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WDM and Directed Star Arboricity

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A digraph is m-labelled if every arc is labelled by an integer in $\{1,\dots,m\}$. Motivated by wavelength assignment for multicasts in optical networks, we introduce and study n-fibre colourings of labelled digraphs. These are colourings of the arcs of D such that at each vertex v, and for each colour α, $\mathrm{in}(v,\alpha)+\mathrm{out}(v,\alpha)\leqslant n$ with $\mathrm{in}(v,\alpha)$ the number of arcs coloured α entering v and $\mathrm{out}(v,\alpha)$ the number of labels l such that there is at least one arc of label l leaving v and coloured with α. The problem is to find the minimum number of colours $\lambda_n(D)$ such that the m-labelled digraph D has an n-fibre colouring. In the particular case when D is 1-labelled, $\lambda_1(D)$ is called the directed star arboricity of D, and is denoted by $\mathrm{dst}(D)$. We first show that $\mathrm{dst}(D)\leqslant 2\Delta^-(D)+1$, and conjecture that if $\Delta^-(D)\geqslant 2$, then $\mathrm{dst}(D)\leqslant 2\Delta^-(D)$. We also prove that for a subcubic digraph D, then $\mathrm{dst}(D)\leqslant 3$, and that if $\Delta^+(D),\Delta^-(D)\leqslant 2$, then $\mathrm{dst}(D)\leqslant 4$. Finally, we study $\lambda_n(m,k)=\max\{\lambda_n(D)\mid D \text{ is } m\text{-labelled and } \Delta^-(D)\leqslant k\}$. We show that if $m\geqslant n$, then $\left\lceil \frac{m}{n}\left\lceil\frac{k}{n}\right\rceil+\frac{k}{n}\right\rceil\leqslant\lambda_n(m,k)\leqslant\left\lceil \frac{m}{n}\left\lceil\frac{k}{n}\right\rceil+\frac{k}{n}\right\rceil+C\frac{m^2\log k}{n}$ for some constant C. We conjecture that the lower bound should be the correct value of $\lambda_n(m,k)$.

1. Introduction

The origin of this paper is the study of wavelength assignment for multicasts in star networks. We are given a star network in which a central node is connected by optical fibres to a set of nodes V. The nodes of V communicate together using a technology called WDM (*wavelength-division multiplexing*), which allows the sending of different signals at the same time through the same fibre but on different wavelengths. The central node or hub is an all-optical transmitter which can redirect a signal arriving from a

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node on a particular wavelength to some (one or more) of the other nodes on the same wavelength. This means that the central node is able to duplicate a message incoming on a wavelength to different fibres without changing its wavelength. Therefore, if a node v sends a multicast to a set of nodes $S(v)$, v should send the message to the central node on a set of wavelengths so that the central node redirects it to each node of $S(v)$ using one of these wavelengths. The aim is to minimize the total number of wavelengths used. We refer to Brandt and Gonzalez [4] for a more complete description of the model and for some partial results. In what follows, we will briefly explain the main contributions of this paper.

We first study the basic case when there is a single fibre between the central node and each node of V and each vertex v sends a single multicast $M(v)$ to a set $S(v)$ of nodes. In this case, the problem becomes equivalent to *directed star colouring*: let D be the digraph with vertex set V such that the outneighbourhood of a vertex v is $S(v)$. We note that D is a digraph and not a multidigraph, *i.e.*, there are no parallel arcs in D, as $S(v)$ is a set. The problem is then to find the smallest k such that there exists a mapping $\phi : A(D) \to \{1, \ldots, k\}$ satisfying the following two conditions.

(i) For each pair of arcs uv and vw, $\phi(uv) \neq \phi(vw)$.
(ii) For each pair of arcs uv and $u'v$, $\phi(uv) \neq \phi(u'v)$.

Such a mapping is called *directed star k-colouring*. The *directed star arboricity* of a digraph D, denoted by $\mathrm{dst}(D)$, is the minimum integer k such that there exists a directed star k-colouring. This notion was introduced by Guiduli in [6], and is an analogue of the *star arboricity* defined by Algor and Alon in [1].

The indegree of a vertex v, $d^-(v)$, corresponds to the number of multicasts that v receives. A sensible assumption on the model is that a node receives a bounded number of multicasts. Hence, Brandt and Gonzalez [4] studied the directed star arboricity of a digraph D with regard to its maximum indegree. The *maximum indegree* of a digraph D, denoted by $\Delta^-(D)$ or simply Δ^- when D is clearly understood from the context, is $\max\{ d^-(v) \mid v \in V(D) \}$. Brandt and Gonzalez showed that $\mathrm{dst}(D) \leqslant \lceil 5\Delta^-/2 \rceil$. This upper bound is tight if $\Delta^- = 1$, because odd circuits have directed star arboricity three. However, as we will show in Section 2, the upper bound can be improved for larger values of Δ^-.

Theorem 1.1. *Every digraph D satisfies* $\mathrm{dst}(D) \leqslant 2\Delta^- + 1$.

We make the following conjecture.

Conjecture 1.2. *Every digraph D with maximum indegree $\Delta^- \geqslant 2$ satisfies* $\mathrm{dst}(D) \leqslant 2\Delta^-$.

This conjecture would be tight, as Brandt [3] showed that for every Δ^- there is an acyclic digraph D_{Δ^-} with maximum indegree Δ^- and $\mathrm{dst}(D_{\Delta^-}) = 2\Delta^-$. His construction is the special case for $n = m = 1$ of the construction given in Proposition 5. We settle Conjecture 1.2 for acyclic digraphs in Section 2. So, combined with Brandt's construction, $2\Delta^-$ is the best bound we can expect for acyclic digraphs.

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Remark. Let us note at this point that we restrict ourselves to simple digraphs, *i.e.*, we allow circuits of length two but multiple arcs are not permitted. When multiple arcs are allowed, all the bounds above do not hold. Indeed, given an integer Δ^-, the multidigraph T_{Δ^-} with three vertices u, v and w, and Δ^- parallel arcs to each of uv, vw and wu satisfies $\mathrm{dst}(T_{\Delta^-}) = 3\Delta^-$. Moreover, this example is extremal since every multidigraph satisfies $\mathrm{dst}(D) \leqslant 3\Delta^-$. This can be shown by induction: pick a vertex v with outdegree at most its indegree. (Such a vertex exists since $\sum_{u\in V(D)} d^+(u) = \sum_{u\in V(D)} d^-(u)$.) If v has no inneighbour, then v is isolated, and we can remove v and apply induction. Otherwise, we consider any arc uv. The colour of uv must be different from the colours of the $d^-(u)$ arcs entering u, the $d^+(v)$ arcs leaving v, and the $d^-(v) - 1$ other arcs entering v, so at most $3\Delta^- - 1$ arcs in total. Hence, we may remove the arc uv, and apply induction to obtain a colouring of $D \setminus uv$. Extending this colouring to uv, we obtain a directed star colouring of D with at most $3\Delta^-$ colours.

Note that to prove Conjecture 1.2 it will be enough to consider the two cases $\Delta^- = 2$ and $\Delta^- = 3$. To see this, let D be a digraph with maximum indegree $\Delta^- \geqslant 2$ and $k = \lfloor \Delta^-/2 \rfloor$. For every vertex v, let $(N_1^-(v), N_2^-(v), \ldots, N_k^-(v))$ be a partition of $N^-(v)$ such that $|N_i^-(v)| \leqslant 2$ for all $1 \leqslant i \leqslant k-1$ and $|N_k^-(v)| \leqslant 2$ if Δ^- is even and $|N_k^-(v)| \leqslant 3$ if Δ^- is odd. Then the digraph D_i with vertex set $V(D)$, and such that $N_{D_i}^-(v) = N_i^-(v)$ for every vertex $v \in V(D)$, has maximum indegree at most two except if $i = k$ and Δ^- is odd, in which case D_k has maximum indegree at most three. If Conjecture 1.2 holds for every D_i then it would also hold for D.

We next consider the directed star arboricity of a digraph with bounded maximum degree. The *degree* of a vertex v is $d(v) = d^-(v) + d^+(v)$. This corresponds to the degree of the vertex in the underlying multigraph. (We have edges with multiplicity two in the underlying multigraph each time there is a circuit of length two in the digraph.) The *maximum degree* of a digraph D, denoted by $\Delta(D)$, or simply Δ when D is clearly understood from the context, is $\max\{d(v), v \in V(D)\}$. Let us denote by $\mu(G)$ the maximum multiplicity of an edge in a multigraph. By Vizing's theorem [11], one can colour the edges of a multigraph with $\Delta(G) + \mu(G)$ colours so that two edges have different colours if they are incident. Since the multigraph underlying a digraph has maximum multiplicity at most two, for any digraph D, $\mathrm{dst}(D) \leqslant \Delta + 2$. We make the following conjecture.

Conjecture 1.3. *Let D be a digraph with maximum degree $\Delta \geqslant 3$. Then* $\mathrm{dst}(D) \leqslant \Delta$.

This conjecture would be tight, since every digraph with $\Delta = \Delta^-$ has directed star arboricity at least Δ. In Section 3, we prove that Conjecture 1.3 holds when $\Delta = 3$.

Theorem 1.4. *Every subcubic digraph has directed star arboricity at most three.*

A first step towards Conjectures 1.2 and 1.3 would be to prove the following weaker statement.

Conjecture 1.5. *Let $k \geqslant 2$ and D be a digraph. If* $\max(\Delta^-, \Delta^+) \leqslant k$ *then* $\mathrm{dst}(D) \leqslant 2k$.

