||v||_X||w||_X}\\&\ge\inf_{v\in X_h}\frac{a^c(v,T_c^\mu v;\mu)+\sum_{q=Q_c+1}^{Q_f}\Theta_q^f(\mu)a_q^f(v,T_c^\mu v)}{||v||_X||T_c^\mu v||_X}\\&\ge\inf_{v\in X_h}\frac{a^c(v,T_c^\mu v;\mu)}{||v||_X||T_c^\mu v||_X}+\inf_{v\in X_h}\frac{\sum_{q=Q_c+1}^{Q_f}\Theta_q^f(\mu)a_q^f(v,T_c^\mu v)}{||v||_X||T_c^\mu v||_X}=\beta_c(\mu)+\varepsilon_{cf}(\mu).\end{aligned}$$

□

We can now propose a two-level algorithm for obtaining lower bounds for the discrete stability factor $\beta_h(\mu)$. First, assume there exists a method for deriving affine expansions (2.2) for any desired tolerance $\delta_f>0$, such as the empirical interpolation method reviewed in Appendix B. Then, choose δ_f sufficiently small that the fine-level affine approximate problem can be considered to be indistinguishable from the true nonaffine problem; typically

[3]This approach is quite general and it allows to properly introduce a hierarchical splitting of the operators (coarse-fine) also dealing with several subdomains [34] and also coupling affine and nonaffine parametrizations in different subdomains. In the case of the affine parametrization the coarse and fine level will be coincident for that part of subdomain.

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this entails choosing Q_f very large, say $\mathcal{O}(10^2)$. Next, we assume that it is possible to find, according to some rule or problem dependent criterion, a good choice of $\delta_c \gg \delta_f$ such that the coarse-level affine approximation (2.1) has a manageable number of terms Q_c, say $\mathcal{O}(10)$, while ideally still satisfying the assumption

$$\delta_c \ll \beta_c^{\mathrm{LB}}(\mu, \delta_c) \quad \text{for all } \mu \in \mathcal{P}, \tag{2.5}$$

being $\beta_c^{\mathrm{LB}}(\mu)$ a lower bound for the coarse stability factor. If this is the case, then we perform the SCM on the coarse-level problem to obtain $\beta_c^{\mathrm{LB}}(\mu)$; by evaluating $\varepsilon_{cf}(\mu)$ and applying the result of Theorem 2.1, we thus obtain a lower bound $\beta_f^{\mathrm{LB}}(\mu)$ also for $\beta_f(\mu)$. Since δ_f was chosen very small, this lower bound also acts as a good estimate for the true discrete stability factor $\beta_h(\mu)$.

Exact evaluation of the correction term $\varepsilon_{cf}(\mu)$ would require the solution of a generalized eigenvalue problem, which is usually not feasible in an online evaluation context. Therefore, we propose in the next section some rigorous and less-than-rigorous approximations $\widetilde{\varepsilon}_{cf}(\mu)$ that allow efficient online evaluation while still giving reasonable lower bounds for the stability factor $\beta_f(\mu)$. The computational cost is reduced in both the offline and online stages: since the computational cost of SCM depends highly on Q, by replacing Q_f with Q_c we potentially greatly reduce the cost of the offline part of the stability lower bound estimation, and if an inexpensive surrogate $\widetilde{\varepsilon}_{cf}(\mu)$ for the correction term can be computed online, then also the online cost of the lower bound computation is slightly decreased. We remark that the a good choice of δ_c that satisfies (2.5) is not in general easy. For simple coercive problems a cheap lower bound surrogate, $0 < \widetilde{\beta}_c^{\mathrm{LB}}(\mu, \delta_c) \le \beta_c^{\mathrm{LB}}(\mu, \delta_c)$, might be available, which together with an iterative procedure can be used to choose δ_c. In noncoercive problems it is possible that for any $\delta_c > 0$ there exists $\mu^* \in \mathcal{P}$ such that $\beta_c^{\mathrm{LB}}(\mu^*, \delta_c) = 0$. In the numerical examples we demonstrate that even for such problems a reasonable truncation of the affinely parametrized problem can sometimes be found.

3. Correction methods in an elliptic coercive case: the Poisson equation

Let us now discuss some possible ways to approximate the correction term $\varepsilon_{cf}(\mu)$ defined in (2.4). For the sake of simplicity, we consider first a (generalized) parametric Poisson equation. Nevertheless, this proposed methodology could be applied in problems where SCM has been exploited for the computation of stability factor lower bounds to certify not only scalar potential flows [20, 33] but also vectorial linear elasticity problems [16] and viscous Stokes flows treated with a penalty approach [12].

Denote by $a(u, v; \nu(\mu, \mathbf{x}))$ the parameter-dependent continuous bilinear form:

$$a(u, v; \nu(\mu, \mathbf{x})) := \int_\Omega \nu(\mu; \mathbf{x}) \nabla u(\mathbf{x}) \cdot \nabla v(\mathbf{x}) \, \mathrm{d}\Omega;$$

the discrete problem is to find $u_h \in X_h$ s.t.

$$a(u_h, v_h; \nu(\mu, \mathbf{x})) = F(v_h; \mu) \quad \forall v_h \in X_h,$$

being $\nu(\mu; \mathbf{x}) \in C^1(\mathcal{P}; L^\infty(\Omega))$ a scalar coefficient function such that $\nu(\mu; \mathbf{x}) \ge \nu_0 > 0$ for all $\mathbf{x} \in \Omega$, $\mu \in \mathcal{P}$, so that the problem is uniformly coercive. In this case the stability factor simplifies to

$$\inf_{v_h \in X_h} \sup_{w_h \in X_h} \frac{a(v_h, w_h; \nu(\mu, \mathbf{x}))}{\|v_h\|_X \|w_h\|_X} = \inf_{v_h \in X} \frac{a(v_h, v_h; \nu(\mu, \mathbf{x}))}{\|v_h\|_X^2} =: \alpha(\nu(\mu)),$$

i.e. the coercivity constant, and it suffices to replace the supremizer in (2.4) with the identity operator, $T_c^\mu \to I$. Moreover, we can identify the coarse- and the fine-level approximations (2.1)–(2.2) of $a(u, v; \nu(\mu, \mathbf{x}))$ as follows:

$$a^c(u, v; \mu) := a(u, v; \nu_c(\mu, \mathbf{x})), \qquad a^f(u, v; \mu) := a(u, v; \nu_f(\mu, \mathbf{x})),$$

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being $\nu_c = \nu_c(\mu, \mathbf{x})$ and $\nu_f = \nu_f(\mu, \mathbf{x})$ the coarse- and fine-level approximation of the parametrized tensor $\nu(\mu; \mathbf{x})$. Thus we obtain the coarse-fine correction term (2.4) of Theorem 2.1 in the simpler form

$$\varepsilon_{cf}(\mu) = \inf_{v \in X_h} \frac{\sum_{q=Q_c+1}^{Q_f} \Theta_q^f(\mu) a_q^f(v, v)}{\|v\|_X^2}. \tag{3.1}$$

We can propose three different estimators for (3.1):

(I) constant correction (CC)

$$\varepsilon_{cf}^{\mathrm{CC}}(\mu) := -\gamma C_a \left(\delta_c + \delta_f\right);$$

(II) global infimum (GI)

$$\varepsilon_{cf}^{\mathrm{GI}}(\mu) := \gamma C_a \inf_{\mathbf{x} \in \Omega} \{\nu_f(\mu, \mathbf{x}) - \nu_c(\mu, \mathbf{x})\};$$

(III) one-point correction (OP)

$$\varepsilon_{cf}^{\mathrm{OP}}(\mu) := -\gamma C_a \left[\delta_f + \hat{\epsilon}_{Q_c}^c(\mu)\right],$$

where

$$C_a = \sup_{v \in X_h} \frac{a(v, v; 1)}{\|v\|_X^2} > 0$$

is the continuity constant of the μ-independent bilinear form $a(v, v; 1)$, $\gamma \in [0, 1]$, and $\hat{\epsilon}_M^c(\mu)$ is defined in (B.3) and it represents a measure of the error committed during the empirical interpolation procedure for a nonaffine parametrization [2] (see Appendix B for details on this procedure). Thus, we can show the following result:

Theorem 3.1. *Assume that the bound* $\max_{\mu \in \mathcal{P}} \|\nu(\mu, \cdot)\|_{L^\infty(\Omega)} \leq C < \infty$ *holds and the one-point estimator of* (B.3) *is computed at the interpolation point* t_{Q_c+1} *s.t.* $\eta := \max_{\mu \in \mathcal{P}} |t_{Q_c+1} - \mu|$. *For the estimators above we have, for all* $\mu \in \mathcal{P}$, *the following inequalities:*

(I)

$$\varepsilon_{cf}^{CC}(\mu) \leq \varepsilon_{cf}(\mu);$$

(II)

$$\varepsilon_{cf}^{GI}(\mu) \leq \varepsilon_{cf}(\mu);$$

(III)

$$\varepsilon_{cf}^{OP}(\mu) \leq \varepsilon_{cf}(\mu) + \gamma C C_a \eta \left(1 + \Lambda_{Q_c}\right),$$

where the Lebesgue constant $\Lambda_{Q_c} \leq 2^{Q_c} - 1$ *is related to the empirical interpolation procedure.*

Proof. We provide the details for the proof of inequality (I), being (II) and (III) based on the same observations.

