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On Minimally 3-Connected Binary Matroids

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We generalize a minimal 3-connectivity result of Halin from graphs to binary matroids. As applications of this theorem to minimally 3-connected matroids, we obtain new results and short inductive proofs of results of Oxley and Wu. We also give new short inductive proofs of results of Dirac and Halin on minimally k-connected graphs for $k \in \{2, 3\}$.

1. Introduction

There are two themes in this paper. The first theme is to generalize results about vertices in minimally 2- and 3-connected graphs to matroids by using the notion of a non-separating cocircuit of a matroid. The second theme is to illustrate the use of induction in proving such results about graphs and matroids. Vertices of degree k appear extensively in a minimally k-connected graph, as evidenced by the following two fundamental results of Dirac [1] and Halin [2, Satz 4], respectively.

Theorem 1.1. *If G is a minimally 2-connected graph with n vertices, then G has at least $\frac{n+4}{3}$ vertices of degree two.* □

Theorem 1.2. *If G is a minimally 3-connected graph with n vertices, then G has at least $\frac{2n+6}{5}$ vertices of degree three.* □

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Short inductive proofs of the previous two theorems are given in Section 3 of this paper. No such proofs have previously appeared in the literature. These proofs rely on the following result. The case $k = 2$ of this theorem and the next can be deduced from a result of Tutte (see Lemmas 2.1 and 2.2). The case $k = 3$ of this theorem is due to Halin [2].

Theorem 1.3. *Let e be an edge of a minimally k-connected graph G where $k \in \{2, 3\}$. If e joins two vertices of degree exceeding k, then G/e is minimally k-connected.* □

Theorem 1.3 is the most useful result in proving theorems about minimally 2- and 3-connected graphs. Hence it is of interest to obtain a generalization of this result for studying minimally 2- and 3-connected matroids. The following result provides such a generalization. The case $k = 3$ of this theorem is due to Oxley [5].

Theorem 1.4. *Let e be an element of a minimally k-connected matroid M where $k \in \{2, 3\}$. If e is not in any k-element cocircuit, then M/e is minimally k-connected.* □

Theorem 1.4 is not a direct generalization of Theorem 1.3 in the sense that a 3-element cocircuit in a graphic matroid need not correspond to a set of edges meeting a vertex of degree three in the associated graph. For this reason, we next discuss the matroid notion that generalizes the concept of a vertex in a graph. This is important when considering minimally k-connected matroids because vertices of degree k are crucial in the study of minimally k-connected graphs. A cocircuit C^* of a connected matroid M is *non-separating* if and only if the deletion $M\backslash C^*$ is connected. Non-separating cocircuits of a matroid with two and three elements, respectively, are called *vertex pairs* and *vertex triads*. Let M be the cycle matroid of a graph G. Suppose that M is 2-connected. If T is a vertex pair of M, then T is the set of edges meeting a vertex of degree two in G. The converse of this statement does not hold. Suppose that M is 3-connected. The set T is a vertex triad of M if and only if T is the set of edges meeting a vertex of degree three in G. Hence vertex pairs and vertex triads of matroids model vertices of degree two and three, respectively, in graphs. Many graph results can be generalized to binary matroids by using vertex triads [10].

The first main result of the paper is given next. It simultaneously generalizes Theorems 1.3 and 1.4 for binary matroids. The proofs of the main results of the paper are given in Section 2.

Theorem 1.5. *Let e be an element of a minimally 3-connected binary matroid M. If e is not in any vertex triad, then M/e is minimally 3-connected.*

We use Theorem 1.5 to prove our next main result. This theorem generalizes a result of Oxley [6] about triads in minimally 3-connected matroids.

Theorem 1.6. *If M is a minimally 3-connected binary matroid with rank exceeding five, then the number of elements in some vertex triad of M is at least $\frac{3}{2}r(M^*) + 3$.*

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The following result of Oxley [6, Lemma 4.4] is an immediate graphic consequence of Theorem 1.6.

Corollary 1.7. *If G is a minimally 3-connected simple graph with at least four vertices, then the number of edges of G that meet a vertex of degree three is at least $\frac{3}{2}(|E(G)|-|V(G)|)+\frac{9}{2}$.* □

The 2-connected analogue of Theorem 1.5 is given next.

