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Horocycle flow orbits and lattice surface characterizations

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Abstract. The orbit closure of any translation surface under the horocycle flow in almost any direction equals its $\mathrm{SL}_2(\mathbb{R})$ orbit closure. This result gives rise to new characterizations of lattice surfaces in terms of the horocycle flow.

1. *Introduction*

A translation surface is a closed, two-real-dimensional manifold M together with a subset $\Sigma \subset M$ consisting of finitely many points such that the restriction to $M \setminus \Sigma$ of each transition map of the manifold is a translation. The group $\mathrm{SL}_2(\mathbb{R})$ acts on the collection of translation surfaces by affinely deforming the charts of a manifold; the horocycle flow is the action of the one-parameter subgroup H consisting of matrices of the form $\left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right)$. A lattice surface is a translation surface whose stabilizer in $\mathrm{SL}_2(\mathbb{R})$ is a lattice (i.e., has finite co-volume in $\mathrm{SL}_2(\mathbb{R})$). We denote the image under $A \subset \mathrm{SL}_2(\mathbb{R})$ of a subset X of translation surfaces in a given stratum by $A \cdot X$, the closure in the stratum of this set by $\overline{A \cdot X}$ and the matrix $\left(\begin{smallmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{smallmatrix}\right)$ by r_θ. The main results of this paper are Theorems 1 and 2.

THEOREM 1. *For any translation surface M*

$$\overline{Hr_\theta \cdot M} = \overline{\mathrm{SL}_2(\mathbb{R}) \cdot M}$$

for (Lebesgue) almost every angle $\theta \in S^1$.

THEOREM 2. *The following are equivalent.*

(i) *M is a lattice surface.*
(ii) *For every angle $\theta \in S^1$, every H-minimal subset of $\overline{Hr_\theta \cdot M}$ is a periodic H-orbit.*
(iii) *There exists a Lebesgue measurable set $Z \subset S^1$ of positive Lebesgue measure such that $\theta \in Z$ implies that every H-minimal subset of $\overline{Hr_\theta \cdot M}$ is a periodic H-orbit.*
(iv) *Every H-minimal subset of $\overline{\mathrm{GL}_2(\mathbb{R}) \cdot M}$ (or of $\overline{\mathrm{SL}_2(\mathbb{R}) \cdot M}$) is a periodic H-orbit.*

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Since $\mathrm{SL}_2(\mathbb{R})$ orbit closures are affine invariant submanifolds [**EMM15**], Theorem 1 implies that horocycle orbit closures in almost every direction also have this 'nice' structure. In particular, these horocycle orbit closures are immersed submanifolds defined by linear equations in period coordinates with real coefficients, and they have an associated $\mathrm{SL}_2(\mathbb{R})$ invariant probability measure that is ergodic with respect to $\mathrm{SL}_2(\mathbb{R})$ and (via the Mauter phenomenon, described in §2.2) ergodic with respect to H. However, the conclusion $\overline{Hr_\theta \cdot M} = \overline{\mathrm{SL}_2(\mathbb{R}) \cdot M}$ may not hold on a measure zero set of angles. To see this, H preserves horizontal saddle connections, and so for any angle θ in which there is a saddle connection on M, $Hr_{-\theta} \cdot M$ is not dense in $\overline{\mathrm{SL}_2(\mathbb{R}) \cdot M}$.

Smillie and Weiss found examples of horocycle orbit closures which: are manifolds with non-empty boundaries; have infinitely generated fundamental group; and at almost every point are described by non-linear equations [**SW14**, **SW**]. Additional examples of non-affine horocycle orbit closures were described by Clavier in [**Cla16**]. All of these examples thus belong to the measure zero set of angles not governed by the conclusion of Theorem 1. The question of finding a characterization of the angles θ for which $\overline{Hr_\theta \cdot M} = \overline{\mathrm{SL}_2(\mathbb{R}) \cdot M}$ is open. Hooper and Weiss have found a sufficient condition: if M is a periodic point for $\left(\begin{smallmatrix} e^t & 0 \\ 0 & e^{-t} \end{smallmatrix}\right)$, then $\overline{H \cdot M} = \overline{\mathrm{SL}_2(\mathbb{R}) \cdot M}$ [**HW15**].

A novel aspect of Theorem 2 is that each of conditions (ii)–(iv) is *sufficient* for the surface to be a lattice surface (i); that each is necessary was either previously known or could be easily deduced from extant conditions. Our proof uses Theorem 1 to show that (ii) (or (iii)) implies (iv). In fact, the full strength of Theorem 1 is not required; we only use that there exists a positive measure set of angles θ for which the horocycle orbit closure equals the $\mathrm{SL}_2(\mathbb{R})$ orbit closure. While various characterizations of lattice surfaces are known, including a characterization in terms of the geodesic flow, Theorem 2 is, to the authors' knowledge, the first characterization of lattice surfaces in terms of the horocycle flow. We reproduce below a list from [**SW10**] of previously known characterizations of lattice surfaces (the reader may consult [**SW10**] for definitions and notation used in the statement of the theorem).

THEOREM 3. [**SW10**] *The following are equivalent.*

- *M is a lattice surface.*
- *M is uniformly completely periodic.*
- *M is uniformly completely parabolic.*
- *M has 'no small triangles'.*
- *(M, Σ) has 'no small virtual triangles'.*
- *$|\mathcal{T}(M)| < \infty$.*
- *The $\mathrm{SL}_2(\mathbb{R})$-orbit of M is closed.*
- *There is a compact subset K of the stratum $\mathcal{H}$ containing M such that for any $\alpha \in \mathrm{SL}_2(\mathbb{R})$, the orbit of $\alpha \cdot M$ under the geodesic flow has non-empty intersection with K.*

A recent preprint by Lanneau, Nguyen and Wright [**LNW**] proves another characterization of lattice surfaces. Namely, M is a lattice surface if and only if M is completely parabolic and $\overline{\mathrm{SL}_2(\mathbb{R}) \cdot M}$ has rank one.

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2. *Background*

2.1. *Translation surfaces, strata, period coordinates.* A *translation surface* is a two-real-dimensional manifold M together with a subset $\Sigma \subset M$ such that the restriction to $M\backslash\Sigma$ of each transition map between charts of M is a translation. We will denote such a translation surface by the pair (M, Σ) or, in cases where the set Σ is clear from the context, simply by M. We will consider only translation surfaces that are of *finite type*—meaning that M is a closed, connected surface of finite genus and Σ is a finite set of distinct points of M—and the reader should interpret any reference to a translation surface as meaning a translation surface of finite type.

The condition that all manifold transition maps of a translation surface are translations implies that the Euclidean metric on the restriction to $M\backslash\Sigma$ of each manifold chart is invariant under transition maps, resulting in a well-defined metric on $M\backslash\Sigma$. Identifying the metric completion of this metric on $M\backslash\Sigma$ yields the *canonical Euclidean metric* on M. Since translations on $\mathbb{R}^2$ also preserve 'direction' (e.g., the 'positive vertical direction'), 'directions' are also well defined on $M\backslash\Sigma$. Consequently, for any direction $\theta \in S^1 = \mathbb{R} \mod 2\pi$, any translation surface has a well-defined foliation $\mathcal{F}_\theta$ of M in direction θ, defined by pulling back the straight-line foliation in direction θ on the manifold charts. One way of reformulating these two observations is to state that the tangent space $T(M\backslash\Sigma)$ has a canonical global trivialization, i.e., $T(M\backslash\Sigma)$ can be identified in a canonical way with $(M\backslash\Sigma) \times \mathbb{R}^2$. Consequently, a translation surface has trivial linear holonomy, i.e., the group of linear maps on the tangent space $T_p(M\backslash\Sigma)$ induced by parallel transport of an element of $T_p(M\backslash\Sigma)$ along closed loops based at p in $M\backslash\Sigma$ is trivial. The *translation flow* in direction θ on a translation surface is the unit speed (with respect to the canonical Euclidean metric) flow along leaves of the foliation $\mathcal{F}_\theta$.

Points in Σ are called *cone points*. The *cone angle* of a cone point is the total Euclidean angle in S around that point. The cone angle of a cone point in a translation surface is of the form $2\pi(1 + n)$ for some non-negative integer n, which is called the *order* of that cone point. A *saddle connection* is a finite-length leaf of the foliation in some direction which has a cone point at both ends. The Gauss–Bonnet theorem implies that the sum of the orders of all cone points of a translation surface is $2g - 2$, where g is the genus of the underlying topological surface.

An orientation-preserving homeomorphism $\psi : (M_1, \Sigma_1) \to (M_2, \Sigma_2)$ such that $\psi(\Sigma_1) = \Sigma_2$ and such that the restriction $\psi|_{M_1\backslash\Sigma_1}$ is affine in each chart is called an *affine isomorphism*. We denote by $D(\psi)$ the linear part (in $\mathrm{GL}_2(\mathbb{R})$) of an affine isomorphism ψ. An affine isomorphism whose linear part is the identity is called a *translation equivalence*. We consider two translation surfaces to be equivalent if there exists a translation equivalence between them.

A *marked translation surface of finite type* is a triple $((Z, \Sigma'), f, (M, \Sigma))$ consisting of a closed, connected topological surface Z of finite genus with a set Σ' of distinct points of Z, a translation surface (M, Σ) of finite type and a homeomorphism $f : Z \to M$ with $f(\Sigma') = \Sigma$. The collection of all marked translation surfaces admits a natural stratification based on the number and orders of the cone points of each surface. As a set, the stratum $\widetilde{\mathcal{H}}(k_1, \ldots, k_n)$ of the space of marked translation surfaces consists of the set of marked translation surfaces $((Z_g, \Sigma'_n), f, (M, \Sigma))$ whose cone points have orders $k_1, \ldots, k_n$.

