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Three related quartic curves in four dimensions.

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Introduction.

1. It has been shown by Pieri*, and independently by James†, that the trisecant planes of a quartic curve in [4] which meet a line meet also another quartic curve intersecting the former in six points. James generalises the theorem and shows that trisecant planes of a quartic curve C, which meet a quartic curve C_1 having six points in common with C, also meet another quartic curve C_2, and that the relation between the three curves is symmetrical. The object of this note is to give a simple proof of this theorem and to discuss the representation by which this proof is effected. The method used is analogous to that of Pieri; an interesting differential method is adopted by C. Segre‡ who shows that the foci of the first and second orders, of any linear system of ∞^2 planes in [4] of which two planes pass through a point, determine a conic and five points respectively, in any plane of the system: the trisecant planes of C meeting C_1 give a particular case of such a system.

Related g_3^1 on normal curves.

2·11. Triads apolar to a quartic form are sets of a cubic involution and may be represented on the rational normal curves in spaces of different dimensions; it is proposed to relate these representations and to consider the correspondence thus arising between elements in the spaces. In [4] if the point α represents the quartic form with reference to the normal curve, the involution is given by the trace on the curve of all trisecant planes through α; and any cubic involution, being determined by two of its triads, may be so represented. In [3] the involution is cut out on the normal curve C^3 by planes through a line a, which is thus in general uniquely related to the point α of [4]. The resulting correspondence between the points of [4] and lines of [3] has certain exceptional elements: the points of a chord of C are all related to one and the same chord of C^3, while to all the lines through a point of C^3 corresponds the same point of C.

This correspondence may be regarded as the projection of C into C^3 from a point O upon C. The trisecant planes from an

* *Gior. di Mat.*, 28 (1890), 209.
† *Proc. Camb. Phil. Soc.*, 21 (1923), 664.
‡ *Rendiconti Lincei*, 30 (1921), 67.

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arbitrary point α in [4] form one system of generating planes of a quadric point cone. Through O there passes one plane of the complementary system, this plane being incident with all the trisecant planes through α, so that these trisecant planes project into planes passing through a line a which contains the projection α' of α. Algebraically, the point α' is given by the first polar with respect to O of the quartic form representing α; and the cubic involution is given by the cubic forms apolar to all the first polars. Thus the points of the line a are represented by the first polars of the form giving α.

Two lines in [3] which intersect lie in one plane containing a triad of C^3 and thus correspond to two points of [4] lying in a trisecant plane of C. Thus the lines meeting a given line a correspond to points of a quadric cone of trisecant planes, vertex α. Moreover two points of a given general line in [4] lie on this cone, so that to the system of points of a line in [4] corresponds the system of lines in [3] which are such that two meet an arbitrary line a. Hence to the points of a line in [4], which meets three chords of C, correspond the lines of a regulus containing three chords of C^3. In particular the points of a director line (which is met by the chords of a quadratic involution on C) correspond to the lines of a regulus consisting entirely of chords of C^3, the ∞^2 director lines of C corresponding to the ∞^2 reguli through C^3.

Six lines of a regulus of general position in [3] contain points of C^3 where the curve meets the quadric containing the regulus; and two lines of the regulus meet an arbitrary line. Thus to the lines of a general regulus in [3] correspond the points of a curve containing six points of C and meeting the quadric cone of trisecant planes from an arbitrary point in two further points. The curve is thus a quartic curve C_1, having six points in common with C. The complementary regulus on the quadric corresponds to another quartic curve C_2 through the same six points; and since every plane through a line of one regulus contains a line of the other, *every trisecant plane of C which meets C_1 meets C_2 also.* In this sense these two curves may be said to be *complementary* with respect to C. In particular, when the regulus contains three chords of C^3, the corresponding curve consists of three chords of C and a line which meets them. We then have the theorem that *every trisecant plane of C which meets an arbitrary line meets also a quartic curve complementary to this line and passing through the six points where the three chords which are incident with the line meet the curve.*

2·12. We may proceed more directly to the relations between the three curves C, C_1, C_2 by stating the correspondence as follows:

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Line a in [3].	Point a in [4].
Lines meeting a.	Points of quadric cone from a containing C.
Lines meeting a and b.	Points of quartic surface of intersection of cones from a and β containing C.
Lines meeting a, b, and c.	Points of residual intersection of quadric cones from a, β, and γ containing C, i.e. points of C_1.

