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Homomorphism-Homogeneous Relational Structures

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For Béla Bollobás on his 60th birthday

We study relational structures (especially graphs and posets) which satisfy the analogue of homogeneity but for homomorphisms rather than isomorphisms. The picture is rather different. Our main results are partial characterizations of countable graphs and posets with this property; an analogue of Fraïssé's theorem; and representations of monoids as endomorphism monoids of such structures.

1. Introduction

A graph G (or more general relational structure) is *homogeneous* if any isomorphism between finite induced subgraphs of G can be extended to an automorphism of G. The homogeneous graphs can be recognized by the fact that their collections of finite subgraphs have the amalgamation property (Fraïssé's theorem). The finite homogeneous graphs were determined by Gardiner [8] and the countably infinite ones by Lachlan and Woodrow [13]. Other determinations of homogeneous structures in various classes include posets (Schmerl [17]), tournaments (Lachlan [12]), permutations (Cameron [3]), and digraphs (Cherlin [4]). These structures are important in many parts of mathematics: see Hubička and Nešetřil [15, 11] for the connection with Ramsey theory, for example.

In this paper we consider what happens if we replace 'isomorphism' in the definition of homogeneity by 'homomorphism'. A homomorphism of a graph, for example, is a

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function which maps vertices to vertices and preserves the edges. A monomorphism is one-to-one homomorphism (see, *e.g.*, [10]).

There are several different conditions. We say that a graph G belongs to the class

HH if every homomorphism from a finite induced subgraph of G into G extends to a homomorphism from G to G;

MH if every monomorphism from a finite induced subgraph of G into G extends to a homomorphism from G to G;

MM if every monomorphism from a finite induced subgraph of G into G extends to a monomorphism from G to G.

Clearly both HH and MM are included in the class MH. So we begin with some structural results for MH graphs. Later we show that it is the class MM in which an analogue of Fraïssé's theory can be developed for arbitrary relational structures.

Proposition 1.1.

(a) *Any disjoint union of complete graphs all of the same size is HH, and hence MH. If a disjoint union of complete graphs is MH, then the complete graphs all have the same size.*

(b) *If an MH graph is disconnected, or if it is finite, then it is a disjoint union of complete graphs of the same size.*

(c) *If an MH graph is connected but not finite, then it has diameter at most* 2 *and contains no finite maximal clique* (*so, in particular, every edge is contained in a triangle*).

Proof. The first part of (a) is clear. Moreover, if an MH graph has two components A and B which are complete, then for $a \in A$ and $b \in B$, the map $a \mapsto b$ extends to a homomorphism which maps A injectively to B, so $|A| \leqslant |B|$. Similarly $|B| \leqslant |A|$. So $|A| = |B|$ by the Cantor–Schröder–Bernstein theorem.

Suppose that G is MH and has a component which is not complete: equivalently, it has an induced path x, y, z. For any two distinct vertices a, b, the map $x \mapsto a$, $z \mapsto b$ extends to a homomorphism, which maps y to a common neighbour of a and b. This shows that G is connected and has diameter 2 and every edge is in a triangle. This proves (b) for disconnected graphs, and also the first part of (c).

Let C be a finite clique in an MH graph and suppose there is a vertex v with degree at least $|C|$. Any injective map from $|C|$ neighbours of v to C extends to a homomorphism, which must map v to a common neighbour of C, so C is not maximal.

Suppose that C is a finite maximal clique. If C is not a component of G, then it contains a vertex v with a neighbour outside C; then the degree of v is at least $|C|$, contrary to the preceding paragraph. This finishes the proof of (c). □

The classification of finite MM graphs is simpler.

Proposition 1.2. *The only finite MM graphs are the complete and null graphs.*

Proof. Suppose that G is a finite MM graph which is neither complete nor null. Let the vertices a and b be adjacent, and the vertices c and d be non-adjacent. Then the map

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$c \mapsto a$, $d \mapsto b$ extends to a monomorphism of G, which strictly increases the number of edges, which is clearly impossible. □

2. Graphs spanned by R

Let R be the countable random graph (the 'Rado graph' [16]). Recall that R is characterized as a countable graph with the property that, if U and V are finite disjoint sets of vertices, there is a vertex z joined to all vertices in U and to none in V. See [2] for more information about R.