(I) Let us remark that $\|\nu_f(\mu, \mathbf{x}) - \nu_c(\mu, \mathbf{x})\|_{L^\infty(\Omega)} \leq \delta_c + \delta_f$ for all $\mu \in \mathcal{P}$. Then, we distinguish three cases:

- if $\nu_c(\mu, \mathbf{x}) - \nu_f(\mu, \mathbf{x}) \geq 0$ uniformly for all $\mathbf{x} \in \Omega$ then

$$-\varepsilon_{cf}(\mu) = -\inf_{v \in X_h} \frac{a(v, v; \nu_f(\mu, \mathbf{x}) - \nu_c(\mu, \mathbf{x}))}{\|v\|_X^2} = \sup_{v \in X_h} \frac{a(v, v; \nu_c(\mu, \mathbf{x}) - \nu_f(\mu, \mathbf{x}))}{\|v\|_X^2}$$
$$\leq \|\nu_f(\mu, \cdot) - \nu_c(\mu, \cdot)\|_{L^\infty(\Omega)} \sup_{v \in X_h} \frac{a(v, v; 1)}{\|v\|_X^2} \leq C_a(\delta_c + \delta_f);$$

- if $\nu_c(\mu, \mathbf{x}) - \nu_f(\mu, \mathbf{x}) \leq 0$ uniformly for all $\mathbf{x} \in \Omega$ then

$$\varepsilon_{cf}(\mu) = \inf_{v \in X_h} \frac{a(v, v; \nu_f(\mu, \mathbf{x}) - \nu_c(\mu, \mathbf{x}))}{\|v\|_X^2} \geq \inf_{\mathbf{x} \in \Omega} \{\nu_f(\mu, \mathbf{x}) - \nu_c(\mu, \mathbf{x})\} \inf_{v \in X_h} \frac{a(v, v; 1)}{\|v\|_X^2} \geq 0;$$

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- in the mixed case, denoting with $[f]_+$ and $[f]_+$ the positive and negative parts of f, we have

$$\varepsilon_{cf}(\mu) = \inf_{v\in X_h} \frac{a(v,v;[\nu_f(\mu,\mathbf{x})-\nu_c(\mu,\mathbf{x})]_+) + a(v,v;[\nu_f(\mu,\mathbf{x})-\nu_c(\mu,\mathbf{x})]_-)}{\|v\|_X^2}$$
$$\geq \inf_{v\in X_h} \frac{a(v,v;[\nu_f(\mu,\mathbf{x})-\nu_c(\mu,\mathbf{x})]_+)}{\|v\|_X^2} + \inf_{v\in X_h} \frac{a(v,v;[\nu_f(\mu,\mathbf{x})-\nu_c(\mu,\mathbf{x})]_-)}{\|v\|_X^2} \geq 0 - \gamma C_a(\delta_c+\delta_f),$$

being $\gamma \in [0,1]$ a factor representing the measure (relative to $|\Omega|$) of the set $\Omega_- = \{\mathbf{x}\in\Omega : \nu_f(\mu,\mathbf{x})-\nu_c(\mu,\mathbf{x})\leq 0,\ \forall\mu\in\mathcal{P}\}$.

(II) Similarly to the previous case, we have

$$\varepsilon_{cf}(\mu) \geq \inf_{v\in X_h} \frac{a(v,v;[\nu_f(\mu,\mathbf{x})-\nu_c(\mu,\mathbf{x})]_-)}{\|v\|_X^2} \geq \gamma C_a \inf_{\mathbf{x}\in\Omega}\{\nu_f(\mu,\mathbf{x})-\nu_c(\mu,\mathbf{x})\},$$

being also in this case $\varepsilon_{cf}(\mu)<0$ since $\inf_{\mathbf{x}\in\Omega}\{\nu_f(\mu,\mathbf{x})-\nu_c(\mu,\mathbf{x})\}<0$.

(III) Applying the *a posteriori* estimates derived in [6], we obtain

$$\sup_{\mathbf{x}\in\Omega}\{\nu_c(\mu,\mathbf{x})-\nu(\mu,\mathbf{x})\} \leq C\eta\,(1+\Lambda_{Q_c}) + \|\nu(\mu,t_{Q_c+1})-\nu_c(\mu,t_{Q_c+1})\|_{L^\infty(\Omega)} = C\eta\,(1+\Lambda_{Q_c}) + \hat{\epsilon}^c_{Q_c}(\mu)$$

and thus

$$\varepsilon_{cf}(\mu) \geq \gamma C_a \inf_{\mathbf{x}\in\Omega}\{\nu_f(\mu,\mathbf{x})-\nu_c(\mu,\mathbf{x})\} = \gamma C_a \inf_{\mathbf{x}\in\Omega}\{[\nu_f(\mu,\mathbf{x})-\nu(\mu,\mathbf{x})]+[\nu(\mu,\mathbf{x})-\nu_c(\mu,\mathbf{x})]\}$$
$$\geq \gamma C_a(-\delta_f - \sup_{\mathbf{x}\in\Omega}\{\nu_c(\mu,\mathbf{x})-\nu(\mu,\mathbf{x})\}) \geq \gamma C_a(-(\delta_f+\hat{\epsilon}^c_{Q_c}(\mu)) - C\eta\,(1+\Lambda_{Q_c})),$$

where the upper bound for the Lebesgue constant is demonstrated in [2]. □

Remark 3.2. The pessimistic bound given by the case (III) of Theorem 3.1 is in fact sharp if $\nu \in L^\infty(\mathcal{P};L^\infty(\Omega))$, see [23]. In practice much better behavior is observed, as we will demonstrate in the numerical examples. By assuming further regularity on $\nu(\mu,\mathbf{x})$, a better *a posteriori* estimate can be derived, as shown in [6].

Remark 3.3. With the current choice of norm (and operator) we have $C_a<1$; moreover, we consider the most pessimistic case where $\gamma=1$, so that $\gamma C_a<1$ and this factor can be dropped in the correction factors. The correction factors presented are in fact more general and take into account the use of different norms.
Moreover, we point out that if $\nu_c(\mu,\mathbf{x})\leq\nu_f(\mu,\mathbf{x})$ uniformly for all $\mathbf{x}\in\Omega$, then $\varepsilon_{cf}(\mu)\geq 0$ and no correction is *a priori* required, since $\beta_f(\mu)\geq\beta_c(\mu)$, so that $\beta_c(\mu)$ is automatically a lower bound of $\beta_f(\mu)$ – although it might be not very accurate.

4. Correction methods in an elliptic noncoercive case: the Helmholtz equation

From a mathematical point of view, noncoercive problems exhibit greater difficulties for stability analysis even in the most simple cases. We consider the scalar Helmholtz problem, denoting in this case $a(u,v;\nu(\mu))$ the following bilinear form:

$$a(u,v;\nu(\mu)) := \int_\Omega \left[\nu(\mathbf{x},\mu)\nabla u\cdot\nabla v - \omega^2 uv\right]\,\mathrm{d}\Omega,$$

where the frequency parameter ω and mesh size parameter h are chosen in such a way that the finite element approximation does not suffer from excessive "pollution"; typically $\omega^2 h<1$ [1]. A specific difficulty related to

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the stability of this noncoercive problem is that at the eigenvalues $\lambda = \omega^2$ of the generalized Laplacian operator $-\nu\triangle$ the problem is not well-posed, and the stability factor becomes singular. Since the spectrum $\Lambda(-\nu\triangle)$ of the operator depends on the parameter μ, it is not always possible to have *a priori* knowledge of the resonance frequencies and therefore the proposed method should be able also to identify resonance frequencies for certain parameter values in the parameter range. Note that in this case the parametric dependence acts only on the elliptic part of the operator, which allows us to estimate (3.1) as in the proof of (II), Theorem 3.1:

$$\varepsilon_{cf}(\mu) = \inf_{v\in X_h} \frac{a^f(v, T_c^\mu v; \mu) - a^c(v, T_c^\mu v; \mu)}{||v||_X ||T_c^\mu v||_X} = \inf_{v\in X_h} \frac{a(v, T_c^\mu v, \nu_f(\mu) - \nu_c(\mu))}{||v||_X ||T_c^\mu v||_X} \geq \gamma C_a \inf_{\mathbf{x}\in\Omega} \{\nu_f(\mu, \mathbf{x}) - \nu_c(\mu, \mathbf{x})\}$$

and thus obtain in theory the same estimators for $\varepsilon_{cf}(\mu)$ as in the coercive case. However, since in practice $\beta_f(\mu) \ll 1$ and even negative for some finite number of parameter points μ, we can expect that the correction terms given in the previous section will be much more sensitive to the choice of δ_c. This proposed method can be used also for vectorial elliptic noncoercive problems [22, 25].

5. Numerical examples of coarse-level bounds for the Poisson equation

In this section we introduce a numerical example of elliptic coercive problem dealing with a parametrized Poisson equation. To demonstrate the effectiveness of the corrections detailed in Sections 3 and 4 we consider a manufactured example where the combination of the standard empirical interpolation (EIM) and the successive constraint (SCM) methods produces poor results. Let $\Omega = (0, 1)^2$ and consider the following scalar parametrized elliptic PDE: find $u_h \in X_h \subset H_0^1(\Omega)$ such that

$$\int_\Omega \nu(\mu, x)\nabla u_h \cdot \nabla v_h \, \mathrm{d}\Omega = \int_\Omega f v_h \, \mathrm{d}\Omega \quad \forall\, v_h \in X_h,$$

where the scalar diffusivity function is

$$\nu(\mu, x) := \exp(\mu_1 + \mu_2)\left[1 + \exp\left(-\frac{(x_1 - \mu_1)^2 + (x_2 - \mu_2)^2}{0.02}\right)\right] \tag{5.1}$$

and $\mu \in \mathcal{P} := [0.4, 0.6]^2$. The coefficient function consists of two parts: a simple exponential scaling factor and a highly localized Gaussian peak which moves around the domain [6]. The first term drives the scaling of the problem and consequently the coercivity constant; the Gaussian function does not greatly contribute to the coercivity constant, but does present difficulties for the EIM because of the difficulty of approximating a localized Gaussian function as a linear combination of localized Gaussians located at different sites. Thus the number of terms in the EIM expansion grows rather rapidly. We performed the SCM on the test problem at both a coarse level, $\delta_c = 10^{-2}$ and $Q_c = 20$, and at a fine level, $\delta_f = 10^{-6}$ and $Q_c = 60$, to obtain the corresponding lower bounds α_{LB}^c and α_{LB}^f. To remove uncertainties related to the random sampling in the standard greedy SCM algorithm, a fixed training sample $|\Xi_{\mathrm{train}}| = 1000$ was used to drive all the tests; moreover, we settled on using a tolerance ratio $\varepsilon_* = 0.25$ such that

$$\rho(\mu) := \frac{\alpha_{\mathrm{UB}}(\mu) - \alpha_{\mathrm{LB}}(\mu)}{\alpha_{\mathrm{UB}}(\mu)} < \varepsilon_*$$

for all $\mu \in \Xi_{\mathrm{train}}$ to mark that the SCM algorithm had obtained a sufficiently good lower bound. This resulted in a maximum number of $K_{\max} = 20$ constraints (see Appendix A) for both the coarse- and fine-level SCMs, but considerably different locations for the constraint points. In Figure 1 the lower bounds α_{LB} for the coarse- and fine-level problems obtained from the SCM are shown, as well as the true coercivity $\alpha^f(\mu)$ of the (fine-level) problem; as expected, it is almost entirely dependent on the scaling term $\exp(\mu_1 + \mu_2)$. Qualitatively there are few differences between the coarse and fine lower bounds, except that the fine level bounds are slightly more pessimistic, due to the extra information coming from the residual terms that are not included in the coarse level expansion.