Thus the quadric cones containing C and having vertices a, β, γ intersect further in C_1 corresponding to the regulus (1) of lines meeting a, b, c, while the complementary regulus (2) containing a, b, c represents the curve C_2, which thus passes through a, β, γ. Moreover, if any other three lines of regulus (2) are taken instead of a, b, c, the same regulus (1) is determined. It follows then that *the curve C_2 is the locus of vertices of quadric cones containing C and C_1**, since any other three points of C_2 may be taken instead of a, β, γ, to define C_1. On each of these cones the planes of one system each meet C three times and C_1 once, while those of the other system each meet C once and C_1 three times, as may be seen by taking a section of the cone by a prime and projecting the curves, from the vertex of the cone, upon this section. It then appears that *the relation between the three curves is symmetrical*, any two being complementary with respect to the third, and each being the locus of vertices of quadric cones through the other two. It also appears that trisecant planes of C which pass through a point of C and meet C_1 generate a quartic cone.

2·13. Returning to the case when C_1 consists of an arbitrary line l and the three chords of C incident with l, we obtain the theorem that the locus of vertices of quadric cones containing C and l is a quartic curve C_2 meeting C in six points. The planes of one system of one of these cones meet l once and C three times, while the planes of the other system meet C once and each of the chords of C which meet l, as is seen by projecting from the vertex of one of the cones upon a prime section, for C projects into a rational quartic curve and l into a generator of the quadric containing this curve, while the generators of one system each meet the curve three times and those of the other system each meet it once. Thus *trisecant planes of C which meet an arbitrary line l meet a quartic curve C_2 also; planes which meet C and the three chords incident with l, each once, meet C_2 also; and trisecant planes of C_2 which meet l meet C also*, as is clear from the general case.

* This locus is not that of the vertices of *all* quadric cones containing C and C_1: there are ∞^1 points, which are intersections of chords of C_1 and C_2, from which these curves project into nodal quartics which are (2, 2) upon a quadric.

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2·14. This result may be regarded as a generalisation of the theorem that planes which meet four arbitrary lines in [4] meet also a fifth associated line. The fifteen line and fifteen point construction of Segre* may be arranged as arising from three degenerate quartic curves: a, b, c with transversal d; a', b', c' with d'; a'', b'', c'' with d''; yielding the incidence diagram

.	d''	d'	a	b	c
d''	.	d	a'	b'	c'
d'	d	.	a''	b''	c''
a	a'	a''	.	g	f
b	b'	b''	g	.	e
c	c'	c''	f	e	.

Equally well in this case may the construction arise from the three degenerate quartic curves a, a', a'' with e; b, b', b'' with f; c, c', c'' with g, giving in all six systems of planes trisecant to one quartic curve and meeting each of the other two†.

2·15. We now consider the case in which the curves C_1 and C_2 are not distinct, that is, when the corresponding reguli in [3] coincide and the quadric containing them is a cone, V. Let C_x denote the curve C_1 when it coincides with C_2. There are ∞^2 such curves through any six points of C. Any generator of the cone V meets any other generator and the corresponding points of C_x then lie in a trisecant plane of C and conversely *a trisecant plane of C which meets C_x meets it again, and thus passes through a chord.* Through the vertex of the cone V passes one chord of C^3 and planes through it contain two generators. The corresponding chord of C is then the vertex of a line-cone of trisecant planes of C which meet C_x in pairs of points determining an involution on C_x. The chord of C is thus a director line of C_x and *the curves C, C_x lie upon a line-cone having a chord of C which is a director line of C_x for vertex.* In [3] the locus of vertices of quadric cones through six points is a quartic surface, so that through the six points of C^3 there are four cones with vertices on an arbitrary line. Hence for an arbitrary point α in [4] there are four curves C_x such that the trisecant planes of C from α are bisecant to all of them.