Proposition 2.1.

(a) *A countable graph contains R as a spanning subgraph if and only if it has the property that any finite set of vertices has a common neighbour.*
(b) *Any graph containing R as a spanning subgraph is HH and MM, and hence MH.*
(c) *If G is an MH-graph which does not contain R as a spanning subgraph, then there is a bound on the size of claws (induced stars $K_{1,n}$) in G.*

Proof. **(a)** The property holds in R, and hence certainly in any graph obtained by adding extra edges.

Conversely, let G be a countable graph satisfying the property. Construct a bijection between R and G by the back-and-forth method, except that in going from R to G we do not insist that non-edges are preserved. In more detail: we define a map $f : R \to G$ recursively. At odd-numbered steps, take the first vertex of R on which f is not yet defined, and map it to a common neighbour of the range of f. At even-numbered stages, take the first vertex v not in the range of f, choose $v' \in R$ such that, for all u in the domain of f, $u \sim v'$ if and only if $f(u) \sim v$.

(b) If the property of (a) holds, then certainly homomorphisms extend: if we have defined f on $v_0, \ldots, v_{n-1}$, then choose $f(v_n)$ to be any vertex adjacent to all of $f(v_0), \ldots, f(v_{n-1})$. Moreover, if f is one-to-one, then so is the extension.

(c) Suppose that G is a countable MH graph which contains claws of unbounded size. Let U be a finite set of vertices, with $|U| = n$. Find a claw $K_{1,n}$ in G and map its independent vertices bijectively to U. The remaining vertex is mapped to a neighbour of U. So G satisfies the condition of (a). □

Corollary 2.2.

(a) *There is a countable graph which is homomorphism-homogeneous but is automorphism-rigid.*
(b) *There is a countable graph which is homomorphism-homogeneous but its complement is homomorphism-rigid.*

Proof. The graph shown in Figure 1 is automorphism-rigid and for every finite subset U there is a vertex joined to no vertex in U. So the complement is also automorphism-rigid and contains R as a spanning subgraph. This proves (a).

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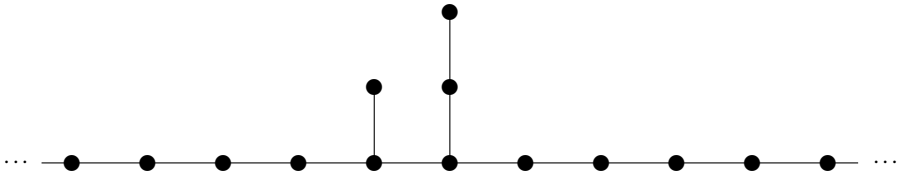


Figure 1.

To prove (b) it suffices to note that there exists a countable homomorphism-rigid graph with all its degrees $\leqslant 3$: see [9]. The complement then contains R as a spanning subgraph. □

Here are two questions which we have not been able to resolve.

Problem 1. Is there a graph which is MH but not HH?

We remark that for more general structures than graphs we prove below that the classes MH and HH are different.

Problem 2. Is there a countable graph which is HH but not a disjoint union of complete graphs and does not contain R as a spanning subgraph?

A positive answer to this problem would yield a graph which contains a finite set of vertices with no common neighbours, and there is a bound on the size of its claws. One famous class of graphs with bounded claw size consists of *line graphs* $L(G)$ of graphs; these contain no 3-claw $K_{1,3}$. We show that at least in this class we obtain no new examples.

Proposition 2.3. *Let G be a finite or countable graph with the property that $L(G)$ is MH. Then G is a disjoint union of stars of the same size* (*and hence $L(G)$ is a disjoint union of complete graphs of the same size*).

Proof. By Proposition 1.1, we can assume that G is infinite and connected with bounded diameter. So G contains a vertex v of infinite degree.