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Given a path γ on a marked translation surface, define the *period coordinate* of γ to be the element of $\mathbb{C}$ given by $\int_\gamma dx + i \int_\gamma dy$. Period coordinates determine a map from a stratum $\widetilde{\mathcal{H}}(k_1, \ldots, k_n)$ of marked translation surfaces to $H^1(Z, \Sigma_n'; \mathbb{C})$ as follows. We may think of an element of $H^1(Z, \Sigma_n'; \mathbb{C})$ as assigning an element of $\mathbb{C}$ to each homotopy (rel Σ_n') class of paths in Z. Given a marked translation surface $((Z_g, \Sigma_n'), f, (M, \Sigma))$, define the corresponding element of $H^1(Z_g, \Sigma_n'; \mathbb{C})$ to be the element that assigns to each relative homotopy class $[\gamma]$ of paths in Z_g the period coordinate of the path $f \circ \gamma$ in M. Thus, period coordinates define a map

$$\widetilde{\mathcal{H}}(k_1, \ldots, k_n) \to H^1(Z_g, \Sigma_n'; \mathbb{R}^2) \simeq \mathbb{C}^{2g+n-1}.$$

This map defines a topology and local coordinates on $\widetilde{\mathcal{H}}(k_1, \ldots, k_n)$, which, with this topology, we refer to as a stratum of the *space of marked translation surfaces*.

The *mapping class group* $\mathrm{Mod}(Z, \Sigma')$ of a closed topological surface Z with a finite set Σ' of (distinct) marked points in Z is the group of isotopy classes of homeomorphisms of (Z, Σ') (homeomorphisms of Z that preserve Σ' as a set). The group $\mathrm{Mod}(Z, \Sigma')$ acts properly discontinuously on each stratum of the space of marked surfaces $\widetilde{\mathcal{H}}(Z, \Sigma')$ by precomposition with the marking map. The *stratum* $\mathcal{H}(k_1, \ldots, k_n)$ of the moduli space of translation surfaces is the quotient space $\mathcal{H}(k_1, \ldots, k_n) := \widetilde{\mathcal{H}}(k_1, \ldots, k_n)/\mathrm{Mod}(Z_g, \Sigma_n')$.

These strata can be disconnected, with up to three connected components, which were classified by Kontsevich and Zorich [**KZ03**]. Pulling back the Lebesgue measure on $\mathbb{C}^{2g+n-1}$ via period coordinates yields a local volume form. It assigns infinite measure to a stratum of the space of translation surfaces. It is standard to restrict to unit-area translation surfaces, a codimension one subset of this space. A modified version of the disintegration of the previous measure onto this subspace gives it finite volume [**Mas82**, **Vee86**]. This measure is called the *Masur–Veech measure*. We will use $\mathcal{H}_1(k_1, \ldots, k_n)$ to denote the stratum of unit-area surfaces, in contrast to the full stratum (of surfaces of any area) $\mathcal{H}(k_1, \ldots, k_n)$.

2.2. *Dynamical preliminaries.* The group $\mathrm{SL}_2(\mathbb{R})$ acts on a stratum $\mathcal{H}_1$ as follows. Given $A \in \mathrm{SL}_2(\mathbb{R})$ and a translation surface (M, Σ), the flat surface given by $A \cdot (M, \Sigma)$ is given by post-composing the charts of M with A. $\mathrm{GL}_2(\mathbb{R})$ acts similarly on $\mathcal{H}$, but does not preserve the area of the translation surfaces. The actions of two subgroups of $\mathrm{SL}_2(\mathbb{R})$ are of particular interest. The *Teichmüller flow* or *geodesic flow* is the action of the one-parameter subgroup consisting of all matrices of the form

$$g_t := \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}, \quad t \in \mathbb{R}.$$

The *horocycle flow* is the action of the one-parameter subgroup consisting of all matrices of the form

$$h_t := \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \quad t \in \mathbb{R}.$$

A translation surface (M, Σ) is said to be:

- *periodic* in direction θ if (M, Σ) admits a cylinder decomposition in direction θ;
- *completely periodic* if (M, Σ) is periodic in every direction in which (S, Σ) has at least one saddle connection;

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- *uniformly completely periodic* if (M, Σ) is completely periodic and there exists $c > 0$ such that, for any direction θ for which (M, Σ) is periodic, the ratio of lengths of any two saddle connections in direction θ is at most c;
- *parabolic* in direction θ if (M, Σ) is periodic in direction θ and the moduli of all the cylinders in direction θ are commensurable;
- *completely parabolic* if M is completely periodic and parabolic in every periodic direction;
- *uniformly completely parabolic* if (M, Σ) is uniformly completely periodic and (M, Σ) is parabolic in every periodic direction.

Let (M, Σ) be a translation surface of finite type. The *Veech group* of (M, Σ) is the stabilizer in S of (M, Σ). A translation surface (S, Σ) is a *lattice surface* if its Veech group is a lattice, i.e., has finite co-volume in $\mathrm{SL}_2(\mathbb{R})$.

THEOREM (The Veech dichotomy) *If (M, Σ) is a lattice surface, then for every direction $\theta \in S^1$, precisely one of the following is true:*
(1) *(M, Σ) admits a cylinder decomposition in direction θ; or*
(2) *the translation flow in direction θ on (M, Σ) is uniquely ergodic.*

The converse to the Veech dichotomy is not true [**SW08**].

THEOREM (The Mautner phenomenon for $\mathrm{SL}_2(\mathbb{R})$) *Let $\mathfrak{H}$ be a Hilbert space and let $\phi : \mathrm{SL}_2(\mathbb{R}) \to U(\mathfrak{H})$ be a continuous unitary representation on $\mathfrak{H}$. Then any element $v \in \mathfrak{H}$ that is invariant under H is also invariant under $\mathrm{SL}_2(\mathbb{R})$.*

(The statement '$\phi : S \to U(\mathfrak{H})$ is a continuous unitary representation on $\mathfrak{H}$' means that ϕ is a homomorphism into the group of unitary automorphisms $\mathbb{U}(\mathfrak{H})$ of $\mathfrak{H}$ such that, for every $v \in \mathfrak{H}$, the element $\phi(g)(v) \in \mathfrak{H}$ depends continuously on $g \in \mathrm{SL}_2(\mathbb{R})$.)

The action of $\mathrm{SL}_2(\mathbb{R})$ on a stratum $\mathcal{H}_1$ with an $\mathrm{SL}_2(\mathbb{R})$-invariant probability measure μ_1 determines a continuous unitary representation on $\mathfrak{H} = L^2(\mathcal{H}_1, \mu_1)$ defined by

$$\alpha \in \mathrm{SL}_2(\mathbb{R}) \mapsto (f \in \mathfrak{H} \mapsto f \circ \alpha \in \mathfrak{H}) \in U(\mathfrak{H}).$$

For flows, ergodicity can be characterized as the condition that 'the only invariant elements of $L^2(\mathcal{H}_1, \mu_1)$ are constant functions'. Consequently, an $\mathrm{SL}_2(\mathbb{R})$-invariant measure on $\mathcal{H}_1$ is ergodic with respect to the action of H if and only if it is ergodic for the $\mathrm{SL}_2(\mathbb{R})$ action.

2.3. *Decompositions of* $\mathrm{SL}_2(\mathbb{R})$.

$$K = \left\{ r_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} : \theta \in S^1 \right\}, \quad A = \left\{ g_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} : t \in \mathbb{R} \right\},$$
$$N = \left\{ h_s = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} : s \in \mathbb{R} \right\}, \quad \overline{N} = \left\{ \hat{h}_s = \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix} : s \in \mathbb{R}. \right\}$$

There are at least three well-known decompositions of $\mathrm{SL}_2(\mathbb{R})$.

Iwasawa decomposition. Every element $g \in \mathrm{SL}_2(\mathbb{R})$ can be written as *kan* for some $k \in K$, $a \in A$ and $n \in N$, and this representation is unique if we require $t \geq 0$ in $a = g_t$.

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Cartan decomposition. Every element $g \in \mathrm{SL}_2(\mathbb{R})$ can be written as kak', for some k, $k' \in K$ and $a \in A$.

Bruhat decomposition. $\mathrm{SL}_2(\mathbb{R}) = \overline{N}AN \cup \iota AN$, where $\iota = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, with explicit formulas

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ c/a & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & 1/a \end{pmatrix} \begin{pmatrix} 1 & b/a \\ 0 & 1 \end{pmatrix} \quad \text{if } a \neq 0$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c & 0 \\ 0 & 1/c \end{pmatrix} \begin{pmatrix} 1 & d/c \\ 0 & 1 \end{pmatrix} \quad \text{if } a = 0.$$

LEMMA 1. *In the Cartan decomposition $h_s = k_s a_s k_s'$, the element a_s leaves every compact set as $s \to \pm\infty$.*

Lemma 1 is an immediate consequence of the compactness of K.

LEMMA 2. *The $\bar{N}AN$ decomposition of r_θ is $\hat{h}_{\tan(\theta)} g_{\log(\cos(\theta))} h_{-\tan(\theta)}$.*

2.4. *Key recent results.*

2.4.1. *Eskin–Mirzakhani–Mohammadi.* Our results build off of several breakthrough results of Eskin, Mirzakhani and Mohammedi.

THEOREM 4. [**EMM15**, Theorem 2.1] *For any flat surface $M \in \mathcal{H}_1$, $\overline{\mathrm{SL}_2(\mathbb{R}) \cdot M}$ is an affine invariant submanifold of $\mathcal{H}_1$.*

Here $\mathcal{H}_1$ is the stratum of unit-area surfaces.

An affine invariant submanifold $\mathcal{M}_1 \subset \mathcal{H}_1$ is the support of an ergodic $\mathrm{SL}_2(\mathbb{R})$-invariant measure ν_1 such that:

(1) $\mathcal{M}_1$ is an immersed submanifold (i.e., $\mathcal{M}_1$ is the image of a manifold $\mathcal{N}$ under a proper continuous map f) and the set of self-intersection points of $\mathcal{M}_1$ is closed and has ν_1-measure zero;
(2) each point of $\mathcal{N}$ has a neighborhood U such that $\mathbb{R}f(U)$ is given by a complex linear subspace defined over $\mathbb{R}$ in period coordinates; and
(3) each point of $\mathcal{N}$ has a neighborhood U such that, if ν is the measure supported on $\mathcal{M} = \mathbb{R}\mathcal{M}_1$ so that $d\mu = d\nu_1\, da$, the restriction of ν to $\mathbb{R}f(U)$ is an affine linear measure in the period coordinates.

Each $\mathrm{SL}_2(\mathbb{R})$ invariant manifold is the support of a unique $\mathrm{SL}_2(\mathbb{R})$-invariant ergodic probability measure. We will refer to this measure as the measure associated with the affine invariant submanifold. (An affine invariant submanifold may properly contain a smaller affine invariant submanifold which has its own measure and whose support is, of course, a proper subset of our initial affine invariant submanifold.)