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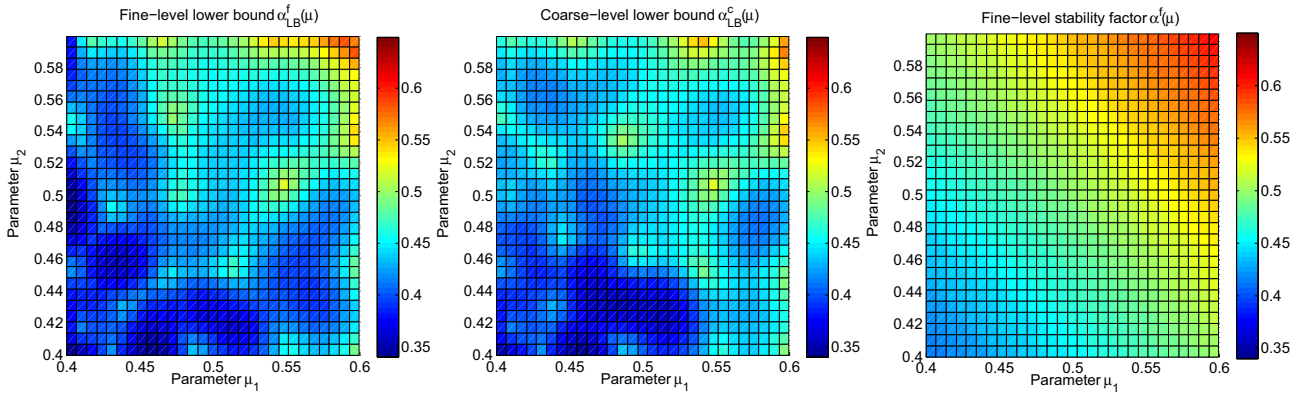


FIGURE 1. Comparison of the lower bounds $\alpha^c_{\mathrm{LB}}(\mu)$ and $\alpha^f_{\mathrm{LB}}(\mu)$ (without corrections) and the true parametric coercivity constant $\alpha^f(\mu)$; lower bounds are obtained from SCM with tolerance ratio $\varepsilon_* = 0.25$ and maximum number of constraints $K_{\max} = 20$.

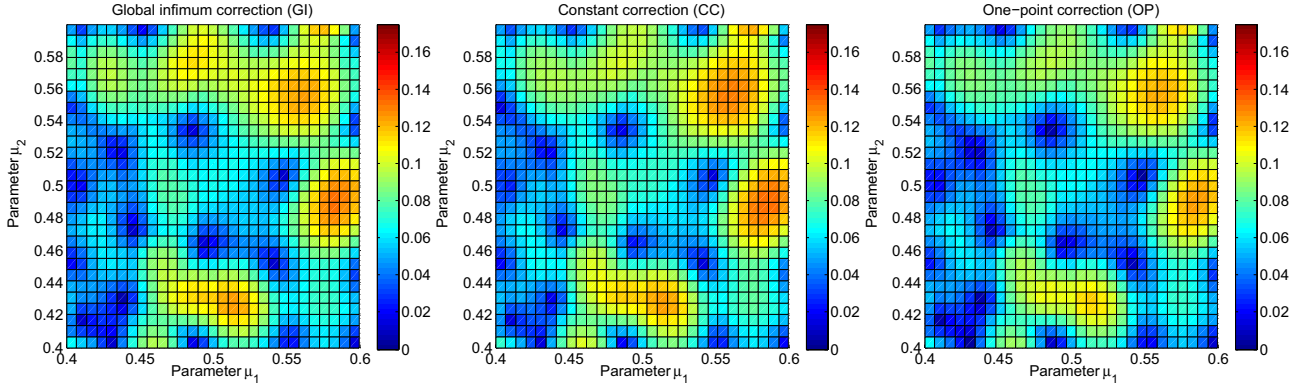


FIGURE 2. Bound gaps between the parametric coercivity constant $\alpha^f(\mu)$ and the corrected lower bounds for $\alpha^c_{\mathrm{LB}}(\mu)$ using the three methods: global infimum (GI, left), constant correction (CC, center), and one-point correction (OP, right); lower bounds are obtained from SCM with tolerance ratio $\varepsilon_* = 0.25$ and maximum number of constraints $K_{\max} = 20$.

We computed the corrections (GI), (CC), and (OP) for the given coarse-level $\alpha^c_{\mathrm{LB}}(\mu)$ to obtain the corrected lower bounds for $\alpha^f(\mu)$. In Figure 2 we plot the bound gap $\alpha^f(\mu) - (\alpha^c_{\mathrm{LB}}(\mu) + \text{correction } \varepsilon_{cf}(\mu))$ for each method. The first observation is that indeed in every case we obtain a lower bound for the coercivity constant, even using (OP) which is not *a priori* guaranteed to give a rigorous lower bound[4]. The (CC) method naturally gives always the largest bound gap, while the (GI) option gives the tightest bounds but requires evaluation of the coefficient function and a costly maximization over the entire domain.

Remark 5.1. At first it seems paradoxical that dropping terms from the EIM expansion would result in more effective error bounds, after all the coarse-level problem is potentially a poor approximation to the original PDE. But when one considers that for Q large the SCM typically needs to increase the number of point constraints $J_{\max}$, the situation becomes clear. A feature of affine expansions obtained from EIM is that the parametric coefficient functions $\Theta_q(\mu)$ decay rapidly to zero as $q \to \infty$. That is to say, the first few terms in the expansion

[4]This lack of rigor is put into evidence in Table 1, where the result reported in the last line shows an effectivity (slightly) <1.

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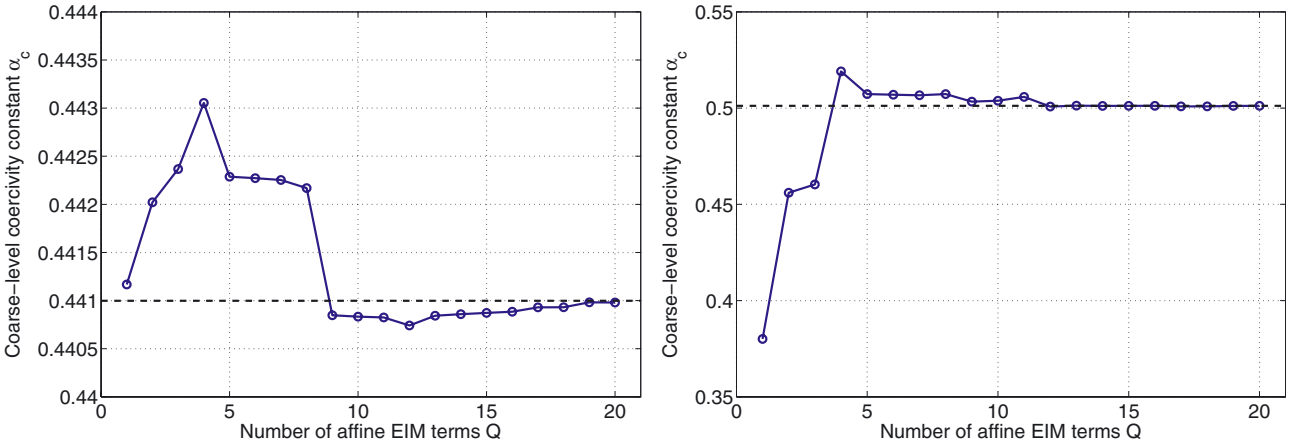


FIGURE 3. Counterexamples demonstrating that the convergence of $\alpha_c(\mu)$ towards $\alpha(\mu)$ (dotted line) can occur either from below or above (Rem. 5.2). Left: coarse-level coercivity constant $\alpha_c(\mu^*)$ for $\mu = (0.55, 0.45)$. Right: coarse-level coercivity constant $\alpha_c(\mu^*)$ for $\mu = (0.44, 0.43)$.

have a disproportionate effect on the problem (and subsequently its coercivity). This property is not considered in any way by the standard SCM algorithm, where all coefficients are treated equal. In a way, the algorithm performs a lot of additional work with low influence constructing constraints to search for lower bounds in the coefficient space of dimension $Q_f = 60$ when it suffices to search just the coefficient space of dimension $Q_c = 20$ consisting of the most significant terms.

Remark 5.2. By properties of the EIM expansion (B.1) we have always $a_1^c(v, v) \geq \alpha_0 \|v\|_X^2$ for some $\alpha_0 > 0$ and $\Theta_1^c(\mu) > 0$, so that the coarse-level approximation taking only the first term will always yield a positive $\alpha_c(\mu)$ for $Q_c = 1$. Two questions arise:

1. Is the coarse approximation $\alpha_c(\mu)$ obtained by taking $Q_c = 1$ always a lower bound for $\alpha(\mu)$?
2. Is the convergence always monotone, that is to say is $\alpha_c(\mu) \leq \alpha_{c'}(\mu)$ (resp. $\alpha_c(\mu) \geq \alpha_{c'}(\mu)$) for all $\mu \in \mathcal{D}$ and $Q_c < Q_{c'}$?

The answer to both questions is negative as demonstrated by Figure 3. We display the convergence of the $\alpha_c(\mu^*)$ for two different choices of the parameter μ^* in the coercive elliptic problem. The questions (1) and (2) are answered positively only in the unlikely case that the EIM expansion happens to give a parametrically coercive problem, which we recall being defined as $\Theta_q^c(\mu) \geq 0$ for all $\mu \in \mathcal{P}$ and $a_q^c(v, v) \geq \alpha_0 \|v\|_X^2$ for all $q = 1, \ldots, Q_c$. In the general case the bilinear forms $a_c^q(\cdot, \cdot)$ are indefinite, and one cannot deduce much about whether the coarse-level coercivity constants are lower bounds or not based on just the signs of the EIM coefficients.

Concerning the computational cost, in the offline stage we have in the SCM a count of $\mathcal{O}(Q)$ eigenproblems, $\mathcal{O}(QK_{\max})$ operations to form $\mathcal{Y}_{\mathrm{UB}}$, and $\mathcal{O}(J_{\max})$ linear programming problems of size $\mathcal{O}(Q)$ to solve. Thus the cost of SCM scales primarily like $\mathcal{O}(QK_{\max})$. Based on previous observations we can play with the two variables, Q and $K_{\max}$, which behave in some sense in opposite way. Increasing $K_{\max}$ will increase the computational cost, but also will monotonically improve the bounds obtained. The number of terms in the expansion Q on the other hand works in almost completely opposite way – as long as $Q_c \geq Q^*$, some cutoff point Q^* at which the correction in the coercivity constant dominates the bounds, it is possible that further increasing Q_c will in fact make the *a posteriori* bounds worse if $K_{\max}$ is kept fixed due to the fact the SCM needs many more point constraints to satisfactorily eliminate the parts of the coefficient space that we are never going to explore. Thus a rule of thumb could be to make Q_c just large enough so that the correction term is of an order of magnitude smaller than $\alpha^f(\mu)$, and then to increase $K_{\max}$ in order to improve the stability factor lower bound.