First we show that G is triangle-free. Suppose that $\{a, b, c\}$ is a triangle in G, and let p, q, r, s be neighbours of v. The map $vp \mapsto ab$, $vq \mapsto bc$, $vr \mapsto ca$ (defined on an induced subgraph of $L(G)$) extends to a homomorphism of $L(G)$, under which vs must map to an edge meeting all three edges of the triangle, which is impossible.

Now we show that any neighbour of v has degree 1. For suppose that p is a neighbour of v which is also adjacent to a vertex x (necessarily not adjacent to v), and let q and r be two further neighbours of v. The map $vq \mapsto vq$, $vr \mapsto vp$, $px \mapsto px$ extends to a homomorphism, which must map vp to an edge containing v and meeting px; this edge cannot be vp, and if it is vx then the graph contains a triangle. So no such vertex can exist.

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Thus, the connected component containing v is an infinite star, and we are done. □

More generally, let $\mathscr{K}(k,l)$ be the class of finite graphs defined as follows, where k and l are integers with $1 < l < k$: the vertex set is an arbitrary set $\mathscr{M}$ of k-sets; two vertices are adjacent if and only if they intersect in at least l points. We call this the $(\geqslant l)$-*intersection graph* of $\mathscr{M}$. (So line graphs form the class $\mathscr{K}(2,1)$. More generally, in the case $l = 1$, we refer to such a graph as the *intersection graph* of the collection of k-sets.)

Now if the k-sets $(X, Y_1, \ldots, Y_r)$ form a claw in the $(\geqslant l)$-intersection graph, then the intersections $X \cap Y_i$ for $1 \leqslant i \leqslant r$ all have size at least l and are all distinct; so certainly there is no 2^k-claw in such a graph. We would like to prove that no countable graph of this type can be homogeneous unless it is a disjoint union of complete graphs. Here is a first step.

Proposition 2.4. *For $k \geqslant 3$, let $\binom{X}{k}$ denote the set of all k-subsets of a countable set. Then the $(\geqslant l)$-intersection graph of $\binom{X}{k}$ is not MH.*

Proof. There always exists a $(\geqslant l)$-intersecting family of k-subsets with the property that no further k-set intersects every set in the family. For example, the set $\binom{Y}{k}$ of all k-subsets of a $(2k-l)$-set Y obviously has this property, since a set containing fewer than k points of Y has intersection smaller than l with some set in $\binom{Y}{k}$.

Now there is also an infinite star (a set of k-sets containing l common points). Now the map taking $\binom{2k-l}{k}$ sets of the star bijectively to $\binom{Y}{k}$ cannot be extended to a further set of the star. □

A final observation about MM graphs:

Proposition 2.5. *Any infinite non-null MM graph contains an infinite complete subgraph.*

Proof. Let G be such a graph.

Suppose first that G is disconnected, and so a union of complete graphs. If the components are finite (of size $n > 1$, say), take two points a, b in different components and map them to two points in the same component; it is clear that no monomorphism can extend this map. So the components are infinite, as required.

The case where G is connected follows immediately from Proposition 1.1(c). □

3. Posets

A homomorphism of posets is a map f such that, if $x < y$, then $f(x) < f(y)$. Now the definitions of the classes HH, MH and MM of posets are exactly as for graphs. Here is the start of an attempt to classify the MH posets.

First, the analogue of the fact that graphs containing R as spanning subgraph have the MM and HH properties also holds here. Let U denote the generic (universal and homogeneous) countable poset. An *extension* of U is a poset P on the same set such that, if $x \leqslant_U y$, then $x \leqslant_P y$.

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Proposition 3.1.

(a) *A countable poset P is an extension of U if and only if it has the following property:*

(†) *for any two finite sets A and B with $A < B$, there is a point z such that $A < z < B$.*

(b) *Any extension of U has the HH and MM properties.*

Proof. **(a)** The argument is similar to that for graphs. Recall that U is characterized by the property that, for any finite disjoint sets A, B, C with $A < B$ and, for all $a \in A$, $b \in B$, $c \in C$, we have $c \not< a$ and $b \not< c$, there is a point z with $A < z < B$ and z and c incomparable for all $c \in C$.