THEOREM 5. [**EMM15**, Theorem 2.6] *Let M be a translation surface, $\overline{\mathrm{SL}_2(\mathbb{R}) \cdot M} := \mathcal{M}$, and let μ be the affine invariant measure associated to $\mathcal{M}$. Let $\phi \in C_c(\mathcal{M})$. Then, for any $\epsilon > 0$ and any interval $I \subset [0, 2\pi)$, there exists T_0 such that $T > T_0$ implies that*

$$\left| \frac{1}{T} \int_0^T \frac{1}{|I|} \int_I \phi(g_t r_\theta M)\, d\theta\, dt - \int_{\mathcal{M}} \phi\, d\mu \right| < \epsilon.$$

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2.4.2. *Minimal sets for the horocycle flow.* The closure in a stratum $\mathcal{H}_1$ of any $\mathrm{SL}_2(\mathbb{R})$-orbit contains an H-orbit closure. Smillie and Weiss [**SW04**] showed that every H-orbit closure contains a *minimal* set for the horocycle flow, and they classified the minimal sets for the horocycle flow. (A *minimal set* for the action of a group A on a space X is a non-empty, closed, A-invariant subset of X that is minimal with respect to inclusion.) We state the following two results only for translation surfaces, but the original results were slightly more general.

PROPOSITION 1. [**SW04**] *Let M be a translation surface that is periodic in the horizontal direction. Let $\mathcal{O} = \overline{H \cdot M}$. Then:*

(1) *M admits a cylinder decomposition $M = C_1 \cup \cdots \cup C_r$, where each C_i is a cylinder whose interior is a union of horizontal core curves;*
(2) *there is an isomorphism between $\mathcal{O}$ and a d-dimensional torus, where d is the dimension of the $\mathbb{Q}$-linear subspace of $\mathbb{R}$ spanned by the moduli of $C_1, \ldots, C_r$. This isomorphism conjugates the H-action on $\mathcal{O}$ with a one-parameter translation flow; and*
(3) *the restriction of the H-action to $\mathcal{O}$ is minimal.*

THEOREM 6. [**SW04**] *If M is a translation surface such that $\overline{H \cdot M}$ is contained in a compact subset of a single stratum, then the flow along any leaf of the horizontal foliation is periodic. In particular, any minimal set for the horocycle flow is as described in Proposition 1.*

2.4.3. *Cylinder deformations.* This characterization of minimal sets for the horocycle flow is a key ingredient in Wright's proof of the 'cylinder deformation theorem' [**Wri15**]. Given a collection of horizontal cylinders $\mathcal{C}$ of a surface M, let $\eta_{\mathcal{C}} \in T_M(\mathcal{M}) \subset H^1(S, \Sigma; \mathbb{C})$ be the derivative (with respect to t) of $h_t^{\mathcal{C}}$ at M in local period coordinates, where $h_t^{\mathcal{C}}$ is the 'cylinder shear' which applies the matrix h_t to the cylinders of $\mathcal{C}$ and leaves the rest of the surface unchanged. Denote by $a_t^{\mathcal{C}}$ the 'cylinder stretch' which applies the matrix $a_t = \left(\begin{smallmatrix} 1 & 0 \\ 0 & e^t \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{R})$ to the cylinders of $\mathcal{C}$ and leaves the rest of the surface unchanged.

Definition 1. Let M be a flat surface and let $\mathcal{M} = \overline{\mathrm{GL}_2(\mathbb{R}) \cdot M}$. Two cylinders of M are:

(1) $\mathcal{M}$-parallel if they are parallel at M and at every nearby surface $M' \in \mathcal{M}$; and
(2) $\mathcal{M}$-collinear if their core curves $\alpha, \beta \in H_1(M, \Sigma; \mathbb{Z})$ have collinear images in $T^M(\mathcal{M})$.

Wright observed [**Wri15**] that, as a consequence of Theorem 4, $\mathcal{M}$-parallel and $\mathcal{M}$-collinear are equivalent notions, i.e., two cylinders are $\mathcal{M}$-parallel if and only if they are $\mathcal{M}$-collinear.

LEMMA 3. [**Wri15**, Lemma 4.11] *For any horizontally periodic surface $M \in \mathcal{M}$ and equivalence class $\mathcal{C}$ of $\mathcal{M}$-parallel horizontal cylinders, $\eta_{\mathcal{C}} \in T^M(\mathcal{M})$.*

Since Theorem 4 asserts that $\mathcal{M}$ is given (locally) by linear equations in period coordinates with real coefficients, a neighborhood of M in $\mathcal{M}$ is identified with a neighborhood of zero in $T_M\mathcal{M}$. Thus, the assertion that $\eta_{\mathcal{C}}$ is in $T^M(\mathcal{M})$ implies that

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sufficiently small deformations of M in the direction (in $T^M(\mathcal{M})$) specified by η remain in the orbit closure $\mathcal{M}$. We denote by p the projection $p : H^1(S, \Sigma; \mathbb{C}) \to H^1(S; \mathbb{C})$ from relative cohomology to absolute cohomology.

THEOREM 7. [**Wri15**, Theorem 1.7] *If* $\dim_{\mathbb{C}} p(T(\mathcal{M})) > 2$*, then there exist translation surfaces in* $\mathcal{M}$ *which are not completely periodic.*

Definition 2. The cylinder rank of an affine invariant submanifold $\mathcal{M}$ is $\frac{1}{2}\dim_{\mathbb{C}} p(T(\mathcal{M}))$.

Definition 3. Let $M \in \mathcal{M}$ be horizontally periodic. The *twist space* of $\mathcal{M}$ at M, $\mathrm{Twist}(M, \mathcal{M})$, is the subspace of $T^M(\mathcal{M})$ consisting of the cohomology classes in $T^M(\mathcal{M})$ which are zero on all horizontal saddle connections.

Definition 4. Let $M \in \mathcal{M}$ be horizontally periodic. The *cylinder preserving space* of $\mathcal{M}$ at M, $\mathrm{Pres}(M, \mathcal{M})$, is the subspace of $T^M(\mathcal{M})$ consisting of the cohomology classes in $T^M(\mathcal{M})$ which are zero on the core curves of all horizontal cylinders.

3. *Horocycle orbits*

The primary goal of this section is to prove Theorem 1. The outline of the proof is as follows.

Step 0 Fix an affine invariant submanifold $\mathcal{M}$ with associated ergodic invariant probability measure μ. We will show that the conclusion of Theorem 1 holds for any translation surface M such that $\mathcal{M} = \overline{\mathrm{SL}_2(\mathbb{R}) \cdot M}$.

Step 1 For any $\epsilon > 0$, define the local metric $d_{\mathcal{K}_\epsilon}$ on $\mathcal{K}_\epsilon$, the ϵ-thick part of the stratum. This metric $d_{\mathcal{K}_\epsilon}$ is chosen so that it is 'bounded above' by $d_{\mathrm{SL}_2\mathbb{R}}$ (Lemma 4) and is consistent with the usual topology on strata.

Step 2 Use the Mautner phenomenon to show that, for any affine invariant manifold $\mathcal{M}$ with associated probability measure μ and sufficiently small $\epsilon, \tilde{\epsilon} > 0$, there exist open sets $U_{L,\epsilon,\tilde{\epsilon},\mathcal{M}}$ of arbitrarily large μ-measure consisting of surfaces M for which $(\bigcup_{s=0}^{L} h_s \cdot M) \cap \mathcal{K}_\epsilon$ is $\tilde{\epsilon}$-dense in $\mathcal{M} \cap \mathcal{K}_\epsilon$.

Step 3 Apply a result of Eskin–Mirzakhani–Mohammadi to show that, for any translation surface M for which $\mathcal{M} = \overline{\mathrm{SL}_2(\mathbb{R}) \cdot M}$, $y > 0$, and for any sufficiently large open set U, there exist arbitrarily large real numbers t so that 'all but y per cent' of the 'circle' (in θ) $g_t r_\theta \cdot M$ is in U.

Step 4 Given such a t, any arc that contains $> y$ per cent of the 'circle' $g_t r_\theta \cdot M$ contains a point, say, u, of $U_{L,\epsilon,\tilde{\epsilon},\mathcal{M}}$. But, for sufficiently big t, such an arc closely approximates the length-L segment of the horocycle flow applied to u. Proposition 4 asserts that small θ-arcs of $g_t r_\theta \cdot M$ (for suitably large t) are $\tilde{\epsilon}$-dense in $\mathcal{K}_\epsilon$. Proposition 5 proves that, for any small θ arc, arbitrarily large such t exist.

Step 5 Use the Cartan decomposition of h_s to turn Proposition 5 into an analogous statement about arcs (in θ) of $h_s r_\theta \cdot M$ for sufficiently large s. Proposition 6 asserts that there are arbitrarily big values of s such that any $2r$-arc of $h_s r_\theta \cdot M$ is $\tilde{\epsilon}$-dense in $\mathcal{M} \cap \mathcal{K}_\epsilon$.

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Step 6A A slight detour from proving Theorem 1—we use the Baire category theorem in conjunction with Proposition 6 to prove the weaker result (Theorem 8) that there is a residual set of directions in which the horocycle flow-orbit closure equals the $\mathrm{SL}_2(\mathbb{R})$-orbit closure. To do this, we show that, for any fixed ϵ, $\tilde{\epsilon}$, the set $G_M(\epsilon, \tilde{\epsilon})$ of angles θ for which Hr_θ is $\tilde{\epsilon}$-dense in $\mathcal{M} \cap \mathcal{K}_\epsilon$ (with respect to $d_{\mathcal{K}_\epsilon}$) is an open dense subset of S^1.

Step 6B Prove that almost every horocycle is dense. This uses slightly more complicated versions of arguments from Steps 5 and 6A.

3.1. *Step 1.* For any fixed stratum of unit-area surfaces $\mathcal{H}_1$ and $\epsilon > 0$, we denote by $\mathcal{K}_\epsilon$ the ϵ-thick part of the stratum $\mathcal{H}_1$. The set $\mathcal{K}_\epsilon$ is the subset of $\mathcal{H}_1$ consisting of all translation surfaces in $\mathcal{H}_1$ which have no saddle connection of length less than ϵ. Any such set $\mathcal{K}_\epsilon$ is compact.