Theorem 1.8. *Let e be an element of a minimally 2-connected matroid M. If e is not in any vertex pair, then M/e is minimally 2-connected.*

It would be natural to provide a 2-connected analogue of Theorem 1.6 using Theorem 1.8. Unfortunately, a minimally 2-connected matroid may not contain any vertex pairs. Examples of such matroids are the cycle matroids of cycles as well as the cycle matroids of triangle-free graphs which consist of a long cycle with a chord.

In Section 3 of the paper are short proofs of the following two results. These results are further applications of Theorems 1.4 and 1.5. The first of these results is due to Wu [10] and the second is due to Oxley [4].

Theorem 1.9. *Let M be a minimally 3-connected binary matroid such that $|E(M)| \geqslant 4$. Then M has at least $\frac{1}{2}r(M^*)+1$ vertex triads.* □

Theorem 1.10. *Let M be a minimally 3-connected matroid such that $|E(M)| \geqslant 4$. Then M has at least $\frac{1}{2}r(M^*)+1$ triads.* □

2. Proofs of the main results

We first give some preliminary results that are used in the proofs of our main theorems. The first result is well known.

Lemma 2.1. *Let M and N be matroids such that $M\backslash e = N$. Suppose that N is n-connected but M is not. Then either e is a coloop of M, or M has a circuit that contains e and has fewer than n elements.* □

The next two connectivity results are due to Tutte [9]. An element e of a 3-connected matroid M is an *essential element* if and only if neither $M\backslash e$ nor M/e are 3-connected.

Lemma 2.2. *If e is an element of a connected matroid M, then $M\backslash e$ or M/e is connected.* □

Lemma 2.3. *Let $\{e,f,g\}$ be a triangle in a 3-connected matroid M with $|E(M)| \geqslant 4$. If neither $M\backslash e$ nor $M\backslash f$ is 3-connected, then e is in a triad with exactly one of f and g. If x is an essential element of M, then x is in a triangle or in a triad of M.* □

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We use the next two structure results of Oxley [5] on minimally 3-connected matroids.

Lemma 2.4. *Let x and y be elements of a k-connected matroid M with $|E(M)| \geqslant 2(k-1)$ for some integer k exceeding one. If $M\backslash x/y$ is k-connected but $M\backslash x$ is not, then $\{x, y\}$ is contained in a k-element cocircuit of M.* □

Lemma 2.5. *Let M be a minimally 3-connected matroid with at least four elements.*

(i) *$|E(M)| \leqslant 3r(M) - 6$ if $r(M) \geqslant 7$, and $|E(M)| \leqslant 2r(M)$ if $r(M) \leqslant 6$.*
(ii) *If M is binary and rank $r \in \{3, 4, 5, 6\}$, then M has exactly $2r$ elements if and only if $M \cong M(W_3), M(W_4), M(W_5), M(W_6)$, or $M(K_{3,4})$.*
(iii) *$|E(M)| \geqslant r(M) + 2$ with equality being attained if and only if $M \cong U_{r,r+2}$.* □

The next three results are due to Wu [10].

Lemma 2.6. *Let $T = \{x, y, z\}$ be a triad of a 3-connected matroid. If $M\backslash x/y$ is 3-connected, then T is a vertex triad.* □

Lemma 2.7. *Let T be a triad of a 3-connected matroid M. If T contains an essential element of M, then T is a vertex triad.* □

Lemma 2.8. *Let T be a triad of a 3-connected matroid M with at least four elements. Suppose that e is an element of M not in T. If $M\backslash T/e$ is connected, but $M\backslash T$ is not, then $M^*|(T \cup e) \cong U_{2,4}$.* □

The following lemma indicates that properties of vertex triads are particularly well suited for study by induction in the class of binary matroids.