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TABLE 1. Effectivities of stability factor lower bounds for $\delta_c = 10^{-2}$ and $Q_c = 20$.

	Effectivity $\alpha_f/\alpha_*^{\mathrm{LB}}$		
	min	avg	max
Coarse	1.0087	1.1587	1.3459
Fine	1.0088	1.1625	1.3350
Coarse + (CC) correction	1.0285	1.1862	1.3769
Coarse + (GI) correction	1.0184	1.1793	1.3818
Coarse + (OP) correction	0.9894^2	1.1588	1.3475

TABLE 2. Offline computational complexity of the SCM in the coarse and fine level.

	Tolerance δ	Affine terms Q	SCM iterations	μ^* points	CPU time (s)
Fine level	5×10^{-6}	59	20	20	2741
Coarse level	1×10^{-2}	20	19	19	1963

In Table 1 we display the computed efficiencies $\alpha_f/\alpha^{\mathrm{LB}}$ of the fine-level lower bound, the uncorrected coarse-level lower bound, and the three different corrected coarse-level lower bounds over a range of 500 different parameter points in $\mu \in \mathcal{P} := [0.4, 0.6]^2$. In Table 2 we present the offline cost of constructing the lower bound surface using the successive constraint method at both the coarse- and fine-level. A speedup factor of 30% in the offline SCM step was observed while still obtaining effective stability factor lower bounds. Online computational costs of coarse stability factors and corrections are the same as the online costs for the fine stability factors, so we are not slowing the online SCM step.

6. Numerical examples of coarse-level bounds for the Helmholtz equation

In this section we introduce a numerical example of elliptic noncoercive problem dealing with a parametrized Helmholtz equation. In particular, we consider the reflection of time-harmonic waves on a stealth aircraft wing profile as in [27]. The computational domain Ω and mesh are displayed in Figure 4a.

A Kármán-Trefftz airfoil [15, 26] reflects an incoming time-harmonic wave with wave vector $\boldsymbol{k} = \omega \boldsymbol{e}_1$ so that the complex amplitude u_h of the reflected wave in the frequency plane satisfies the Helmholtz equation: find $u_h \in X_h \subset H_0^1(\Omega)$ such that

$$\int_\Omega \nu(\mu; \mathbf{x}) \nabla u_h \cdot \nabla v_h \, \mathrm{d}\Omega - \omega^2 \int_\Omega u_h v_h \, \mathrm{d}\Omega = 0 \quad \text{for all } v_h \in X_h,$$

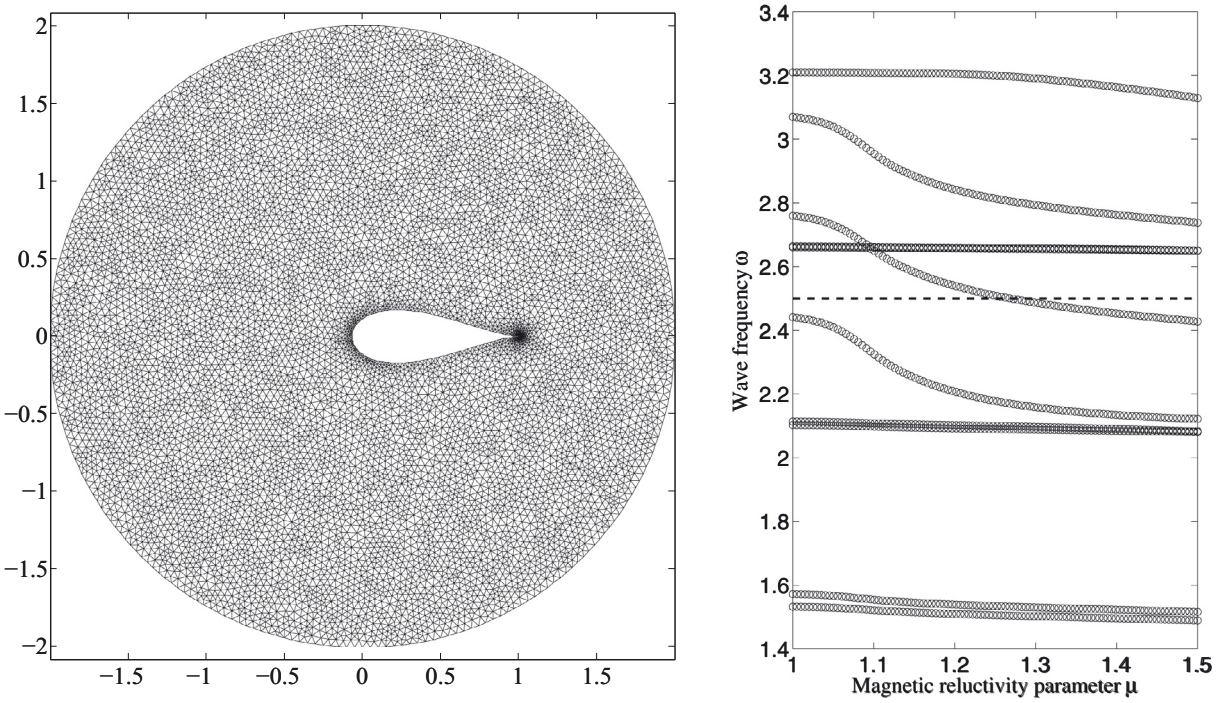
with Dirichlet condition $u_h|_{\Gamma_{foil}} = -\exp(i\boldsymbol{k}\cdot\mathbf{x})$ on the airfoil, and for simplicity the natural boundary condition is assumed on the far-field boundary. For the frequency we fixed $\omega = 2.5$. The magnetic reluctivity ν is chosen as the parametric function

$$\nu(\mu; z) = 0.9 \cdot H_\epsilon(|\kappa^{-1}(z)|; \mu) + 0.1$$

where the real plane is identified with the complex numbers, $\mathbf{x} \mapsto z$, so that $\kappa : \mathbb{C} \to \mathbb{C}$ is a Kármán-Trefftz map defined in the complex plane as

$$z = \kappa(\zeta) := \left[\left(\frac{\zeta - 1}{\zeta + \frac{1}{2}} \right)^2 - 1 \right]^{-1},$$

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(a) (b)

FIGURE 4. (a) Computational domain and mesh for the Helmholtz scattering problem; (b) approximate resonance lines in the (μ, ω)-plane for the Helmholtz example.

and the smoothed radial step function H_ϵ is defined as

$$H_\epsilon(r) := \begin{cases} 0, & r \le \mu - \epsilon \\ \frac{1}{2}\left[1 + \frac{r-\mu}{\epsilon} + \frac{1}{\pi}\sin\frac{\pi(r-\mu)}{\epsilon}\right], & \mu - \epsilon < r < \mu + \epsilon \\ 1, & r \ge \mu + \epsilon \end{cases}$$

with constant $\epsilon = 0.1$. For the parameter range $\mu \in [1, 1.5]$ this rather complicated coefficient function models an absorbing coat of paint on the surface of the airfoil, where $\mu - 1$ is the thickness of the layer of coating [27]. Approximating the nonaffinely parametrized coefficient function with the empirical interpolation procedure gives an expansion with rapidly increasing number of terms. For the coarse-level tolerance we chose $\delta_c =$ 1e-3 and for the fine-level tolerance $\delta_f =$ 1e-5. The corresponding number of affine terms were $Q_c = 27$ and $Q_f = 119$.

The location of the resonance frequencies depends on μ in quite a complicated way. In Figure 4b we display the approximate locations of resonances in the (μ, ω) plane. For the fixed frequency choice of $\omega = 2.5$ (dashed horizontal line) we expect to find only one resonance point, near $\mu \approx 1.27$. A requirement for any algorithm for computing (approximations of) stability factors is that they must correctly identify points of resonance, because in practical engineering design the presence of undiscovered resonances can lead to catastrophic results [13]. For comparison purposes we computed the stability factor lower bounds using both the lower bounds for the stability factors of the coarse- and fine-level approximations obtained by the natural norm SCM with a uniform training sample $|\Xi_{\text{train}}| = 1500$ and a stopping ratio $\rho < 0.25$. In Figure 5a we display the true fine-level stability factor $\beta_f(\mu)$ and the fine level lower bound $\beta_f^{\text{LB}}(\mu)$. Sample points where no positive lower bound for the stability factor were obtained are denoted by vertical lines.

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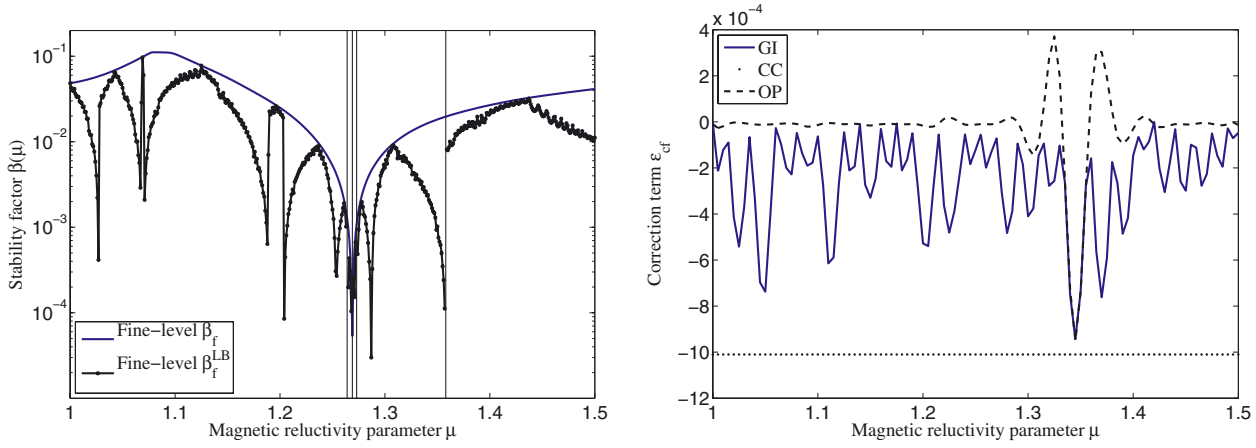


FIGURE 5. (a) Stability constant $\beta_f(\mu)$ and lower bound for the fine-level approximation of the Helmholtz problem with δ_f = 1e-5; (b) the three correction terms $\varepsilon_{cf}^{\mathrm{GI}}$, $\varepsilon_{cf}^{\mathrm{CC}}$, and $\varepsilon_{cf}^{\mathrm{OP}}$.