LEMMA 4. *For any $\epsilon > 0$ (and $\mathcal{H}_1$), there exist a metric $d_{\mathcal{K}_\epsilon}$ on $\mathcal{K}_\epsilon$ and real number $r_0 > 0$ such that $d_{\mathrm{SL}_2\,\mathbb{R}}(A, \mathrm{Id}) < r_0$ implies that*

$$d_{\mathcal{K}_\epsilon}(A \cdot M, M) < d_{\mathrm{SL}_2\,\mathbb{R}}(A, \mathrm{Id})$$

for all M, $AM \in \mathcal{K}_\epsilon$.

By $d_{\mathrm{SL}_2\,\mathbb{R}}$ we mean a left-invariant metric on $\mathrm{SL}_2(\mathbb{R})$†. Our proof of Lemma 4 also uses a metric d_{op} on $\mathrm{SL}_2(\mathbb{R})$; by $d_{\mathrm{op}}(A_1, A_2)$ we mean the operator norm of $(A_1 - A_2)$ acting on $\mathbb{R}^2$.

Proof. Period coordinates yield local metrics modeled on Euclidean space on open 'patches' covering the stratum $\mathcal{H}_1$. By the compactness of $\mathcal{K}_\epsilon$, we may restrict our attention to a finite collection of patches U_j that cover $\mathcal{K}_\epsilon$. For each patch U_j, we may pick a list of paths $\gamma_1^j, \ldots, \gamma_n^j$ in the underlying topological marked surface and define the 'Euclidean metric' $d_{\mathrm{Euc}\,U_j}$ on this patch to be the Euclidean distance on $(\mathbb{R}^2)^n$ pulled back to U_j via the identification of M with $(\mathrm{hol}(\gamma_1^j), \ldots, \mathrm{hol}_M(\gamma_n^j))$, where $\mathrm{hol}_M(\gamma_i^j) \in \mathbb{R}^2$ is the x- and y-components of the path γ_i in the surface M, i.e., $\mathrm{hol}_M(\gamma_i^j) = (\int_{\gamma_i^j} dx, \int_{\gamma_i^j} dy)$ in M.

We first define a metric $D_{\mathcal{K}_\epsilon}$ on $\mathcal{K}_\epsilon$ by using the infimum of the lengths (with respect to any combination of the Euclidean metrics on patches) of all rectifiable paths connecting two points. Since $\mathcal{K}_\epsilon$ is compact, there is a minimal $c > 0$ so that, for every $M \in \mathcal{K}_\epsilon$, a c-neighborhood of M in $\mathcal{K}_\epsilon$ with respect to the metric $D_{\mathcal{K}_\epsilon}$ is contained within a single patch. For each $M \in \mathcal{K}_\epsilon$, define $U(M) \in \{U_j\}_j$ to be such a patch, and define $d_{\mathrm{Euc}\,U(M)}$ to be the Euclidean metric on the patch $U(M)$.

Hence, for any $M \in \mathcal{K}_\epsilon$ and any $A \in \mathrm{SL}_2(\mathbb{R})$ such that $A \cdot M$ is in the $D_{\mathcal{K}_\epsilon}$ c-ball centered at M,

$$D_{\mathcal{K}_\epsilon}(M, A \cdot M) \leq \min\{d_{\mathrm{Euc}\,U(M)}(M, A \cdot M), d_{\mathrm{Euc}\,U(AM)}(M, A \cdot M)\}. \tag{1}$$

For such an A, $d_{\mathrm{Euc}\,U(M)}(M, A \cdot M)$ is precisely the Euclidean distance in $(\mathbb{R}^2)^n$ between

$$\left(\mathrm{hol}_M(\gamma_1^{U(M)}), \ldots, \mathrm{hol}_M(\gamma_n^{U(M)})\right)$$

† See [**EW11**, §9.3.2] for the construction of a left-invariant metric.

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and

$$\big(A \cdot \mathrm{hol}_M(\gamma_1^{U(M)}), \ldots, A \cdot \mathrm{hol}_M(\gamma_n^{U(M)})\big).$$

Thus, $d_{\mathrm{Euc}\, U(M)}(M, A \cdot M)$ equals the Euclidean norm of the vector

$$\big((A - \mathrm{Id}) \cdot \mathrm{hol}_M(\gamma_1^{U(M)}), \ldots, (A - \mathrm{Id}) \cdot \mathrm{hol}_M(\gamma_n^{U(M)})\big),$$

which is less than or equal to the product

$$d_{\mathrm{op}}(A, \mathrm{Id}) \cdot \big|\mathrm{hol}_M(\gamma_1^{U(M)}), \ldots, \mathrm{hol}_M(\gamma_n^{U(M)})\big|.$$

Similarly, $d_{\mathrm{Euc}\, U(A\cdot M)}(M, A \cdot M)$ is less than or equal to

$$d_{\mathrm{op}}(A, \mathrm{Id}) \cdot \big|\mathrm{hol}_M(\gamma_1^{U(A\cdot M)}), \ldots, \mathrm{hol}_M(\gamma_n^{U(A\cdot M)})\big|.$$

Hence, from inequality (1),

$$D_{\mathcal{K}_\epsilon}(M, A \cdot M) \le d_{\mathrm{op}}(A, \mathrm{Id}) \cdot \min\big\{\big|\mathrm{hol}_M(\gamma_1^{U(M)}), \ldots, \mathrm{hol}_M(\gamma_n^{U(M)})\big|, \big|\mathrm{hol}_M(\gamma_1^{U(A\cdot M)}), \ldots, \mathrm{hol}_M(\gamma_n^{U(A\cdot M)})\big|\big\}. \tag{2}$$

We now define a scaled version $d_{\mathcal{K}_\epsilon}$ of the metric $D_{\mathcal{K}_\epsilon}$. By [**EW11**, Lemma 9.12], there is a neighborhood V of Id in $\mathrm{SL}_2(\mathbb{R})$ on which $d_{\mathrm{SL}_2\mathbb{R}}$ and d_{op} are Lipschitz equivalent. Hence there exists $\eta > 0$ such that $d_{\mathrm{op}}(A, \mathrm{Id}) < \eta \cdot d_{\mathrm{SL}_2\mathbb{R}}$ for $A \in V$. Now let

$$\rho = \sup_j \sup_{N \in U_j} \big\{\big|\mathrm{hol}_N(\gamma_1^{U_j}), \ldots, \mathrm{hol}_N(\gamma_n^{U_j})\big|\big\}.$$

By compactness of $\mathcal{K}_\epsilon$ and continuity of hol, ρ is finite. So define the metric $d_{\mathcal{K}_\epsilon}$ on $\mathcal{K}_\epsilon$ by

$$d_{\mathcal{K}_\epsilon} = \frac{1}{\eta\rho} D_{\mathcal{K}_\epsilon}.$$

Then, from equation (2), for any $M \in \mathcal{K}_\epsilon$ and any $A \in V$ such that $A \cdot M$ is in the $D_{\mathcal{K}_\epsilon}$ c-ball centered at M,

$$d_{\mathcal{K}_\epsilon}(M, A \cdot M) \le \frac{1}{\eta} \cdot d_{\mathrm{op}}(A, \mathrm{Id}) \le d_{\mathrm{SL}_2\mathbb{R}}(A, \mathrm{Id}).$$

For each $M \in \mathcal{K}_\epsilon$, there exists a real number $r(M) > 0$ such $d_{\mathrm{SL}_2\mathbb{R}}(A, \mathrm{Id}) < r(M)$ implies that AM is in the $D_{\mathcal{K}_\epsilon}$ c-ball centered at M. Now, by compactness, we may fix $r_0 > 0$ such that $r_0 < r(M)$ for all $M \in \mathcal{K}_\epsilon$ and the r_0-ball about Id with respect to $d_{\mathrm{SL}_2\mathbb{R}}$ is contained in V. □

3.2. *Step 2.*

Definition 5. We say a surface M is $(L, \epsilon, \tilde{\epsilon})$-*nice for* h_s if

$$\left(\bigcup_{s=0}^{L} h_s \cdot M\right) \cap \mathcal{K}_\epsilon$$

is $\tilde{\epsilon}$-dense in

$$\overline{\mathrm{SL}_2(\mathbb{R}) \cdot M} \cap \mathcal{K}_\epsilon$$

with respect to the metric $d_{\mathcal{K}_\epsilon}$. We define $(L, \epsilon, \tilde{\epsilon})$-nice for $\hat{h}_s$ similarly.

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PROPOSITION 2. *Let μ be an $\mathrm{SL}_2(\mathbb{R})$-invariant, ergodic, Borel, probability measure on a stratum $\mathcal{H}$. Then, for any sufficiently small $\epsilon, \tilde{\epsilon} > 0$,*

$$\lim_{L\to\infty} \mu(\{M \in \mathcal{H} : (L, \epsilon, \tilde{\epsilon})\text{-nice for } h_s\}) = 1.$$

Proof. Let $\mathcal{M}$ denote the support of μ and fix $\epsilon, \tilde{\epsilon} > 0$.

The Mautner phenomenon implies that since μ is ergodic with respect to the action of $\mathrm{SL}_2(\mathbb{R})$, it is also ergodic with respect to the horocycle flow. Hence, any measurable set $W \subset \mathcal{H}$ with positive μ-measure has that $H \cdot W$ has full measure. Cover $\mathcal{K}_\epsilon \cap \mathcal{M}$ by a collection of open balls in $\mathcal{K}_\epsilon$ (with the subspace topology) of radius $\tilde{\epsilon}/2$ with respect to $d_{\mathcal{K}_\epsilon}$. Since $\mathcal{M}$ is the support of μ, each of these $\tilde{\epsilon}/2$-balls has positive μ measure.

By the compactness of $\mathcal{K}_\epsilon$, we may take a finite, non-empty, subcover of $\mathcal{K}_\epsilon \cap \mathcal{M}$ consisting of a finite subset of these $\tilde{\epsilon}/2$-balls $B_1, \ldots, B_m$. Let

$$G = \bigcap_{j=1}^{m} \bigcup_{s\in\mathbb{R}^+} h_{-s} \cdot B_j.$$

As a finite intersection of sets of full μ-measure, G has full μ-measure. So, for every $\epsilon > 0$, there exists L so that

$$\mu\left(\bigcap_{j=1}^{m} \bigcup_{s\in[0,L]} h_{-s} \cdot B_j \right) > 1 - \epsilon$$

and, by construction, the hororcyle orbit in $\mathcal{K}_\epsilon$ of each point in G passes within distance $\tilde{\epsilon}$ of every point in $\mathcal{M} \cap \mathcal{K}_\epsilon$. □

COROLLARY 1. *Let μ be an $\mathrm{SL}_2(\mathbb{R})$-invariant, ergodic, Borel, probability measure on a stratum $\mathcal{H}_1$, and let $\tilde{\epsilon}, \epsilon > 0$. Denote by $\mathcal{M}$ the support of μ. Then there exists a non-empty open set $U \subset \mathcal{H}_1$ and a real number $L > 0$ such that $M \in U$ implies that*

$$\left(\bigcup_{s\in[0,L]} h_s \cdot M \right) \cap \mathcal{K}_\epsilon$$

is $\tilde{\epsilon}$-dense in $\mathcal{M} \cap \mathcal{K}_\epsilon$. Moreover, for any $y > 0$, the set U may be chosen so that $\mu(U) > 1 - y$.