Lemma 2.9. *Let M and N be 3-connected binary matroids such that $M/e = N$. If T is a vertex triad of N, then T is a vertex triad of M.*

Proof. The set T is a triad of M. Suppose that $M\backslash T$ is not connected. The matroid $M/e\backslash T = N\backslash T$ is connected since T is a vertex triad of N. Thus Lemma 2.8 implies that $M^*|(T \cup e) \cong U_{2,4}$. This is a contraction because M is binary. Thus $M\backslash T$ is connected. Hence T is a vertex triad of M. □

Proof of Theorem 1.8. The matroid M/e is connected by Lemma 2.2. Suppose that M/e is not minimally connected. Then there exists an element f of M/e such that $M/e\backslash f$ is connected. Since $M/e\backslash f$ is connected, but $M\backslash f$ is not, Lemma 2.4 implies that $\{e, f\}$ is a cocircuit of M. Thus $M\backslash e, f$ is equal to the connected matroid $M/e\backslash f$. Thus e is in a vertex pair $\{e, f\}$ of M; a contradiction. Thus M/e is minimally connected. □

We now prove the main results of the paper.

Proof of Theorem 1.5. If e is essential, then Lemma 2.3 and the minimal 3-connectivity of M imply that e is in a triad T of M. By Lemma 2.7, T is a vertex triad; a contradiction.

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Figure 1 Matrix A

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & a_1 & a_2 & a_3 & a_4 \end{bmatrix}$$

Thus M/e is 3-connected. Suppose that M/e is not minimally 3-connected. Then there exists an element f of M/e such that $M/e\backslash f$ is 3-connected. By Lemma 2.4, e and f are in a triad. By Lemma 2.6, this triad is a vertex triad; a contradiction. Thus M/e is minimally 3-connected. □

Proof of Theorem 1.6. We prove the result by induction on $r(M)$. Suppose that $r(M) = 6$ and the result does not hold. Then Lemma 2.5 parts (i) and (iii) and the fact that M is binary imply that $9 \leqslant |E(M)| \leqslant 12$. If M has nine elements, then M^* is a 3-connected binary matroid with nine elements and rank three. This is a contradiction because the seven element Fano-matroid is the largest 3-connected binary matroid with rank three. If M has twelve elements, then Lemma 2.5(ii) implies that M is isomorphic to $M(W_6)$ or $M(K_{3,4})$ and the theorem holds; a contradiction. Hence M has ten or eleven elements.

Suppose M has ten elements. Then M has fewer than $\frac{3}{2}r^*(M) + 3 = \frac{3}{2}(4) + 3 = 9$ elements that are in some vertex triad. Thus not every element of M is in a vertex triad. It follows from Theorem 1.5 that there exists an element e of M such that M/e is minimally 3-connected. Any element of M/e that is in a vertex triad of M/e is in a vertex triad of M by Lemma 2.9. Hence the nine-element matroid M/e has at most eight elements in some vertex triad. Thus Theorem 1.5 implies that there is an element f of M/e such that $M/e, f$ is a minimally 3-connected matroid with eight elements and rank four. Lemma 2.5(ii) implies that $M/e, f \cong M(W_4)$. There is a binary matrix A representing M/e as given in Figure 1. The columns of this matrix are labelled by $\{1, 2, 3, 4, f, 5, 6, 7, 8\}$, respectively, where column $f = (0, 0, 0, 0, 1)^T$ of A corresponds to element f of M/e. There are three non-isomorphic 3-connected binary matroids with a contraction isomorphic to $M(W_4)$. These may be obtained by assuming that $a = (a_1, a_2, a_3, a_4)$ is $v_{13} = (1, 0, 1, 0)$, $v_{234} = (0, 1, 1, 1)$, or $v_{1234} = (1, 1, 1, 1)$. The eight elements of the set $\{1, 2, \ldots, 8\}$ are all in vertex triads in $M/e, f \cong M(W_4)$. Thus these eight elements are in vertex triads of M/e by Lemma 2.9. Since M/e has at most eight such elements, f is not in a vertex triad of M/e. Suppose that $a = v_{13}$. Then $T = \{1, 2, f\}$ is a triad of M/e. The sets $\{3, 4, 5, 6, 7\}$ and $\{3, 4, 8\}$ are circuits of $M/e\backslash T$ which meet and whose union is the ground set of the matroid. The relation on the ground set of a matroid, that two elements are related if and only if they lie in a common circuit, is transitive. Hence every pair of elements of $M/e\backslash T$ is in a circuit. Thus $M/e\backslash T$ is connected. Hence f is in the vertex triad T of M/e; a contradiction. Likewise, if $a = v_{234}$, then let T be the triad $\{2, 8, f\}$ of M/e. It follows that $M/e\backslash T$ is connected, from considering the circuits $\{1, 3, 6, 7\}$ and $\{1, 4, 5\}$; a contradiction. Hence $a = v_{1234}$. Then we obtain a contradiction by letting T be the triad $\{1, 3, f\}$ of M/e and considering the circuits $\{2, 4, 5, 6\}$ and $\{2, 4, 7, 8\}$ of $M/e\backslash T$. Thus M has eleven elements.