The curve C_x is the locus of vertices of quadric cones through C and C_x. Hence, since from any point O on C_x the curves project into a quartic and cubic curve upon a quadric in [3], there is one chord of C_x through O which meets C. Thus *the quadric cones with vertices on C and passing through C_x are all line-cones having these chords as vertices.*

* H. F. Baker, *Principles of Geometry*, 4 (1925), 114.

† This figure arises significantly by projection from three points of itself of the V_3^6 discussed by James (*loc. cit.*).

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It may be remarked that if C_x is regarded as fundamental, and a (1, 1) correspondence is set up between its trisecant planes and the planes of [3] by projecting C_x, from a point of itself, into C^3, then the director line which is a chord of C is represented in [3] by a regulus containing C^3, while the curve C is represented by the same regulus—as in fact is any quartic curve C in [4] lying on the line-cone having the line vertex for a chord. Trisecant planes of C_x which meet the director line, or any quartic curve C, do not meet any further quartic curve.

2·21. We now show that *the system of* ∞^2 *planes* ψ, considered above, trisecant to C and meeting C_1 and C_2, *envelopes a quartic primal* M. The planes of the system correspond to tangent planes of the quadric Q containing the reguli which represent C_1 and C_2. Two tangent planes of Q may be drawn from a general line a and these coincide when a touches Q. Thus two planes of the system pass through an arbitrary point α; these coincide when α corresponds to a line touching Q, and in this case α is a focus of the first order of the system, and determines a point of the primal in question. But four of the system of lines touching Q belong to an arbitrary regulus, and hence in [4] four points of the locus enveloped by planes of the system are incident with an arbitrary line (or with any quartic curve having six points in common with C). The envelope is thus a quartic primal M. Further, if a is a generator of Q, ∞^1 tangent planes of Q pass through it and the corresponding point α is a focus of the second order of the system, ∞^1 planes of the system passing through any such focus, and the curve C_1 (or C_2) is thus a nodal curve of M.

2·22. Since the lines which lie in a tangent plane of Q and pass through the point of contact are each coplanar with a single chord of C^3, the corresponding chord of C is the vertex of a line-cone containing C and all the points of M which lie in the plane of the system which corresponds to the tangent plane of Q. Thus M is touched by any plane of the system along a conic, the foci of the second order in the plane being the five points of the conic which lie three on C and one on each of C_1 and C_2, respectively.

The curves C, C_1, C_2 are thus nodal curves on M; and since a plane of the system ψ_1 meets M in a plane quartic with five double points at the points in which it meets the nodal curves it also must touch M along a conic. Hence *each of the systems* ψ, ψ_1, ψ_2 *is a generating system of the same envelope* M; in fact, as may be seen immediately from a quadric cone through C and C_1, any prime containing a plane of ψ (and thus touching M) contains also a plane of ψ_1 and ψ_2, the point of contact of the prime being on each of the conics of M in these planes. In general the primal M has then three systems of generating planes, while in the case

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when the three quartic curves are degenerate the primal M is the usual Segre envelope of a quartic locus, and has the six systems (2·14) as generating planes.