Proof sketch. Apply Proposition 2 to obtain a real number $L > 0$ so that the set

$$V := \left\{ M \in \mathcal{M} : M \text{ is } \left(L, \epsilon, \frac{\tilde{\epsilon}}{2}\right)\text{-nice for } h_s \right\}$$

has μ measure at least $1 - y$. By the equicontinuity of the h_s for all $0 \le s \le L$, some open set $U \supset V$ has the property that, for every $M \in U$,

$$\bigcup_{s\in[0,L]} h_s \cdot M$$

is $\tilde{\epsilon}$-dense in $\mathcal{M} \cap \mathcal{K}_\epsilon$. □

Definition 6. We will use the notation $U_{L,\epsilon,\tilde{\epsilon},\mathcal{M}}$ to denote a set U that satisfies the conclusions of Corollary 1 for the parameters $L, \epsilon, \tilde{\epsilon} > 0$ and affine invariant manifold $\mathcal{M}$.

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3.3. *Step 3.*

PROPOSITION 3. *Let $\mathcal{M}$ be an affine invariant submanifold of $\mathcal{H}$ with associated $\mathrm{SL}_2(\mathbb{R})$-invariant ergodic Borel probability measure μ and let U be an open set in $\mathcal{H}$ such that $\mu(U) \geq 1 - \epsilon/2$. Then, for every M so that $\overline{\mathrm{SL}_2(\mathbb{R}) \cdot M} = \mathcal{M}$, there exist arbitrarily large $t \in \mathbb{R}$ such that*

$$\lambda(\{\theta : g_t r_\theta \cdot M \in U\}) > (1 - \epsilon),$$

where λ is Lebesgue measure (with total mass 1) on S^1.

Proof. We will obtain this as a consequence of [**EMM15**, Theorem 2.6] (Theorem 5 of this paper). Indeed, choose $\phi \in C_C(\mathcal{M})$ so that ϕ is non-negative, supported on U, $\|\phi\|_\infty = 1$ and $\int_{\mathcal{M}} \phi \, d\mu > 1 - \frac{3}{4}\epsilon$. By Theorem 2.6, there exist arbitrarily large T so that

$$\frac{1}{T} \int_0^T \int_0^{2\pi} \phi(g_t r_\theta \cdot M) \, d\theta \, dt > \left(1 - \frac{7}{8}\epsilon\right).$$

Since

$$\int_0^{2\pi} \phi(g_t r_\theta \cdot M) \, d\theta \leq 1$$

for all $t \in \mathbb{R}$, it follows that

$$\left| \left\{ t \leq T : \int_0^{2\pi} \phi(g_t r_\theta \cdot M) \, d\theta \, dt > 1 - \epsilon \right\} \right| \geq \frac{1}{8}\epsilon T.$$

For any t in this set, by our choice of ϕ,

$$\lambda(\{\theta : g_t r_\theta \cdot M \in U\}) > 1 - \epsilon,$$

and the proposition follows. □

3.4. *Step 4.* The next lemma shows that under g_t arcs of 'circles' (in θ) track horocycles. This is used multiple times in our argument to show that arcs of 'circles' pushed by g_t are often dense or even approximately uniformly distributed. This is well known.

LEMMA 5. *For any $\delta, \epsilon > 0$, there exist real numbers $t_0 > 0$, $\pi/2 > \psi_0 > -\pi/2$ such that if $t > t_0$, $g_t r_\psi \cdot M \in \mathcal{K}_{2\epsilon}$ and $|\psi| < \psi_0$, then $h_{-e^{2t}\tan(\psi)} g_t \cdot M \in \mathcal{K}_\epsilon$ and*

$$d_{\mathcal{K}_\epsilon}(g_t r_\psi \cdot M, h_{-e^{2t}\tan(\psi)} g_t \cdot M) < \delta.$$

Proof. Recall that

$$g_t r_\theta = \hat{h}_{e^{-2t}\tan(\theta)} g_{\log(\cos(\theta))} h_{-e^{2t}\tan(\theta)} g_t.$$

By Lemma 4, if θ is small enough and t is large enough, then, for any $M \in \mathcal{K}_{2\epsilon}$, $\hat{h}_{e^{-2t}\tan(\theta)} g_{\log(\cos(\theta))} \cdot M \in \mathcal{K}_\epsilon$ and

$$d_{\mathcal{K}_\epsilon}(\hat{h}_{e^{-2t}\tan(\theta)} g_{\log(\cos(\theta))} \cdot M, M) < d_{\mathrm{SL}_2(\mathbb{R})}(\hat{h}_{e^{-2t}\tan(\theta)} g_{\log(\cos(\theta))}, \mathrm{Id}) < \delta. \qquad \square$$

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PROPOSITION 4. *Let M be any translation surface and let $\mathcal{M} = \overline{\mathrm{SL}_2(\mathbb{R}) \cdot M}$. For all sufficiently small $\tilde{\epsilon}$, ϵ, $y > 0$ and $L \in \mathbb{R}^+$, there exists $b_{\tilde{\epsilon},\epsilon,L,\mathcal{M}}$ so that if $t > b_{\tilde{\epsilon},\epsilon,L,\mathcal{M}}$ and*

$$\lambda(\{\theta : g_t r_\theta \cdot M \in U_{L,\epsilon,\tilde{\epsilon}/2,\mathcal{M}}\}) > 1 - \frac{y}{2\pi}, \tag{3}$$

then

$$\left(\bigcup_{\theta \in (a-y, a+y)} g_t r_\theta \cdot M \right) \cap \mathcal{K}_\epsilon \tag{4}$$

is $\tilde{\epsilon}$-dense in $\mathcal{M} \cap \mathcal{K}_\epsilon$ for all $a \in S^1$.

Proof. Because

$$\lambda\left(\left(a - \frac{y}{2}, a + \frac{y}{2}\right)\right) = \frac{y}{2\pi},$$

there exists $\phi \in (a - y/2, a + y/2)$ so that $g_t r_\phi \cdot M \in U_{L,\epsilon,\tilde{\epsilon}/2,\mathcal{M}}$. By the definition of $U_{L,\epsilon,\tilde{\epsilon}/2,\mathcal{M}}$,

$$\left(\bigcup_{s \in [0,L]} h_s g_t r_\phi \cdot M \right) \cap \mathcal{K}_\epsilon$$

is $\tilde{\epsilon}/2$-dense in $\mathcal{K}_\epsilon \cap \mathcal{M}$. By Lemmas 4 and 5, if t is large enough, depending on L and $s \in [0, L]$, there exists $\theta_{s,t} \in (-3Le^{-2t}, 3Le^{-2t})$ so that

$$d_{\mathcal{K}_\epsilon}(h_s g_t r_\phi \cdot M, g_t r_{(\phi+\theta_{s,t})} \cdot M) < \frac{\tilde{\epsilon}}{2}.$$

For sufficiently large values of t, $3e^{-2t} < y/2$ and $\phi + \theta_{s,t} \in (a - y, a + y)$. The proposition follows. □

PROPOSITION 5. *Fix a translation surface M and let $\mathcal{M} = \overline{\mathrm{SL}_2(\mathbb{R}) \cdot M}$. Then, for any ϵ, $\tilde{\epsilon}$, $y > 0$, there exists arbitrarily large $t \in \mathbb{R}$ such that the set*

$$\left(\bigcup_{\theta \in (a-y, a+y)} g_t r_\theta \cdot M \right) \cap \mathcal{K}_\epsilon$$

is $\tilde{\epsilon}$-dense in $\mathcal{M} \cap \mathcal{K}_\epsilon$ for any $a \in S^1$.

Proof. By Corollary 1, for any $\epsilon, \tilde{\epsilon} > 0$, there exists $L > 0$ and an open set $U_{L,\epsilon,\tilde{\epsilon}/2,\mathcal{M}}$ satisfying the conclusions of Corollary 1 with

$$\mu(U_{L,\epsilon,\tilde{\epsilon}/2,\mathcal{M}}) > 1 - \frac{y}{4\pi}.$$

By Proposition 3, there exist arbitrarily large real numbers t for which

$$\lambda(\{\theta : g_t r_\theta \cdot M \in U_{L,\epsilon,\tilde{\epsilon}/2,\mathcal{M}}\}) > 1 - \frac{y}{2\pi}.$$

Thus the assumptions of Proposition 4 are satisfied. Proposition 5 follows. □

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3.5. *Step 5.*

LEMMA 6. *For any $\epsilon, \tilde{\epsilon} > 0$, $Y \in \mathrm{SL}_2(\mathbb{R})$, $\mathcal{M}$ there exist real numbers $\delta, \tilde{\delta} > 0$ so that if S is a $\tilde{\delta}$-dense subset of $\mathcal{M} \cap \mathcal{K}_\delta$ (with respect to $d_{\mathcal{K}_\delta}$), then $Y \cdot S$ is $\tilde{\epsilon}$-dense in $\mathcal{M} \cap \mathcal{K}_\epsilon$ (with respect to $d_{\mathcal{K}_\epsilon}$).*

Proof. Let $\{p_1, \ldots, p_n\}$ be a finite set of points in $\mathcal{K}_\epsilon$ so that if $\{q_1, \ldots, q_n\}$ is a finite set of points in $\mathcal{K}_\epsilon$ with $d_{\mathcal{K}_\epsilon}(p_i, q_i) < \tilde{\epsilon}/4$, then $\{q_1, \ldots, q_n\}$ is $\tilde{\epsilon}/2$-dense in $\mathcal{K}_\epsilon$. Consider $Y^{-1}B(p_i, \tilde{\epsilon}/4)$. There exists $\delta, \tilde{\delta} > 0$ so that all of these sets are contained in $\mathcal{K}_\delta$ and, for each i, there is a $d_{\mathcal{K}_\delta}$ metric ball of radius $\tilde{\delta}$ contained in $Y^{-1}B_{d_{\mathcal{K}_\epsilon}}(p_i, \tilde{\epsilon}/4)$ (because this is an open set in period coordinates and therefore also open with respect to any $d_{\mathcal{K}_*}$ metric). □

PROPOSITION 6. *For any flat surface M, $\epsilon, \tilde{\epsilon}, r > 0$, there exist arbitrarily large $s \in \mathbb{R}$ so that, for all $a \in S^1$,*

$$\left(\bigcup_{\theta \in (a-r, a+r)} h_s r_\theta \cdot M \right) \cap \mathcal{K}_\epsilon$$

is $\tilde{\epsilon}$-dense in $\overline{\mathrm{SL}_2(\mathbb{R}) \cdot M} \cap \mathcal{K}_\epsilon$.