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Figure 2 Matrix B

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & a_1 & a_2 & a_3 & a_4 & a_5 \end{bmatrix}$$

Figure 3 Matrix Z^*

$$H = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

The matroid M has fewer than $\frac{3}{2}r^*(M) + 3 = \frac{3}{2}(5) + 3 = \frac{21}{2}$ elements that are in some vertex triad. Hence not every element of M is in a vertex triad. It follows from Theorem 1.5 that there exists an element e of M such that M/e is minimally 3-connected. By Lemma 2.5(ii), $M/e \cong M(W_5)$. Since each of the ten elements of $M/e \cong M(W_5)$ is in a vertex triad of M, e is the unique element of M that is not in a vertex triad of M. There is a binary matrix B representing M as given in Figure 2. The columns of this matrix are labelled by $\{1, 2, 3, 4, 5, e, 6, 7, 8, 9, 10\}$, respectively, where column $e = (0, 0, 0, 0, 0, 1)^T$ of B corresponds to the element e of M.

There are five non-isomorphic 3-connected binary matroids that have a contraction that is isomorphic to $M(W_5)$. These may be obtained by assuming that $a = (a_1, a_2, a_3, a_4, a_5)$ is $v_{13} = (1, 0, 1, 0, 0)$, $v_{123} = (1, 1, 1, 0, 0)$, $v_{124} = (1, 1, 0, 1, 0)$, $v_{1234} = (1, 1, 1, 1, 0)$, or $v_{12345} = (1, 1, 1, 1, 1)$. Suppose that $a = v_{13}$. Then $T = \{1, 2, e\}$ is a triad of M. It follows from considering the circuits $\{3, 5, 6, 7, 8\}$, $\{4, 5, 10\}$, and $\{3, 4, 9\}$ that $M\backslash T$ is connected. Thus e is in the vertex triad T of M; a contradiction. We obtain a contradiction in the remainder of the proof in a similar manner. Suppose that $a = v_{123}$. Then $T = \{1, 8, e\}$ is a triad of M containing e. It follows from considering the circuits $\{2, 5, 6, 7\}$, $\{3, 5, 9, 10\}$, and $\{3, 4, 9\}$ that T is a vertex triad containing e; a contradiction. Suppose that $a = v_{124}$. Then $T = \{1, 9, e\}$ is a triad of M. It follows from considering the circuits $\{2, 5, 6, 7\}$, $\{2, 3, 8\}$, and $\{4, 5, 10\}$ that T is a vertex triad; a contradiction. Suppose that $a = v_{1234}$. Then $T = \{1, 3, e\}$ is a triad of M. It follows from considering the circuits $\{2, 4, 8, 9\}$, $\{2, 5, 6, 7\}$, and $\{4, 5, 10\}$ that T is a vertex triad of M; a contradiction. Hence $a = v_{12345}$. We next show that $M\backslash e$ is 3-connected. Let Z be the binary matrix representing $M\backslash e$ obtained from B by deleting column e and replacing row i by row i + row 6 for $i \in \{1, 5\}$. Then the dual matrix Z^* of Z represents $(M\backslash e)^*$ and is given in Figure 3. The columns of this matrix are $\{7, 8, 9, 10, 1, 2, 3, 4, 5, 6\}$, respectively. The vector matroid of Z^* is isomorphic to $M(K_5)$. Thus $(M\backslash e)^*$ is 3-connected. Hence $M\backslash e$ is 3-connected. This contradicts that M is minimally 3-connected. Hence the result holds if $r(M) = 6$.

