Primes, polygons and the roots of unity

C. H. REID

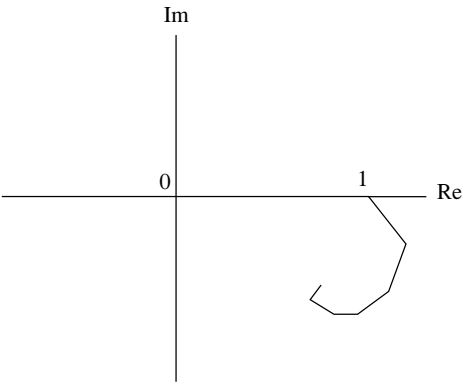
In this Article the author hopes to present an unusual view of the Riemann zeta landscape when $\mathrm{Re}\,(s) > 0$, to show how the prime numbers are discoverable by the altered configuration, and to show a correlation amongst the primes, the regular polygons and a set of complex numbers with real part $\frac{1}{2}$.

This is done by adding a term to the existing formula such that

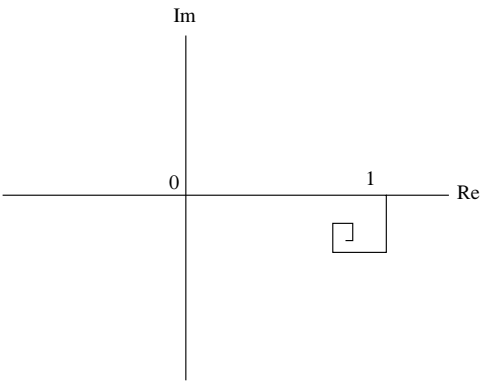
$$f(s) = \sum_{n=1}^{\infty} \left(\frac{1}{n^s}\right)^{\frac{n-1}{\ln(n)}}.$$

To avoid dividing zero by zero we might agree that the exponent in the first term (when $n = 1$) is given by $\lim_{n \to \infty} \dfrac{n-1}{\ln(n)} = 1$.

The term has the effect of equalising the angles of the spirals of convergence, so for example things that might look like this:



come to look like this:

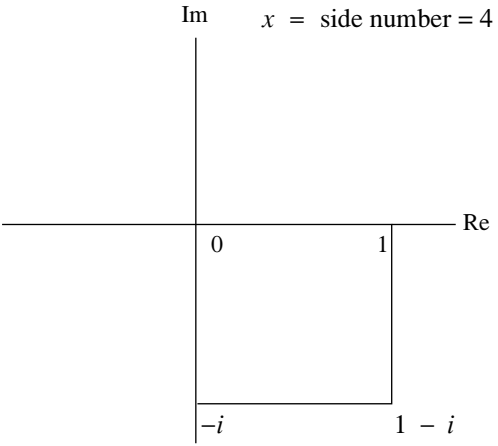


https://doi.org/10.1017/mag.2025.10131 Published online by Cambridge University Press

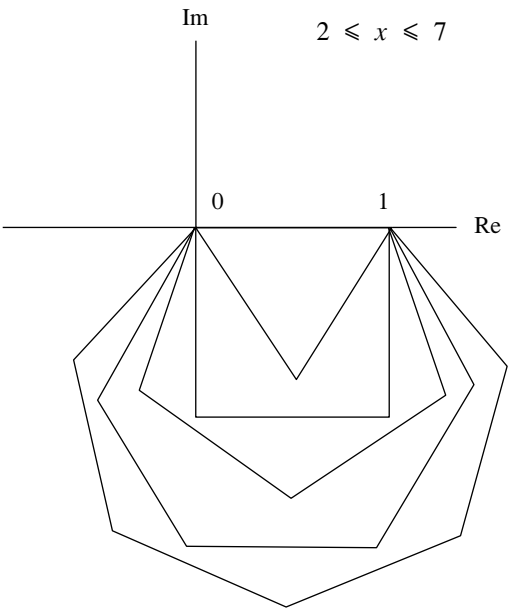
The series converges for $\mathrm{Re}\,(s) > 0$, becomes periodic when s is an xth root of unity, and diverges when $\mathrm{Re}\,(s) < 0$.

$$\text{Let } s = 0 + \frac{2\pi i}{x} \text{ and } g(x) = \sum_{n=1}^{\infty}\left(\frac{1}{n^{(0+2\pi i)/x}}\right)^{(n-1)/\ln(n)} = \sum_{n=1}^{\infty} e^{\left((n-1)\frac{-2\pi i}{x}\right)}.$$

$$g(4) = \sum_{n=1}^{\infty} e^{(n-1)\frac{-\pi i}{2}} = 1 - i - 1 + i + 1 - i - 1 + i + \ldots$$



