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Order continuous measures

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1. *Objective.* It is possible to define order convergence on the vector lattice of all continuous functions of compact support on a locally compact topological space. Every measure is a linear form on this vector lattice. The object of this paper is to prove that a measure is such that every set of the first category of Baire has measure zero if and only if the measure is a linear form which is continuous in the order convergence.

2. *Order convergence.* This paragraph is a summary of the results required on order convergence. For further details see (3) or (4).

As the vector lattices to be considered are not usually complete, it is necessary to employ a definition of order convergence which is valid in incomplete vector lattices rather than the more usual one which requires the lattice to be complete. The net (or directed set) $\{x_\alpha | \alpha \in \mathrm{A}\}$ is defined to be *order convergent* to x if and only if there is a net $\{y_\alpha | \alpha \in \mathrm{A}\}$ indexed by the same set A such that

$$\text{(i)} \quad y_\alpha \leqslant y_\beta \quad \text{if} \quad \alpha \succ \beta,$$
$$\text{(ii)} \quad \wedge\{y_\alpha | \alpha \in \mathrm{A}\} = 0,$$
$$\text{(iii)} \quad |x_\alpha - x| \leqslant y_\alpha \quad \text{for all} \quad \alpha \in \mathrm{A}.$$

A net $\{y_\alpha | \alpha \in \mathrm{A}\}$ satisfying (i) and (ii) is said to *decrease to zero.*

With this definition of convergence it is easy to show that if x_α converges to x then so does any subnet of $\{x_\alpha | \alpha \in \mathrm{A}\}$ and that the algebraic operations of the vector lattice are all continuous. It is not, however, true that every net which does not converge to x has a subnet no subnet of which is convergent to x. Consequently this convergence is not one that can be obtained from a topology defined in terms of open sets. Nevertheless, it is possible to define real continuous linear forms on the vector lattice as those linear forms which map all nets convergent to x into nets of real numbers convergent to the image of x.

A linear form on a vector lattice is said to be *order bounded* if, and only if, it maps all sets bounded above and below in the lattice ordering into bounded sets. Such linear forms are just those which are the difference of two order-preserving linear forms. The set of all order bounded linear forms is itself a complete vector lattice, if it is ordered by taking the order-preserving forms as the positive elements.

Lemma 1. *If a linear form on a vector lattice is order continuous it is order bounded.*

Proof. If f is an unbounded linear form it maps some interval $\{x | |x| \leqslant a\}$ of the vector lattice onto an unbounded set. Thus there is a sequence $\{x_n\}$ such that $\langle f, x_n \rangle = n$

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and $|x_n| \leqslant a$. Then $\{x_n/n\}$ is order convergent to zero, since $\{a/n\}$ decreases to zero; yet $\langle f, x_n/n\rangle = 1$.

LEMMA 2. *A bounded linear form f on a vector lattice is order continuous if and only if the net $\{\langle|f|, y_\alpha\rangle|\alpha \in A\}$ converges to zero whenever $\{y_\alpha|\alpha \in A\}$ decreases to zero.*

Proof. If $\{x_\alpha|\alpha \in A\}$ is order convergent to x, there is a net $\{y_\alpha|\alpha \in A\}$ decreasing to zero such that $|x_\alpha - x| \leqslant y_\alpha$. Now

$$|\langle f, x_\alpha - x\rangle| \leqslant \langle|f|, |x_\alpha - x|\rangle \leqslant \langle|f|, y_\alpha\rangle$$

and by hypothesis the last term tends to zero. Hence $\langle f, x_\alpha - x\rangle$ tends to zero and so f is order continuous.

To prove the converse let f be order continuous and $\{y_\alpha|\alpha \in A\}$ decrease to zero. Now

$$\langle|f|, y_\alpha\rangle = \sup\{\langle f, z\rangle| \, |z| \leqslant y_\alpha\}$$

by a well-known formula ((4), p. 64). If $\{\epsilon_\alpha|\alpha \in A\}$ is any net of strictly positive real numbers which tends to zero, there is a z_α for each $\alpha \in A$ such that $|z_\alpha| \leqslant y_\alpha$ and $\langle|f|, y_\alpha\rangle \leqslant \langle f, z_\alpha\rangle + \epsilon_\alpha$. It follows that the right-hand side of the last inequality tends to zero and so the net $\{\langle|f|, y_\alpha\rangle|\alpha \in A\}$ converges to zero.

It follows at once from this lemma that, if f is an order continuous linear form and g is an order bounded linear form with $|g| \leqslant |f|$, then g is also order continuous. In particular, if f is order continuous, so is $|f|$. It is also true that, if F is a set of order continuous linear forms and $\bigvee F$ exists in the lattice of all order bounded linear forms, then $\bigvee F$ is order continuous.