Suppose that the rank of M exceeds six and the theorem holds for minimally 3-connected binary matroids with rank r, when $6 \leqslant r < r(M)$. Suppose that each element of

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M in a vertex triad. Then Lemma 2.5(i) implies that $3r(M)-6 \geqslant |E(M)| = r(M)+r(M^*)$. Thus $r(M) \geqslant \frac{1}{2}r(M^*) + \frac{6}{2}$. Hence $|E(M)| = r(M) + r(M^*) \geqslant \frac{3}{2}r(M^*) + 3$ and the result holds. Hence we may assume that there exists an element e of M that is not in any vertex triad. By Theorem 1.5, M/e is minimally 3-connected. By the induction hypothesis, M/e has at least $\frac{3}{2}r((M/e)^*) + 3 = \frac{3}{2}r(M^*) + 3$ elements in vertex triads. It follows from Lemma 2.9 that M has at least $\frac{3}{2}r(M^*) + 3$ elements in vertex triads. This completes the proof of Theorem 1.6. □

3. Some applications

In this section, we give short inductive proofs of Theorems 1.1, 1.2, 1.9, and 1.10. The proof of Theorem 1.9 is an application of Theorem 1.5. In order to prove Theorems 1.1 and 1.2, we first prove the following two slightly stronger results of Oxley [4]. The number of vertices of degree k in a graph G will be denoted by $v_k(G)$. The number of edges and vertices in the graph is denoted by m and n, respectively.

Theorem 3.1. *Let G be a minimally 2-connected graph with $m \geqslant 4$. Then $v_2(G) \geqslant m-n+2$.*

Proof. If $n = 4$, then G is a cycle and the result holds. Suppose that n exceeds four and that G is a minimally 2-connected graph that is a counterexample to the theorem statement with a minimum number of vertices. Suppose there is an edge e of G that joins two vertices of degree exceeding two. Then Theorem 1.3 implies that G/e is minimally 2-connected. By assumption, $v_2(G/e) \geqslant (m-1)-(n-1)+2 = m-n+2$. Since $v_2(G) \geqslant v_2(G/e)$, the result holds for G; a contradiction. Hence each edge of G is incident to a vertex of degree 2. Then $m-n+2 > v_2(G) \geqslant \frac{m}{2}$. Thus $m > 2n-4$. Hence $v_2(G) \geqslant \frac{m}{2} > \frac{2n-4}{2} = n-2$. Then $4n-8 < 2m = \sum_{v\in V(G)} \deg(v) \leqslant 2(n-1)+n-1 = 3n-3$. Hence $4n-8 < 3n-3$ implies that $n < 5$; a contradiction. This completes the proof of Theorem 3.1. □

Theorem 3.2. *Let G be a minimally 3-connected graph. Then $v_3(G) \geqslant \frac{1}{2}(m-n+3)$.*

Proof. Suppose that $n \in \{4, 5\}$. Then G is isomorphic to the wheel graph with three or four spokes, W_3 or W_4, and the theorem holds.

Suppose that n exceeds five and G is a minimally 3-connected graph that is a counterexample to the theorem statement with a minimum number of vertices. Suppose that there exists an edge e of G that joins two vertices of degree exceeding three. Then Theorem 1.3 implies that G/e is minimally 3-connected. By assumption, $v_3(G/e) \geqslant \frac{1}{2}((m-1)-(n-1)+3) = \frac{1}{2}(m-n+3)$. Since $v_3(G) \geqslant v_3(G/e)$, the result holds; a contradiction. Hence each edge of G is incident to a vertex of degree three. Then $\frac{1}{2}(m-n+3) > v_3(G) \geqslant \frac{m}{3}$. Hence $m > 3n-9$. Thus $v_3(G) \geqslant \frac{m}{3} > n-3$. Hence G has at most two vertices with degree greater than three. If there are two such vertices, then these two vertices are not adjacent, by previous remarks. Moreover, if there are two such vertices and these vertices both have degree $n-2$, then the graph obtained by deleting these two vertices consists of a disconnected graph which is a matching. Thus G is not 3-connected;

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a contradiction. Hence, if G has two vertices of degree greater than three, then these two vertices do not both have degree $n-2$. Thus $2m = \sum_{v \in V(G)} \deg(v) < 3(n-2) + 2(n-2)$. Hence $2m \leqslant 5n-11$. Thus $m \leqslant \frac{5n-11}{2}$. Hence $3n-9 < m \leqslant \frac{5n-11}{2}$. Thus $n < 7$. It follows that $n = 6$. Then $m > 3n-9 = 9$ so that $m \geqslant 10$. We obtain a contradiction as $10 \leqslant m \leqslant \frac{5n-11}{2} = \frac{19}{2}$. This completes the proof of Theorem 3.2. □

We next show that Theorems 1.1 and 1.2 are easy consequences of Theorems 3.1 and 3.2.