2·31. The correspondence used above is founded by relating g_3^1 on normal curves of [3] and [4]. In [2] a g_3^1 on the normal conic consists of triads self-polar with respect to a definite inpolar conic, which may therefore be related to the line a in [3], and the point α in [4]. If α represents the quartic form $(at)^4$, the line a is given by

$$\begin{vmatrix} a_0 & a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 & a_4 \end{vmatrix}$$

and the inpolar conic by $(\xi a)^2(\eta a)^2 = 0$, where ξ, η are equivalent symbols, or as an order conic by

$$\begin{vmatrix} 0 & x_0 & x_1 & x_2 \\ x_0 & a_0 & a_1 & a_2 \\ x_1 & a_1 & a_2 & a_3 \\ x_2 & a_2 & a_3 & a_4 \end{vmatrix} = 0.$$

The coefficients of this latter conic which are the elements of the reciprocal determinant of the catalecticant, viz.

$$\begin{vmatrix} A_0 & A_1 & A_2 \\ A_1 & A_5 & A_3 \\ A_2 & A_3 & A_4 \end{vmatrix},$$

may be taken as the coordinates of the line a in [3]; in fact they are the usual coordinates, p_{ij}, save in the case of A_5, which is equal to $p_{04} + p_{13}$. A linear relation in these coefficients expresses

(1) that the inpolar conics are outpolar to a given conic envelope,

(2) that the lines a belong to a linear complex,

(3) that the points α lie on a quadric passing through C.

When the linear complex is special, the quadric in [4] is a cone, and the conic envelope (being inpolar to all inpolar conics having a particular triad of C^2 for self-polar triad) is triangularly inscribed in C^2.

In particular the fundamental complex of C^3 is $A_5 = 4A_2$, which is the condition that the inpolar conic is also outpolar to C^2, and being $a_0a_4 - 4a_1a_3 + 3a_2^2 = 0$ is the condition that A lies on the fundamental quadric of C; thus $(xy)^2 = 0$ corresponds to $(xy)^3 = 0$ and to $(xy)^4 = 0$.

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2·32. Accordingly we express the correspondence in the following symbolic form. The general conic polarity in [2],

$$(xb)^2(yb)^2 + \lambda(xy)^2 = 0, \tag{1}$$

the general linear complex in [3],

$$(xy)(xb)^2(yb)^2 + \lambda(xy)^3 = 0, \tag{2}$$

and the polarity associated with a general quadric through C in [4],

$$(xy)(xb)^2(yb)^2 + \lambda(xy)^4 = 0, \tag{3}$$

each having the same coefficients, induce by their conjugate points on the normal curves the same symmetrical (2, 2) correspondence

$$(tb)^2(t'b)^2 + \lambda(tt')^2 = 0,$$

and in this sense may be taken to correspond to one another. Following the argument adopted in discussing the porism of Poncelet, this (2, 2) correspondence becomes involutory in the case $\lambda = \lambda_0$, where $6\lambda_0{}^2 = (bb')^4$, and its triads are apolar to a quartic form $(at)^4$ having $(bt)^4$ as its Hessian. [The ambiguity of sign in λ_0 indicates that there are two forms with the same Hessian.] When $\lambda = \lambda_0$, the conic (1) is inpolar, its envelope equation being $(\xi a)^2(\eta a)^2 = 0$; the complex (2) is special with its axis meeting the tangents to C^3 at $(bt)^4 = 0$, and the quadric (3) is a cone through the four tangents to C at $(bt)^4 = 0$. Hence the correspondence, by means of one value of λ_0, reduces to the former correspondence between an inpolar conic, a line and a point.

Moreover the equation of any conic envelope (σ) may be taken as

$$(\xi c)^2(\eta c)^2 + 2\mu(\xi\eta)^2 = 0.$$

If $6\lambda\mu + (bc)^4 = 0$, the conic σ is inpolar to the conic (1) which corresponds to the complex (2); and this in turn, under the same condition, is conjugate to the complex

$$(\xi\eta)(\xi c)^2(\eta c)^2 + \mu(\xi\eta)^3 = 0,$$

which thus corresponds to the arbitrary envelope σ. When this latter complex is special the conic σ is triangularly inscribed in the normal conic; in fact it is the reciprocal of the normal conic with respect to an inpolar conic (1).