In Table 3 we display the computed efficiencies of the fine-level lower bound $\beta_f/\beta_f^{\mathrm{LB}}$, the uncorrected coarse-level lower bound $\beta_f/\beta_c^{\mathrm{LB}}$, and the three different corrected coarse-level lower bounds $\beta_f/(\beta_c^{\mathrm{LB}}+\varepsilon_i)$ with $i = CC, GI, OP$, over a range of 500 different parameter points in $\mu \in [1, 1.5]$. These quantities measure the quality of the proposed lower bounds: to be *reliable*, we shall require efficiencies ≥ 1; to be *effective*, we desire efficiencies as close as possible to 1. We also display the total number of sample points in the range where a positive lower bound was not obtained. In this case the coarse-level lower bound without any corrections actually behaves the best, even better than the fine-level lower bound. Again we attribute this to the convergence difficulties faced by the SCM when a large number of affine terms $Q_f = 119$ are used. The coarse lower bound β_c^{LB} turns out to be reliable in this case, but it is important to realize that *a priori* we cannot make such an assumption. Therefore, it is necessary to correct the coarse-level lower bound to obtain a lower bound that is rigorously reliable. We remark that for this particular problem the stability factor lower bound is for several parameter points below the tolerance used for the coarse-level approximation, $\delta_c \gg \beta_c^{\mathrm{LB}}(\mu)$, so that the constant correction method (CC) behaves poorly. Using this method a large number of sample points do not have a positive lower bound, and thus "ghost resonances" are predicted. The global infimum method (GI) is much better in comparison. In this test case the best correction turns out to be the one-point method (OP), which adds no failed points while remaining both reliable and on the average more effective than the (GI) method with a considerably less expensive online evaluation cost. However, in general we cannot expect the (OP) method to give reliable lower bounds and for this reason the (GI) method is the preferable one for noncoercive problems with resonances. In Table 4 we present the offline cost of constructing the lower bound surface using the successive constraint method at both the coarse- and fine-level. For this problem the coarse-level offline procedure gives a computational time reduction of 60%; moreover, by choosing the (GI) method in this case the stability factor lower bounds are also reasonable since they do not substantially differ from the ones that would be obtained by SCM when applied to the fine-level approximation.

7. CONCLUSION

The proposed methodology may represent a considerable reduction in terms of computational times and complexity in running offline expensive steps to prepare for the online evaluation of parametric stability factors. These quantities are crucial ingredients for error bounds certifying the rapid solution of parametric PDEs approximated for example by a reduced basis method.

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TABLE 3. Effectivities of stability factor lower bounds for δ_c =1e-3 and $Q_c = 27$.

	Effectivity $\beta_f/\beta_*^{\mathrm{LB}}$			# failed points
	min	avg	max	
Coarse	1.0001	4.47	177.17	7
Fine	1.0000	5.54	271.01	4
Coarse + (CC) correction	1.0133	11.1	2698.1	39
Coarse + (GI) correction	1.0007	9.75	773.83	9
Coarse + (OP) correction	1.0003	9.34	972.17	7

TABLE 4. Offline computational complexity of the successive constraint method.

	Tolerance δ	Affine terms Q	SCM iterations	μ^* points	CPU time (s)
Fine level	10^{-5}	119	297	16	14 567
Coarse level	10^{-3}	27	297	17	5816

Our tests on elliptic linear scalar problems both in the coercive (Poisson) and in the noncoercive (Helmholtz) case underline that the correction in the stability factor computed by the global infimum (GI) option is reliable and accurate and it represents a good compromise in terms of effectivity. The first option with a constant correction (CC) factor is better for the coercive test problem, while the third one based on a one point (OP) correction factor seems better in the noncoercive test problem, but affecting a bit the effectivity. The improvement aspects we dealt with are related only to stability factors computations. Other recent works addressing different needs of improvement in terms of offline procedure and performances for certified reduced order modelling by reduced basis method are related to the study of better sampling strategies in the parametric space [7,8] and to a two-step approximation [9] recalling again the need of a coarse and fine approximation level.

Important improvements are expected in the application of this methodology in nonlinear problems with nonaffine complex geometric parametrizations [24] governed by Navier-Stokes equations. A further activity of interest in this field is the development of the same coarse-fine approach to parametric lower bounds for the Brezzi inf-sup constant. This would lead to a different error bound [4,39] for the Stokes problem, with respect to the one based on the lower bound for the Babuška inf-sup constant proposed in [35] and considered in this work with a scalar linear elliptic problem.

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APPENDIX A. DESCRIPTION OF THE SUCCESSIVE CONSTRAINT METHOD

In this Appendix we review the successive constraint method used for the estimation of lower and upper bounds of stability factors. This algorithm has been first introduced in [17] for both coercive and noncoercive problems, more deeply analyzed in [34] in the coercive case and afterwards improved in [5]. A general version using the so-called "natural norm" [37] has been analyzed in [18]. Some modifications *e.g.* to get rid of some user-dependent parameters or to enhance its robustness have been recently proposed in [38,40].

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We recall here the basic ingredients of this second version in the more general case of noncoercive problems used for example in saddle point problems such as the Stokes case [22,35]; the simpler coercive case can be seen as a particular instance where the stability factor is just the coercivity constant.

Our goal is to build a lower bound for the inf-sup stability factor

$$\beta(\mu) := \inf_{v\in X_h} \sup_{w\in X_h} \frac{a(v,w,\mu)}{\|v\|_X \|w\|_X} = \inf_{v\in X_h} \frac{a(v,w,\mu)}{\|v\|_X \|T^\mu v\|_X},$$

being $w = T^\mu v$ the supremizer operator $T^\mu : X_h \to X_h$ defined as

$$(T^\mu v, w)_X = a(v,w;\mu), \qquad \text{for all } w \in X_h.$$

The *natural norm* approach we have adopted is based on the patching of some local (or *surrogate*) inf-sup stability factors properly computed for a set of J parameter values $\mathcal{S} = \{\mu^{1*}, \dots, \mu^{J*}\}$. In order to motivate this approach, let us analyze the discrete version of the problem; in particular, we shall observe that the computation of the stability factor

$$\beta(\mu) := \inf_{v_h\in X_h} \sup_{w_h\in X_h} \frac{a(v_h,w_h,\mu)}{\|v_h\|_X \|w_h\|_X}$$

in the discrete case can be formulated as finding a minimum eigenvalue, since

$$\beta^2(\mu) = \left(\inf_{v_h\in X_h} \sup_{w_h\in X_h} \frac{a(v_h,w_h;\mu)}{||v_h||_X ||w_h||_X} \right)^2 = \left(\inf_{u_h\in X_h} \frac{||T^\mu v_h||_X}{||v_h||_X} \right)^2 = \inf_{v_h\in X_h} \frac{||T^\mu v_h||_X^2}{||v_h||_X^2}.$$

Let us denote with $v_h = \sum_{j=1}^{\mathcal{N}} v_j \varphi_j$ a generic element of the discrete space X_h, being $\{\varphi_j\}_{j=1}^{\mathcal{N}}$ a FE basis of X_h and $(\mathbf{v})_j = v_j$ its vector representation. By introducing the discrete inner product $(v_h, w_h)_X = \mathbf{v}_h^T \mathbb{X} \mathbf{w}_h$ being $\mathbb{X}_{ij} = (\varphi_i, \varphi_j)_X$ with the Cholesky decomposition $\mathbb{X} = \mathbb{H}^T\mathbb{H}$, we obtain the following eigenvalue problem in matrix form: find $(\beta^2(\mu), \mathbf{v}_h)$, $\mathbf{v}_h \neq 0$, s.t.

$$\left(\mathbb{H}^{-T}\mathbb{A}(\mu)\mathbb{X}^{-1}\,\mathbb{A}(\mu)\mathbb{H}^{-1}\right)\mathbf{v}_h = \beta^2(\mu)\,\mathbf{v}_h \qquad \text{for each } \mathbf{v}_h \neq 0, \tag{A.1}$$

being $A_{ij}(\mu) = a(\varphi_j, \varphi_i; \mu)$. The original version of SCM proposed in [17] deals with affine coercive operators under the assumption (AP), and thus features a complexity of order $\mathcal{O}(Q)$; its extension to affine noncoercive operators – and then to the solution of problem (A.1) – is straightforward, even if this implies a complexity which is of order $\mathcal{O}(Q^2)$ – too cumbersome for problems with larger Q like the ones coming from EIM. The "natural norm" approach overcomes this limitation by means of a different strategy, based on the computation of a lower bound for a surrogate inf-sup stability factor $\beta_{\mu^*}(\mu)$. We define, for a fixed parameter value μ^*,

$$\beta^S_{\mu^*}(\mu) = \inf_{v\in X_h} \sup_{w\in X_h} \frac{a(v,w,\mu)}{\|w\|_X \|T^{\mu^*} v\|_X}; \tag{A.2}$$

since we assume that $\beta(\mu) > 0$ for all $\mu \in \mathcal{D}$, $\|T^{\mu^*}\cdot\|_X$ is a well-defined norm (which is equivalent to $\|\cdot\|_X$ in a neighborhood $\mathcal{P}_{\mu^*} \ni \mu^*$), called *natural norm*. A lower bound for the surrogate $\beta^S_{\mu^*}(\mu)$ is thus given by

$$\beta^{SLB}_{\mu^*}(\mu) = \inf_{v\in X_h} \frac{a(v, T^{\mu^*} v; \mu)}{\|T^{\mu^*} v\|_X^2}, \tag{A.3}$$

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thanks to the definition of the supremizer operator. It is also possible to show [37] that $\beta^{SLB}_{\mu^*}(\mu)$ is a good approximation to $\beta_{\mu^*}(\mu)$ when $\mu \to \mu^*$, *i.e.* $|\beta^{SLB}_{\mu^*}(\mu) - \beta_{\mu^*}(\mu)| = \mathcal{O}(|\mu - \mu^*|^2)$ when $\mu \to \mu^*$. Following the same analogy introduced before, $\beta^{SLB}_{\mu^*}(\mu)$ can be seen as the solution of the following eigenproblem in matrix form: find the smallest $\beta_{\mu^*}(\mu)$ such that

$$\left(\mathbb{H}\,\mathbb{A}^{-1}(\mu^*)\mathbb{A}(\mu)\mathbb{H}^{-1}\right)\mathbf{v}_h = \beta_{\mu^*}(\mu)\,\mathbf{v}_h \quad \text{for each } \mathbf{v}_h \neq 0. \tag{A.4}$$

The point is that, unlike the version (A.1), for fixed μ^* the operator on the right-hand-side of (A.4) contains only Q terms. Moreover, since it can be shown [37] that $\beta(\mu^*)\beta^{SLB}_{\mu^*}(\mu) \leq \beta(\mu)$, it is sufficient to compute a lower bound $\beta^{\mathrm{LB}}_{\mu^*}(\mu) \leq \beta^{SLB}_{\mu^*}(\mu)$ for the surrogate (A.3) and some $\beta(\mu^*)$ on the selected μ^*, and then translate it into a lower bound for $\beta(\mu)$.