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This conjecture holds and is far from being tight for large values of k. Indeed, Guiduli [6] showed that if $\max(\Delta^-, \Delta^+) \leqslant k$, then $\mathrm{dst}(D) \leqslant k + 20 \log k + 84$. Guiduli's proof is based on the fact that, when both out- and indegrees are bounded, the colour of an arc depends on the colour of few other arcs. This bounded dependency allows the use of the Lovász Local Lemma. This idea was first used by Algor and Alon [1] for the star arboricity of undirected graphs. We also note that Guiduli's result is (almost) tight, since there are digraphs D with $\max(\Delta^-, \Delta^+) \leqslant k$ and $\mathrm{dst}(D) \geqslant k + \Omega(\log k)$ (see [6]).

As for Conjecture 1.2, it is quite straightforward to check that it is sufficient to prove Conjecture 1.5 for $k = 2$ and $k = 3$. In Section 4, we prove that Conjecture 1.5 holds for $k = 2$. By the above remark, this implies that Conjecture 1.5 holds for all even values of k.

Theorem 1.6. *Let D be a digraph. If $\Delta^- \leqslant 2$ and $\Delta^+ \leqslant 2$, then $\mathrm{dst}(D) \leqslant 4$. In particular, Conjecture 1.5 holds for all even values of k.*

Next, we study the more general and more realistic problem in which every vertex of V is connected to the hub by n optical fibres. Moreover, each node may send several multicasts. We let $M_1(v), \ldots, M_{s(v)}(v)$ denote the $s(v)$ multicasts sent by node v. For $1 \leqslant i \leqslant s(v)$, the set of nodes to which the multicast $M_i(v)$ is sent is denoted by $S_i(v)$. The problem is still to find the minimum number of wavelengths used considering that all fibres are identical.

We model this as a problem on *labelled* digraphs. We construct a multidigraph D on vertex set V. For each multicast $M_i(v) = (v, S_i(v))$, $v \in V$, $1 \leqslant i \leqslant s(v)$, we add the set of arcs $A_i(v) = \{vw, w \in S_i(v)\}$ with label i. The label of an arc $\vec{a}$ is denoted by $l(\vec{a})$. Thus, for every ordered pair (u, v) of vertices and label i there is at most one arc uv labelled by i. If each vertex sends at most m multicasts, there are at most m labels on the arcs. Such a digraph is said to be *m-labelled*. One wishes to find an *n-fibre wavelength assignment* of D, that is, a mapping $\Phi : A(D) \to \Lambda \times \{1, \ldots, n\} \times \{1, \ldots n\}$ in which every arc uv is associated to a triple $(\lambda(uv), f^+(uv), f^-(uv))$ such that the following hold.

(i) For each pair of arcs uv and vw, $(\lambda(uv), f^-(uv)) \neq (\lambda(vw), f^+(vw))$.
(ii) For each pair of arcs uv and $u'v$, $(\lambda(uv), f^-(uv)) \neq (\lambda(u'v), f^-(u'w))$.
(iii) For each pair of arcs vw and vw', $(\lambda(vw), f^+(vw)) \neq (\lambda(vw'), f^+(vw'))$ if $l(vw) \neq l(vw')$.

Here Λ is the set of available wavelengths, $\lambda(uv)$ corresponds to the wavelength of uv, and $f^+(uv)$ and $f^-(uv)$ are the fibres used in u and v, respectively. We can describe the above equations as follows.

- Condition (i) corresponds to the requirement that an arc entering v and an arc leaving v should have either different wavelengths or different fibres.
- Condition (ii) corresponds to the requirement that two arcs entering v should have either different wavelengths or different fibres.
- Condition (iii) corresponds to the requirement that two arcs leaving v with different labels have either different wavelengths or different fibres.

The problem is to find the minimum cardinality $\lambda_n(D)$ of Λ such that there exists an *n-fibre wavelength assignment* of D.

The crucial part of an n-fibre wavelength assignment is the function λ which assigns colours (wavelengths) to the arcs. It must be an *n-fibre colouring*, that is, a function

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$\phi : A(D) \to \Lambda$ such that at each vertex v, for each colour $\omega \in \Lambda$, $\text{in}(v,\omega) + \text{out}(v,\omega) \leqslant n$, where $\text{in}(v,\omega)$ denotes the number of arcs coloured by ω entering v and $\text{out}(v,\omega)$ denotes the number of labels l such that there exists an arc leaving v coloured by ω. Once we have an n-fibre colouring, we can easily find a suitable wavelength assignment. For every vertex v and every colour ω, this is done by assigning a different fibre to each arc of colour ω entering v, and to each set of arcs of colour ω of the same label that leave v. We conclude that $\lambda_n(D)$ is the minimum number of colours such that there exists an n-fibre colouring.

We are particularly interested in $\lambda_n(m,k) = \max\{\lambda_n(D) \mid D \text{ is } m\text{-labelled and } \Delta^-(D) \leqslant k\}$, that is, the maximum number of wavelengths that may be necessary if there are n fibres, and each node sends at most m multicasts and receives at most k multicasts. In particular, $\lambda_1(1,k) = \max\{\text{dst}(D) \mid \Delta^-(D) \leqslant k\}$. (So our above-mentioned results show that $2k \leqslant \lambda_1(1,k) \leqslant 2k+1$.) Brandt and Gonzalez showed that for $n \geqslant 2$ we have $\lambda_n(1,k) \leqslant \lceil \frac{k}{n-1} \rceil$. In Section 5, we study the case when $n \geqslant 2$ and $m \geqslant 2$. We show in Proposition 5 and Theorem 5.8 that:

$$\text{if } m \geqslant n, \quad \text{then} \quad \left\lceil \frac{m}{n}\left\lceil \frac{k}{n}\right\rceil + \frac{k}{n}\right\rceil \leqslant \lambda_n(m,k) \leqslant \left\lceil \frac{m}{n}\left\lceil \frac{k}{n}\right\rceil + \frac{k}{n}\right\rceil + C\frac{m^2 \log k}{n}$$

for some constant C. We conjecture that the lower bound is the correct value of $\lambda_n(m,k)$ when $m \geqslant n$. We also show in Proposition 5 and Proposition 5 that:

$$\text{if } m < n, \quad \text{then} \quad \left\lceil \frac{m}{n}\left\lceil \frac{k}{n}\right\rceil + \frac{k}{n}\right\rceil \leqslant \lambda_n(m,k) \leqslant \left\lceil \frac{k}{n-m}\right\rceil.$$

The lower bound generalizes the results of Brandt and Gonzalez [4], which established this inequality in the particular cases when $k \leqslant 2$, $m \leqslant 2$ and $k = m$. The digraphs used to show this lower bound are all acyclic. We show that if $m \geqslant n$ then this lower bound is tight for acyclic digraphs. Moreover, the above-mentioned digraphs have large outdegree. Generalizing the result of Guiduli [6], we show that for an m-labelled digraph D with both in- and outdegree bounded by k, few colours are needed when $m \geqslant n$:

$$\lambda_n(D) \leqslant \frac{k}{n} + C'\frac{m^2 \log k}{n}$$

for some constant C'.

Finally, in Section 6, we consider the complexity of finding the directed star arboricity of a digraph, and prove that, unsurprisingly, this is an $\mathcal{NP}$-hard problem. More precisely, we show that determining the directed star arboricity of a digraph with in- and outdegree at most two is $\mathcal{NP}$-complete. We then give a very short proof of a theorem of Pinlou and Sopena [9], showing that *acircuitic directed star arboricity of subcubic graphs is at most four* (see Section 6 for the definitions).

2. Directed star arboricity of digraphs with bounded indegrees

In this section, we give the proof of Theorem 1.1 and settle Conjecture 1.2 for acyclic digraphs.

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An *arborescence* is a connected digraph in which every vertex has indegree one except one, called the *root*, which has indegree zero. A *forest* is the disjoint union of arborescences. A *star* is an arborescence in which the root dominates all the other vertices. A *galaxy* is a forest of stars. Clearly, every colour class of a directed star colouring is a galaxy. Hence, the directed star arboricity of a digraph D is the minimum number of galaxies into which $A(D)$ may be partitioned.

It is easy to see that a forest has directed star arboricity at most two. Hence, an idea to prove Conjecture 1.2 would be to show that every digraph has an arc-partition into Δ^- forests. However, this statement is false. Indeed, a theorem of Frank [5] (see also Chapter 53 of [10]) characterizes all digraphs which have an arc-partition into k forests. Let $D = (V, A)$. For any $U \subset V$, the digraph induced by the vertices of U is denoted $D[U]$.

Theorem 2.1 (A. Frank). *A digraph $D = (V, A)$ has an arc-partition into k forests if and only if $\Delta^-(D) \leqslant k$ and, for every $U \subset V$, the digraph $D[U]$ has at most $k(|U| - 1)$ arcs.*

This theorem implies that every digraph D has an arc-partition into $\Delta^- + 1$ forests. Indeed, for any $U \subset V$, $\Delta^-(D[U]) \leqslant \min\{\Delta^-, |U| - 1\}$, so $D[U]$ has at most $\min\{\Delta^-, |U| - 1\} \times |U| \leqslant (\Delta^- + 1)(|U| - 1)$ arcs. Hence, every digraph has directed star arboricity at most $2\Delta^- + 2$.

Corollary 2.2. *Every digraph D satisfies* $\mathrm{dst}(D) \leqslant 2\Delta^- + 2$.