Proof. For each $A \in \mathrm{SO}_2$, denote by δ_A and $\tilde{\delta}_A$ the constants given by Lemma 6 for ϵ. Set

$$\delta = \sup_{A \in \mathrm{SO}_2} \delta_A,$$
$$\tilde{\delta} = \inf_{A \in \mathrm{SO}_2} \tilde{\delta}_A.$$

Observe that because rotating does not change the length of the systole, δ_A is independent of $A \in \mathrm{SO}_2$ and by Lemma 4, $\tilde{\delta}_A$ varies continuously with A. So these numbers are positive by the compactness of SO_2. Let $t_0 > 0$ be a constant as given by Proposition 5 using parameters $\delta, \tilde{\delta}, r$ for $\epsilon, \tilde{\epsilon}, y$, respectively. Thus, for any $a \in S^1$,

$$\left(\bigcup_{\theta \in (a-r, a+r)} g_{t_0} r_\theta \cdot M \right) \cap \mathcal{K}_\delta$$

is $\tilde{\delta}$-dense in $\mathcal{M} \cap \mathcal{K}_\delta$.

Choose $s \in \mathbb{R}$ so that, in the Cartan decomposition of h_s, $h_s = r_{\theta_1} g_{t_0} r_{\theta_2}$ for some $\theta_1, \theta_2 \in S^1$. Now

$$\begin{aligned} \bigcup_{\theta \in (a-r, a+r)} h_s r_\theta \cdot M &= \bigcup_{\theta \in (a-r, a+r)} r_{\theta_1} g_{t_0} r_{\theta_2} r_\theta \cdot M \\ &= r_{\theta_1} \left(\bigcup_{\theta \in (a+\theta_2-r, a+\theta_2+r)} g_{t_0} r_\theta \cdot M \right). \end{aligned} \tag{5}$$

The set $\bigcup_{\theta \in (a+\theta_2-r, a+\theta_2+r)} h_s r_\theta \cdot M$ is $\tilde{\delta}$-dense in $\mathcal{M} \cap \mathcal{K}_\delta$ so, by our choice of $\delta, \tilde{\delta}$,

$$r_{\theta_1} \left(\bigcup_{\theta \in (a+\theta_2-r, a+\theta_2+r)} g_{t_0} r_\theta \cdot M \right)$$

is $\tilde{\epsilon}$-dense in $\mathcal{M} \cap \mathcal{K}_\epsilon$. Consequently, equation (5) implies that $\bigcup_{\theta \in (a-r, a+r)} h_s r_\theta \cdot M$ is $\tilde{\epsilon}$-dense in $\mathcal{M} \cap \mathcal{K}_\epsilon$. □

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3.6. *Step 6A: a residual set of directions.* Recall that a subset A of a topological space X is said to be *meager* if it can be expressed as the union of countably many nowhere dense subsets of X, and a *residual set* is the complement of a meager set.

PROPOSITION 7. *Let M be a flat surface and let $\mathcal{M} = \overline{\mathrm{SL}_2(\mathbb{R}) \cdot M}$. For every $\epsilon > 0$, the set*

$$G_M(\epsilon, \tilde{\epsilon}) := \{\theta \in S^1 : r_\theta \cdot M \text{ is } (\infty, \epsilon, \tilde{\epsilon})\text{-nice for } h_s\}$$

contains an open dense subset of S^1.

Proof. It suffices to show that, for every $\phi \in S^1$ and $r > 0$, the set $G_M(\epsilon, \tilde{\epsilon}) \cap (\phi - r, \phi + r)$ contains a non-empty open set.

Apply Corollary 1 to obtain $L > 0$ and a set $U_{L,\epsilon,\tilde{\epsilon},\mathcal{M}}$. Since $U_{L,\epsilon,\tilde{\epsilon},\mathcal{M}}$ is open, for any sufficiently small $y > 0$, if S is any y-dense subset of $\mathcal{K}_\epsilon$, then S intersects $U_{L,\epsilon,\tilde{\epsilon},\mathcal{M}}$. Applying Proposition 6 for parameters ϵ and y, we obtain $s \in \mathbb{R}$ so that

$$A := \bigcup_{\theta \in (\phi - r, \phi + r)} h_s r_\theta \cdot M$$

is y-dense in $\overline{\mathrm{SL}_2(\mathbb{R}) \cdot M} \cap \mathcal{K}_\epsilon$. Thus, A has non-empty intersection with the open set $U_{L,\epsilon,\tilde{\epsilon},\mathcal{M}}$. However, since $h_s r_\theta \cdot M$ is continuous in θ, the interval $(\phi - r, \phi + r)$, in fact, contains an open set of angles θ such that $h_s r_\theta \cdot M \in U_{L,\epsilon,\tilde{\epsilon},\mathcal{M}}$. All of these angles θ are in $G_M(\epsilon, \tilde{\epsilon})$. □

THEOREM 8. *For any flat surface M, there exists a residual set $A \subset S^1$ such that*

$$\overline{Hr_\theta \cdot M} = \overline{\mathrm{SL}_2(\mathbb{R}) \cdot M}$$

for all $\theta \in A$.

Proof. This follows from Proposition 7 and the Baire category theorem. □

3.7. *Hitting many sets—Proposition 8.*

PROPOSITION 8. *For any $\epsilon > 0$, translation surface M so that $\overline{\mathrm{SL}_2(\mathbb{R}) \cdot M} = \mathcal{M}$ with associated affine invariant probability measure μ, and a finite collection of open sets $W_1, \ldots, W_n$ with $\mu(W_i) > 0$ for all i, there exists arbitrarily large $t \in \mathbb{R}$ such that, for each i,*

$$\lambda(\{\theta \in S^1 \mid g_t r_\theta \cdot M \in W_i\}) > \mu(W_i) - \epsilon.$$

Definition 7. For any real numbers $L, \epsilon > 0$, affine invariant probability measure μ on the stratum $\mathcal{H}$, and a finite collection of open sets $W_1, \ldots, W_n$ in $\mathcal{H}$, define

$$\tilde{U}^{\mu}_{L,\epsilon}(W_1, \ldots, W_n) := \left\{M' \in \mathcal{H} \;\middle|\; \lambda(\{s \in [0, L) | h_s \cdot M' \in W_i\}) > L \cdot (\mu(W_i) - \epsilon) \text{ for all } i = 1, \ldots, n\right\}.$$

When the sets $W_1, \ldots, W_n$ and the measure μ are clear from the context, we will sometimes write simply $\tilde{U}_{L,\epsilon}$.

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LEMMA 7. *Fix $\epsilon > 0$, an affine invariant probability measure μ on $\mathcal{H}$ and finitely many open sets $W_1, \ldots, W_n$ in $\mathcal{H}$ with $\mu(W_i) > 0$ for all i. Then*

$$\lim_{L\to\infty} \mu(\tilde{U}^{\mu}_{L,\epsilon}(W_1, \ldots, W_n)) = 1.$$

Moreover, for each $L > 0$, $\tilde{U}^{\mu}_{L,\epsilon}(W_1, \ldots, W_n)$ is open.

Proof. By the ergodicity of H with respect to μ (via the Mautner phenomenon),

$$\lim_{L\to\infty} \mu(\tilde{U}_{L,\epsilon}) = 1.$$

Because H acts continuously and all the W_i are open, $\tilde{U}_{L,\epsilon}$ is also open. Indeed, for each $M' \in \tilde{U}_{L,\epsilon}$ and i, there exists $V_i \subset W_i$ so that an open neighborhood of V_i is contained in W_i and

$$\lambda(\{s \in [0, L) \mid h_s \cdot M' \in V\}) > L \cdot (\mu(W_i) - \epsilon).$$

By the continuity of h_s, the lemma follows. □

We need the following easy and well-known result in measure theory.

LEMMA 8. *Let U be an open set with finite measure for a Borel measure ν, and let $\delta > 0$. Then there exists an open set $V \subset U$ and a real number $c > 0$ so that $\nu(U\backslash V) < \delta$ and $B(x, c) \subset U$ for all $x \in V$.*

Proof. Set $S_\epsilon = \{x \in U \mid B(x, \epsilon) \subset U\}$. Since U is open, $\bigcup_{n\in\mathbb{N}} S_{1/n} = U$. So $\nu(U) = \lim_{n\to\infty} \nu(S_{1/n})$. Hence there exists $n \in \mathbb{N}$ such that $\nu(S_{1/n}) > \nu(U) - \delta$. □

Proof of Proposition 8. Choose $\tilde{\epsilon} > 0$ so that $\mu(W_i\backslash\mathcal{K}_{\tilde{\epsilon}}) < \epsilon/3$ for all i. For each i, apply Lemma 8 to choose an open subset $V_i \subset W_i$ and a real number $c_i > 0$ such that the c_i-neighborhood (with respect to $d_{\mathcal{K}_{\tilde{\epsilon}}}$) of V_i is contained in $W_i \cap \mathcal{K}_{\tilde{\epsilon}}$ and $\mu(W_i\backslash V_i) < 2\epsilon/3$.

Then, by Lemma 7, choose a real number $L > 0$ so that

$$\mu(\tilde{U}_{L,\epsilon}(V_1, \ldots, V_n)) > 1 - \frac{\epsilon}{3}.$$

By Proposition 3, there exist arbitrarily large numbers $T > 0$ such that

$$\lambda(\{\theta \in S^1 \mid g_T r_\theta \cdot M \in \tilde{U}_{L,\epsilon}(V_1, \ldots, V_n)\}) > 1 - \epsilon. \tag{6}$$

Consider such T, θ and the segment of angles $[0, \tan^{-1}(e^{-2T}L)]$.

