105.04 Polynomials with no real zeros

Polynomials are encountered in many areas of mathematics and have been studied extensively for at least 200 years, with special attention paid to properties of their zeros expressed in terms of their coefficients. Many results about polynomials can be found in the encyclopedic works by Milovanović, Mitrinović and Rassias ([1]) and Rahman and Schmeisser ([2]), and also in the book by Barbeau ([3], which is less formal and contains many interesting exercises and problems.

Of particular interest are localisation results for zeros of complex polynomials. Since these are often expressed only in terms of the moduli or absolute values of the coefficients, it seemed natural to see if geometric properties of the coefficients, namely, their distribution in the complex plane, could perhaps also be used to provide information on the zeros. It is this line of thought that led to the result below, which is a sufficient condition for a polynomial not to have real zeros. Such knowledge could be important when solving linear differential equations, where an absence of real zeros of the characteristic equation indicates oscillatory solutions (stable or unstable).

We first state two definitions and then two simple lemmas from which the sufficient condition follows immediately.

A *convex combination* of $a_1, \ldots, a_m$ is a linear combination of the form $\varepsilon_1 a_1 + \ldots + \varepsilon_m a_m$, where $\varepsilon_j \geqslant 0$ and $\sum_{j=1}^{m} \varepsilon_j = 1$. If $\varepsilon_j > 0, j = 1, \ldots, m$, then the convex combination is *strict*.

The *convex hull* of $a_1, \ldots, a_m$ is the smallest convex set that encloses them. It consists of all their possible convex combinations.

Lemma 1: If the coefficients of the complex polynomial

$$p(z) = a_k z^{n_k} + a_{k-1} z^{n_{k-1}} + \ldots + a_1 z^{n_1} + a_0,$$

with $n_k > n_{k-1} > \ldots > n_1 > n_0 = 0$ and $a_k a_0 \neq 0$, lie on one side of a line or on a line through the origin in the complex plane and at least one coefficient does not lie on the line, then that polynomial does not have positive zeros.

Proof: Assume that p has a zero $\zeta > 0$. Then $\dfrac{p(\zeta)}{\sigma} = 0$, where $\sigma = \sum_{j=0}^{k} \zeta^{n_j}$. Since $\dfrac{\zeta^{n_j}}{\sigma} > 0$ and $\displaystyle\sum_{j=0}^{k} \frac{\zeta^{n_j}}{\sigma} = 1$, this means that the origin is a strict convex combination of the coefficients, and it therefore lies in the relative interior of their convex hull, as illustrated on the left in Figure 1 for a polynomial of degree 6. This is clearly not possible under the given conditions on the coefficients and we conclude that p has no positive zeros.

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Lemma 2: Let $p(z) = a_k z^{n_k} + a_{k-1} z^{n_{k-1}} + \ldots + a_1 z^{n_1} + a_0$ be a complex polynomial with $n_k > n_{k-1} > \ldots > n_1 > n_0 = 0$ and $a_k a_0 \neq 0$. If the coefficients of the even and odd powers of z lie on different sides of a line or on a line through the origin in the complex plane and at least one coefficient does not lie on that line, then that polynomial does not have negative zeros.

Proof: Define $J_1 = \{j : n_j \text{ is even}\}$ and $J_2 = \{j : n_j \text{ is odd}\}$. Assume that p has a zero $-\zeta$, with $\zeta > 0$. Then $p(-\zeta) = 0$ is given by

$$p(-\zeta) = \sum_{j \in J_1} a_j \zeta^{n_j} + \sum_{j \in J_2} (-a_j)\, \zeta^{n_j} = 0. \tag{1}$$

However, because the coefficients of even and odd powers of z lie on, or on different sides of, a line through the origin, the coefficients a_j with $j \in J_1$ and $-a_j$ with $j \in J_2$ all lie on, or on the same side of, that line, as illustrated on the right in Figure 1 for a polynomial of degree 6, where the coefficients corresponding to even and odd powers of z are indicated as filled and unfilled dots, respectively. Since $\zeta > 0$, and with the given conditions on the coefficients, Lemma 1 then implies that (1) cannot hold from which it follows that p has no negative zeros.

Theorem 1: Let $p(z) = a_k z^{n_k} + a_{k-1} z^{n_{k-1}} + \ldots + a_1 z^{n_1} + a_0$ be a complex polynomial with $n_k > n_{k-1} > \ldots > n_1 > n_0 = 0$ and $a_k a_0 \neq 0$, such that the origin and its coefficients are not all collinear. If the coefficients lie on the same side of a line or on a line through the origin in the complex plane and if the coefficients of the even and odd powers of z lie on different sides, respectively, of a line or on a line through the origin, then p does not have real zeros.

Proof: Since the origin and the coefficients are not all collinear, we have from Lemma 1 that p does not have positive zeros, and Lemma 2 implies that p also does not have negative zeros. Consequently, p does not have real zeros. The convex hulls corresponding to Lemma 1 and Lemma 2 are illustrated in Figure 2 for a polynomial of degree 6 with the same conventions as in Figure 1.