Theorem 1.1 states that $\mathrm{dst}(D) \leqslant 2\Delta^- + 1$. The idea to prove this theorem is to show that every digraph has an arc-partition into Δ^- forests and a galaxy G. To do so, we prove a stronger result, Lemma 2.3 below.

We need some extra definitions. A *sink* is a vertex with outdegree 0. A *source* is a vertex with indegree 0. A multidigraph D will be called *k-nice* if $\Delta^- \leqslant k$, and if the tails of parallel arcs, if any, are sources. A *k-decomposition* of D is an arc-partition into k forests and a galaxy G such that every source of D is isolated in G. Let u be a vertex of D. A k-decomposition of D is *u-suitable* if no arc of G has head u.

Lemma 2.3. *Let u be a vertex of a k-nice multidigraph D. Then D has a u-suitable k-decomposition.*

Proof. We proceed by induction on $n + k$ by considering (strong) connectivity of D.

(1) If D is not a connected graph, we apply induction on every component.

(2) If D is strongly connected, every vertex has indegree at least one. (Recall that there are no parallel arcs.) Let v be an outneighbour of u. There exists a spanning arborescence T with root v which contains all the arcs with tail v. Let D' be the digraph obtained from D by removing the arcs of T and v. Observe that D' is $(k-1)$-nice. By induction, it has a u-suitable $(k-1)$-decomposition $(F_1, \ldots, F_{k-1}, G)$. Note that each F_i, for $1 \leqslant i \leqslant k-1$, T and G contain all the arcs of D except those with head v. By construction, $G' = G \cup uv$

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is a galaxy since no arc of G has head u. Let $u_1, \ldots, u_{l-1}$ be the inneighbours of v distinct from u, where $l \leqslant k$. Let $F'_i = F_i \cup u_i v$, for all $1 \leqslant i \leqslant l-1$. Each F'_i is a forest, so $(F_1, \ldots, F_{k-1}, T, G')$ is a u-suitable k-decomposition of D.

(3) In the only remaining case, D is connected but not strongly connected. We consider a terminal strongly connected component D_1 of D. Set $D_2 = D \setminus D_1$. Let u_1 and u_2 be two vertices of D_1 and D_2, respectively, such that u is one of them.

If D_2 has a unique vertex v (thus $u_2 = v$), since D is connected and D_1 is strong, there exists a spanning arborescence T of D with root v. Now $D' = D \setminus A(T)$ is a $(k-1)$-nice multidigraph, so by induction it has a u-suitable $(k-1)$-decomposition. Adding T to this decomposition, we obtain a u-suitable k-decomposition. If D_2 has more than one vertex, it admits a u_2-suitable k-decomposition $(F_1^2, \ldots, F_k^2, G^2)$, by induction. Moreover, the digraph D'_1 obtained by contracting D_2 to a single vertex v is k-nice and so has a u_1-suitable k-decomposition $(F_1^1, \ldots, F_k^1, G^1)$. Moreover, since v is a source, it is isolated in G^1. Hence $G = G^1 \cup G^2$ is a galaxy. We now let F_i be the union of F_i^1 and F_i^2 by replacing the arcs of F_i^1 with tail v by the corresponding arcs in D. Then $(F_1, \ldots, F_k, G)$ is a k-decomposition of D which is suitable for both u_1 and u_2. □

Theorem 1.1 is an immediate consequence of Lemma 2.3.

2.1. Acyclic digraphs

It is not very hard to show that $\mathrm{dst}(D) \leqslant 2\Delta^-$ when D is acyclic, but we will prove this result in a more constrained way. For $n \leqslant p$, a *cyclic n-interval* of $\{1, 2, \ldots, p\}$ is a set of n consecutive numbers modulo p. Now, for the directed star colouring, we will insist that for every vertex v, the (distinct) colours used to colour the arcs with head v are chosen in a *cyclic k-interval* of $\{1, 2, \ldots, 2k\}$. Thus, the number of possible sets of colours used to colour the entering arcs of a vertex v drastically falls from $\binom{2k}{d^-(v)}$ when every set is *a priori* possible, to at most $2k \times \binom{k}{d^-(v)}$. Note that having consecutive colours on the arcs entering a vertex corresponds to having consecutive wavelengths on the link between the corresponding node and the central one. This may be of importance for issues related to grooming in optical networks. For details about grooming, we refer the reader to the two comprehensive surveys [7] and [8].

Theorem 2.4. *Let D be an acyclic digraph with maximum indegree k. Then D admits a directed star $2k$-colouring such that, for every vertex, the colours assigned to its entering arcs are included in a cyclic k-interval of $\{1, 2, \ldots, 2k\}$.*

To prove this theorem, we first state and prove the following result on sets of distinct representatives.

Lemma 2.5. *Let $I_1, \ldots, I_k$ be k not necessarily distinct cyclic k-intervals of $\{1, 2, \ldots, 2k\}$. Then $I_1, \ldots, I_k$ admit a set of distinct representatives forming a cyclic k-interval.*

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Proof. We consider $I_1,\dots,I_k$ to be a set of p *distinct* cyclic k-intervals $I_1,\dots,I_p$ with respective multiplicity $m_1,\dots,m_p$ such that $\sum_{i=1}^{p} m_i = k$. Such a system will be denoted by $((I_1,m_1),\dots,(I_p,m_p))$. We shall prove the existence of a cyclic k-interval J, such that J can be partitioned into p subsets J_i, $1 \leqslant i \leqslant p$, such that $|J_i| = m_i$ and $J_i \subset I_i$. This proves the lemma (by associating distinct elements of J_i to each copy of I_i).

We proceed by induction on p. The result holds trivially for $p = 1$. We have to deal with the following two cases.

(1) There exist i and j such that $|I_j \setminus I_i| = |I_i \setminus I_j| \leqslant \max(m_i,m_j)$. Suppose without loss of generality that $i < j$ and $m_i \geqslant m_j$. We apply the induction hypothesis to $((I_1,m_1),\dots,(I_i,m_i+m_j),\dots,(I_{j-1},m_{j-1}),(I_{j+1},m_{j+1}),\dots,(I_p,m_p))$, in order to find a cyclic interval J', such that J' admits a partition into subsets J'_r, such that, for any r different from i and j, the set $J'_r \subset I_r$ is a subset of size m_r, and $J'_i \subset I_i$ is of size $m_i + m_j$. We now partition J'_i into two sets J_i and J_j with respective size m_i and m_j, in such a way that $(I_i \setminus I_j) \cap J'_i \subseteq J_i$. Note that this is possible precisely because of our assumption $|I_j \setminus I_i| = |I_i \setminus I_j| \leqslant m_i$. Since $J_i \subset I_i$ and $J_j \subset I_j$, this refined partition of J' is the desired one.

(2) For any i,j, we have $|I_j \setminus I_i| = |I_i \setminus I_j| \geqslant \max(m_i,m_j) + 1$. Each I_i intersects exactly $2m_i - 1$ other cyclic k-intervals on fewer than m_i elements. Since there are $2k$ cyclic k-intervals in total and $\sum_{i=1}^{p}(2m_i - 1) = 2k - p < 2k$, we conclude the existence of a cyclic k-interval J which intersects each I_i in an interval of size at least m_i.

Let us prove that one can partition J in the desired way. By Hall's matching theorem, it suffices to prove that, for every subset $\mathcal{I}$ of $\{1,\dots,p\}$, we have $|\bigcup_{i\in\mathcal{I}} I_i \cap J| \geqslant \sum_{i\in\mathcal{I}} m_i$.

Suppose for the sake of contradiction that a subset $\mathcal{I}$ of $\{1,\dots,p\}$ violates this inequality. Such a subset will be called *contracting*. Without loss of generality, we assume that $\mathcal{I}$ is a contracting set with minimum cardinality and that $\mathcal{I} = \{1,\dots,q\}$. Observe that by the choice of J we have $q \geqslant 2$. The set $K := \bigcup_{i\in\mathcal{I}} I_i \cap J$ consists of one or two intervals of J, each containing one extremity of J. By the minimality of $\mathcal{I}$, K must be a single interval. (If not, one would take $\mathcal{I}_1$, resp. $\mathcal{I}_2$, all the elements of $\mathcal{I}$ which contains the first, resp. the second, extremity of J. Then one of $\mathcal{I}_1$ or $\mathcal{I}_2$ would be contracting.) Thus, one of the two extremities of J is in every I_i, $i \in \mathcal{I}$. Without loss of generality, we may assume that $(I_1 \cap J) \subset (I_2 \cap J) \subset \cdots \subset (I_q \cap J)$. Now, for every $2 \leqslant i \leqslant q$, we have $|I_i \setminus I_{i-1}| = |(I_i \cap J) \setminus (I_{i-1} \cap J)| \geqslant \max(m_i,m_{i-1}) + 1 \geqslant m_i + 1$. But $|\bigcup_{i\in\mathcal{I}} I_i \cap J| = |(I_1 \cap J)| + \sum_{i=2}^{q} |(I_i \cap J) \setminus (I_{i-1} \cap J)|$. So $|\bigcup_{i\in\mathcal{I}} I_i \cap J| \geqslant \sum_{i=1}^{q} m_i + q - 1$, which is a contradiction. □

Proof of Theorem 2.4. We use induction on the number of vertices, the result being trivial if D has one vertex. Suppose now that D has at least two vertices. Then D has a sink x. By the induction hypothesis, $D \setminus x$ has a directed star $2k$-colouring c such that for every vertex, the colours assigned to its entering arcs are included in a cyclic k-interval. Let $v_1,v_2,\dots,v_l$ be the inneighbours of x in D, where $l \leqslant k$ because $\Delta^-(D) \leqslant k$. For each $1 \leqslant i \leqslant l$, let I'_i be a cyclic k-interval which contains all the colours of the arcs with head v_i. We set $I_i = \{1,\dots,2k\} \setminus I'_i$. Clearly, I_i is a cyclic k-interval and the arc v_ix can be coloured by any element of I_i. By Lemma 2.5, $I_1,\dots,I_l$ have a set of distinct

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representatives included in a cyclic $2k$-interval J. Hence, assigning J to x, and colouring the arc $v_i x$ by the representative of I_i gives a directed star $2k$-colouring of D. □

Theorem 2.4 is tight: Brandt [3] showed that for every k there is an acyclic digraph such that $\Delta^-(D_k) = k$ and $\mathrm{dst}(D_k) = 2k$. His construction is the special case for $n = m = 1$ of the construction given in Proposition 5.