Now construct a bijection from U to P which preserves comparability as follows. Enumerate P and U. Suppose that f has been defined on $u_1, \ldots, u_n \in U$. If n is even, choose the first unused point x of U; let A and B be the subsets of $\{u_1, \ldots, u_n\}$ consisting of points less than and greater than x respectively. Then $A < B$, so $f(A) < f(B)$; by (†), there is a point z such that $A < z < B$, and we can map x to z. If n is odd, choose the first unused point y of P, and choose a point $z \in P$ incomparable with all of $u_1, \ldots, u_n$; then map z to y.

(b) Let P be an extension of U, and f a homomorphism between finite subsets of P. Now we can extend f to all of P as follows. Suppose that f has been defined on $p_1, \ldots, p_n$. Let A and B be the sets of elements of $\{p_1, \ldots, p_n\}$ which are respectively less than and greater than p_{n+1}. As before, choose z with $f(A) < z < f(B)$, and map p_{n+1} to z. The extension can be taken to be one-to-one everywhere except possibly on the points where f was initially defined. □

Now we gather a few facts about MH posets. Clearly any antichain is HH and MM.

Proposition 3.2. *Let P be a countable poset which is MH but not an antichain.*

(a) *Any maximal chain in P is dense and without endpoints.*

(b) *If P is disconnected then it is a disjoint union of incomparable chains each isomorphic to the rationals.*

(c) *If there is a 2-element antichain in P which has an upper bound, then any finite antichain has an upper bound.*

Proof. **(a)** By assumption, there exist a and b with $a < b$. Extending the map $a \mapsto x$ for any x, we see that there is an element above x. (Dually there is an element below x.) In particular, there is a 3-element chain $a < b < c$. Now, if $x < y$, extending the map $a \mapsto x$, $c \mapsto y$, we see that the image of b is an element z with $x < z < y$.

(b) If P has a component which is not a chain, then it contains a 2-element antichain with either an upper or a lower bound. By the MH property, every 2-element antichain has a bound; so there is only one connected component.

(c) As in (b), if some 2-element antichain has an upper bound, then so does every 2-element antichain. Now, inductively, let $a_1, \ldots, a_n$ be an antichain. Let b_i be an upper bound of a_i and a_{i+1} for $i = 1, \ldots, n-1$. Now the set of maximal elements among $b_1, \ldots, b_{n-1}$ is an

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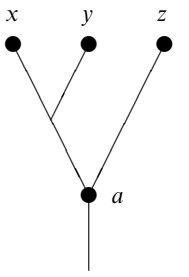


Figure 2.

antichain of size smaller than n; by induction it has an upper bound c, which is also an upper bound for $a_1, \dots, a_n$. □

Consequently all homogeneous posets have the MM property. A bit surprisingly there is another example (which is also not an extension of the generic poset U, but is more analogous to a claw-free graph). Recall that a *tree order* is a poset in which the elements below any given element x form a chain. Tree orders may have large symmetry groups (see Droste [5]) while not being homogeneous (in the usual sense). We prove that they may have the MM property. Let T be a countable tree order satisfying the following additional properties.

(a) T is dense.
(b) T has neither maximal nor minimal element.
(c) If an element a of T is an infimum of a finite subset A of T then $a \in A$ (in other words: infima and minimal elements of finite subsets coincide).
(d) T is infinitely branching (*i.e.*, for every $x < y$ there exists an antichain $z_i; i = 1, 2, \dots$, with $x < z_i$ and $y \not\leqslant z_i$ for every $i = 1, 2, \dots$).

An example of T can be obtained as follows. First we split the set of rational numbers in two dense sets, say D, D'. We form an infinite tree of copies of D by adding infinitely many branching copies of D at any element of D' and continuing recursively in this way.

We have the following.

Proposition 3.3.