The SCM procedure we adopt for the computation of a *global* lower bound, *i.e.* valid for each $\mu \in \mathcal{D}$, can be seen as the combination between two main ingredients: (i) the construction of a local lower bound $\beta^{\mathrm{LB}}_{\mu^*}(\mu)$ upon a given parameter value $\mu^* \in \mathcal{S}$, being $\mathcal{S} = \{\mu^{1*}, \ldots, \mu^{J*}\}$ a set of J parameter values properly (and iteratively) sampled, and (ii) the combination of the local lower bounds computed upon each $\mu^* \in \mathcal{S}$.

A.1 Construction of a local lower bound $\beta^{\mathrm{LB}}_{\mu^*}(\mu)$

Let us analyze the construction of a local lower bound (A.3) for the surrogate inf-sup stability factor (A.2), considering a chosen μ^* value; since this surrogate problem is coercive, the standard successive constraint method [17] can be used. Let us denote $\Xi_{\mathrm{train}} \subset \mathcal{D}$ a very rich training sample, playing the role of $\mathcal{D}$ throughout the algorithm. First of all, we rewrite the eigenvalue problem (A.3) as the following minimization (*linear programming*) problem:

$$\beta^{SLB}_{\mu^*}(\mu) = \inf_{\mathbf{y}\in\mathcal{Y}_*} J^{\mathrm{obj}}(\mathbf{y};\mu), \tag{A.5}$$

being $J^{\mathrm{obj}}(\mathbf{y};\mu)$ the following linear objective functional:

$$J^{\mathrm{obj}}(\mathbf{y};\mu) = \sum_{q=1}^{Q} \Theta^q(\mu) y_q, \qquad \text{with } \mathbf{y} = (y_1, \ldots, y_Q),$$

and $\mathcal{Y}_* \subset \mathbb{R}^Q$ the following constraint set (exploiting the affine decomposition of $a(\cdot,\cdot;\mu)$):

$$\mathcal{Y}_* = \left\{ \mathbf{y} \in \mathbb{R}^Q \ : \ \exists\, w_{\mathbf{y}} \in X \text{ s.t. } y_q = \frac{a^q(w_{\mathbf{y}}, T^{\mu^*} w_{\mathbf{y}})}{\|T^{\mu^*} w_{\mathbf{y}}\|^2_X}, \quad 1 \leq q \leq Q \right\}.$$

The goal is to build a sequence of suitable relaxed problems of the original LP problem (A.5) by seeking the minimum of the objective on a descending sequence of larger sets, built by adding successively linear constraints. In order to define this sequence, let us consider the following steps:

1. *Bounding box construction.* In order to guarantee *a priori* that all relaxations which will be considered are well-posed, we construct once for all a (continuity) bounding box given by

$$B_{\mu^*} = \prod_{q=1}^{Q} \left[-\frac{\gamma_q}{\beta(\mu^*)}, \frac{\gamma_q}{\beta(\mu^*)} \right],$$

being $\beta(\mu^*)$ the solution of the eigenproblem (A.1) computed for $\mu = \mu^*$ (equivalently given by (A.2)) and γ_q the (μ^*-independent) continuity factor of the bilinear form $a^q(\cdot,\cdot)$, given by

$$\gamma_q = \sup_{v\in X} \sup_{w\in X} \frac{a^q(w,v)}{\|v\|_X \|w\|_X}, \quad 1 \leq q \leq Q.$$

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2. *Relaxed LP problem.* Given a properly selected constraints sample (or SCM sample) $\mathcal{C}_k^* = \{\mu_1^*, \dots, \mu_k^*\}$ associated to μ^*, compute the (surrogate) lower bounds $\beta_{\mu^*}^{SLB}(\mu')$ defined by (A.3), for each $\mu' \in \mathcal{C}_k^*$; then, define the relaxation set

$$\mathcal{Y}_*^{\mathrm{LB}}(\mathcal{C}_k^*) = \left\{ \mathbf{y} \in \mathbb{R}^Q \ : \mathbf{y} \in B_{\mu^*} \ \middle| \ \sum_{q=1}^{Q} \Theta^q(\mu') y_q \geq \beta_{\mu^*}^{SLB}(\mu'), \quad \forall \mu' \in \mathcal{C}_k^* \right\}$$

by selecting a set of additional linear constraints associated to $\mathcal{C}_k^*$. It is fundamental to observe that since $\mathcal{Y}_* \subset \mathcal{Y}_*^{\mathrm{LB}}(\mathcal{C}_k^*)$ – for the proof, see [18] – the solution of the following relaxed problem,

$$\beta_{\mu^*}^{\mathrm{LB}}(\mu) \equiv \beta_{\mu^*}^{\mathrm{LB}}(\mu; \mathcal{C}_k^*) = \inf_{\mathbf{y} \in \mathcal{Y}_*^{\mathrm{LB}}(\mathcal{C}_k^*)} J^{\mathrm{obj}}(\mathbf{y}; \mu), \qquad \forall \mu \in \mathcal{D}_{\mu^*} \tag{A.6}$$

gives the desired *local lower bound*. In fact, we have that $\beta_{\mu^*}^{SLB}(\mu) \geq \beta_{\mu^*}^{\mathrm{LB}}(\mu)$, being the minimum taken over a larger set; we omit the specification of the set $\mathcal{C}_k^*$ for the sake of simplicity where no ambiguity occurs. We remark that problem (A.6) has to be solved for each $\mu \in \Xi_{\mathrm{train}}$.

3. *Selection of the successive constraint.* The last step deals with the selection of the set $\mathcal{C}_k^*$, which is performed by means of a *greedy* procedure. In order to measure the quality of the lower bounds, we need to introduce an upper bound, defined as follows:

$$\beta_{\mu^*}^{\mathrm{UB}}(\mu) \equiv \beta_{\mu^*}^{\mathrm{UB}}(\mu; \mathcal{C}_k^*) = \inf_{\mathbf{y} \in \mathcal{Y}_*^{\mathrm{UB}}(\mathcal{C}_k^*)} J^{\mathrm{obj}}(\mathbf{y}; \mu), \qquad \forall \mu \in \mathcal{D}_{\mu^*}, \tag{A.7}$$

being $\mathcal{Y}_*^{\mathrm{UB}}(\mathcal{C}_k^*)$ the set given by

$$\mathcal{Y}_*^{\mathrm{UB}}(\mathcal{C}_k^*) = \left\{ \tilde{\mathbf{y}} \in \mathbb{R}^Q \ : \ \tilde{\mathbf{y}} = \arg\min_{\mathbf{y} \in \mathcal{Y}_*} J^{\mathrm{obj}}(\mathbf{y}; \mu'), \quad \forall \mu' \in \mathcal{C}_k^* \right\}.$$

Since $\mathcal{Y}_*^{\mathrm{UB}}(\mathcal{C}_k^*) \subset \mathcal{Y}_*$ – see [17] for the proof – (A.7) is in fact an upper bound for $\beta_{\mu^*}^{\mathrm{SLB}}(\mu)$, *i.e.* $\beta_{\mu^*}^{\mathrm{SLB}}(\mu) \leq \beta_{\mu^*}^{\mathrm{UB}}(\mu)$; observe that (A.7) is just an enumeration problem. Finally, we can show how to add the *successive constraint*, by means of a *(local) greedy procedure*. Starting from an arbitrarily chosen $\mathcal{C}_1^* = \{\mu_1^*\}$, at step k we enrich the set $\mathcal{C}_k^* = \{\mu_1^*, \dots, \mu_k^*\}$, by means of the value μ_{k+1}^* given by

$$\mu_{k+1}^* = \arg\max_{\mu \in \Xi_{\mathrm{train}}} \rho(\mu; \mathcal{C}_k^*) \equiv \frac{\beta_{\mu^*}^{\mathrm{UB}}(\mu; \mathcal{C}_k^*) - \beta_{\mu^*}^{\mathrm{LB}}(\mu; \mathcal{C}_k^*)}{\beta_{\mu^*}^{\mathrm{UB}}(\mu; \mathcal{C}_k^*)};$$

i.e. choosing the element corresponding to the largest ratio $\rho(\mu; \mathcal{C}_k^*)$ over Ξ_{train}. The stopping criterium for this *successive* enrichment is given by $\rho(\mu; \mathcal{C}_k^*) \leq \varepsilon_*$, *i.e.* the procedure for the local lower bound finishes when the largest ratio is under a chosen SCM (local) tolerance $\varepsilon_* \in (0, 1)$. At the end of this procedure, we end up with K constraints, corresponding to the set $\mathcal{C}_K^* = \{\mu_1^*, \dots, \mu_K^*\}$.

A.2 Computation of a global lower bound

We now need to translate the local lower bound $\beta_{\mu^*}^{\mathrm{LB}}(\mu)$, computed upon a selected value μ^*, to a global lower bound. We shall make a distinction between the iterative procedure by which we "cover" the parameter space $\mathcal{D}$ and the relationship between the local and the global lower bounds.

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Let us start from this second point; the output of the coverage procedure are the set $\mathcal{S} = \{\mu^{1*}, \dots, \mu^{J*}\}$, $J \leq J_{\max}$ and the associated SCM samples $\mathcal{C}^{j*}_{K(j)} = \{\mu^{j*}_1, \dots, \mu^{j*}_{K(j)}\}$, for any $j = 1, \dots, J$, where $K(j) < K_{\max}$ is the number of constraints points related to each $\mu^{j*} \in \mathcal{S}$. The global lower bound for $\beta(\mu)$ can be defined (see [18] for the proof) as

$$\beta^{\mathrm{LB}}(\mu) = \beta(\mu^{\sigma*})\beta^{\mathrm{LB}}_{\mu^{\sigma*}}\left(\mu; \mathcal{C}^{\sigma*}_{K(\sigma)}\right), \quad \sigma \equiv \sigma(\mu) = \arg\max_{j \in \{1,\dots,J\}} \beta(\mu^{j*})\beta^{\mathrm{LB}}_{\mu^{j*}}\left(\mu; \mathcal{C}^{j*}_{K(j)}\right). \tag{A.8}$$

In practice, for each μ the global lower bound is given by the maximum among the products between the stability factors $\beta(\mu^{s*})$ and the local lower bounds $\beta^{\mathrm{LB}}_{\mu^{j*}}(\mu; \mathcal{C}^{j*}_{K(j)})$, corresponding to the selected $\{\mu^{1*}, \dots, \mu^{J*}\}$. Previous equation also implicitly defines the subdomains $\mathcal{D}_{\mu^{j*}}$:

$$\mathcal{D}_{\mu^{*j}} = \left\{\mu \in \mathcal{D} \ : \ \beta(\mu^{j*})\beta^{\mathrm{LB}}_{\mu^{j*}}\left(\mu; \mathcal{C}^{j*}_{K(j)}\right) \geq \beta(\mu^{j'})\beta^{\mathrm{LB}}_{\mu^{j'}}\left(\mu; \mathcal{C}^{j'}_{K(j')}\right)., \quad \forall j' \in \{1, \dots, J\}\right\}.$$

We remark that the global lower bound $\beta^{\mathrm{LB}}(\mu)$ given by this method interpolates $\beta(\mu)$ at each $\mu^* \in \mathcal{S}$, being $\beta^{\mathrm{LB}}(\mu^*) = \beta(\mu^*)$ in these cases.