3. Directed star arboricity of subcubic digraphs

Recall that a *subcubic* digraph is a graph with degree at most three. In this section, we give the proof of Theorem 1.4, which states that the directed star arboricity of a subcubic digraph is at most three.

To do so, we need to establish some preliminary lemmas which will enable us to extend a partial directed star colouring into a directed star colouring of the whole digraph. To state these lemmas, we need the following definition. Let $D = (V, A)$ be a digraph and S be a subset of $V \cup A$. Suppose that each element x of S is assigned a list $L(x)$. A colouring c of S is an *L-colouring* if $c(x) \in L(x)$ for every $x \in S$.

Lemma 3.1. *Let C be a circuit in which every vertex v receives a list $L(v)$ of two colours among $\{1,2,3\}$ and each arc $\vec{a}$ receives the list $L(\vec{a}) = \{1,2,3\}$. The following two statements are equivalent.*

- *There is no L-colouring c of the arcs and vertices such that $c(x) \neq c(xy)$, $c(y) \neq c(xy)$, and $c(xy) \neq c(yz)$, for all arcs xy and yz.*
- *C is an odd circuit and all the vertices have the same list.*

Proof. Assume first that every vertex is assigned the same list, say $\{1,2\}$. If C is odd, it is a simple matter to check that we can not find the desired colouring. Indeed, among two consecutive arcs, one has to be coloured 3. If C is even, we colour the vertices by 1 and the arcs alternately by 2 and 3.

Now assume that $C = x_1 x_2 \cdots x_k x_1$ and x_1 and x_2 are assigned different lists. Say $L(x_1) = \{1,2\}$ and $L(x_2) = \{2,3\}$. We colour the arc x_1x_2 by 3, the vertex x_2 by 2 and the arc x_2x_3 by 1. Then we colour x_3, x_3x_4, …, x_k greedily. It remains to colour x_kx_1 and x_1. One of two cases may occur: if we can colour x_kx_1 by 1 or 2, we do it and colour x_1 by 2 or 1 respectively; otherwise the set of colours assigned to x_k and $x_{k-1}x_k$ is $\{1,2\}$. Hence, we colour x_kx_1 with 3, x_1 by 1, and recolour x_1x_2 by 2 and x_2 by 3. □

Lemma 3.2. *Let D be a subcubic digraph with no vertex of outdegree two and indegree one. Suppose that every arc $\vec{a}$ has a list of colours $L(\vec{a}) \subset \{1,2,3\}$ such that the following hold.*

- *If the head of $\vec{a}$ is a sink s (in which case $\vec{a}$ will be called a final arc), $|L(\vec{a})| \geqslant d^-(s)$.*
- *If $\vec{a}$ is not a final arc and the tail of $\vec{a}$ is a source (in which case $\vec{a}$ will be called an initial arc), $|L(\vec{a})| \geqslant 2$.*
- *In all the other cases, $|L(\vec{a})| = 3$.*

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In addition, assume that the following hold.

- *If a vertex is the head of at least two initial arcs $\vec{a}$ and $\vec{b}$, the union of the lists of colours $L(\vec{a})$ and $L(\vec{b})$ contains all three colours.*
- *If all the vertices of an odd circuit are the tails of initial arcs, the union of the lists of colours of these initial arcs contains all three colours.*

Then D has a directed star L-colouring.

Proof. We colour the graph inductively. Consider a terminal strong component C of D. Since D has no vertex with indegree one and outdegree two, C induces either a singleton or a circuit.

(1) Assume that C is a singleton v which is the head of a unique arc $\vec{a} = uv$. If u has indegree zero, we colour $\vec{a}$ with a colour of its list. If u has indegree one, and thus total degree two, we colour $\vec{a}$ by the colour of its list and remove this colour from the list of the arc with head u. If u is the head of $\vec{e}$ and $\vec{f}$, observe that $L(\vec{e})$ and $L(\vec{f})$ have at least two colours and their union has all three colours. To conclude, we colour $\vec{a}$ with a colour in its list, remove this colour from $L(\vec{e})$ and $L(\vec{f})$, remove $\vec{a}$, and split u into two vertices, one with head $\vec{e}$ and the other with head $\vec{f}$. Now, we choose different colours for the arcs $\vec{e}$ and $\vec{f}$ in their respective lists to form the new list $L(\vec{e})$ and $L(\vec{f})$.

(2) Assume that C is a singleton v which is the head of several arcs, including $\vec{a} = uv$. In this case, we reduce $L(\vec{a})$ to a single colour, remove this colour from the other arcs with head v and split v into v_1, which becomes the head of $\vec{a}$, and v_2 which becomes the head of the other arcs.

(3) Assume that C is a circuit. Every arc entering C has a list of at least two colours. We can apply Lemma 3.1 to conclude. □

Proof of Theorem 1.4. Assume for the sake of contradiction that the digraph D has directed star arboricity more than three and is minimum for this property with respect to the number of arcs. Observe that D has no source, otherwise we simply delete it with all its incident arcs, apply induction and extend the colouring. This is possible since arcs leaving from a source can be coloured arbitrarily. Let D_1 be the subdigraph of D induced by the vertices of indegree at most 1. We denote by D_2 the digraph induced by the other vertices, and by $[D_i, D_j]$ the set of arcs with tail in D_i and head in D_j. We claim that D_1 contains no even circuit. If not, we simply remove the arcs of this even circuit, apply induction. We can extend the colouring to the arcs of the even circuit since every arc of the circuit has two colours available.

A *critical set* of vertices of D_2 is either a vertex of D_2 with indegree at least two in D_1, or an odd circuit of D_2 having all its inneighbours in D_1. Observe that critical sets are disjoint. For every critical set S, we select two arcs entering S from D_1, called *selected arcs* of S.

Let D' be digraph induced by the arc set $A' = A(D_1) \cup [D_2, D_1]$. We now define a *conflict graph* on the arcs of D' as follows.

- Two arcs xy, yv of D' are in conflict, called *normal conflict* at y.

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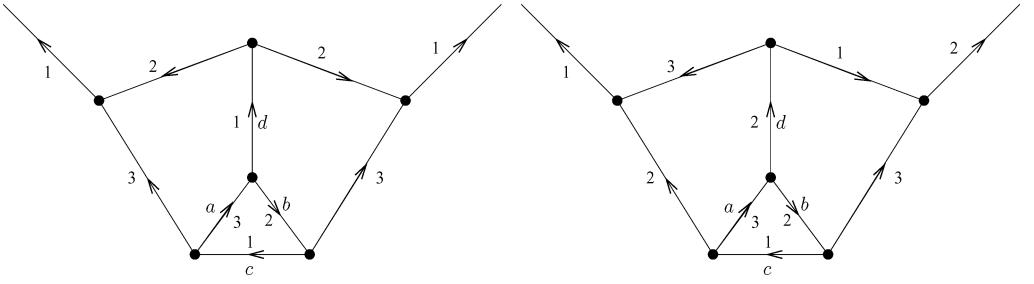


Figure 1. A K_4 in the conflict graph and the two ways of extending the colouring.

- Two arcs xy, uv of D' are also in conflict if there exists two selected arcs of the same set S with tails y and v. These conflicts are called *selected conflicts* at y and v.

Let us analyse the structure of the conflict graph. Observe first that an arc is in conflict with three arcs: one normal conflict at its tail and at most two (normal or selected) at its head. We claim that there is no K_4 in the conflict graph. For the sake of contradiction, suppose there is one. This means that there are four arcs $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ pairwise in conflict. Since each of these arcs have degree three in K_4, each of these arcs should have a normal conflict at its tail, and so the digraph induced by these four arcs contains a circuit. This circuit cannot be of even length (two or four) so it has to be of length three. It follows that the four arcs $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ are as in Figure 1 below (modulo a permutation of the labels). Let D^* be the digraph obtained from D by removing the arcs $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ and their four incident vertices. By minimality of D, D^* admits a directed star 3-colouring which can be extended to D as depicted in Figure 1, depending on whether the two leaving arcs are coloured the same or differently. This proves the claim.

Brooks' theorem asserts that every subcubic graph without K_4 is 3-colourable. So the conflict graph admits a 3-colouring c. This gives a colouring of the arcs of D'. Let D'' be the digraph obtained from D, and let L be the list-assignment on the arcs of D'' defined simultaneously as follows.

- Remove the arcs of D_1 from D.
- Assign to each arc of $[D_2, D_1]$ the singleton list containing the colour it has in D'.
- For each arc uv of $[D_1, D_2]$, there is a unique arc tu in $A(D')$. Assign to uv the list $L(uv) = \{1, 2, 3\} \setminus c(tu)$.
- Assign the list $\{1, 2, 3\}$ to the other arcs.
- If there are vertices with indegree one and outdegree two (they were in D_1), split each of them into one source of degree two and a sink of degree one.