Each x, with the exceptions of 1 and 2, traces out a regular polygon.



https://doi.org/10.1017/mag.2025.10131 Published online by Cambridge University Press

$g(1)$	1	$+1$	$+1$	$+1$	$+1$	$+1$	$+1$
$g(2)$	1	-1	$+1$	-1	$+1$	-1	$+1$
$g(3)$	1	$-\frac{1}{2}-\frac{\sqrt{3}}{2}i$	$-\frac{1}{2}+\frac{\sqrt{3}}{2}i$	1	$-\frac{1}{2}-\frac{\sqrt{3}}{2}i$	$-\frac{1}{2}+\frac{\sqrt{3}}{2}i$	1
$g(4)$	1	$-i$	-1	i	1	$-i\ldots$	-1
$g(5)$	1	$+0.31-0.95i$	$-0.81-0.59i$	$-0.81+0.59i$	$+0.31+0.95i$	1	$+0.31-0.95i$
$g(6)$	1	$+\frac{1}{2}-\frac{\sqrt{3}}{2}i$	$-\frac{1}{2}-\frac{\sqrt{3}}{2}i$	-1	$-\frac{1}{2}+\frac{\sqrt{3}}{2}i$	$+\frac{1}{2}+\frac{\sqrt{3}}{2}i$	1
$g(7)$	1	$+0.62-0.78i$	$-0.223-0.97i$	$-0.9-0.434i$	$-0.9+0.43i$	$-0.223-0.96i$	$+0.62+0.78i$

The partial sums of $g(x)$, $ps(x)$, reveal the 0's of the function. $ps(1)$ lists the integers and $ps(2)$ recreates Grandi's series.

Partial sums

$ps(1)$	1	2	3	4	5	6	7
$ps(2)$	1	**0**	1	**0**	1	**0**	1
$ps(3)$	1	$\frac{1}{2}-\frac{\sqrt{3}}{2}i$	**0**	1	$\frac{1}{2}-\frac{\sqrt{3}}{2}i$	**0**	1
$ps(4)$	1	$1-i$	$-i$	**0**	1	$1-i$	$-i$
$ps(5)$	1	$1.309-0.95i$	$\frac{1}{2}-1.538i$	$-0.309-0.95i$	**0**	1	$1.309-0.95i$
$ps(6)$	1	$\frac{3}{2}-\frac{\sqrt{3}}{2}i$	$1-\sqrt{3}i$	$0-\sqrt{3}i$	$-\frac{1}{2}-\frac{\sqrt{3}}{2}i$	**0**	1
$ps(7)$	1	$1.623-0.78i$	$1.4-1.757i$	$\frac{1}{2}-2.19i$	$-0.4-1.757i$	$-0.623-0.782i$	**0**

The term $\dfrac{e^s}{e^s-1}$ gives the centres of the polygons.

We now show $$\sum_{n=1}^{\infty}\left(\frac{1}{n^s}\right)^{\frac{n-1}{\ln(n)}} \equiv \frac{e^s}{e^s-1}.$$

(a) Let $$\left(\frac{1}{n^s}\right)^{\frac{n-1}{\ln(n)}} = e^X.$$

Then $$X = \ln\left(\frac{1}{n^s}\right)^{\frac{n-1}{\ln(n)}} = \frac{(n-1)}{\ln(n)}\ln\left(\frac{1}{n^s}\right) = \frac{(n-1)}{\ln(n)}\ln\left(n^{-s}\right)$$
$$= -s\frac{(n-1)}{\ln(n)}\ln(n) = -s(n-1) = s(1-n).$$

Therefore $$\left(\frac{1}{n^s}\right)^{\frac{n-1}{\ln(n)}} = e^{s(1-n)}$$

and $$\sum_{n=1}^{\infty}\left(\frac{1}{n^s}\right)^{\frac{n-1}{\ln(n)}} = \sum_{n=1}^{\infty} e^{s(1-n)}.$$

https://doi.org/10.1017/mag.2025.10131 Published online by Cambridge University Press

(b) Let $$\sum_{n=1}^{\infty} e^{s(1-n)} = S,$$

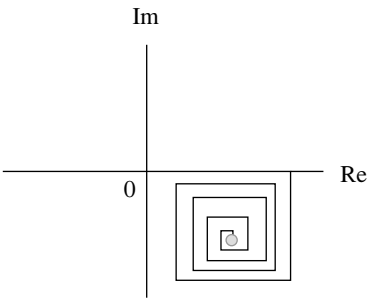
i.e. $$S = 1 + \frac{1}{e^s} + \frac{1}{e^{2s}} + \frac{1}{e^{3s}} + \dots,$$