We now discuss the procedure by which we select the set $\mathcal{S} = \{\mu^{1*}, \dots, \mu^{J*}\}$ and the associated SCM samples; also in this case, we use a *(global) greedy procedure*, which encapsulates the local ones used for the construction of each SCM sample. Starting from a chosen μ^{1*}, we set $\mathcal{S} = \{\mu^{1*}\}$ and initialize the corresponding SCM sample $\mathcal{C}^{1*}_1 = \{\mu^{1*}_1\}$, being $\mu^{1*}_1 = \mu^{1*}$. At step j, we have

$$\mathcal{S}^{(j-1)} = \{\mu^{1*}, \dots, \mu^{(j-1)*}\}, \qquad \mathcal{C}^{s*}_{K(s)} = \{\mu^{s*}_1, \dots, \mu^{s*}_{K(s)}\}, \qquad s = 1, \dots, j-1,$$

(through the construction of the local lower bounds around $\mu^{1*}, \dots, \mu^{(j-1)*}$) and

$$\mu^{j*} = \arg\min_{\mu \in \Xi_{\mathrm{train}}} \beta^{\mathrm{LB}}_{\mu^{(j-1)*}}\left(\mu; \mathcal{C}^{(j-1)*}_{K(j-1)}\right)$$

i.e. the new μ^{j*} is selected by taking the minimum over Ξ_{train} of the local lower bound computed w.r.t. the previous $\mu^{(j-1)*}$. Then, we build the *covered set*

$$\mathcal{R}_j = \left\{\mu \in \Xi_{\mathrm{train}} \ \middle| \ \beta^{\mathrm{LB}}_{\mu^{j*}}(\mu; \mathcal{C}^{j*}_1) > 0\right\}$$

and start the procedure for the construction of the local lower bound (upon μ^{j*}): for $k = 1, \dots, K(j)$, we build iteratively the set $\mathcal{C}^{j*}_k$ and compute the *actual covered set*

$$\mathcal{R}^{\mathrm{act}}_{j,k} = \left\{\mu \in \Xi_{\mathrm{train}} \ \middle| \ \beta^{\mathrm{LB}}_{\mu^{j*}}(\mu; \mathcal{C}^{j*}_k) > 0\right\},$$

checking at each step k if the current μ^{j*} does not give the possibility to increase the coverage, *i.e.* if $\mathcal{R}^{\mathrm{act}}_{j,k} \backslash \mathcal{R}_j = \emptyset$, and the stopping criterium $\rho(\mu; \mathcal{C}^{j*}_k) \leq \varepsilon_*$ is fulfilled. If these conditions are not fulfilled ($k < K(j)$), we keep on adding linear constraints, and setting $\mathcal{R}_j = \mathcal{R}^{\mathrm{act}}_{j,k}$; instead, if they are verified ($k = K(j)$), we stock the (local) covered set, by putting $\Xi_{\mathrm{train}} := \Xi_{\mathrm{train}} \setminus \mathcal{R}_j$, and seek for the subsequent $\mu^{(j+1)*}$. The global procedure ends up when all the train sample has been covered, *i.e.* when $\Xi_{\mathrm{train}} = \emptyset$. For the reader's convenience, we sum up the

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local/global procedure in the following schematic algorithm:

$\mathcal{S}^{(1)} = \{\mu^{1*}\},\ \ \mathcal{C}_1^{1*} = \{\mu_1^{1*}\},\ \ \mu_1^{1*} = \mu^{1*}$
for $j = 1 : J_{\max}$
  $\mathcal{C}_1^{j*} = \{\mu_1^{j*}\},\ \ \mu_1^{j*} = \mu^{j*}$
  $\mathcal{R}_j = \left\{\mu \in \Xi_{\text{train}} \;\middle|\; \beta^{\text{LB}}_{\mu^{j*}}(\mu; \mathcal{C}_1^{j*}) > 0\right\}$
  for $k = 1 : K_{\max}$
    compute the lower bound (A.6): $\beta^{\text{LB}}_{\mu^{j*}}(\mu; \mathcal{C}_k^{j*})$
    compute the upper bound (A.7): $\beta^{\text{UB}}_{\mu^{j*}}(\mu; \mathcal{C}_k^{j*})$
    add the successive constraint: $\mu_{k+1}^* = \arg\max_{\mu \in \Xi_{\text{train}}} \rho(\mu; \mathcal{C}_k^*)$
    set $\mathcal{C}_{k+1}^{j*} = \mathcal{C}_k^{j*} \cup \mu_{k+1}^*;\quad \mathcal{R}^{\text{act}}_{j,k+1} = \left\{\mu \in \Xi_{\text{train}} \;\middle|\; \beta^{\text{LB}}_{\mu^{j*}}(\mu; \mathcal{C}_{k+1}^{j*}) > 0\right\};$
    if $\mathcal{R}^{\text{act}}_{j,k+1} \setminus \mathcal{R}_j = \emptyset$ **and** $\rho(\mu; \mathcal{C}_k^{j*}) \le \varepsilon_*$
      set $K(j) = k;\quad \mathcal{C}^{j*}_{K(j)} = \mathcal{C}^{j*}_{k+1};\quad \Xi_{\text{train}} := \Xi_{\text{train}} \setminus \mathcal{R}^{\text{act}}_{j,k+1};$
      exit for
    else
      $\mathcal{R}_j = \mathcal{R}^{\text{act}}_{j,k+1};\quad$ **set** $k = k + 1;$
    end
  end
  if $\Xi_{\text{train}} \neq \emptyset$
    $\mu^{(j+1)*} = \arg\min_{\mu \in \Xi_{\text{train}}} \beta^{\text{LB}}_{\mu^{j*}}(\mu; \mathcal{C}^{j*}_{K(j)})$
    set $\mathcal{S}^{(j+1)} = \mathcal{S}^j \cup \mu^{(j+1)*};\quad j = j + 1;$
  else
    set $J = j; \mathcal{S} = \mathcal{S}^{(j)};$
    return
  end
end

Appendix B. Description of the empirical interpolation method

The empirical interpolation method (EIM) is a model reduction scheme that recovers the assumption of affine parametric dependence in nonaffinely parametrized operators (*e.g.* linear, bilinear forms, etc.). In the case of a nonaffinely parametrized bilinear form $a(v, w; \mu)$, the latter is replaced by an affinely parametrized approximation of the form

$$a(v, w; \mu) = \sum_{q=1}^{Q} \Theta^q(\mu) a^q_{\text{EIM}}(v, w) + \varepsilon_{\text{EIM}}(v, w; \mu), \tag{B.1}$$

where the error term ε_{EIM} needs to be controlled to an acceptable tolerance. The approximation is obtained by direct application of the EIM to the (nonaffinely) parametrized functions or tensors which appear in the original operators. We provide a short presentation of the EIM procedure based on [2]. Let us denote by $g(\mathbf{x}, \mu) \in C^0(\mathcal{D}; L^\infty(\Omega))$ a scalar function depending on both the spatial coordinates $\mathbf{x}$ and the parameters vector μ in a nonaffine way; the extension to tensors through an element-wise procedure is straightforward. The goal is to find an approximate expansion under the form

$$g_M(\mathbf{x}, \mu) = \sum_{j=1}^{M} \Theta^j(\mu) \zeta_j(\mathbf{x}), \tag{B.2}$$

where $\Theta^j(\mu)$, $j = 1, \ldots, M$ are M parameter-dependent functions and $\zeta_j(\mathbf{x})$, $j = 1, \ldots, M$ are M parameter-independent functions, denoted also *shape functions*. Being an interpolation procedure, the EIM procedure

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seeks a sequence of (nested) sets of interpolation points $T_M = \{p_1, \ldots, p_M\}$ (*magic points*), with $p_j \in \Omega$ for each $j = 1, \ldots, M$, and a set of shape functions $\zeta_j(\mathbf{x})$, in order to compute the expansion (B.2) by solving the following Lagrange interpolation problem:

$$\sum_{j=1}^{M} \mathbb{B}^M_{i,j} \Theta^j(\mu) = g(\mathbf{t}_i, \mu), \quad \forall\, i = 1, \ldots, M,$$

being the interpolation matrix $\mathbb{B}^M \in \mathbb{R}^{M\times M}$ defined as $(\mathbb{B}^M)_{ij} := \zeta_j(\mathbf{t}_i)$, for each $i, j = 1, \ldots, M$. Let us denote by $\Xi^{\mathrm{EIM}}_{\mathrm{train}} \subset \mathcal{D}$ a large training set, $M_{\max}$ the maximum number of terms, $\varepsilon^*_{\mathrm{EIM}}$ a fixed tolerance and select an initial parameter value μ^1. The EIM procedure [2] is as follows:

$\zeta_1(\mathbf{x}) := g(\mathbf{x}, \mu^1)$; **compute** $\mathbf{p}_1 := \arg\operatorname{ess\,sup}_{\mathbf{x}\in\Omega} |\zeta_1(\mathbf{x})|$;
$\mathbf{q}_1 = \zeta_1(\mathbf{x})/\zeta_1(\mathbf{p}_1)$; $G_1 := \operatorname{span}(\zeta_1)$, **set** $\mathbb{B}^1_{11} = 1$;
for $M = 2$: $M_{\max}$
  solve *(linear programming problem)*
    $\mu^M := \arg\max_{\mu\in\Xi^{\mathrm{EIM}}_{\mathrm{train}}} \inf_{v\in G_{M-1}} ||g(\cdot,\mu) - v||_{L^\infty(\Omega)}$
  set $\zeta_M(\mathbf{x}) := g(\mathbf{x}, \mu^M)$, $G_M := \operatorname{span}(\zeta_1, \ldots, \zeta_M)$
  solve $\sum_{j=1}^{M-1} \sigma_j^{M-1} q_j(\mathbf{t}_i) = \zeta_M(\mathbf{t}_i)$, $i = 1, \ldots, M-1$;
  compute *(residual)* $r_M(\mathbf{x}) := \zeta_M(\mathbf{x}) - \sum_{j=1}^{M-1} \sigma_j^{M-1} \zeta_j(\mathbf{x})$;
  compute $\mathbf{p}_M := \arg\operatorname{ess\,sup}_{\mathbf{x}\in\Omega} |r_M(\mathbf{x})|$;
  set $\mathbf{q}_M(\mathbf{x}) = r_M(\mathbf{x})/r_M(\mathbf{p}_M)$, $\mathbb{B}^M_{ij} = q_j(\mathbf{t}_i)$, $i, j = 1, \ldots, M$;
  solve *(interpolation problem)*
    $\sum_{j=1}^{M-1} \mathbb{B}^M_{ij} \Theta^j_M = g(\mathbf{t}_i, \mu)$, $\forall\, i = 1, \ldots, M-1$;
  if $\max_{\mu\in\Xi^{\mathrm{EIM}}_{\mathrm{train}}} \inf_{v\in G_M} ||g(\cdot,\mu) - v||_{L^\infty(\Omega)} < \varepsilon^*_{\mathrm{EIM}}$
    $M_{\max} = M - 1$;
  end;
end.