Proof of Theorem 1.1. It follows from the Handshaking Lemma that $2m \geqslant 2v_2(G) + 3(n - v_2(G))$. Thus $m \geqslant \frac{1}{2}(3n - v_2(G))$. It follows from substituting this last inequality into the result of Theorem 3.1 that $v_2(G) \geqslant m - n + 2 \geqslant \frac{1}{2}(3n - v_2(G)) - n + 2$. It follows from solving for $v_2(G)$ in the inequality $v_2(G) \geqslant \frac{1}{2}(3n - v_2(G)) - n + 2$ that $v_2(G) \geqslant \frac{n+4}{3}$. □

Proof of Theorem 1.2. It follows from the Handshaking Lemma that $2m \geqslant 3v_3(G) + 4(n - v_3(G))$. Thus $m \geqslant \frac{1}{2}(4n - v_3(G))$. It follows from substituting this last inequality into the result of Theorem 3.2 that $v_3(G) \geqslant \frac{1}{2}(m - n + 3) \geqslant \frac{1}{2}(\frac{1}{2}(4n - v_3(G)) - n + 3)$. Thus $v_3(G) \geqslant \frac{2n+6}{5}$. □

The usefulness of Theorem 1.5 as an inductive tool for proving results on minimally 3-connected matroids is illustrated by the following short proof. The number of vertex triads in a matroid M is denoted by v_3.

Proof of Theorem 1.9. Evidently, M has rank at least three. If M has rank three, then Lemma 2.5 can be used to show that M is isomorphic to $M(W_3)$. The result holds for this matroid.

Assume that M is a counterexample to the theorem of minimum rank. Then the rank of M exceeds three. Suppose that there is an element e of M not in a vertex triad. By Theorem 1.5, M/e is minimally 3-connected. By assumption M, M/e has at least $\frac{1}{2}r^*(M/e)+1$ vertex triads. Each vertex triad of M/e is a vertex triad of M by Lemma 2.9. Since $r^*(M/e) = r^*(M)$, the result holds for M; a contradiction. Hence each element of M is in a vertex triad. Thus a counting argument implies that $|E(M)| \leqslant 3v_3$. Thus $\frac{1}{3}|E(M)| \leqslant v_3 < \frac{1}{2}r^*(M)+1$. From substituting $|E(M)|-r(M)$ for $r^*(M)$ into the inequality $\frac{1}{3}|E(M)| < \frac{1}{2}r^*(M)+1$, we obtain that $|E(M)| \geqslant 3r(M)-5$. It follows from Lemma 2.5(i) that $r(M) \leqslant 6$ and $|E(M)| \leqslant 2r(M)$. But $v_3 \geqslant \frac{1}{3}|E(M)| \geqslant \frac{1}{3}(3r(M)-5) = r(M) - \frac{5}{3}$ implies that $v_3 \geqslant r(M)-1$. Thus $\frac{1}{2}r^*(M)+1 > v_3 \geqslant r(M)-1$. It follows from substituting $|E(M)|-r(M)$ for $r^*(M)$ into the inequality $\frac{1}{2}r^*(M)+1 > r(M)-1$ that $|E(M)| > 3r(M)-4$. Hence $2r(M) \geqslant |E(M)| > 3r(M)-4$. This implies that $r(M) < 4$; a contradiction. This completes the proof of Theorem 1.9. □

Proof of Theorem 1.10. Evidently, M has rank at least three. If M has rank three, then Lemma 2.5 and a result of Oxley [7, 11.2.19] can be used to show that M is isomorphic to $U_{3,5}$, $M(W_3)$, or the rank-3 whirl. The result holds for these matroids. The remainder of the proof is the same as that of the proof of Theorem 1.9 except that Theorem 1.4 should be invoked in this proof instead of Theorem 1.5. □

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