*Math. Proc. Camb. Phil. Soc.* (2005), **138**, 55 © 2005 *Cambridge Philosophical Society* 
DOI: 10.1017/S0305004104008060 *Printed in the United Kingdom*

# An indecomposable $PD_3$-complex whose fundamental group has infinitely many ends

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(*Received* 27 *August* 2003; *revised* 1 *December* 2003)

*Abstract*

We show that $S_3 *_{Z/2Z} S_3$ satisfies Turaev's criterion to be the fundamental group of an orientable $PD_3$-complex. Since this group has infinitely many ends but is indecomposable as a free product this provides a negative answer to a question of Wall.

## 1. *Introduction*

In his seminal paper on Poincaré duality complexes Wall asked whether a $PD_3$-complex whose fundamental group is a free product is a connected sum, and whether the fundamental group of an indecomposable $PD_3$-complex must have finitely many ends [**8**]. (These are well-known consequences of the Sphere Theorem for 3-manifolds.) The first question was answered affirmatively by Turaev [**7**], while Crisp showed that any counterexample to the second question must have a virtually free fundamental group [**1**].

Turaev showed that a pair $(\pi, w_1 : \pi \to Z/2Z)$ is realizable as the fundamental group and orientation character of a $PD_3$-complex if and only if the augmentation ideal $I(\pi)$ is projective homotopy equivalent to $\mathrm{Coker}(\overline{M}^{tr})$, where $M$ is a presentation matrix for $I(\pi)$ and the bar denotes the involution of $\mathbb{Z}[\pi]$ determined by $w_1$ [**7**]. Modules $I$ and $J$ over a ring $R$ are are projective homotopy equivalent if and only if $I \oplus P \cong J \oplus Q$ for some finitely generated projective $R$-modules $P$ and $Q$, by [**7**, lemma 2·2]. (See also [**4**].) Turaev's result is more precise, in that he also characterizes the image of the fundamental class of the $PD_3$-complex in $H_3(\pi; \mathbb{Z}^{w_1})$.

We shall apply Turaev's criterion to show that $S_3 *_{Z/2Z} S_3$ is the fundamental group of an orientable $PD_3$-complex, thus providing a negative answer to Wall's second question. Thus there are indecomposable infinite groups which are fundamental groups of $PD_3$-complexes but are not 3-manifold groups. It remains an open question whether every $PD_3$-complex with torsion-free fundamental group is homotopy equivalent to a 3-manifold. (It has long been known that a finite group is the fundamental group of a $PD_3$-complex if and only if it has cohomological period dividing 4 [**6**]. Our calculations below were prompted by a similar but simpler verification of Turaev's criterion for $S_3$, which is the smallest finite group with cohomological period 4, and which is not a 3-manifold group [**5**].)

https://doi.org/10.1017/S0305004104008060 Published online by Cambridge University Press

## 2. *Selection of the example*

Let $X$ be a $PD_3$-complex and let $\pi = \pi_1(X)$. In [**3**] it was shown that if $\pi$ has a nontrivial finite normal subgroup then $\pi \cong Z \oplus (Z/2Z)$. As an infinite virtually cyclic group is an extension of $Z$ or $Z/2Z * Z/2Z$ by a finite normal subgroup [**8**], it follows that $Z$ and $Z \oplus (Z/2Z)$ are the only indecomposable, infinite, virtually cyclic groups which are fundamental groups of $PD_3$-complexes.

Since $\pi$ is finitely presentable it is accessible, by [**2**, theorem VI·6·3]. In other words $\pi \cong \pi\mathcal{G}$, where $\mathcal{G}$ is a finite graph of finitely generated groups such that all vertex groups have at most one end and all edge groups are finite. If $X$ is indecomposable then all edge groups are nontrivial, since $\pi$ is then indecomposable with respect to free product [**7**]. Crisp used the presentation of $H^1(\pi; \mathbb{Z}[\pi])$ derived from the Chiswell–Mayer–Vietoris sequence associated to such a graph of groups structure to show that if $X$ is indecomposable and not aspherical then $\pi$ is virtually free [**1**]. We assume this henceforth, and also that $\pi$ is not virtually cyclic. Then the graph underlying $\mathcal{G}$ has at least one edge, and all vertex groups are finite.

The simplest such groups are the generalized free products $A *_C B$, where $A$ and $B$ are finite groups. As we may assume that $C$ is not normal in $A *_C B$, the "natural" candidates for the simplest indecomposable virtually free group on which to test Turaev's criterion are perhaps $S_3 *_{Z/2Z} Z/4Z$ and $S_3 *_{Z/2Z} S_3$. The first of these groups contains the second as its unique subgroup of index 2; moreover it may be excluded by arguments related to those of [**1**] and [**3**]. Thus we shall consider the group $S_3 *_{Z/2Z} S_3$.