(a) *T is not homogeneous (in the usual sense).*
(b) *T has the MM property.*

Proof. The reason why T is not homogeneous in the ordinary sense can be seen from Figure 2.

(Of course a is just a reference point which does not belong to the tree T.) It is clear that no automorphism can interchange x with z and fix y. (But there is a monomorphism that will do this: simply map the points below x, y, z to suitable points below a.)

To prove (b), it suffices to prove that for any $x \notin A$ any monomorphism $f : A \longrightarrow B$ may be extended to a monomorphism $f' : A' \longrightarrow B'$ where $A' = A \cup \{x\}$. This may be

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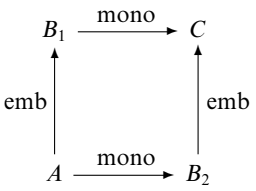


Figure 3.

seen as an extension of the above example. Given $x \notin A$ denote by A_x the set of all $z \in A$ such that $x \leqslant z$. Let $B_x = \{f(y); y \in A_x\}$. Consider the infimum a of the set $A_x \cup B_x$ (it may not belong to T). If $a \geqslant x$ then we put $f'(x) = x$. If $a < x$ then necessarily $B_x \setminus A_x$ is non-empty. In this situation we distinguish two cases.

- If there is an element z of A satisfying $a < z < x$, we choose maximal such z, and we let $f'(x)$ be the element above $f(z)$ close enough to be distinct from all of A and B.
- If there is no element of A satisfying $a < z < x$ we let $f'(x)$ be an element below a which is such that $f'(x) > z$ for all $z < a, z \in A \cup B$. □

4. Homogeneity and amalgamation

The definitions of the classes HH, MH and MM work in the same way for arbitrary relational structures as for graphs or posets. We are going to develop a characterization of MM structures in general. As usual, the *age* of a structure is the class of finite structures embeddable in it; and the *joint embedding property* or JEP is as usual: $\mathscr{C}$ has the JEP if any two members of $\mathscr{C}$ can be embedded in some member of $\mathscr{C}$.

The *mono-amalgamation property* (for short, MAP) of a class $\mathscr{C}$ of finite relational structures is the following assertion.

For any $A, B_1, B_2 \in \mathscr{C}$, and any maps $f_i : A \to B_i$ (for $i = 1, 2$) such that f_1 is an embedding (an isomorphism to an induced substructure) and f_2 a monomorphism, there exists $C \in \mathscr{C}$ and monomorphisms $g_i : B_i \to C$ for $i = 1, 2$ such that $g_1 \circ f_1 = g_2 \circ f_2$ and g_2 is an embedding (the diagram in Figure 3 commutes).

Note the asymmetry between B_1 and B_2!

The *mono-extension property* of a structure M with age $\mathscr{C}$ is the following property.

If $B \in \mathscr{C}$ and A is an induced substructure of B, then every monomorphism $A \to M$ extends to a monomorphism $B \to M$.

Now our analogue of Fraïssé's theorem is the following result.

Proposition 4.1.

(a) *A countable structure is MM if and only if it has the mono-extension property.*
(b) *The age of any MM-structure has the mono-amalgamation property.*
(c) *If a class $\mathscr{C}$ of finite relational structures is isomorphism-closed, closed under induced substructures, has only a countable number of isomorphism classes, and has the JEP and the MAP, then there is a countable MM structure M whose age is equal to $\mathscr{C}$.*

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Proof. **(a)** If M has the mono-extension property, then clearly any monomorphism from a finite substructure of M can be extended (one point at a time) to a monomorphism of M.

Conversely, suppose that M is an MM structure. Let B be a structure in the age of M and $A \subseteq B$: without loss of generality, $B \subseteq M$. Suppose that f is any monomorphism from A into M. Then f extends to a monomorphism g of M, whose restriction to B is the required monomorphism $B \to M$.

(b) Suppose that M is an MM structure and let A, B_1, B_2, f_1, f_2 be as in the hypothesis of the mono-amalgamation property, with A, B_1, B_2 in the age of M. As in (a), we can assume that $B_1, B_2 \subseteq M$ and f_1 is the identity on A. Now extend f_2 to a monomorphism g of M; let $C = B_1 g$, g_1 the restriction of g to B_1, and g_2 the identity on B_2.