Note that there is a trivial one-to-one correspondence between $A(D'')$ and $A(D) \setminus A(D')$. By the definition of the conflict graph and D'', one can easily check that D'' and L satisfies the condition of Lemma 3.2. Hence D'' admits a directed star L-colouring whose union with c is a directed star 3-colouring of D, a contradiction. The proof of Theorem 1.4 is now complete. □

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4. Directed star arboricity of digraphs with maximum in- and outdegree two

The goal of this section is to prove Theorem 1.6: *Every digraph with outdegree and indegree at most two has directed star arboricity at most four*. However, the class of digraphs with in- and outdegree at most two is certainly not an easy class with respect to directed star arboricity, as we will show in Section 6.1.

In order to prove Theorem 1.6, it suffices to show that D contains a galaxy G which spans all the vertices of degree four. Indeed, if this is true, then $D' = D - A(G)$ has maximum degree at most 3 and by Theorem 1.4, $\mathrm{dst}(D') \leqslant 3$. So $\mathrm{dst}(D) \leqslant 4$. Hence Theorem 1.6 is directly implied by the following lemma.

Lemma 4.1. *Let D be a digraph with maximum indegree and outdegree two. Then D contains a galaxy which spans the set of vertices with degree four.*

To prove this lemma, we need some preliminaries. Let V be a set. An *ordered digraph* on V is a pair $(\leqslant, D)$ satisfying the following conditions.

- $\leqslant$ is a partial order on V.
- D is a digraph with vertex set V.
- D contains the Hasse diagram of $\leqslant$ (*i.e.*, when $x \leqslant y \leqslant z$ implies $x = y$ or $y = z$, then xz is an arc of D).
- If xy is an arc of D, the vertices x, y are $\leqslant$-comparable.

The arcs xy of D thus belong to two different types: the *forward arcs*, when $x \leqslant y$, and the *backward arcs*, when $y \leqslant x$.

Lemma 4.2. *Let $(\leqslant, D)$ be an ordered digraph on V. Assume that every vertex is the tail of at most one backward arc and at most two forward arcs, and that the indegree of every vertex of D is at least two, except possibly one vertex x with indegree one. Then D contains two arcs $\gamma\alpha$ and $\beta\lambda$ such that $\alpha \leqslant \beta \leqslant \gamma$, $\beta \leqslant \lambda$ and $\gamma \not\leqslant \lambda$, all four vertices being distinct except possibly $\alpha = \beta$.*

Proof. For the sake of contradiction, let us consider a counter-example with minimum $|V|$. An *interval* is a subset I of V which has a minimum m and a maximum M such that $I = \{z : m \leqslant z \leqslant M\}$. An interval I is *good* if every arc with tail in I and head outside I has tail M and every backward arc in I has tail M.

Let I be an interval of D. The digraph D/I obtained from D by *contracting* I is the digraph with vertex set $(V \setminus I) \cup \{v_I\}$ such that xy is an arc if and only either $v_I \notin \{x, y\}$ and $xy \in A(D)$, or $x = v_I$ and there exists $x_I \in I$ such that $x_I y \in A(D)$, or $y = v_I$ and there exists $y_I \in I$ such that $xy_I \in A(D)$.

Similarly, the binary relation $\leqslant_{/I}$ obtained from $\leqslant$ by *contracting* I is the binary relation on $(V \setminus I) \cup \{v_I\}$ such that $x \leqslant_{/I} y$ if and only if either $v_I \notin \{x, y\}$ and $x \leqslant y$, or $x = v_I$, and there exists $x_I \in I$ such that $x_I \leqslant y$, or $y = v_I$, and there exists $y_I \in I$ such that $x \leqslant y_I$. We claim that if I is good then $\leqslant_{/I}$ is a partial order. Indeed, suppose it is not: then there are two elements u and t such that $u \leqslant_{/I} v_I$, $v_I \leqslant t$ and $u \not\leqslant_{/I} t$. Then $M \not\leqslant t$. Let $\alpha \in I$ be the maximal element of I such that $\alpha \leqslant t$, let λ be a successor of α in I, and

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let γ be a successor of α not in I (it exists as $t \notin I$ and $\alpha \leqslant t$ and maximal in I with this property). Then λ and γ are incomparable, and $\alpha\gamma$ and $\alpha\lambda$ are in the Hasse diagram of $\leqslant$. Because I is good, it follows that $\gamma\alpha$ and $\alpha\lambda$ are arcs of D, which is impossible as D is supposed to be a counter-example.

Hence, if I is a good interval, then $(\leqslant_{/I}, D/I)$ is an ordered digraph. Note that if $x \leqslant_{/I} v_I$, then $x \leqslant M$ with M the maximum of I. The crucial point is that if I is a good interval of D for which the conclusion of Lemma 4.2 holds for $(\leqslant_{/I}, D/I)$, then it holds for $(\leqslant, D)$. Indeed, suppose there exists two arcs $\gamma\alpha$ and $\beta\lambda$ of D/I such that $\alpha \leqslant_{/I} \beta \leqslant_{/I} \gamma$, $\beta \leqslant_{/I} \lambda$, and $\gamma \not\leqslant_{/I} \lambda$. Note that since I is good, we have $v_I \neq \gamma$. Let M be the maximum of I.

If $v_I \notin \{\alpha, \beta, \gamma, \lambda\}$, then $\gamma\alpha$ and $\beta\lambda$ gives the conclusion for D.

If $v_I = \alpha$, then γM is an arc. Let us show that $M \leqslant \beta$. Indeed, let x be a maximal vertex in I such that $x \leqslant \beta$ and let y be a minimal vertex such that $x \leqslant y \leqslant \beta$. Since the Hasse diagram of $\leqslant$ is included in D, xy is an arc, and so $x = M$ (since I is good). Thus γM and $\beta\lambda$ are the desired arcs.

If $v_I = \beta$, then $M\lambda$ is an arc and $\alpha \leqslant M$, so $\gamma\alpha$ and $M\lambda$ are the desired arcs.

If $v_I = \lambda$, then there exists $\lambda_I \in I$ such that $\beta\lambda_I$, so $\gamma\alpha$ and $\beta\lambda_I$ are the desired arcs.

Hence, to get a contradiction, it is sufficient to find a good interval I such that $(\leqslant_{/I}, D/I)$ satisfies the hypotheses of Lemma 4.2.

Observe that there are at least two backward arcs. Indeed, if there are two minimal elements for $\leqslant$, there are at least three backward arcs entering these vertices (since one of them can be x). And if there is a unique minimum m, by letting m' minimal in $V \setminus m$, at least two arcs are entering m, m'.

Let M be a vertex which is the tail of a backward arc and which is minimal for $\leqslant$ for this property. Since two arcs cannot have the same tail, M is not the maximum of $\leqslant$ (if any). Let Mm be the backward arc with tail M.

We claim that the interval J with minimum m and maximum M is good. Indeed, by the definition of M, no backward arc has its tail in $J \setminus \{M\}$. Moreover, any forward arc $\beta\lambda$ with its tail in $J \setminus \{M\}$ and its head outside J would give our conclusion (with $\alpha = m$ and $\gamma = M$), a contradiction.

Now consider a good interval I with maximum M which is maximal with respect to inclusion. We claim that if $x \in I$, then there is at least one arc entering I, and if $x \notin I$, there are at least two arcs entering I with different tails.

Call m_1 the minimum of I and m_2 any minimal element of $I \setminus m_1$. First assume that x is in I. There are at least three arcs with heads m_1 or m_2. One of them is m_1m_2, one of them can be with tail M, but there is still one left with tail not in I. Now assume that x is not in I. There are at least two arcs with heads m_1 or m_2 and tails not in I. If the tails are different, we are done. If the tails are the same, say v, observe that vm_1 and vm_2 are both backward or both forward (otherwise v would be in I). Since both cannot be backward, both vm_1 and vm_2 are forward. Hence the interval with minimum v and maximum M is a good interval, contradicting the maximality of I. This proves the claim.

This in turn implies that $(\leqslant_{/I}, D/I)$ satisfies the hypotheses of Lemma 4.2, yielding a contradiction. □

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Proof of Lemma 4.1. Let G be a galaxy of D which spans a maximum number of vertices of degree four. Suppose for the sake of contradiction that some vertex x with degree four is not spanned.

An *alternating path* is an oriented path ending at x, starting by an arc of G, and alternating with arcs of G and arcs of $A(D) \setminus A(G)$. We denote by $\mathcal{A}$ the set of arcs of G which belong to an alternating path.

Claim 1. *Every arc of $\mathcal{A}$ is a component of G.*

Proof of claim. Indeed, if uv belongs to $\mathcal{A}$, it starts some alternating path P. Thus, if u has outdegree more than one in G, the digraph with the set of arcs $A(G)\triangle A(P)$ is a galaxy and spans $V(G) \cup x$. □

Claim 2. *There are no circuits alternating arcs of $\mathcal{A}$ and arcs of $A(D) \setminus \mathcal{A}$.*

Proof of claim. Assume that there is such a circuit C. Consider a shortest alternating path P starting with some arc of $\mathcal{A}$ in C. Now the digraph with arcs $A(G)\triangle(A(P) \cup A(C))$ is a galaxy which spans $V(G) \cup x$, contradicting the maximality of G. □

We now endow $\mathcal{A} \cup x$ with a partial order structure by letting $a \leqslant b$ if there exists an alternating path starting at a and ending at b. The fact that this relation is a partial order relies on Claim 2. Observe that x is the maximum of this order.