By Lemma 5, choosing ϕ close enough to zero and t large enough,

$$d_{\mathcal{K}_\epsilon}(g_t r_{\theta+\phi} \cdot M, h_{-e^{2t}\tan(\phi)} g_t r_\theta \cdot M) < \min c_i.$$

If T is large enough so that this estimate holds, then, for each i,

$$\begin{aligned}&\lambda(\{\theta \in [0, \tan^{-1}(e^{-2T}L)] : g_T r_\theta \cdot M \in W_i\})\\ &\quad > \tan^{-1}(e^{-2T}L)(\lambda(V_i) - \epsilon)\frac{\min\{|\tan^{-1}(\theta)'| : \theta \in [0, \tan^{-1}(e^{-2T}L)]\}}{\max\{|\tan^{-1}(\theta)'| : \theta \in [0, \tan^{-1}(e^{-2T}L)]\}}.\end{aligned}$$

Because the derivative of $\tan^{-1}$ is continuous and is one at zero, if T is large enough, this is at least $(\lambda(V_i) - 2\epsilon)\tan^{-1}(e^{-2T}L) \leq (\lambda(W_i) - 3\epsilon)\tan^{-1}(e^{-2T}L)$. So, for any large enough T satisfying equation (6),

$$\lambda(\{\theta : g_T r_\theta \cdot M \in W_i\}) > \lambda(W_i) - 4\epsilon.$$

Proposition 8 follows. □

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3.8. *Almost every horocycle is dense.* Before using Proposition 8 to prove Theorem 1, we require some set-up and a lemma to use the Cartan decomposition to relate g_t and h_s.

Definition 8. Let M be a translation surface, let $\mathcal{M} = \overline{\mathrm{SL}_2(\mathbb{R}) \cdot M}$ and let $\epsilon, \tilde{\epsilon}, \epsilon_3 > 0$. Let $U_{L,\epsilon,\tilde{\epsilon},\mathcal{M}}$ be as in Definition 6. We may choose $\xi > 0$ such that

$$\mu(\mathcal{K}_\xi \cap U_{L,\epsilon,\tilde{\epsilon},\mathcal{M}}) > \mu(U_{L,\epsilon,\tilde{\epsilon},\mathcal{M}}) - \frac{\epsilon_3}{2}.$$

We may now choose an open set $V \subset \mathcal{K}_\xi$ so that $V \subset U_{L,\epsilon,\tilde{\epsilon},\mathcal{M}}$,

$$\mu(V) > \mu(U_{L,\epsilon,\tilde{\epsilon},\mathcal{M}}) - \epsilon_3$$

and there exists $c > 0$ so that a c-neighborhood of V (with respect to $d_{\mathcal{K}_\xi}$) is contained in $U_{L,\epsilon,\tilde{\epsilon},\mathcal{M}}$. Let $\{\psi_1, \ldots, \psi_n\}$ be a c-dense subset of S^1. We say that a real number t is a *spreading time* (for M with parameters $\epsilon, \tilde{\epsilon}, \epsilon_3, \xi, V, c, \{\psi_i\}_i$) if, for each $i = 1, \ldots, n$,

$$\lambda(\{\theta \in S^1 \mid g_t r_\theta \cdot M \in r_{-\psi_i} \cdot V\}) > \mu(V) - \epsilon_3.$$

The next lemma uses the notation above.

LEMMA 9. *Let s, t be real numbers such that t is a spreading time (for M with parameters $\epsilon, \tilde{\epsilon}, \epsilon_3, \xi, V, c, \{\psi_i\}_i$) and $h_s = r_{\phi_1} g_t r_{\phi_2}$ for some $\phi_1, \phi_2 \in S^1$. Then*

$$\lambda(\{\theta \in S^1 \mid h_s r_\theta \cdot M \in U_{L,\epsilon,\tilde{\epsilon},\mathcal{M}}\}) > \mu(V) - \epsilon_3 > \mu(U_{L,\epsilon,\tilde{\epsilon},\mathcal{M}}) - 2\epsilon_3.$$

Proof. Since the set $\{\psi_1, \ldots, \psi_n\}$ is c-dense in S^1, there exist j, with $1 \le j \le n$, and $\rho \in \mathbb{R}$ with $|\rho| < c$ such that $\phi_1 = \psi_j + \rho$. Then

$$\begin{aligned}\{\theta \in S^1 \mid h_s r_\theta \cdot M \in U_{L,\epsilon,\tilde{\epsilon},\mathcal{M}}\} &= \{\theta \in S^1 \mid r_{\phi_1} g_t r_{\phi_2} r_\theta \cdot M \in U_{L,\epsilon,\tilde{\epsilon},\mathcal{M}}\} \\ &= \{(\theta - \phi_2) \in S^1 \mid r_{\psi_j+\rho} g_t r_\theta \cdot M \in U_{L,\epsilon,\tilde{\epsilon},\mathcal{M}}\}.\end{aligned}$$

Let $G = \{\theta \in S^1 \mid g_t r_\theta \cdot M \in r_{-\psi_j} V\}$. Observe that $\theta \in G$ implies that $r_{\psi_j} g_t r_\theta \cdot M = h_s r_{\theta+\theta_2} \cdot M \subset V$, and since a c-neighborhood of V (in the $d_{\mathcal{K}_\xi}$ metric) is contained in $U_{L,\epsilon,\tilde{\epsilon},\mathcal{M}}$, this, in turn, implies that

$$r_{\psi_j+\rho} g_t r_\theta \cdot M = h_s r_{\theta+\theta_2} \cdot M \in U_{L,\epsilon,\tilde{\epsilon},\mathcal{M}},$$

by Lemma 4. Since t the is spreading time,

$$\lambda(G) > \mu(V) - \epsilon_3 > \mu(U_{L,\epsilon,\tilde{\epsilon},\mathcal{M}}) - 2\epsilon_3. \qquad \square$$

Proof of Theorem 1. It suffices to show that, for every $\epsilon, \tilde{\epsilon}, y > 0$,

$$\lambda(\{\theta \in S^1 \mid Hr_\theta \cdot M \text{ is } \tilde{\epsilon}\text{-dense in } \mathcal{K}_\epsilon\}) > 1 - y.$$

Fix $\epsilon, \tilde{\epsilon}, y > 0$. By Corollary 1, we may pick the set $U_{L,\epsilon,\tilde{\epsilon},\mathcal{M}}$ big enough and $\epsilon_3 > 0$ small enough so that $\mu(U_{L,\epsilon,\tilde{\epsilon},\mathcal{M}}) - 2\epsilon_3 > 1 - y$. We may then pick parameters $\xi, V, c, \{\psi_i\}_i$ as in Definition 8. By Proposition 8, there exists a spreading time t (for M with parameters $\epsilon, \tilde{\epsilon}, \epsilon_3, \xi, V, c, \{\psi_i\}_i$). By Lemma 9, there exists $s \in \mathbb{R}$ so that

$$\lambda(\{\theta \in S^1 \mid h_s r_\theta \cdot M \in U_{L,\epsilon,\tilde{\epsilon},\mathcal{M}}\}) > 1 - y.$$

Choosing $R = L + s$,

$$\lambda\left(\left\{\theta \in S^1 \;\middle|\; \bigcup_{0 \le \ell < R} h_\ell r_\theta \cdot M \text{ is } \tilde{\epsilon}\text{-dense in } \mathcal{K}_\epsilon\right\}\right) > 1 - y.$$

Theorem 1 follows. $\square$

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4. *New characterizations of lattice surfaces*

The goal of this section is to prove Theorem 2. For the convenience of the reader, we reproduce here the four conditions that Theorem 2 asserts are equivalent.

(I) M is a lattice surface.

(II) For every angle $\theta \in S^1$, every H-minimal subset of $\overline{Hr_\theta \cdot M}$ is a periodic H-orbit.

(III) There exists a positive Lebesgue measure set of angles $\Theta \subset S^1$ such that $\theta \in \Theta$ implies that every H-minimal subset of $\overline{Hr_\theta \cdot M}$ is a periodic H-orbit.

(IV) Every H-minimal subset of $\overline{\mathrm{GL}_2(\mathbb{R}) \cdot M}$ (or of $\overline{\mathrm{SL}_2(\mathbb{R}) \cdot M}$) is a periodic H-orbit.

Proof of Theorem 2. Lemmas 10–16 establish the following circle of implications.

$$\mathrm{I} \to \mathrm{II} \to \mathrm{III} \to \mathrm{IV} \to \mathrm{I}.$$

□

LEMMA 10. *(I) implies (II) implies (III).*

Proof. To show that (I) implies (II), fix a lattice surface M and angle θ. Let L be any surface in any H-minimal subset A of $\overline{Hr_{-\theta} \cdot M}$. Since a lattice surface has, by definition, a closed $\mathrm{SL}_2(\mathbb{R})$ orbit, L is the image of M under some element of $\mathrm{SL}_2(\mathbb{R})$, and hence L is also a lattice surface. By Theorem 6, L is periodic in the horizontal direction and, by Theorem 3, L is uniformly completely parabolic, so the moduli of all the horizontal cylinders of L are commensurable. Hence, by Proposition 1, A is the periodic H-orbit of L.

That (II) implies (III) is trivial. □

We use Theorem 1 to prove that (III) implies (IV).

LEMMA 11. *(III) implies (IV).*

Proof. Assume that M is a flat surface for which (III) holds. Let $\mathcal{A}$ be an H-minimal subset of $\overline{\mathrm{GL}_2(\mathbb{R}) \cdot M}$. Then $\mathcal{A}$ has the form $\mathcal{A} = \overline{H \cdot N}$ for some flat surface $N \in \overline{\mathrm{GL}_2(\mathbb{R}) \cdot M}$. We may assume without loss of generality that N is in $\overline{\mathrm{SL}_2(\mathbb{R}) \cdot M}$ (all the surfaces in $\mathcal{A}$ necessarily have the same area, so we may rescale them to have the same area as that of M, and this subset of rescaled surfaces will still be an H-minimal set). By Theorem 1, there exists an angle θ such that $\overline{Hr_{-\theta} \cdot M} = \overline{\mathrm{SL}_2(\mathbb{R}) \cdot M}$. Thus

$$\mathcal{A} \subset \overline{\mathrm{SL}_2(\mathbb{R}) \cdot N} \subset \overline{\mathrm{SL}_2(\mathbb{R}) \cdot M} = \overline{Hr_{-\theta} \cdot M}.$$

Then (III) implies that $\mathcal{A}$ is a periodic H-orbit. □

LEMMA 12. *Condition (IV) implies that for any surface $N \in \overline{\mathrm{GL}_2(\mathbb{R}) \cdot M}$:*

(1) *N is parabolic in every periodic direction; and*

(2) *N is completely periodic.*

Proof. To see that N is parabolic in every periodic direction, suppose that N is periodic in direction θ but the moduli of the cylinders (in direction θ) are not rationally related. Then $Hr_{-\theta} \cdot N$ is a H-minimal set that is not periodic, which contradicts (IV).