(c) We build the structure in stages; even and odd numbered stages achieve different parts of the construction. Suppose that the finite structure $M_i \in \mathscr{C}$ has been constructed at stage i.

If i is even, we can use the JEP to find a structure M_{i+1} containing M_i and any given structure $A \in \mathscr{C}$.

If i is odd, we select a pair (A, B) of structures in $\mathscr{C}$ with $A \subseteq B$. Now a given monomorphism $A \to M'$ can be extended to a monomorphism $B \to M''$ for some $M'' \supseteq M'$, by the MAP. Applying this successively for each monomorphism $A \to M_i$, we obtain the structure M_{i+1} so that every monomorphism $A \to M_i$ extends to a monomorphism $B \to M_{i+1}$.

Arranging the stages so that every structure in $\mathscr{C}$ occurs at some even stage and every pair (A, B) at infinitely many odd stages, we finally build a countable structure M. Every finite substructure of M is contained in one of the finite structures M_i and thus belongs to $\mathscr{C}$ by the induced substructure property. It follows also that $\mathscr{C}$ is the age of M with the mono-extension property. Hence M is an MM-structure. □

We say that a class $\mathscr{C}$ having the properties of part (c) of the theorem is a *mono-Fraïssé class*, and that an MM structure with age $\mathscr{C}$ is a *mono-limit* of $\mathscr{C}$. Unlike the usual form of Fraïssé's theorem, it is not the case that a class satisfying the mono-amalgamation property has a unique mono-limit (up to isomorphism). Indeed, there are many examples of graphs containing R as spanning subgraph whose age is the class of all finite graphs.

However, any two such structures M and M' must bear a certain resemblance to each other. For example, there is a monomorphism from M to M' and *vice versa*. In fact, more is true; we can characterize this equivalence as follows.

Let M and M' be two structures. We say that M and M' are *mono-equivalent* if

- $\mathrm{Age}(M) = \mathrm{Age}(M')$;
- every embedding of a finite substructure A of M into M' extends to a monomorphism from M to M', and *vice versa* (with M and M' reversed).

This turns out to be the relation which replaces isomorphism in our version of Fraïssé's theorem.

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Proposition 4.2.

(a) *Suppose that M and M' are mono-equivalent structures. If M is an MM structure, then M' is an MM structure too.*

(b) *Conversely, if M and M' are MM structures with $\mathrm{Age}(M) = \mathrm{Age}(M')$, then they are mono-equivalent.*

Proof. **(a)** Suppose that M and M' are equivalent and that M has the MM property. Take $A, B \in \mathrm{Age}(M')$ with $A \subseteq B$, and let $f : A \to M'$ be a monomorphism. By assumption, we may assume that $B \subseteq M$.

Let A' be the image of f. Since $\mathrm{Age}(M) = \mathrm{Age}(M')$, we can find a copy A'' of A' within M; in other words, there is a monomorphism $\phi : A \to A''$ and an isomorphism $g : A'' \to A'$ such that $g \circ \phi = f$. Since M has the MM property, the monomorphism ϕ extends to a monomorphism $\phi^* : M \to M$. Let $B'' = \phi^*(B)$. Also, by assumption, the isomorphism g extends to a monomorphism $g^* : M \to M'$. Now the restriction of $g^* \circ \phi^*$ to B is a monomorphism $B \to M'$ extending the given monomorphism f. So M' has the mono-extension property, and hence it is an MM structure.

(b) Suppose that M and M' are MM structures with the same age. Let A be a finite substructure of M and $f : A \to M'$ an embedding. For any $B \supseteq A$, the mono-extension property in M' allows us to extend f to a monomorphism $B \to M'$. So there is a monomorphism $M \to M'$ extending f. Thus M and M' are equivalent. □

The above proof that the MM property for M implies that of M' uses only half of the definition of equivalence. Let us say that $M \preceq M'$ holds if

- $\mathrm{Age}(M) \supseteq \mathrm{Age}(M')$, and
- any embedding of a finite substructure of A into M' extends to a monomorphism from M to M'.