We also construct a digraph $\mathcal{D}$ on vertex set $\mathcal{A} \cup x$ with all arcs $uv \to st$ such that us or vs is an arc of D (and $uv \to x$ such that ux or vx is an arc of D).

Claim 3. *The pair $(\mathcal{D}, \leqslant)$ is an ordered digraph. Moreover, an arc of $\mathcal{A}$ is the tail of at most one backward arc and two forward arcs, and x is the tail of at most two backward arcs.*

Proof of claim. The fact that the Hasse diagram of $\leqslant$ is contained in $\mathcal{D}$ follows from the fact that if $uv \leqslant st$ belongs to the Hasse diagram of $\leqslant$, there is an alternating path starting by $uvst$, in particular, the arc vs belongs to D, and thus $uv \to st$ in $\mathcal{D}$.

Suppose that $uv \to st$; then vs or us is an arc of D. If vs is an arc, because there is no alternating circuit, st follows uv on some alternating path, and so $uv \leqslant st$. In this case, $uv \to st$ is forward.

If us is an arc of D, we claim that $st \leqslant uv$. Indeed, if an alternating path P starting at st does not contain uv, the galaxy with arcs $(A(G)\triangle A(P)) \cup \{us\}$ spans $V(G) \cup x$, contradicting the maximality of G. In this case, $uv \to st$ is backward.

It follows that an arc uv of $\mathcal{A}$ is the tail of at most one backward arc (since this arc and uv are the two arcs leaving u in D), and uv is the tail of at most two forward arcs (since v has outdegree at most two). Furthermore, since x has outdegree at most two, it follows that x is the tail of at most two backward arcs. □

Claim 4. *The indegree of every vertex of $\mathcal{D}$ is two.*

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Proof of claim. Let uv be a vertex of $\mathcal{D}$ which starts an alternating path P. If u has indegree less than two, and thus does not belong to the set of vertices of degree four, the galaxy with arcs $A(G)\triangle A(P)$ spans more vertices of degree four than G, a contradiction. Let s and t be the two inneighbours of u in D. An element of $\mathcal{A}\cup x$ should contain s, since otherwise, the galaxy with arcs $(A(G)\triangle A(P))\cup\{su\}$ spans $V(G)\cup x$ and contradicts the maximality of G. Similarly an element of $\mathcal{A}\cup x$ contains t.

Observe that the same element of $\mathcal{A}\cup x$ cannot contain both s and t (either the arc st or the arc ts), otherwise the arcs su and tu would be both backward or forward, which is impossible. □

At this stage, in order to apply Lemma 4.2, we just need to ensure that the backward outdegree of every vertex is at most one. Since the only element of $\mathcal{D}$ which is the tail of two backward arcs is x, we simply delete any of these two backward arcs. The indegree of a vertex of $\mathcal{D}$ decreases by one, but we are still fulfilling the hypothesis of Lemma 4.2.

Hence, according to this lemma, $\mathcal{D}$ contains two arcs $\gamma\alpha$ and $\beta\lambda$ such that $\alpha\leqslant\beta\leqslant\gamma$, $\beta\leqslant\lambda$ and $\gamma\nleqslant\lambda$. Recall that $\alpha,\beta,\gamma,\lambda$ are elements of $\mathcal{A}\cup x$. In particular, there is an alternating path P containing α,β,λ (in this order) which does not contain γ. Setting $\alpha=\alpha_1\alpha_2$ and $\gamma=\gamma_1\gamma_2$, note that the backward arc $\gamma\alpha$ corresponds to the arc $\gamma_1\alpha_1$ in D. We reach a contradiction by considering the galaxy with arcs $(A(G)\triangle A(P))\cup\{\gamma_1\alpha_1\}$ which spans $V(D')\cup x$. The proof of Lemma 4.1 is now complete. □

5. Multiple fibres

In this section we consider the general problem with $n\geqslant 2$ fibres, and give lower and upper bounds on $\lambda_n(m,k)$. Let us start by proving a lower bound on $\lambda_n(m,k)$.

Proposition 5.1. *For all $m,n,k\in\mathbb{N}$, we have $\lambda_n(m,k)\geqslant\lceil\frac{m}{n}\lceil\frac{k}{n}\rceil+\frac{k}{n}\rceil$.*

Proof. Consider the following m-labelled digraph $G_{n,m,k}$ with vertex set $X\sqcup Y\sqcup Z$ such that:

- $|X|=k$, $|Y|=k2^{(m+1)k}$ and $|Z|=m\binom{|Y|}{k}$,
- for any $x\in X$ and $y\in Y$, there is an arc xy (of whatever label),
- for every set S of k vertices of Y and any integer $1\leqslant i\leqslant m$, there is a vertex z_S^i in Z which is dominated by all the vertices of S via arcs labelled i.

Suppose there exists an n-fibre colouring of $G_{n,m,k}$ with $c<\lceil\frac{m}{n}\lceil\frac{k}{n}\rceil+\frac{k}{n}\rceil$ colours. For $y\in Y$ and $1\leqslant i\leqslant m$, let $C_i(y)$ be the set of colours assigned to the arcs labelled i leaving y. For $0\leqslant j\leqslant n$, let P_j be the set of colours used on j arcs entering y (and necessarily with two different fibres). Then $\sum_{j=0}^{n}j|P_j|=k$ as k arcs enter y. Moreover, $\sum_{j=0}^{n}|P_j|=c$, since $(P_0,P_1,\ldots,P_n)$ is a partition of the set of colours. Now each colour of P_j may appear in at most $n-j$ of the $C_i(y)$, so

$$\sum_{i=1}^{m}|C_i(y)|\leqslant\sum_{j=0}^{n}(n-j)|P_j|=n\sum_{j=0}^{n}|P_j|-\sum_{j=0}^{n}j|P_j|=cn-k.$$

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Because $|Y| > (k-1)2^{cm}$, there is a set S of k vertices y of Y having the same m-tuple $(C_1(y),\dots,C_m(y)) = (C_1,\dots,C_m)$. Without loss of generality, we may assume $|C_1| = \min\{|C_i| \mid 1 \leqslant i \leqslant m\}$. Hence $|C_1| \leqslant \frac{cn-k}{m}$. But the vertex z_S^1 has indegree k, so $|C_1| \geqslant \frac{k}{n}$. Since $|C_1|$ is an integer, we have $\lfloor \frac{cn-k}{m} \rfloor \geqslant |C_1| \geqslant \lceil \frac{k}{n} \rceil$. So $c \geqslant \frac{m}{n}\lceil \frac{k}{n} \rceil + \frac{k}{n}$. Since c is an integer, we get $c \geqslant \lceil \frac{m}{n}\lceil \frac{k}{n} \rceil + \frac{k}{n} \rceil$, a contradiction. □

Note that the graph $G_{n,m,k}$ is acyclic. The following lemma shows that, if $m \geqslant n$, one cannot expect better lower bounds by considering acyclic digraphs. Indeed, $G_{n,m,k}$ is the m-labelled acyclic digraph with indegree at most k for which an n-fibre colouring requires more colours.

Lemma 5.2. *Let D be an acyclic m-labelled digraph with $\Delta^- \leqslant k$. If $m \geqslant n$, then $\lambda_n(D) \leqslant \lceil \frac{m}{n}\lceil \frac{k}{n} \rceil + \frac{k}{n} \rceil$.*

Proof. Since D is acyclic, its vertex set admits an ordering $(v_1, v_2, \dots, v_p)$ such that if $v_j v_{j'}$ is an arc, then $j < j'$.

By induction on q, we shall find an n-fibre colouring of $D[\{v_1,\dots,v_q\}]$ together with sets $C_i(v_r)$ of $\lceil \frac{k}{n} \rceil$ (potential) colours, for $1 \leqslant i \leqslant m$ and $1 \leqslant r \leqslant q$, such that assigning a colour in $C_i(v_r)$ to an arc labelled i leaving v_r (in the future) will fulfil the condition of an n-fibre colouring at v_r.

Starting the process is easy. We may let $C_i(v_1)$ be any family of $\lceil \frac{k}{n} \rceil$-sets such that a colour appears in at most n of them.

Suppose now that we have an n-fibre colouring of $D[\{v_1,\dots,v_{q-1}\}]$, and that, for any $1 \leqslant i \leqslant m$ and $1 \leqslant r \leqslant q-1$, the set $C_i(v_r)$ is already determined. Let us colour the arcs entering v_q. Each of these arcs $v_r v_q$ may be assigned one of the $\lceil \frac{k}{n} \rceil$ colours of $C_{l(v_r v_q)}(v_r)$. Since a colour may be assigned to n arcs (using different fibres) entering v_q, one can assign a colour and a fibre to each such arc. It remains to determine the sets $C_i(v_q)$, $1 \leqslant i \leqslant m$.

For $0 \leqslant j \leqslant n$, let P_j be the set of colours assigned to j arcs entering v_q. Let $N = \sum_{i=0}^{n}(n-j)|P_j|$ and $(c_1, c_2, \dots, c_N)$ be a sequence of colours such that each colour of P_j appears exactly $n-j$ times and consecutively. For $1 \leqslant i \leqslant m$, set $C_i(v_q) = \{c_a \mid a \equiv i \bmod m\}$. As $n \leqslant m$, a colour appears at most once in each $C_i(v_q)$. Moreover, $N = n\lceil \frac{m}{n}\lceil \frac{k}{n} \rceil + \frac{k}{n} \rceil - k \geqslant m\lceil \frac{k}{n} \rceil$. So for $1 \leqslant i \leqslant m$, $|C_i(v_q)| \geqslant \lceil \frac{k}{n} \rceil$. □

Lemma 5.2 shows that the lower bound of Proposition 5 is tight for acyclic digraphs. In fact, we conjecture that the lower bound remains tight for digraphs in general.