## 3. *Verifying Turaev's criterion for* $S_3 *_{Z/2Z} S_3$

The group $\pi = S_3 *_{Z/2Z} S_3$ has a presentation $\langle a, b, c, \mid a^2, aba = b^2, aca = c^2 \rangle$. Let $B$ and $C$ be the cyclic subgroups generated by the images of $b$ and $c$, respectively, and let $\pi'$ be the commutator subgroup of $\pi$. Then $\pi' \cong B * C \cong Z/3Z * Z/3Z$ and $\pi/\pi' = Z/2Z$ is generated by the image of $a$ (so $\pi$ is a semidirect product of $Z/2Z$ with $Z/3Z * Z/3Z$).

If $\pi$ were a nontrivial free product $G * H$ then we could assume that $G$ is cyclic, by the Grushko–Neumann Theorem and the fact that $\pi$ is generated by 3 elements. Then $\pi/\pi' \cong G \oplus H/H'$ and so $H$ would be perfect. But then $\pi'$ would also be perfect, which is clearly not so. Thus $\pi$ is indecomposable.

Let $\Gamma = \mathbb{Z}[\pi]$ and let $\varepsilon: \Gamma \to \mathbb{Z}$ be the augmentation homomorphism. The canonical involution on $\Gamma$ is defined by $\bar{g} = g^{-1}$, for all $g \in \pi$. The augmentation ideal $I(\pi) = \mathrm{Ker}(\varepsilon)$ is generated as a left $\Gamma$-module by the elements $a-1$, $b-1$ and $c-1$. It is also generated by $e = a-1$, $f = (a-1)(b-1)$ and $g = (a-1)(c-1)$, since $b-1 = ab(b-1)e + b^2 f$ and $c-1 = ac(c-1)e + c^2 g$. Moreover these generate complementary direct summands. For if $p = p_1 + p_2 a$ with $p_1, p_2 \in \mathbb{Z}[\pi']$ then $p(a-1) = (p_1 - p_2)(a-1)$. Therefore if $xe + yf + zg = 0$ in $\Gamma$ we may assume that $x, y, z$ are in $\mathbb{Z}[\pi']$. Comparing coefficients of $a$ and 1 on either side of this equation gives $x + y(b^2-1) + z(c^2-1) = 0$ and $x + y(b-1) + z(c-1) = 0$. Hence $yb(b-1) + zc(c-1) = 0$ in $\mathbb{Z}[\pi']$. But $b-1$ and $c-1$ generate complementary summands in $I(B * C)$, so $yb(b-1) = zc(c-1) = 0$. Hence $y(b-1) = z(c-1) = 0$ and so $x = 0$ and $yf = y(b-1)(a + ba - 1) = zg = z(c-1)(a + ac - 1) = 0$. Therefore $xe = yf = zg = 0$ and $I(\pi) \cong \Gamma/\Gamma(a+1) \oplus \Gamma/\Gamma(a+1, \beta) \oplus \Gamma/\Gamma(a+1, \gamma)$, where $\beta = b^2 + b + 1$ and $\gamma = c^2 + c + 1$. (Note that $a\beta = \beta a$ and $a\gamma = \gamma a$.)

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Let $N$ be the obvious $5\times 3$ presentation matrix for $I(\pi)$ corresponding to this decomposition as a direct sum of cyclic modules. (Thus $I(\pi)\cong\Gamma^3/\Gamma^5 N$, where we view the free left $\Gamma$-modules $\Gamma^n$ as modules of *row* vectors.) Let $U=\mathrm{Coker}(\overline{N}^{tr})=\Gamma^5/\Gamma^3\overline{N}^{tr}$. The entries of $N$ are invariant under the canonical involution of $\Gamma$ and $\Gamma^3\overline{N}^{tr}$ is generated by the vectors $R_1=(a+1,0,0,0,0)$, $R_2=(0,a+1,\beta,0,0)$ and $R_3=(0,0,0,a+1,\gamma)$. Define homomorphisms $\lambda_b,\lambda_c\colon\Gamma^5\to\Gamma$ by $\lambda_b(v,w,x,y,z)=w(ab^2+b+1)-x(a+1)$ and $\lambda_c(v,w,x,y,z)=y(ac^2+c+1)-z(a+1)$. Then $\lambda_b(R_1)=\lambda_b(R_2)=0$ and $\lambda_c(R_1)=\lambda_c(R_3)=0$, so $\lambda_b$ and $\lambda_c$ factor through $U$. Moreover $\lambda_b(0,a,a+b^2,0,0)=-1$ and $\lambda_c(0,0,0,a,a+c^2)=-1$, so they together define an epimorphism $\rho\colon U\to\Gamma^2$. Let $[v,w,x,y,z]$ denote the image of $(v,w,x,y,z)$ in $U$, for all $(v,w,x,y,z)\in\Gamma^5$.