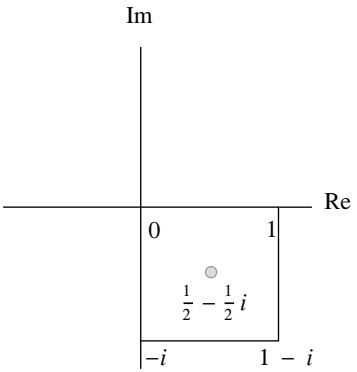
which is a geometric series with common ratio $r = e^{-s}$.

Therefore $$S = \frac{1}{1 - e^{-s}} = \frac{e^s}{e^s - 1},$$

which means $$\sum_{n=1}^{\infty} \left(\frac{1}{n^s}\right)^{\frac{n-1}{\ln(n)}} = \sum_{n=1}^{\infty} e^{s(1-n)} = S = \frac{e^s}{e^s - 1}$$

and therefore $$\sum_{n=1}^{\infty} \left(\frac{1}{n^s}\right)^{\frac{n-1}{\ln(n)}} \equiv \frac{e^s}{e^s - 1}.$$





$$\sum_{n=1}^{17} \left(\frac{1}{n^{\left(\frac{1}{2}+\frac{\pi i}{2}\right)}}\right)^{\frac{n-1}{\ln(n)}} = 0.731 - 0.443i \qquad \sum_{n=1}^{\infty} \left(\frac{1}{n^{\left(0+\frac{\pi i}{2}\right)}}\right)^{\frac{n-1}{\ln(n)}} \equiv \frac{e^{\pi i/2}}{e^{\pi i/2} - 1} = \frac{1}{2} - \frac{1}{2}i.$$

The left-hand sides show the spirals and polygons and the right-hand sides give the points of convergence of the spirals and the centres of the polygons. Both sides go to infinity when $s = 0$.

The centrepoints of the polygons can also be found with a trigonometric method. Using $x = 3$ as an example:

$$g(3) = \sum_{n=1}^{\infty} \left(\frac{1}{n^{(0+2\pi i)/3}}\right)^{\frac{n-1}{\ln(n)}}.$$

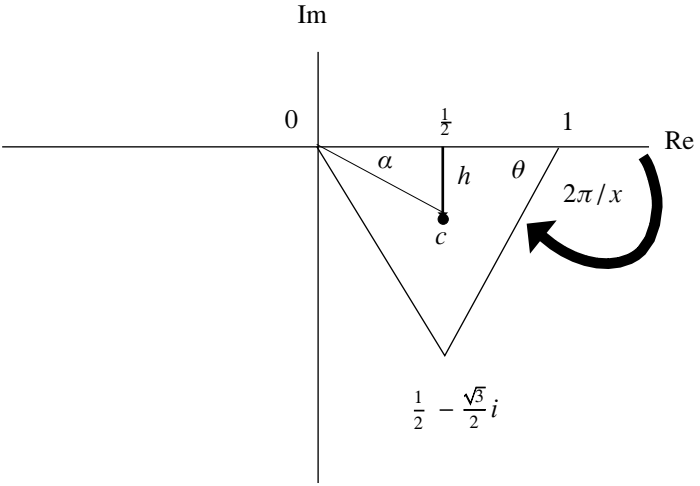
$$\theta = \pi - \frac{2\pi}{x} = \pi - \frac{2\pi}{3}$$

$$\alpha = \frac{\theta}{2} = \frac{\pi}{2} - \frac{\pi}{x} = \frac{\pi}{2} - \frac{\pi}{3} = \frac{\pi}{6}$$

$$h = \frac{1}{2}\tan\alpha = \frac{1}{2}\tan\left(\frac{\pi}{2} - \frac{\pi}{3}\right)$$

$$= \frac{1}{2}\tan\frac{\pi}{6} = \frac{\sqrt{3}}{6}.$$

https://doi.org/10.1017/mag.2025.10131 Published online by Cambridge University Press



The distance h corresponds to the imaginary co-ordinate of the centre c. As the polygons are situated with their bases between 0 and 1 on the real axis, the centres will all have real part $\frac{1}{2}$. When $x = 3$

$$c = \frac{1}{2} - \frac{\sqrt{3}}{6}i.$$

And in general $$c = \frac{1}{2} - \frac{1}{2}\tan\left(\frac{\pi}{2} - \frac{\pi}{x}\right)i.$$