2·33. Confining ourselves first to the relation between [3] and [4] we see that to a linear complex in [3] corresponds a quadric through C in [4], and hence *to a collineation or correlation in* [3] *corresponds a birational transformation converting quadrics through C into quadrics through C.* A linear relation between the coefficients of $(at)^4$ is a quadratic relation in $A_0, A_1, \ldots, A_5$, so that to a prime in [4] corresponds a quadratic complex in [3] which

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contains all the chords of C^3 inasmuch as the prime contains ∞^2 points on chords, and thus the quadratic complex is a tetrahedral complex having as the vertices of its tetrahedron the four points corresponding to the four points of C on the prime, and as its cross-ratio that of the four points of the prime on C. *To any collineation in* [4] *then corresponds a birational correspondence in* [3] *transforming the lines of such tetrahedral complexes into one another.*

From this more general point of view the former results follow immediately: to a linear congruence in [3] corresponds a V_2^4 through C and four of its chords; to a regulus in [3] which is formed by the common lines of three linear complexes corresponds C_1, the residual intersection of three of the quadrics, and to the complementary regulus formed by the common lines of the conjugate complexes corresponds C_2, given similarly by complementary quadrics. To the special complexes of one system correspond cones with vertices on the curve associated with the other system.

Related g_4^1 on normal curves.

3·1. Analogous results may be found by using the relation in [2], but it is of more interest to consider the correspondence between [3] and [4] arising from similarly relating g_4^1 on the normal curves. In [3] a g_4^1 consists of tetrads on C^3 self-polar for an inpolar quadric, which is thus related to a plane in [4] given by the matrix

$$\begin{vmatrix} a_0 & a_1 & a_2 & a_3 & a_4 \\ a_1 & a_2 & a_3 & a_4 & a_5 \\ a_2 & a_3 & a_4 & a_5 & a_6 \end{vmatrix},$$

where the inpolar quadric is $(\xi a)^3(\eta a)^3 = 0$, or as an order quadric is

$$\begin{vmatrix} 0 & x_0 & x_1 & x_2 & x_3 \\ x_0 & a_0 & a_1 & a_2 & a_3 \\ x_1 & a_1 & a_2 & a_3 & a_4 \\ x_2 & a_2 & a_3 & a_4 & a_5 \\ x_3 & a_3 & a_4 & a_5 & a_6 \end{vmatrix},$$

its coefficients thus being the elements of the reciprocal determinant

$$\begin{vmatrix} A_0 & A_1 & A_2 & A_3 \\ A_1 & A_7 & A_8 & A_4 \\ A_2 & A_8 & A_9 & A_5 \\ A_3 & A_4 & A_5 & A_6 \end{vmatrix},$$

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These coefficients may be taken as coordinates of a plane, A_7, A_8, A_9 being each the sum of two of the ordinary Grassmannian coordinates; and then a linear relation in these coefficients expresses

(i) that the planes in [4] form a linear complex,

(ii) that the quadric in [3] is outpolar to a certain quadric Q.

Since any plane in the singular prime of the complex belongs to it, there is a tetrad on C^3 such that every quadric having it as self-polar tetrad is outpolar to Q; it is therefore the unique tetrahedron circumscribed to Q. Further, if the complex is special, consisting of all the planes meeting a line (axis of the special complex), then every prime through this line is singular and there is a g_4^2 of tetrads circumscribing Q. The three neutral pairs of this g_4^2, given by the chords of C which meet the axis, present three chords of C^3 which belong to a regulus of Q. This may be described as the case for which Q is *tetrahedrally inscribed* in C^3; it appears that the lines of the regulus meet the faces of any one of the ∞^2 circum-tetrahedra in four points whose cross-ratio is that of the four vertices of the tetrahedron on C^3.

