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A large deviation local limit theorem

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1. *Introduction and results*

The following elegant one-sided large deviation result is given by S. V. Nagaev in [2].

THEOREM 0. *Suppose that* $\{S_n, n \geqslant 0\}$ *is a random walk whose increments* X_i *are independent copies of* X, *where* $\mathbb{E}(X) = 0$ *and*

$$Pr\{X > x\} \sim x^{-\alpha} L(x) \quad as \quad x \to +\infty,$$

and where $1 < \alpha < \infty$ *and* L *is slowly varying at* ∞. *Then for any* $\epsilon > 0$ *and uniformly in* $x \geqslant \epsilon n$

$$Pr\{S_n > x\} \sim n Pr\{X > x\} \quad as \quad n \to \infty.$$

It is the purpose of this note to point out that for lattice-valued random walks there is an analogous local limit theorem.

THEOREM 1. *Suppose that* $\{S_n, n \geqslant 0\}$ *is a random walk on the integers with* $\mathbb{E}(X) = \mu$ *finite and such that*

$$Pr\{X = m\} \sim L(m)\, m^{-(1+\alpha)} \quad as \quad m \to \infty,$$

where $1 < \alpha < \infty$ *and* L *is slowly varying at* ∞. *Then for any* $\epsilon > 0$ *and uniformly in* $m \geqslant (\mu + \epsilon) n$

$$Pr\{S_n = m\} \sim n Pr\{X = [m - n\mu]\} \quad as \quad n \to \infty.$$

2. *Proof*

We prove Theorem 1 in the case $\mu = 0$, only minor changes being required when $\mu \neq 0$. With $\epsilon > 0$ fixed throughout, we have to show that with $p_n = Pr\{X = n\}$

$$p_m \sim L(m)/m^{\alpha+1} \quad \text{as} \quad m \to \infty, \tag{1}$$

implies that, uniformly for $m \geqslant \epsilon n$,

$$Pr\{S_n = m\} \sim n p_m \quad \text{as} \quad n \to +\infty. \tag{2}$$

If we write $\Psi(x) = Pr\{X > x\}$ then (1) implies that

$$\Psi(x) \sim \alpha^{-1} L(x)/x^{\alpha} \quad \text{as} \quad x \to +\infty, \tag{3}$$

so that Theorem 0 applies, and, uniformly in $x \geqslant \epsilon n$,

$$Pr\{S_n > x\} \sim n\Psi(x) \quad \text{as} \quad n \to +\infty. \tag{4}$$

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The technique used in [2] involves partitioning the event $\{S_n > x\}$ according to how many of the events $\{X_i \geqslant x/(\log x)^3\}$ occur, and our proof follows the same lines. Specifically, with $l = m/(\log m)^3$ and

$$A_k = \{\text{exactly } k \text{ of } X_i \geqslant l, 1 \leqslant i \leqslant n \text{ occur}\},$$

we put $q_k = Pr\{A_k \wedge \{S_n = m\}\}$ for $k = 0$, $k = 1$, and $q_2 = Pr\{\{\bigcup_2^n A_k\} \wedge \{S_n = m\}\}$, so that

$$Pr\{S_n = m\} = q_0 + q_1 + q_2. \tag{5}$$

Now

$$\begin{aligned} q_2 &= Pr\{S_n = m, \text{ 2 or more of } \{X_1, X_2, \ldots, X_n\} \geqslant l\} \\ &\leqslant {}^nC_2 Pr\{S_n = m, X_n \geqslant l; X_{n-1} \geqslant l\} \\ &= {}^nC_2 \sum_r Pr\{S_{n-2} = r\} Pr\{X_1 \geqslant l, X_2 \geqslant l, X_1 + X_2 = m - r\} \\ &\leqslant {}^nC_2 \sup_i Pr\{X_1 \geqslant l, X_2 \geqslant l, X_1 + X_2 = i\} \\ &\leqslant {}^nC_2 Pr\{X_2 \geqslant l\} \sup_{k \geqslant l} Pr\{X_1 = k\}. \end{aligned}$$

Since (1) implies that $\sup_{k \geqslant l} Pr\{X_1 = k\} \sim p_l$, it follows easily that

$$q_2 = o(np_m) \quad \text{as} \quad n \to +\infty. \tag{6}$$

As we are assuming $\mathbb{E}(X) = 0$, there exists a sequence $n' \to +\infty$ with $n'/n \to 0$ such that

$$Pr\{|S_n| > n'\} \to 0 \quad \text{as} \quad n \to +\infty. \tag{7}$$

The crux of Nagaev's proof is that there also exists a sequence $m' \to +\infty$ with $m'/m \to 0$ such that, for all sufficiently large n

$$Pr\{S_n \geqslant m', M_n < l\} \leqslant \exp\{-(\log m)^2\}, \tag{8}$$

where $M_n = \max\{X_1, X_2, \ldots, X_n\}$. It follows immediately that

$$q_0 \leqslant Pr\{S_n \geqslant m', M_n < l\} = o(np_m) \quad \text{as} \quad n \to +\infty, \tag{9}$$

and if

$$q_1^{(1)} = Pr\{S_n = m, X_n \geqslant l, M_{n-1} < l, S_{n-1} > m'\}$$

then

$$q_1^{(1)} \leqslant Pr\{S_{n-1} \geqslant m', M_{n-1} < l\} = o\{p_m\} \quad \text{as} \quad n \to +\infty.$$

Also if

$$q_1^{(3)} = Pr\{S_n = m, X_n \geqslant l, M_{n-1} < l, S_{n-1} < -n'\}$$

we have, in view of (7),

$$\begin{aligned} q_1^{(3)} &\leqslant \sum_{j < -n'} Pr\{S_{n-1} = j, X_n = m - j\} \leqslant Pr\{S_{n-1} < -n'\} \sup_{j > n'} Pr\{X_1 = m + j\} \\ &= o(p_m) \quad \text{as} \quad n \to \infty. \end{aligned}$$

Since $n'/m \to 0$ and $m'/m \to 0$ it follows from (1) that $Pr\{X_n = m - j\} \sim p_m$ uniformly for $-n' \leqslant j \leqslant m'$; since $l = o(m)$ it follows that, with

$$q_1^{(2)} = Pr\{S_n = m, X_n \geqslant l, M_{n-1} < l, -n' \leqslant S_{n-1} \leqslant m'\}$$

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we have

$$q_1^{(2)} \sim Pr\{M_{n-1} < l, -n' \leqslant S_{n-1} \leqslant m'\} p_m \quad \text{as} \quad n \to +\infty.$$

From (3) we have

$$Pr\{M_{n-1} < l\} = 1 + O(n\Psi(l)) = 1 + o(1) \quad \text{as} \quad n \to +\infty,$$

so in view of (7) and (8) we have $q_1^{(2)} \sim p_m$. But clearly

$$q_1 = n\,Pr\{S_n = m, X_n \geqslant l, M_{n-1} < l\} = n\{q_1^{(1)} + q_1^{(2)} + q_1^{(3)}\},$$

so we have shown that

$$q_1 \sim np_m \quad \text{as} \quad n \to +\infty. \tag{10}$$

Together with (6) and (9) this establishes (2), as required.

3. *Remarks*

(i) Although we have not specifically assumed aperiodicity in Theorem 1 it is clear that any random walk satisfying the hypotheses of Theorem 1 has $Pr\{X = m\} > 0$ for all sufficiently large m and hence is aperiodic. A version of the result in the case of periodic walks can easily be formulated.

(ii) An application of Theorem 1 is given in [1] and indeed it was this that motivated the present work.

REFERENCES

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