This relation between structures defines a partial order on the set of equivalence classes. (The reverse ordering of ages looks strange, but consider graphs containing R as spanning subgraph: intuitively, the more extra edges we add, the smaller the age becomes.) Then the proof of Proposition 4.2(a) actually shows that, if M is an MM structure and $M \preceq M'$, then M' is also an MM structure.

We mention one fact about this order for graphs.

Proposition 4.3. *Let R be the Rado graph. Then a countable graph G satisfies $R \preceq G$ if and only if R is a spanning subgraph of G.*

Proof. Since R is universal, the condition $\mathrm{Age}(R) \supseteq \mathrm{Age}(G)$ is trivial.

Suppose that R is a spanning subgraph of G, and let $f : A \to G$ be an embedding of a finite subgraph A of R into G. Since there is a common neighbour of $f(A)$ in G by Proposition 2.1, it is always possible to extend f to a monomorphism on one extra point.

Conversely, suppose that $R \preceq G$, and let U be a finite set of vertices in G. Let A be a subgraph of R isomorphic to the subgraph U (by means of the isomorphism f), and z a common neighbour of A in R. The map f extends to a monomorphism from R to G

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(by assumption), and the image of z is a common neighbour of U. So G contains R as a spanning subgraph, again by Proposition 2.1. □

On the other hand, it is not true that if M and M' are MM structures and $\mathrm{Age}(M) \supseteq \mathrm{Age}(M')$, then $M \preceq M'$. For example, let M be the Rado graph R, and M' the disjoint union of two infinite complete graphs. The map taking a non-edge in M to a non-edge in M' clearly cannot be extended.

5. Algebraic classes

We say that a class $\mathscr{K}$ of finite structures has the ME-property if the classes of monomorphisms and embeddings coincide for $\mathscr{K}$.

Proposition 5.1. *Let $\mathscr{K}$ be a mono-Fraïssé class of finite structures with the ME-property. Then any two mono-limits of $\mathscr{K}$ are isomorphic.*

Examples:

- all algebraic classes (containing only function and constant symbols)
- tournaments (see below).

Commentary. Clearly any monomorphism between finite algebras is an embedding. For if ρ is an r-ary operation in the algebra and f a monomorphism, then clearly $\rho(f(x_1),\dots,f(x_r)) = f(\rho(x_1,\dots,x_r))$.

Now, given any loopless directed graph D, it is possible to define an algebra on the vertex set of D with a single binary operation $\cdot$ by the rules

$$x \cdot x = x,$$

$$\text{and if } x \neq y, \text{ then } x \cdot y = \begin{cases} x & \text{if there is an arc } x \to y, \\ y & \text{otherwise.} \end{cases}$$

Any monomorphism of this algebra is a digraph embedding. Note that not all digraph monomorphisms are algebra monomorphisms! If the digraph is a tournament then the two types of monomorphisms coincide: see [14].

This example shows that, even in the case of algebras of a given type, we may have $2^{\aleph_0}$ non-isomorphic MM-structures (in this case, the algebras of the homogeneous digraphs determined by Cherlin [4]).

A more familiar example of this phenomenon occurs with abelian groups. For any set Π of prime numbers, the direct product of the countable abelian group of exponent p for all $p \in \Pi$ is homogeneous.

The above example of tournament algebras (sometimes called *quasitrivial algebras*) also yield an example of structures showing MH $\neq$ HH: Consider the algebra A_T corresponding to the universal homogeneous tournament T. This algebra is obviously an MM structure. However it fails to be an HH structure as every finite tournament may be extended to a finite tournament which is *simple* (*i.e.*, does not have any nontrivial congruence); see [6, 14]. (This may be seen as follows: if T_0 is a non-simple finite

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tournament, $f : T_0 \to T_0$ a homomorphism which is not one-to-one, $T_0 \subset T_1$ an inclusion with T_1 a simple tournament, then no homomorphism $f : T_0 \to T_0$ which is not one-to-one can be extended to a homomorphism $T \to T$.)