Now consider the case when the g_4^1 consists of three fixed points and one variable point. The inpolar quadric is a conic on the plane of the fixed points, and as an order quadric is this plane taken twice; and this is outpolar to an arbitrary quadric Q when it touches it. But this g_4^1 on C gives a trisecant plane which belongs to the system of planes which also meet the two complementary quartic curves which correspond to the reguli on Q. Thus the system of planes ψ belongs to a linear complex corresponding to the quadric Q. *The system ψ may thus be defined as a system of trisecant planes of C which belongs to a linear complex,* just as the Segre system (2·14) is defined as a system of planes meeting three lines and belonging to a linear complex. When the complex is special the planes meet a quartic curve and a line which is the axis of the complex (2·13).

3·2. This linear complex is obtained more explicitly by means of the methods of 2·32.

The equation of any quadric envelope σ is

$$(\xi c)^3 (\eta c)^3 + 9 (\xi \eta)^2 (\xi r) (\eta r) = 0,$$

where $(ct)^6 = 0$ gives the points of contact of osculating planes touching σ. If $(bc)^6 + 10 (rs)^2 = 0$, this quadric σ is inpolar to the quadric

$$(xb)^3 (yb)^3 + (xy)^2 (xs) (ys) = 0, \tag{4}$$

which corresponds to the complex

$$(xy) (xb)^3 (yb)^3 + (xy)^3 (xs) (ys) = 0 \tag{5}$$

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in that it gives the same symmetrical (3, 3) correspondence on the normal curve, viz.

$$(tb)^3(t'b)^3 + (tt')^2(ts)(t's) = 0.$$

Also under the same condition the complex (5) is conjugate to the linear complex Σ of planes

$$(\xi\eta)(\xi c)^3(\eta c)^3 + 4(\xi\eta)^3(\xi r)(\eta r) = 0,$$

which thus corresponds to σ in the sense that any line complex conjugate to Σ corresponds to any quadric inpolar to σ. The points of contact of osculating planes of Σ are given by $(ct)^6 = 0$.

When the (3, 3) correspondence is involutory, giving a quartic involution, the quadric (4) is inpolar and the complex (5) consists of all lines meeting the plane whose primes cut out the quartic involution on C, i.e. a plane of the conjugate complex Σ. In order that the (3, 3) correspondence may be involutory, there are three conditions which determine the coefficients of the quadratic $(st)^2$ in terms of the sextic $(bt)^6$. These forms are in this case, save for numerical factors, the third and first transvectants of two quartic forms, and the relation* which must be satisfied identically may be written

$$(b, b)^4 - (b, s)^2 = s^2.$$

Thus the intersections of the singular prime of Σ, or the vertices of the one tetrahedron inscribed in C^3 and circumscribed to σ, are given by

$$(cc')^4(ct)^2(c't)^2 - 4(cr)^2(ct)^4 = 16(rt)^2(rt)^2,$$

and, if this equation is satisfied identically, Σ is special, and σ is tetrahedrally inscribed.

Some interesting results follow for particular cases. If σ itself is an inpolar quadric its equation is $(\xi c)^3(\eta c)^3 = 0$, and the vertices of the tetrahedron circumscribing it and inscribed in C^3 are given by $(cc')^4(ct)^2(c't)^2 = 0$. When this inpolar quadric is tetrahedrally inscribed, $(ct)^6$ is the sextic covariant of a quartic form and the points $(ct)^6 = 0$ are the three pairs of points AA', BB', CC' on the chords belonging to one regulus of σ; the osculating planes at A and A' meet in another line of this regulus, so that the three axes and their associated chords lie on σ, and are polar in the fundamental complex of C^3. *The quadric is thus tetrahedrally circumscribed, as well as inscribed, for the curve, and is both inpolar and outpolar.* On the other hand† if $(ct)^6 \equiv 0$, the quadric σ is inscribed in the cubic developable, the tangents at $(rt)^2 = 0$ being generators; the complex Σ contains all the osculating planes of C, and its singular prime is bitangent at $(rt)^2 = 0$. The ∞^3 planes which are common to all the ∞^2 complexes are such that their primes cut out

* Gordan, *Invariantentheorie*, 2 (1887), 298; Meyer, *Apolarität* (1883), 273.
† Cf. Wood, *Twisted Cubic* (1913), 33.