The side number x is recoverable from the imaginary part of c

$$x = \frac{2\pi}{\pi - 2\arctan\left(2\,\mathrm{Im}\,(c)\right)}.$$

We may now say

$$\sum_{n=1}^{\infty}\left(\frac{1}{n^s}\right)^{\frac{n-1}{\ln(n)}} \equiv \frac{e^s}{e^s - 1} = \frac{1}{2} - \frac{1}{2}\tan\left(\frac{\pi}{2} - \frac{\pi}{x}\right)i$$

$$\text{when } s = 0 + \frac{2\pi i}{x} \text{ and } x > 2.$$

The primes are discoverable using the partial sums:

ps (1)	1	2	3	4	5	6	7
ps (2)	1	**0**	1	**0**	1	**0**	1
ps (3)	1	$\frac{1}{2} - \frac{\sqrt{3}}{2}i$	**0**	1	$\frac{1}{2} - \frac{\sqrt{3}}{2}i$	**0**	1
ps (4)	1	$1 - i$	$-i$	**0**	1	$1 - i$	$-i$
ps (5)	1	$1.309 - 0.95i$	$\frac{1}{2} - 1.538i$	$-0.309 - 0.95i$	**0**	1	$1.309 - 0.95i$
ps (6)	1	$\frac{3}{2} - \frac{\sqrt{3}}{2}i$	$1 - \sqrt{3}i$	$0 - \sqrt{3}i$	$\frac{1}{2} - \frac{\sqrt{3}}{2}i$	**0**	1
ps (7)	1	$1.623 - 0.78i$	$1.4 - 1.757i$	$\frac{1}{2} - 2.19i$	$-0.4 - 1.757i$	$-0.623 - 0.782i$	**0**

https://doi.org/10.1017/mag.2025.10131 Published online by Cambridge University Press

We see that:

ps(2) places a 0 at 2 and at every even integer after that

ps(3) places a 0 at 3 and at every third integer after that

ps(4) places a 0 at 4, however ps(2) has already done that so we remove ps(4) from the list

ps(5) places a 0 at 5 and at every fifth integer after that

ps(6) places a 0 at 6, however ps(2) has already done that so we remove ps(6) from the list

ps(7) places a 0 at 7 and at every seventh integer after that.

After removing the unwanted series from the list, we are left with ps(1) and ps(p),where p is prime

ps (1)	1	2	3	4	5	6	7…
ps (2)	1	**0**	1	**0**	1	**0**	1
ps (3)	1	$\frac{1}{2} - \frac{\sqrt{3}}{2}i$	**0**	1	$\frac{1}{2} - \frac{\sqrt{3}}{2}i$	**0**	1
ps (5)	1	$1.309 - 0.95i$	$\frac{1}{2} - 1.538i$	$-0.309 - 0.95i$	**0**	1	$1.309 - 0.95i$
ps (7)	1	$1.623 - 0.78i$	$1.4 - 1.757i$	$\frac{1}{2} - 2.19i$	$-0.4 - 1.757i$	$-0.623 - 0.782i$	**0**

In each case the 0's fall at a prime and at all multiples of that prime, i.e. for all p the 0's are at kp where $k \in Z^+$.

We may now say

$$\sum_{n=1}^{\infty} \left(\frac{1}{n^z}\right)^{(n-1)/\ln(n)} \equiv \frac{e^s}{e^s - 1} = \frac{1}{2} - \frac{1}{2}\tan\left(\frac{\pi}{2} - \frac{\pi}{p}\right)i$$

$$\text{when } s = 0 + \frac{2\pi i}{p} \text{ and } p > 2.$$

The one-to-one relationship between prime numbers (with the exception of 2) and the centres c of prime number sided regular polygons is given by

$$c = \frac{1}{2} - \frac{1}{2}\tan\left(\frac{\pi}{2} - \frac{\pi}{p}\right)i \text{ and } p = \frac{2\pi}{\pi - 2\arctan\left(2\,\mathrm{Im}\,(c)\right)}.$$

Grandi's series, which corresponds to $p = 2$, can be evaluated as 1/2.

10.1017/mag.2025.10131 © The Authors, 2025

Published by Cambridge University Press
on behalf of The Mathematical Association

C. H. REID
The Workshops,
1A Lawrence Street,
Belfast, BT7 1LE

https://doi.org/10.1017/mag.2025.10131 Published online by Cambridge University Press