Now, suppose there is a direction θ in which N has at least one cylinder and at least one minimal component. Without loss of generality, we will assume that θ is the

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positive horizontal direction. By Theorem 6, fix a surface $N' \in \overline{H \cdot N} \subset \overline{\mathrm{GL}_2(\mathbb{R}) \cdot M}$ with a horizontal cylinder decomposition, and let $\mathcal{C}$ be the cylinders in N' which 'came from' the horizontal cylinders of N. Clearly, the horizontal cylinders of N' which are not in $\mathcal{C}$ are not in the same $\mathcal{M}$-equivalence class as any cylinder of $\mathcal{C}$. Thus, by Lemma 3, $\eta_{\mathcal{C}} \in T_{N'}(\mathcal{M})$. Following the approach in [**Wri15**], for some small $\epsilon > 0$, there exists a surface $N'' \in \mathcal{M}$ corresponding to $[\omega] + \epsilon i \eta_{\mathcal{C}}$, where $[\omega]$ is element of $H^1(S, \Sigma; \mathbb{C})$ corresponding to N', that is horizontally periodic and such that the moduli of the cylinders of N'' that came from $\mathcal{C}$ are not rationally related to the moduli of the cylinders of N'' that did not come from $\mathcal{C}$. Hence N'' is horizontally periodic but not parabolic; by the conclusion of the previous paragraph, this is a contradiction. □

LEMMA 13. *Condition (IV) implies that, for any surface* $N \in \overline{\mathrm{GL}_2(\mathbb{R}) \cdot M} = \mathcal{M}$ *and for any direction* θ *that is periodic for* N, N *has a unique* $\mathcal{M}$*-equivalence class of cylinders in direction* θ.

Proof. Suppose that N has at least two $\mathcal{M}$-equivalence classes of cylinders in a periodic direction θ. Without loss of generality, assume that θ is the positive horizontal direction. By Lemma 3, we can stretch the cylinders in one of these equivalence classes vertically while keeping the rest of the surface unchanged to obtain a horizontally periodic surface $N' \in \mathcal{M}$ that has at least two horizontal cylinders whose moduli are not rationally related. By Lemma 12, this is a contradiction. □

LEMMA 14. *Condition (IV) implies that if* S *is any horizontally periodic (not necessarily regular) surface in* $\mathcal{M}$, *then* $\mathrm{Twist}(S, \mathcal{M}) = \mathrm{Pres}(S, \mathcal{M})$.

Remark 1. The definitions (from [**Wri15**]) of $\mathrm{Twist}(S, \mathcal{M})$ and $\mathrm{Pres}(S, \mathcal{M})$ use $T^S\mathcal{M}$, which exists only if S is a regular point (i.e., non-self-intersection point) of $\mathcal{M}$. However, it is easy to extend these notions to the case when S is a self-intersection point of $\mathcal{M}$; in this case, we interpret $T^S(\mathcal{M})$ to be the 'union' of the 'tangent spaces' to $\mathcal{M}$ at S and define $\mathrm{Twist}(S, \mathcal{M})$ and $\mathrm{Pres}(S, \mathcal{M})$ accordingly. In [**Wri15**], Wright shows that for any regular point $S \in \mathcal{M}$ that is horizontally periodic, $\mathrm{Twist}(S, \mathcal{M})$ is contained in $\mathrm{span}_{\mathbb{R}}(\eta_{C_i^S})_{i=1}^n$, where $C_1^S, \ldots, C_n^S$ are the horizontal cylinders of S, and if $\mathrm{Pres}(S, \mathcal{M})$ properly contains $\mathrm{Twist}(S, \mathcal{M})$, then there is a surface $S' \subset \mathcal{M}$ that has more horizontal cylinders than S does. Lemma 14 checks that this result still holds if S is not a regular point with our generalized notion of $\mathrm{Twist}(S, \mathcal{M})$ and $\mathrm{Pres}(S, \mathcal{M})$.

Proof. The definition of $\mathcal{M}$-equivalence classes of cylinders in $\mathcal{S}$ is phrased in terms of all nearby surfaces to S in $\mathcal{M}$ and thus still makes sense for non-singular points S of $\mathcal{M}$. Wright's lemma [**Wri15**] asserting that the deformation corresponding to η_C (where C is a $\mathcal{M}$-equivalence class of cylinders) remains in the tangent space at S still holds when we interpret tangent space to mean the union of the tangent spaces at S. Thus, Lemma 13 holds even when S is a self-intersection point of $\mathcal{M}$.

Now suppose that condition (IV) holds for some horizontally periodic surface S but $\mathrm{Twist}(S, \mathcal{M}) \neq \mathrm{Pres}(S, \mathcal{M})$. Then there is some horizontal saddle connection whose length can be changed without altering the absolute cohomology of S. By slightly tipping this horizontal saddle connection so that it is not horizontal (which we may do because of

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the fact [**EMM15**] that the tangent space is linear in *complex-valued* period coordinates), we obtain a nearby surface in the orbit closure that has more horizontal cylinders. This surface is in the orbit closure and clearly has more than one $\mathcal{M}$-equivalence class (it has the cylinders from S and the new cylinder); by Lemma 13, this is a contradiction. □

LEMMA 15. *Condition (IV) implies that the self-intersection set of $\mathcal{M}$ is empty and*

$$\dim_{\mathbb{C}}(T^N(\mathcal{M})) = 2$$

for any point $N \in \mathcal{M}$.

Proof. Without loss of generality, assume that N has a cylinder decomposition in the horizontal direction. Let $C_1, \ldots, C_n$ be the horizontal cylinders of N. We will represent N as a collection of polygons whose edges are glued together. Each cylinder C_i can be represented by a parallelogram embedded in $\mathbb{R}^2$, whose 'top' and 'bottom' sides are horizontal and whose 'left' and 'right' sides (which are parallel, but not necessarily vertical) are glued together via a translation of the form $(x, y) \mapsto (x + c_i, y)$, where c_i is the length of the core curve of cylinder C_i.

Let $\gamma_1, \ldots, \gamma_k$ be a list of the horizontal saddle connections in N that are in the horizontal sides of these parallelograms. We then pick n additional saddle connections $\gamma_{k+1}, \ldots, \gamma_{k+n}$ that are the saddle connections representing the non-horizontal sides of the parallelograms. Then the relative homology classes $[\gamma_1], \ldots, [\gamma_{k+n}]$ constitute a spanning set for $H_1(N, \Sigma; \mathbb{C})$, since the dual cohomology classes in $H^1(N, \Sigma; \mathbb{C})$ completely determine the geometry of N.

Since $C_1, \ldots, C_n$ constitute the unique $\mathcal{M}$-equivalence class of cylinders in the horizontal direction, there is a system of linear equations relating the cohomology classes of the core curves of $C_1, \ldots, C_n$ that collectively implies that if we slightly deform N while remaining in the orbit closure $\mathcal{M}$, we must keep these core curves all parallel (i.e., we may rotate the complex values assigned to the core curves by cohomology about the origin in $\mathbb{C}$ by a uniform rotation) but we may scale all their lengths by a real number. Therefore, since $\mathrm{Twist}(N, \mathcal{M}) = \mathrm{Pres}(N, \mathcal{M})$ by Lemma 14, any deformation which remains in the orbit closure will scale and rotate the complex numbers assigned by cohomology to $[\gamma_1], \ldots, [\gamma_k]$ uniformly (i.e., uniformly multiply them by a complex number). Thus a single complex parameter governs the cohomology values of $[\gamma_1], \ldots, [\gamma_k]$.

Now, for any given values for $[\gamma_1], \ldots, [\gamma_k]$, suppose there are two or more (complex) degrees of freedom in assigning values to $[\gamma_{k+1}], \ldots, [\gamma_{k+n}]$ (saddle connections comprising the non-vertical sides of the parallelograms). Then we could remain in the orbit closure $\mathcal{M}$ by, in particular, multiplying the complex number representing one of these saddle connections by any real number α while multiplying the complex number representing a different one of these saddle connections by any other real number β. Since we are assuming that $[\gamma_1], \ldots, [\gamma_k]$ are constant, to each of the γ_{k+i} there is a cylinder whose modulus depends only on γ_{k+1}. In particular, we could pick α and β so that the moduli of the two corresponding cylinders are not commensurable. The H-orbit of the resulting surface would not be periodic, which would contradict (IV). Hence, there is a single complex parameter that governs the values of $[\gamma_{k+1}], \ldots, [\gamma_{k+n}]$.

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We have shown that, if we are to remain in the orbit closure in a neighborhood of N, we have two complex degrees of freedom in prescribing the cohomology values of $[\gamma_1], \ldots, [\gamma_{k+n}]$, which completely determine the geometry of the surface: we can multiply $[\gamma_1], \ldots, [\gamma_k]$ uniformly by a complex number, and we can multiply $[\gamma_{k+1}], \ldots, [\gamma_{k+n}]$ uniformly by a complex number. Consequently, N is a regular point, and $\dim_{\mathbb{C}}(T^N(\mathcal{M})) = 2$. □

LEMMA 16. *(IV) implies (I).*

Proof. By Theorem 3, it suffices to show that (IV) implies that $\mathrm{GL}_2(\mathbb{R}) \cdot M$ is closed. Suppose there exists $X \in \mathcal{M} \setminus \mathrm{GL}_2(\mathbb{R}) \cdot M$. Then X is a regular point of $\mathcal{M}$ by Lemma 15, and $\dim_{\mathbb{C}} T^X\mathcal{M} > 2$. By Lemma 15, this is impossible. □

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