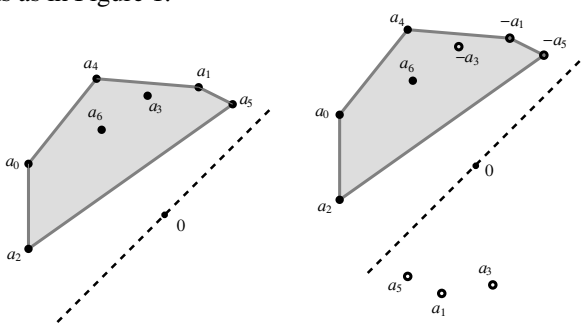


FIGURE 1: Convex hull for Lemma 1 (left) and Lemma 2 (right)

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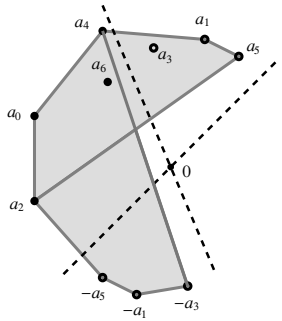


FIGURE 2: Convex hulls for Theorem 1

Examples and Remarks

(1) Applying Theorem 1 to a linear polynomial $a_1z + a_0$ is not very interesting as its zero $-\dfrac{a_0}{a_1}$ is imaginary as long as a_0 and a_1 are not real multiples of each other, i.e., as long as they are not collinear with the origin. In such a case they always lie on the same side of some line through the origin with another line separating them. Theorem 1 then confirms that the zero cannot be real.

(2) A simple quadratic equation already complicates matters. The zeros of $z^2 + a_1z + a_0$ are easily computed as $\dfrac{1}{2}\left(-a_1 \pm \sqrt{a_1^2 - 4a_0}\right)$, but the conditions on a_1 and a_0 that make this expression real or imaginary are not obvious, as is shown by the quadratic $z^2 + (1 + i)z + \dfrac{1}{4} + \dfrac{i}{2}$, which has one real and one imaginary zero, namely, $-\dfrac{1}{2}$ and $-\dfrac{1 + i}{2}$.

(3) The quadratic $z^2 - z + i$ satisfies the conditions of Theorem 1, since $1, -1$ and i all lie on or above the real axis without all being collinear with the origin, and i and 1 lie on and to the right of the imaginary axis, respectively, while -1 lies to the left of it. Consequently, this quadratic only has imaginary zeros, namely,

$$\frac{1}{2}\left(\left(1 \pm \sqrt{\frac{1 + \sqrt{17}}{2}}\right) - \left(\pm\sqrt{\frac{-1 + \sqrt{17}}{2}}\right)i\right).$$

(4) The conditions of Theorem 1 for the absence of real zeros are sufficient, but they are not necessary, as is demonstrated by the quadratic $2z^2 + (i + 1)z + 2i$ with zeros $\frac{1}{4}\left(\mp\sqrt{7} - 1\right) + \frac{1}{4}\left(\pm\sqrt{7} - 1\right)i$. Here, the coefficients $2, i + 1$ and $2i$ all lie on one side of a line through the origin, but 2 and $2i$ cannot be separated from $i + 1$ by a line through the origin, and yet there are no real zeros.

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Conclusion

We have derived a sufficient condition for a complex polynomial not to have any real zeros by using geometric properties of its coefficients, and illustrated the result with a few examples.

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105.05 Simpler derivation of Bring radical

The infinite series solution of the single-parameter Bring quintic equation of the form $x^5 + x = a$ (with a real) is known as the Bring radical. The solution is derived by making use of the Lagrange inversion theorem [1, 2]. In this Note, we obtain the same result by an alternative (simpler) method, for which prior knowledge of the Lagrange inversion theorem is not required. Consider the following single-parameter Bring quintic equation,

$$x^5 + x = a, \tag{1}$$

where a is real. Since the polynomials of odd degree in x (with real coefficients) cross the x-axis at least once, we note that (1) has at least one real solution, which we plan to determine using a new method. Notice that (1) can be rearranged as $\frac{x}{a} = \frac{1}{1 + x^4}$, implying $0 < \frac{x}{a} < 1$, and further $0 < |x| < 1$ for $|a| < 1$. Let $|a| < 1$; so, we can write (1) as $x = a(1 + x^4)^{-1}$ and expand it using the binomial theorem to obtain

$$x = a\left(1 - x^4 + x^8 - x^{12} + x^{16} - x^{20} + x^{24} - \ldots\right). \tag{2}$$

Raising (1) to the power four yields

$$a^4 = x^4 + 4x^8 + 6x^{12} + 4x^{16} + x^{20}.$$

Eliminating x^4 from (2) using the above expression results in

$$x = a\left(1 - a^4 + 5x^8 + 5x^{12} + 5x^{16} + x^{24} - x^{28} + x^{32} - \ldots\right). \tag{3}$$

Raising (1) to the power eight yields

$$a^8 = x^8 + 8x^{12} + 28x^{16} + 56x^{20} + 70x^{24} + 56x^{28} + 28x^{32} + 8x^{36} + x^{40}.$$

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