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a *syzygetic* involution on C; corresponding to such planes are the quadrics (4) which are both inpolar and outpolar to C^3, and these as we have seen above are tetrahedrally inscribed and circumscribed for the curve.

3·3. There are six planes of the system ψ which are incident with a given plane: for the given plane is represented by a g_4^1 on C^3, the faces of whose tetrads touch a cubic developable, so that six of the faces touch the quadric Q. Thus the planes of the system meet any prime section in a (2, 6) congruence contained in a tetrahedral complex; a plane in the singular prime contains four lines of the faces of the tetrahedron, and thus the congruence in this prime is (2, 2), the intersection of a linear and a tetrahedral complex.

A focus of the first order on a plane of the system ψ appears in the prime section as a focus of the congruence, while a focus of the second order appears as a node of the focal surface. Hence for a general prime the focal surface of the congruence* is the quartic surface section of M, with 12 nodes where the prime meets C, C_1, C_2, and the congruences given by the systems ψ, ψ_1, ψ_2 are confocal. For a prime containing a plane of ψ, and thus of ψ_1 and of ψ_2 also, the congruence is (2, 5) and the focal surface has 13 nodes, one of these being the point of contact of the prime with M. Lastly, for the singular prime which contains four planes of ψ, and thus four of ψ_1 and of ψ_2 also, the focal surface is a Kummer's quartic surface, the four further nodes (other than on C, C_1, C_2) being introduced by points of contact of the prime with M, and completing the configuration of four Rosenhain tetrahedra mutually inscribed and circumscribed in pairs. Hence *the three linear complexes of planes which contain respectively the planes of* ψ, ψ_1, ψ_2 *have one and the same singular prime and this prime is quadritangent to the primal* M.

Through any plane pass six primes containing planes of the system ψ, and thus M is of class six (2·2).

Further results are obtained for trisecant planes of C which belong to two linear complexes. Such planes correspond to the planes of a quartic developable touching two quadrics. They form a primal of order eight. The focus of a plane corresponds to a line of the developable. Since eight lines of the developable meet an arbitrary line, the locus of the focus meets a quadric cone in eight points and C in twenty-four points, and is thus an elliptic curve of order sixteen. Also corresponding to the fact that the planes of the developable touch four conics, we have the result that the planes of the system ψ which belong to a second linear complex meet four lines which lie each in a trisecant plane of C.

Relation of the main theorem to [5].

4·1. A simple Cremona transformation between two fivefold

* Cf. Jessop, *Line Complex* (1903), 280, for order of the singularities.

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spaces S and S' exhibits two Veronese surfaces V, V_1, which give in section curves C, C_1, whose related C_2 arises from the construction. A conic in the plane when regarded as a class conic is represented by a point (x) of S; the same conic regarded as an order conic is represented by a point (x') of S'. This gives rise to a $(1, 1)$ correspondence between the points of S and of S', wherein to the points of a prime of S' correspond points of a quadric in S which passes through a Veronese surface V, while to the points of a prime in S correspond in S' the points of a quadric through a Veronese surface V'.

The conics of a linear ∞^2 system are represented in S' by primes through a plane and in S by ∞^2 quadrics through V. Thus a plane π_1 in S' corresponds to the residual intersection of these quadrics, which is another Veronese surface, V_1. The locus of intersection of the line pairs of the system of conics is represented by the elliptic cubic curve (π_1, J), where J is the cubic primal upon which V is double; and in S by (V_1, V), which is thus an elliptic sextic curve. In particular, conics having a common self-conjugate triangle form a linear ∞^2 system whether regarded as order or class conics; whence it follows that to a trisecant plane of V corresponds a trisecant plane of V'.