Conjecture 5.3. $\lambda_n(m,k) = \lceil \frac{m}{n}\lceil \frac{k}{n} \rceil + \frac{k}{n} \rceil$.

We now establish an upper bound on $\lambda_n(m,k)$ for general digraphs. Note that the graphs $G_{n,m,k}$ require many colours but have very large outdegree. We first give an upper bound on $\lambda_n(D)$ for m-labelled digraphs with bounded in- and outdegree. In this case, one can show that few colours are needed. This is derived from the following theorem of Guiduli.

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Theorem 5.4 (Guiduli [6]). *If* $\Delta^-, \Delta^+ \leqslant k$, *then* $\mathrm{dst}(D) \leqslant k + 20\log k + 84$. *Moreover, D admits a directed star colouring with* $k + 20\log k + 84$ *colours such that, for each vertex v, there are at most* $10\log k + 42$ *colours assigned to its leaving arcs.*

As we will show below, Guiduli's theorem can be extended to the following statement for m-labelled digraphs.

Theorem 5.5. *Let*

$$f(n,m,k) = \left\lceil \frac{k + (10m^2+5)\log k + 80m^2 + m + 21}{n} \right\rceil$$

and let D be an m-labelled digraph with $\Delta^-, \Delta^+ \leqslant k$. *Then* $\lambda_n(D) \leqslant f(n,m,k)$. *Moreover, D admits an n-fibre colouring with* $f(n,m,k)$ *colours such that, for each vertex v and each label l, the number of colours assigned to the arcs labelled l and leaving v is at most* $g(m,k) = \lceil (10m+5)\log k + 40m + 21 \rceil$.

As one can notice, Theorem 5.5 in the case $n = m = 1$ is slightly better than Theorem 5.4 (for $\Delta^-, \Delta^+ \leqslant k$, Theorem 5.5 gives $\mathrm{dst}(D) \leqslant k + 15\log k + 102$). But this is superficial and is only due to the upper bound given in Lemma 1.1, which is better than the upper bound 3Δ used by Guiduli. Indeed, the methods are identical.

We recall the following definition: given a family of sets $\mathcal{F} = (A_i, i \in I)$, a *transversal* of $\mathcal{F}$ is a family of distinct elements $(t_i, i \in I)$ with $t_i \in A_i$ for all $i \in I$.

Lemma 5.6. *Let D be an m-labelled digraph with* $\Delta^- \leqslant k$. *Suppose that for each vertex v, there are m disjoint lists* $L_v^1, \ldots, L_v^m$ *of c colours, each being a subset of* $\{1, \ldots, k+c\}$. *If, for each vertex v, the family* $\{L_y^i \mid yx \in E(D)$ *and* yx *is labelled* $i\}$ *has a transversal, then there is a 1-fibre colouring of D with* $k + (2m^2+1)c + m$ *colours such that, for each vertex v and each label l, at most* $(2m+1)c + 1$ *colours are assigned to arcs labelled l that leave v.*

Proof. Using the transversal to colour the entering arcs at each vertex, we obtain a colouring with few conflicts. Indeed, there is no conflict between arcs entering a same vertex. So the only possible conflicts are between an arc entering a vertex v and an arc leaving v. Since arcs leaving v use at most mc colours (those of $L_v^1 \cup \cdots \cup L_v^m$), there are at most mc arcs entering v having the same colour as an arc leaving c. Removing such entering arcs for every vertex v, we obtain a digraph D' for which the colouring with the $k + c$ colours is a 1-fibre colouring. We now want to colour the arcs of $D - D'$ with few extra colours. Consider a label $1 \leqslant l \leqslant m$ and let D'_l be the digraph induced by the arcs of $D - D'$ labelled l. Then D'_l has indegree at most mc. By Theorem 1.1, we can partition D'_l into $2mc + 1$ star forests. Thus D can be 1-fibre coloured with $k + c + m(2mc+1)$ colours. Moreover, in the colouring described above, arcs labelled l which leave a vertex v have a colour in L_v^l or corresponding to one of the $2mc+1$ star forests of D'_l. So at most $(2m+1)c + 1$ colours are assigned to arcs labelled l leaving v. □

We will also need the following theorem.

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Theorem 5.7 (Alon, McDiarmid and Reed [2]). *Let k and c be positive integers with $k \geqslant c \geqslant 5\log k + 20$. Choose independent random subsets $S_1, \ldots, S_k$ of $X = \{1, \ldots, k+c\}$ as follows. For each i, choose S_i by performing c independent uniform samplings from X. Then the probability that $S_1, \ldots, S_k$ do not have a transversal is at most $k^{3-\frac{c}{2}}$.*

Proof of Theorem 5.5. It suffices to prove the result for $n = 1$. Indeed, we can extend a 1-fibre colouring satisfying the conditions of the theorem into an n-fibre colouring satisfying the conditions by replacing all the colours $qn + r$ with $1 \leqslant r \leqslant n$ by the colour $q + 1$ on fibre r.

Let $c = \lceil 5\log k + 20 \rceil$. We can assume $k \geqslant mc$. For all vertices x, select mc different ordered elements $e_1, e_2, \ldots, e_{mc}$ independently and uniformly. For all $1 \leqslant i \leqslant m$, let $L_x^i = \{e_{ci+1}, \ldots, e_{c(i+1)}\}$. Each set has the same distribution as a set of c elements chosen uniformly and independently.

Let A_x be the event that the family $\{L_y^i \mid yx \in E(D)$ and yx is labelled $i\}$ fails to have a transversal. By Theorem 5.7, $P(A_x) \leqslant k^{3-c/2}$. Furthermore, the event A_x is independent of all A_y for which there is no vertex z such that both zx and zy are in $E(D)$. It follows that the dependency graph for these events has degree at most k^2, and so we can apply the Lovász Local Lemma to obtain that there exists a family of lists satisfying conditions of Lemma 5.6. This lemma gives the desired colouring. □

For general digraphs, when we do not have $\Delta^-, \Delta^+ \leqslant k$, we may use the following trick to obtain an upper bound. Any digraph D may be decomposed into an acyclic digraph D_a and an Eulerian digraph D_e (*i.e.*, in D_e, for every vertex v, $d^-_{D_e}(V) = d^+_{D_e}(v)$). (To see this, consider an Eulerian subdigraph D_e of D which has a maximum number of arcs. Then the digraph $D_a = D - D_e$ is necessarily acyclic.) Hence, by Lemma 5.2 (applied to D_a) and Theorem 5.5 (applied to D_e), we have:

$$\text{if } m \geqslant n, \quad \text{then } \lambda_n(D) \leqslant \left\lceil \frac{m}{n} \left\lceil \frac{k}{n} \right\rceil + \frac{k}{n} \right\rceil + f(n,m,k),$$

for $f(n,m,k)$ the function given in Theorem 5.5. But, as we will now show, it is possible to lessen this bound by roughly $\frac{k}{n}$.

Theorem 5.8. *If $m \geqslant n$, then*

$$\lambda_n(m,k) \leqslant \left\lceil \frac{m}{n} \left\lceil \frac{k}{n} \right\rceil + \frac{k}{n} \right\rceil + 2m \frac{\lceil (10m+5)\log k + 40m + 21 \rceil}{n}.$$

Proof. Let D be an m-labelled digraph with $\Delta^-(D) \leqslant k$. Consider a decomposition of D into an acyclic digraph D_a and an Eulerian digraph D_e. We first apply Theorem 5.5 to find an n-fibre colouring of the arcs of D_e with $f(n,m,k)$ colours such that, in addition, at most $g(m,k)$ colours are assigned to the arcs leaving each vertex.

We shall extend the n-fibre colouring of D_e to the arcs of D_a in a way similar to the proof of Lemma 5.2. That is, we will assign to each vertex v, sets $C_i(v)$, $1 \leqslant i \leqslant m$ of $\lceil \frac{k}{n} + mg(m,k) \rceil$ colours such that an arc labelled i leaving v will be labelled using a colour in $C_i(v)$.

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Let $(v_1, \ldots, v_n)$ be an ordering of the vertices of A such that if $v_j v_{j'}$ is an arc then $j < j'$. We start to build the $C_i(v_1)$ with the colours assigned to the leaving arcs of v_1 labelled i. The vertex v_1 has at most k entering arcs. Each of them forbid one *type* (colour, fibre). In the colouring of D_e induced by Theorem 5.5, there are at most $mg(m,k)$ types assigned to the arcs leaving v_1. So there are at least $\lceil \frac{mk}{n^2} \rceil + \lceil \frac{k}{n} \rceil + 2m\frac{g(m,k)}{n} - k - mg(m,k) \geqslant m\lceil \frac{k}{n} \rceil + mg(m,k)$ types unused at vertex v_1. Since $m \geqslant n$, we can partition these types into m sets of size at least $\frac{k}{n}$ such that no two types having the same colour are in the same set. These sets are the $C_i(v_1)$.

Suppose that the sets have been defined for v_1 up to v_{q-1}, and that all the arcs $v_i v_j$ for $i < j < q$ have been assigned a colour. We now give a colour to each arc $v_i v_q$ for $i < q$.