Given an approximation $g_M(x, \mu)$, $M < M_{\max}$, we denote the *one point (OP) error estimator* the following quantity (very inexpensive to compute):

$$\hat{\epsilon}_M(\mu) = |g(t_{M+1}; \mu) - g_M(t_{M+1}; \mu)|, \tag{B.3}$$

corresponding to the difference between the function and the interpolant at the point t_{M+1}, which gives the largest residual $r_M(x)$. While not rigorous as *a posteriori* error bound, this quantity proves to be an intuitive measure of the error committed by the EIM procedure [2]. Advances in error bounds developments have been presented in [6, 14, 23, 28].

References

[1] I. Babuška and S.A. Sauter, Is the pollution effect of the FEM avoidable for Helmholtz equation considering high wave numbers? *SIAM Rev.* **42** (2000) 451–484.
[2] M. Barrault, Y. Maday, N.C. Nguyen and A.T. Patera, An 'empirical interpolation' method: application to efficient reduced-basis discretization of partial differential equations. *C. R. Acad. Sci. Paris, Sér. I Math.* **339** (2004) 667–672.
[3] S.C. Brenner and L.R. Scott, *The mathematical theory of finite element methods*, 2nd edition. Springer (2002).
[4] F. Brezzi and M. Fortin, Mixed and Hybrid Finite Element Methods. Springer Series in *Comput. Math.* **15** (1991).

https://doi.org/10.1051/m2an/2012016 Published online by Cambridge University Press

[5] Y. Chen, J. Hesthaven, Y. Maday and J. Rodriguez, A monotonic evaluation of lower bounds for inf-sup stability constants in the frame of reduced basis approximations. *C. R. Acad. Sci. Paris, Sér. I Math.* **346** (2008) 1295–1300.
[6] J.L. Eftang, M.A. Grepl and A.T. Patera, *A posteriori* error bounds for the empirical interpolation method. *C. R. Acad. Sci. Paris, Sér. I Math.* **348** (2010) 575–579.
[7] J.L. Eftang, A.T. Patera and E.M. Rønquist, An "hp" certified reduced basis method for parametrized elliptic partial differential equations. *SIAM J. Sci. Comput.* **32** (2010) 3170–3200.
[8] J.L. Eftang, D.J. Knezevic and A.T. Patera, An "hp" certified reduced basis method for parametrized parabolic partial differential equations. *Math. Comput. Model. Dyn.* **17** (2011) 395–422.
[9] J.L. Eftang, D.B.P. Huynh, D.J. Knezevic and A.T. Patera, A two-step certified reduced basis method. *J. Sci. Comput.* **51** (2012) 28–58.
[10] A. Ern and J.-L. Guermond, *Theory and practice of finite elements*. Springer-Verlag, New York (2004).
[11] L.C. Evans, *Partial Differential Equations*. Amer. Math. Soc. (1998).
[12] A.L. Gerner and K. Veroy, Reduced basis *a posteriori* error bounds for the stokes equations in parametrized domains: a penalty approach. *Math. Mod. Methods Appl. Sci.* **21** (2011) 2103–2134.
[13] D. Green and W.G. Unruh, The failure of the Tacoma bridge: a physical model. *Am. J. Phys.* **74** (2006) 706–716.
[14] M.A. Grepl, Y. Maday, N.C. Nguyen and A.T. Patera, Efficient reduced-basis treatment of nonaffine and nonlinear partial differential equations. *ESAIM: M2AN* **41** (2007) 575–605.
[15] A. Holt and M. Landahl, *Aerodynamics of wings and bodies*. Dover New York (1985).
[16] D.B.P. Huynh and G. Rozza, Reduced basis method and *a posteriori* error estimation: application to linear elasticity problems (2011). Submitted.
[17] D.B.P Huynh, G. Rozza, S. Sen and A.T. Patera, A successive constraint linear optimization method for lower bounds of parametric coercivity and inf-sup stability costants. *C. R. Acad. Sci. Paris, Sér. I Math.* **345** (2007) 473–478.
[18] D.B.P. Huynh, D. Knezevic, Y. Chen, J. Hesthaven and A.T. Patera, A natural-norm successive constraint method for inf-sup lower bounds. *Comput. Methods Appl. Mech. Eng.* **199** (2010) 1963–1975.
[19] D.B.P. Huynh, N.C. Nguyen, A.T. Patera and G. Rozza, Rapid reliable solution of the parametrized partial differential equations of continuum mechanics and transport. Available on http://augustine.mit.edu.
[20] T. Lassila and G. Rozza, Parametric free-form shape design with PDE models and reduced basis method. *Comput. Methods Appl. Mech. Eng.* **199** (2010) 1583–1592.
[21] T. Lassila and G. Rozza, Model reduction of semiaffinely parametrized partial differential equations by two-level affine approximation. *C. R. Math. Acad. Sci. Paris, Ser. I* **349** (2011) 61–66.
[22] T. Lassila, A. Quarteroni and G. Rozza, A reduced basis model with parametric coupling for fluid-structure interaction problems. *SIAM J. Sci. Comput.* **34** (2012) A1187–A1213.
[23] Y. Maday, N.C. Nguyen, A.T. Patera and G.S.H. Pau, A general multipurpose interpolation procedure: the magic points. *Commun. Pure Appl. Anal.* **8** (2009) 383–404.
[24] A. Manzoni, A. Quarteroni and G. Rozza, Model reduction techniques for fast blood flow simulation in parametrized geometries. *Int. J. Numer. Methods Biomed. Eng.* (2011). In press, DOI: 10.1002/cnm.1465.
[25] A. Manzoni, A. Quarteroni and G. Rozza, Shape optimization of cardiovascular geometries by reduced basis methods and free-form deformation techniques. *Int. J. Numer. Methods Fluids* (2011). In press, DOI: 10.1002/fld.2712.
[26] L.M. Milne-Thomson, *Theoretical aerodynamics*. Dover (1973).
[27] B. Mohammadi and O. Pironneau, *Applied shape optimization for fluids*. Oxford University Press (2001).
[28] N.C. Nguyen, *A posteriori* error estimation and basis adaptivity for reduced-basis approximation of nonaffine-parametrized linear elliptic partial differential equations. *J. Comput. Phys.* **227** (2007) 983–1006.
[29] N.C. Nguyen, G. Rozza and A.T. Patera, Reduced basis approximation and *a posteriori* error estimation for the time-dependent viscous Burgers equation. *Calcolo* **46** (2009) 157–185.
[30] A.T. Patera and G. Rozza, *Reduced Basis Approximation and a Posteriori Error Estimation for Parametrized Partial Differential Equation*. Version 1.0, Copyright MIT (2006), to appear in (tentative rubric) *MIT Pappalardo Graduate Monographs in Mechanical Engineering* (2009).
[31] C. Prud'homme, D.V. Rovas, K. Veroy and A.T. Patera, A mathematical and computational framework for reliable real-time solution of parametrized partial differential equations. *ESAIM: M2AN* **36** (2002) 747–771.
[32] A. Quarteroni, G. Rozza and A. Manzoni, Certified reduced basis approximation for parametrized partial differential equations in industrial applications. *J. Math. Ind.* **1** (2011).
[33] G. Rozza, Reduced basis approximation and error bounds for potential flows in parametrized geometries. *Commun. Comput. Phys.* **9** (2011) 1–48.
[34] G. Rozza, D.B.P. Huynh and A.T. Patera, Reduced basis approximation and *a posteriori* error estimation for affinely parametrized elliptic coercive partial differential equations. *Arch. Comput. Methods Eng.* **15** (2008) 229–275.
[35] G. Rozza, D.B.P. Huynh and A. Manzoni, *Reduced basis approximation and a posteriori error estimation for Stokes flows in parametrized geometries: roles of the inf-sup stability constants.* Technical Report 22.2010, MATHICSE (2010). Online version available at: http://cmcs.epfl.ch/people/manzoni.
[36] G. Rozza, T. Lassila and A. Manzoni, Reduced basis approximation for shape optimization in thermal flows with a parametrized polynomial geometric map, in *Spectral and High Order Methods for Partial Differential Equations. Selected papers from the ICOSAHOM'09 Conference*, Trondheim, Norway, edited by J.S. Hesthaven and E.M. Rønquist. *Lect. Notes Comput. Sci. Eng.* **76** (2011) 307–315.

https://doi.org/10.1051/m2an/2012016 Published online by Cambridge University Press

[37] S. Sen, K. Veroy, P. Huynh, S. Deparis, N.C. Nguyen and A.T. Patera, "Natural norm" *a posteriori* error estimators for reduced basis approximations. *J. Comput. Phys.* **217** (2006) 37–62.

[38] S. Vallaghe, A. Le-Hyaric, M. Fouquemberg and C. Prud'homme, A successive constraint method with minimal offline constraints for lower bounds of parametric coercivity constant. *C. R. Acad. Sci. Paris, Sér. I Math.* (2011). Submitted.

[39] J. Xu and L. Zikatanov, Some observation on Babuška and Brezzi theories. *Numer. Math.* **94** (2003) 195–202.

[40] S. Zhang, Efficient greedy algorithms for successive constraints methods with high-dimensional parameters. *C. R. Acad. Sci. Paris, Sér. I Math.* (2011). Submitted.

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