If $sf=0$ then $s=p(a+1)+q\beta$ for some $p,q\in\Gamma$, so $(0,0,s(a-1),0,0)=q(a-1)R_2$. Similarly if $tg=0$ then $(0,0,0,0,t(a-1))=q'R_3$ for some $q'\in\Gamma$. Hence there is a well-defined homomorphism $\theta\colon I(\pi)\to U$ given by $\theta(re+sf+tg)=[r,0,s(a-1),0,t(a-1)]$. Clearly $\rho\theta=0$. If $\rho([v,w,x,y,z])=0$ there are $w_0,x_1,x_2\in\mathbb{Z}[\pi']$ with $w=w_0(a+1)$, $x=x_1+x_2a$ and $x_1+x_2=w_0\beta$, and $y_0,z_1,z_2\in\mathbb{Z}[\pi']$ with $y=y_0(a+1)$, $z=z_1+z_2a$ and $z_1+z_2=y_0\gamma$. But then $[v,w,x,y,z]=\theta(ve+x_2f+z_2g)$. Hence $\theta$ maps $I(\pi)$ onto $\mathrm{Ker}(\rho)$. If $\theta(re+sf+tg)=0$ then $re=0$, $s(a-1)=s_0\beta$ for some $s_0\in\Gamma$ such that $s_0(a+1)=0$ and $t(a-1)=t_0\gamma$ for some $t_0\in\Gamma$ such that $t_0(a-1)=0$. But then $s_0=s_1(a-1)$ and $t_0=t_1(a-1)$ for some $s_1,t_1\in\mathbb{Z}[\pi']$, and so $sf=tg=0$ also. Hence $\theta$ is a monomorphism and so $U\cong\Gamma^2\oplus\mathrm{Ker}(\rho)\cong\Gamma^2\oplus I(\pi)$. Since this is projective homotopy equivalent to $I(\pi)$ the group $\pi$ satisfies Turaev's criterion with $w_1$ trivial, and so is the fundamental group of an orientable $PD_3$-complex.

Is this $PD_3$-complex determined up to homotopy equivalence by its fundamental group $\pi=S_3*_{Z/2Z}S_3$? (There are 18 possible classes in $H_3(\pi;\mathbb{Z})\cong(Z/3)^2\oplus(Z/2Z)$ to consider in applying Turaev's classification.) The index 2 subgroup $\pi'$ is the fundamental group of the 3-manifolds $L(3,1)\sharp L(3,1)$ and $L(3,1)\sharp -L(3,1)$. (These may be distinguished by the torsion linking forms on the first homology groups.) Is there an easily described involution on one of these with homotopy quotient a $PD_3$-complex? Note finally that Turaev's criterion may be used (with the standard presentation of $I(\pi)$) to show that $\pi$ is not the group of a nonorientable $PD_3$-complex.

*Acknowledgements.* I would like to thank the referee for his suggestions and in particular for reminding me that $L(3,1)$ admits no orientation reversing self homotopy equivalence.

REFERENCES

[1] J. S. Crisp. The decomposition of 3-dimensional Poincaré complexes. *Comment. Math. Helv.* **75** (2000), 232–246.
[2] W. Dicks and M. J. Dunwoody. *Groups acting on Graphs.* Cambridge Studies in Advanced Mathematics, no. 17, (Cambridge University Press, 1989).
[3] J. A. Hillman. On 3-dimensional Poincaré duality complexes and 2-knot groups. *Math. Proc. Camb. Phil. Soc.* **114** (1993), 215–218.
[4] P. Hilton. *Homotopy Theory and Duality* (Gordon and Breach Science Publishers, 1965).
[5] J. W. Milnor. Groups which act on $S^n$ without fixed points. *Amer. J. Math.* **79** (1957), 623–630.
[6] R. G. Swan. Periodic resolutions for finite groups. *Ann. of Math.* **72** (1960), 267–291.
[7] V. G. Turaev. Three-dimensional Poincaré complexes: homotopy classification and splitting. *Math. USSR-Sb.* **67** (1990), 261–282.
[8] C. T. C. Wall. Poincaré complexes: I. *Ann. of Math.* **86** (1967), 213–245.

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