A repeated point is a class conic whose order conic may be any line pair intersecting at the point. Thus to the tangent plane at a point of V corresponds a conic on V'; and to a conic on V and its plane corresponds a tangent plane of V' and its point of contact.

Now let ω be a prime of S corresponding to a quadric Ω of S'; the points of contact in S' of the ∞^1 primes through a plane π_1 which touch Ω form a conic γ_2 to which corresponds in S a quartic curve Γ_2, while (π_1, Ω) is a conic γ_1 to which corresponds (V_1, ω), a quartic curve Γ_1. Further, any generating plane of Ω which meets γ_1 meets γ_2, since these conics lie in polar planes of Ω. But any generating plane of one system of Ω is trisecant to V': therefore any trisecant plane of V which lies in ω and meets Γ_1 meets Γ_2 also. To the six points in which ω meets (V_1, V) correspond six conic-planes of V' and these are generating planes of Ω which meet γ_1 and therefore γ_2. Thus Γ_2 passes through the six points common to Γ_1 and Γ (the section of V by ω), and Γ, Γ_1, Γ_2 are related like the curves C, C_1, C_2, above.

This view of the theorem is immediately connected with the former if Ω is regarded as the quadric in [5] whose points represent lines in [3]. Then γ_1 and γ_2 are the complementary reguli of the former work. To a line in ω corresponds a conic on Ω whose plane is trisecant to V', and this represents a regulus containing three chords of the cubic curve whose chords are represented by points of V'. To a tangent plane of V corresponds a conic of V' (which must be a conic of Ω) and the tangent plane meets the section in

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a director line of Γ; thus to a director line in ω corresponds a conic of V' which represents a regulus containing ∞^1 chords of the cubic curve.

It is to be noted that there is no theorem analogous to that relating to Γ, Γ_1, Γ_2 concerning Veronese surfaces V, V_1, V_2. The generalisation of the theorem pertains to ruled quartic surfaces in [5], as is shown in the following section.

4·2. Chords of an elliptic sextic curve in [5] which join pairs of points of a g_2^1 on the curve are generators of a ruled quartic surface R. Three pairs of points in a prime give rise to three such involutions and thus to three surfaces R, R_1, R_2 which are such that any three generators g, g_1, g_2, one on each surface, lie in a prime. Expressed in terms of elliptic parameters the involutions are given by $u+v=c$, $u_1+v_1=c_1$, $u_2+v_2=c_2$, where $c+c_1+c_2=0$. The solid determined by two of the three generators g_1, g_2 meets R in a cubic curve residual to g, and *any solid meeting R in a cubic curve and passing through a generator of R_1 passes also through a generator of R_2*. So a prime section of the figure cuts R, R_1, R_2 in quartic curves C, C_1, C_2 through six points such that any trisecant plane of C which meets C_1 meets C_2 also. By varying the parameters separately it is seen that the envelope of the ∞^3 system of primes containing the three generators is the locus of the single transversal of these generators, since this transversal is the intersection of the three solids (g, g_1), (g_1, g_2), (g_2, g). When g_1, g_2 are fixed the locus of the transversal is a regulus through the cubic curve on R in the solid (g_1, g_2). In any prime section the prime (g, g_1, g_2) appears as the prime [3] containing one plane of each system ψ, ψ_1, ψ_2, and the regulus as the repeated conic of M on the plane (2·2). The locus of the transversal is a primal M_4^4*; six primes of the system enveloping it pass through an arbitrary plane, as may be verified immediately by examining the case of a plane trisecant to the curve.

* This is the primal which is obtained by James (*loc. cit.*) as the section of a sextic locus in [7].

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