3. *Measures.* Denote by $C(\Omega)$ the vector lattice of all real valued continuous functions of compact support on a locally compact Hausdorff topological space Ω. This vector lattice is only complete if Ω is extremally disconnected. The set of all measures on Ω can be defined as the set of all order bounded linear forms on $C(\Omega)$. It follows that every order continuous linear form on $C(\Omega)$ is a measure.

In the following theorem Ω is restricted to be a σ-compact space; that is Ω must be expressible as the union of a countable set of compact subsets.

THEOREM. *A linear form on $C(\Omega)$, where Ω is a locally compact σ-compact Hausdorff topological space, is order continuous if and only if it is a measure such that every set of the first category is of measure zero.*

Proof. Let N be a relatively compact nowhere dense subset of Ω. Take A to be the set of all positive elements of $C(\Omega)$, whose value on N exceeds 1, ordered by the lattice ordering of $C(\Omega)$ reversed. Let y_α be the function α for each $\alpha \in A$. $\{y_\alpha|\alpha \in A\}$ is then a net. Since Ω is locally compact and Hausdorff and $\overline{N}$ is compact, it follows ((2), p. 146) that, given any $t \in \Omega$ not in $\overline{N}$, there is a continuous function taking values in the closed interval [0, 1] which takes the value one on $\overline{N}$ and zero outside a compact set not including t. Since this function is in A, it follows that, if z is any positive lower bound of $\{y_\alpha|\alpha \in A\}$ in $C(\Omega)$, it must be a function which is zero outside $\overline{N}$. But, since z is continuous, the set of points at which it exceeds zero is open. Hence, since the interior of N is empty, z must be zero and $\{y_\alpha|\alpha \in A\}$ must decrease to zero in $C(\Omega)$. From this it

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follows that $\int_\Omega y_\alpha d\mu$ tends to zero for every positive order continuous measure μ. Since $y_\alpha(t) \geqslant 1$ for all $t \in N$, $\mu(N) \leqslant \int_\Omega y_\alpha d\mu$ for all $\alpha \in \mathrm{A}$ and so N is of measure zero with respect to μ.

If K is any subset of Ω of the first category, it can (since Ω is σ-compact) be expressed as the union of a countable number of nowhere dense relatively compact sets and so K is of measure zero with respect to μ. If ν is any order continuous measure, positive or otherwise, $|\nu|$ is positive and order continuous and so the measure with respect to $|\nu|$ of any set of the first category is zero; hence its measure with respect to ν is also zero.

Now to prove the converse let μ be a positive measure such that every set of the first category has measure zero with respect to μ and let $\{y_\alpha | \alpha \in \mathrm{A}\}$ decrease to zero. Take $\{r_n\}$ to be a sequence of strictly positive numbers tending to zero and let

$$D_n = \{t | y_\alpha(t) \geqslant r_n \text{ for all } \alpha \in \mathrm{A}\}.$$

Now since D_n is the intersection of the closed sets $\{t | y_\alpha(t) \geqslant r_n\}$, it is closed. Suppose that D_n contains an interior point s. Let G be a relatively compact open neighbourhood of s in D_n. Since Ω, being locally compact and Hausdorff, is completely regular, we can find $z \in C(\Omega)$ so that $z(s) = r_n, z(t) = 0$ if t is not in G and $0 \leqslant z(t) \leqslant r_n$ for all $t \in \Omega$. Then z is a lower bound of $\{y_\alpha | \alpha \in \mathrm{A}\}$ in $C(\Omega)$, but is strictly greater than zero, which is impossible. Thus D_n has no interior points and so is nowhere dense. Now $\{t | \inf\{y_\alpha(t) | \alpha \in \mathrm{A}\} \neq 0\} = \bigcup_{n=1}^{\infty} D_n$ which is a set of the first category and so is of measure zero with respect to μ. Hence $y_\alpha(t)$ decreases to zero for almost all t. To show that $\int_\Omega y_\alpha d\mu$ tends to zero we need only consider those functions for which α succeeds β for some fixed $\beta \in \mathrm{A}$. All these functions have their support included in the compact support of y_β. It follows from Dini's theorem, (1), chapter X, § 5, Theorem 1, that $\{y_\alpha | \alpha \in \mathrm{A}\}$ tends to zero almost uniformly and so $\int_\Omega y_\alpha d\mu$ tends to zero.

If ν is any measure, positive or not, which gives measure zero to every set of the first category and has Jordan decomposition $\nu = \nu_+ - \nu_-$, then ν_+ and ν_- both give measure zero to every set of the first category and so are order continuous by what has just been proved. It follows that ν is also order continuous.

REFERENCES

(1) Bourbaki, N. *Topologie générale* (Hermann; Paris, 1949).
(2) Kelley, J. L. *General topology* (Van Nostrand; New Jersey, 1955).
(3) McShane, E. J. *Order-preserving maps and integration processes* (Princeton; 1953).
(4) Nakano, H. *Modulared semi-ordered linear spaces* (Maruzen; Tokyo, 1950).

https://doi.org/10.1017/S030500410003766X Published online by Cambridge University Press