There are k_e arcs entering v_q in D_e which are already coloured. So it remains to give a colour to $k_a \leqslant k - k_e$ arcs. Each uncoloured arc may be assigned a colour in a list of size at least $\lceil \frac{k}{n} + mg(m,k) \rceil$. This gives a choice between $n\lceil \frac{k}{n} + mg(m,k) \rceil$ different types. k_e types are forbidden by the entering arcs in D_e while at most $mg(m,k)$ types are forbidden by the leaving arcs in D_e. Hence, there remain at least $n\lceil \frac{k}{n} + mg(m,k) \rceil - k_e - mg(m,k) \geqslant k_a$ types for the entering arcs of D_a. So one can assign distinct available colours to each of the k_a arcs entering v_q. We then build the $C_i(v_q)$ as we did for v_1.

Once this process is finished, we obtain an n-fibre colouring of D using $\lceil \frac{mk}{n^2} \rceil + \lceil \frac{k}{n} \rceil + 2m\frac{g(m,k)}{n}$ colours. □

Theorem 5.8 gives an upper bound on $\lambda_n(m,k)$ when $m \geqslant n$. We now give an upper bound for the case $m < n$.

Proposition 5.9. If $m < n$ then $\lambda_n(m,k) \leqslant \lceil \frac{k}{n-m} \rceil$.

Proof. Let D be an m-labelled digraph with $\Delta^- \leqslant k$. We should show the existence of a proper n-fibre colouring with $\lceil \frac{k}{n-m} \rceil$. For each vertex v, we give to each of its entering arcs a colour such that none of the colours is used more than $n-m$ times. This is possible since there are at most $k \leqslant (n-m)\lceil \frac{k}{n-m} \rceil$ arcs entering v. So we now have $\text{in}(v,\lambda) \leqslant n-m$. Moreover, each arc vw is given a colour by w. Since D is m-labelled, a colour λ can be used to colour an arc of at most m different labels, *i.e.*, $\text{out}(v,\lambda) \leqslant m$. Consequently $\text{in}(v,\lambda) + \text{out}(v,\lambda) \leqslant n$. This gives a proper n-fibre colouring. □

6. Concluding remarks

One question arising naturally from the previous sections is the complexity of calculating $\lambda_n(D)$ for an m-labelled digraph D. As we will show in the first subsection, unsurprisingly, this problem is $\mathcal{NP}$-hard even for the simpler problem of directed star arboricity and even for restricted class of digraphs of in- and outdegree bounded by two. We end this section by showing how a similar approach to the one in Section 3 allows us to give a very short proof of a recent result of Pinlou and Sopena [9].

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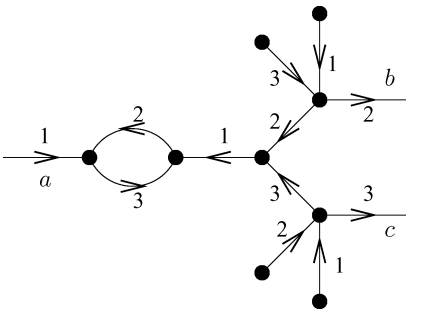


Figure 2. The graph H and one of its directed star 3-colourings.

6.1. Complexity

The digraphs with directed star arboricity one are the galaxies, so one can decide in polynomial time if $\mathrm{dst}(D) = 1$. Deciding whether or not $\mathrm{dst}(D) = 2$ is also easy, since we just have to check that the conflict graph (with vertex set the arcs of D, two distinct arcs xy, uv being in conflict when $y = u$ or $y = v$) is bipartite. However, for larger values, as expected, it is $\mathcal{NP}$-complete to decide if a digraph has directed star arboricity at most k. This is illustrated by the next result.

Theorem 6.1. *The following decision problem is $\mathcal{NP}$-complete:*
INSTANCE: A digraph D with $\Delta^+(D) \leqslant 2$ and $\Delta^-(D) \leqslant 2$.
QUESTION: Is $\mathrm{dst}(D)$ at most 3?

Proof. The proof is by a reduction from 3-edge-colouring of 3-regular graphs, which is known to be $\mathcal{NP}$-complete.

Let G be a 3-regular graph. It is easy to see that G admits an orientation D such that every vertex has in- and outdegree at least one (*i.e.*, D does not have neither sink nor source).

Let D' be the digraph obtained from D by replacing every vertex with indegree one and outdegree two by the subgraph H depicted in Figure 2, which has also one entering arc (namely $\vec{a}$) and two leaving arcs ($\vec{b}$ and $\vec{c}$). It is quite easy to check that in any directed star 3-colouring of H, the three arcs $\vec{a}$, $\vec{b}$ and $\vec{c}$ get different colours. Moreover, if these three arcs are precoloured with three different colours, we can extend this to a directed star 3-colouring of H. Such a colouring with $\vec{a}$ coloured 1, $\vec{b}$ coloured 2 and $\vec{c}$ coloured 3 is given in Figure 2. Furthermore, in a directed star 3-colouring, a vertex with indegree two and outdegree one must have its three incident arcs coloured differently. So $\mathrm{dst}(D') = 3$ if and only if G is 3-edge colourable. □

6.2. Acircuitic directed star arboricity

A directed star colouring is *acircuitic* if there is no *bicoloured* circuits, *i.e.*, circuits for which only two colours appear on its arcs. The *acircuitic directed star arboricity* of a

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digraph D is the minimum number k of colours such that there exists an acircuitic directed star k-colouring of D.

In this last section, we give a short alternative proof of the following theorem.

Theorem 6.2 (Pinlou and Sopena [9]). *Every subcubic oriented graph has acircuitic directed star arboricity at most* 4.

Indeed, it is possible to apply our Theorem 1.4 directly to derive this theorem. However, there is a shorter proof using the following lemma.

Lemma 6.3. *Let D be an acyclic subcubic digraph. Let L be a list-assignment on the arcs of D such that, for every arc uv, $|L(uv)| \geqslant d(v)$. Then D admits a directed star L-colouring.*

Proof. We prove the result by induction on the number of arcs of D; the result holds trivially if D has no arcs.

Since D is acyclic, it has an arc xy with y a sink. Let ω be a colour in $L(xy)$. For any arc $\vec{a}$ distinct from xy, set $L'(\vec{a}) = L(\vec{a}) \setminus \{\omega\}$ if $\vec{a}$ is incident to xy (and thus has head in $\{x, y\}$ since y is a sink), and $L'(\vec{a}) = L(\vec{a})$ otherwise. Then in $D' = D - xy$, we have $|L'(uv)| \geqslant d(v)$ for any arc $uv \neq xy$. Hence, by induction hypothesis, D' admits a directed star L'-colouring that can be extended to a directed star L-colouring of D by colouring xy with ω. □

Proof of Theorem 6.2. Let V_1 be the set of vertices of outdegree at most one and $V_2 = V \setminus V_1$. Every vertex of V_2 has outdegree at least two (and so indegree at most one).

Let M be the set of arcs with tail in V_1 and head in V_2. We colour all the arcs of M with colour 4. Moreover, for every circuit C in $D[V_1]$ or in $D[V_2]$, we choose an arc $\vec{a}(C)$ and colour it by 4. Note that, by definition of V_1 and V_2, the arc $\vec{a}(C)$ is not incident to any arc of M, and in addition, C is the unique circuit containing $\vec{a}(C)$. Let us denote by M_4 the set of all arcs coloured by 4. It is easily seen that M_4 is a matching and $D - M_4$ is acyclic.

We shall now find a directed star colouring of $D - M_4$ with colours $\{1, 2, 3\}$ that does not create any bicoloured circuit. In any colouring of the arcs, if such a circuit existed, 4 would be one of its colour because $D - M_4$ is acyclic, and moreover, all the arcs of this circuit coloured by 4 would be in M, because each arc in $M_4 \setminus M$ is in a unique circuit and this unique circuit has a unique arc coloured by 4. Hence we just have to be careful when dealing with arcs in the digraph induced by the end-vertices of the arcs of M.

Let us denote the arcs of M by $x_i y_i$, $1 \leqslant i \leqslant p$, and set $X = \{x_i, 1 \leqslant i \leqslant p\}$ and $Y = \{y_i, 1 \leqslant i \leqslant p\}$ (we have then $x_i \in V_1$ and $y_i \in V_2$). Let E' be the set of arcs with tail in Y and head in X. Let H be the graph with vertex set E' such that an arc $y_i x_j$ is adjacent to an arc $y_k x_l$ if:

(a) either $k = l$,

(b) or $j = k$ and $i > j$ and $l > j$.

Since a vertex of X has indegree at most two and a vertex of Y has outdegree at most two, H has maximum degree three. Moreover, H contains no K_4, because two arcs of

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E' with same tail y_k are not adjacent in H. Hence, by Brooks' theorem, H has a vertex colouring with colours $\{1,2,3\}$, and this colouring corresponds to a colouring c of the arcs of E'. Since (a) is satisfied, c is a directed star colouring. Moreover, this colouring creates no bicoloured circuits. Indeed, a circuit contains a subpath $y_i x_j y_j x_l$, with $i > j$ and $k > j$, whose three arcs are coloured differently by (b).

Let $D' = D - (M_4 \cup E')$. For any arc uv in D', let $L(uv) = \{1,2,3\} \setminus \{c(wv) \mid wv \in E'\}$. The set $L(uv)$ is the set of colours in $\{1,2,3\}$ that may be assigned to uv without creating any conflict with the already coloured arcs. The digraph D' is acyclic and $|L(uv)| \geqslant d(v)$, so by Lemma 6.3, it admits a directed star L-colouring. We infer that D has an acircuitic directed star colouring with colours in $\{1,2,3,4\}$ and the theorem follows. In addition, we note that in this colouring, the arcs coloured by 4 form a matching. □

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