6. Representing closed monoids

There is a natural topology on X^X, for a countable set X, namely the product topology induced from the discrete topology on X. Thus the basic open sets are of the form

$$\{f \in X^X : f(x_i) = y_i \text{ for } i = 1, \dots, n\},$$

where $x_1, \dots, x_n, y_1, \dots, y_n \in X$ and $x_1, \dots, x_n$ are distinct. It is known that, in the induced topology on the symmetric group $\mathrm{Sym}(X)$, a permutation group G is closed if and only if it is the automorphism group of a homogeneous relational structure on X (see [1]). A similar observation holds here.

Proposition 6.1.

(a) *A submonoid S of X^X is closed in the product topology on X^X if and only if S is the monoid* $\mathrm{End}(M)$ *of endomorphisms of an HH relational structure M on X.*

(b) *A submonoid S of the monoid of one-to-one maps $X \to X$ is closed in the product topology if and only if S is the monoid of monomorphisms of an MM relational structure M on X.*

Proof. **(a)** First note that $\mathrm{End}(M)$ is always closed in X^X. In the reverse direction assume that S is a fixed submonoid of X^X. For each n, and each $\overline{x} \in X^n$, we take an n-ary relation $R_{\overline{x}}$ defined by

$$R_{\overline{x}}(\overline{y}) \Leftrightarrow (\exists s \in S)(\overline{y} = s(\overline{x})).$$

Let M be the relational structure with relations $R_{\overline{x}}$ for all n-tuples $\overline{x}$ (and all n). We claim that S acts as endomorphisms of M, that M is HH, and $\mathrm{End}(M) = S$.

For the first point, take $s \in S$ and $\overline{y} \in X^n$ such that $R_{\overline{x}}(\overline{y})$ holds; we must show that $R_{\overline{x}}(s(\overline{y}))$ holds. But $\overline{y} = s'(\overline{x})$ for some $s' \in S$; then $s(\overline{y}) = ss'(\overline{x})$, so the assertion is true.

Next, let f be a homomorphism between finite subsets of X, say $f(x_i) = y_i$ for $i = 1, \dots, n$. Let $\overline{x} = (x_1, \dots, x_n)$ and $\overline{y} = (y_1, \dots, y_n)$. Now S is a monoid and so contains the identity mapping. Thus, by definition, $R_{\overline{x}}(\overline{x})$ holds. Since f is a homomorphism, $R_{\overline{x}}(\overline{y})$ holds. So by definition, there exists $s \in S$ such that $s(\overline{x}) = \overline{y}$. Now s is an endomorphism of M extending f. So M is HH.

Finally, to show that $\mathrm{End}(M) = S$, we know already that $S \subseteq \mathrm{End}(M)$ and have to prove the reverse inclusion. We must take $h \in \mathrm{End}(M)$ and show that every basic neighbourhood of h contains an element of S, so that h is a limit point of S. Since S is assumed closed, we conclude that $h \in S$. Now each n-tuple $\overline{x}$ defines a basic neighbourhood of h, consisting of all functions g such that $g(\overline{x}) = h(\overline{x})$. Now $R_{\overline{x}}(\overline{x})$ holds; since h is a homomorphism, $R_{\overline{x}}(h(\overline{x}))$ also holds, and by definition of $R_{\overline{x}}$ this means that there exists $s \in S$ with $h(\overline{x}) = s(\overline{x})$, as required.

(b) The proof of this is entirely analogous, replacing homomorphisms by monomorphisms. □

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The relational structures constructed in the proof have infinitely many relations of each arity. It would be interesting to recognize the monoids which are the endomorphism monoids (or monomorphism monoids) of homogeneous structures with only finitely many relations of each arity (these would be the analogue of the closed *oligomorphic* permutation groups, [1]), or even those with only finitely many relations altogether.

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