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ON THE LAST EXIT TIMES FOR SPECTRALLY NEGATIVE LÉVY PROCESSES

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Abstract

Using a new approach, for spectrally negative Lévy processes we find joint Laplace transforms involving the last exit time (from a semiinfinite interval), the value of the process at the last exit time, and the associated occupation time, which generalize some previous results.

Keywords: Spectrally negative Lévy process; last exit time; Gerber–Shiu function; occupation time

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Secondary 60K30

1. Introduction

The first passage times, also known as the first exit times, have been studied extensively for stochastic processes. They find many applications in risk theory since the first exit time corresponds to the ruin time of a risk process. The joint distributions of quantities related to the ruin time, often referred as Gerber–Shiu functions, are very interesting in the study of risk models. For spectrally negative Lévy processes, those first passage time related quantities can often be expressed in terms of the associated scale functions.

The last passage time is also interesting in risk theory. In a risk model, it can be regarded as the final recovery time after which there will be no more ruin; see [4] for early work in this respect. A Laplace transform for the last exit time from the positive part of the real line was first obtained in [3] for a spectrally negative Lévy process using an identity between the density of the first exit time and the marginal density of the Lévy process. Applying the strong Markov property and the memoryless property for exponential random variables, in [2] Laplace transforms were also found for the last exit times (before an independent exponential time) from the positive part and the negative part of the real line, respectively. In both [2] and [3] the Laplace transforms are expressed in terms of the Laplace exponent and scale functions for the associated spectrally negative Lévy process.

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In this paper we further consider the joint distribution concerning the last time when a spectrally negative Lévy process leaves a semiinfinite interval. More precisely, for a spectrally negative Lévy process running up to an independent exponential time, we find the joint Laplace transform of the occupation time until the last exit time, the last exit time, and the value of the process at the last exit time. They can be treated as Gerber–Shiu function like quantities for the last exit time of a spectrally negative Lévy process and generalize the previous results in [2] and [3]. In particular, we show that downward creeping occurs at the last exit time if and only if the process X has a nontrivial Brownian component.

We note the very recent work of Li and Cai [7] in which the authors address the joint distribution involving the the last exit time, the joint occupation time up to the last exit time, and the value of the process prior to the last exit time. The authors adopt an interesting time reversal argument to associate the last exit time for X with the first exit time for its dual process and use previous results on occupation times. The joint distributions involving overshoot and creeping of the process at the last exit time are also considered in the most recent version of [7].

Our approach is different from those in [2], [3], and [7]. We use the Poisson approach of [1] and [9] to find Laplace transforms of the occupation times. To overcome difficulties due to the nonstoping last exit time and the infinite activity of the Lévy processes, we adopt a perturbation approach, similar to that of [14], to approximate the last exit times by first exit times over certain events. We then apply the fluctuation theory for spectrally negative Lévy processes to express the approximating Laplace transforms in terms of scale functions and Laplace exponents before taking limits to reach the desired results. Our arguments have the flavor of those in [1].

We remark that similar results on the last exit times can also be considered for other stochastic processes, provided that the associated exit problems are solvable and the potential measure allows an explicit expression. For example, we believe similar results can also be obtained for time homogeneous diffusion processes; see [12], [13] and the references therein for an approach of 'progressive enlargement of filtration' and [8] for a perturbation approach.

In the next section we present some preliminary results for spectrally negative Lévy processes. The main results and their proofs are provided in Section 3. Some corollaries and discussions are also given in this section. Section 4 contains several examples with discussions.

2. Preliminaries

In this section we collect some known fluctuation results for spectrally negative Lévy processes to prepare for future proofs. Let a process $X \equiv (X_t)_{t\geq 0} \equiv (X(t))_{t\geq 0}$ be a spectrally negative Lévy process, i.e. a Lévy process with right-continuous sample paths and with no positive jumps. Then, for any $t > 0$,

$$\frac{1}{t}\log \mathbb{E}e^{\lambda X_t} = \psi(\lambda)$$

with

$$\psi(\lambda) = a\lambda + \frac{1}{2}\sigma^2\lambda^2 + \int_{-\infty}^{0} (e^{\lambda x} - 1 - \lambda x \mathbf{1}_{\{x>-1\}})\Pi(dx),$$

where Π is a σ-finite measure on $(-\infty, 0)$ satisfying $\int_{-\infty}^{0}(x^2 \wedge 1)\Pi(dx) < \infty$. We further suppose that process X does not have monotone samples. Write $\Phi\colon [0,\infty) \mapsto [0,\infty)$ for the right inverse function of ψ, i.e.

$$\Phi(p) := \sup\{\lambda \geq 0\colon \psi(\lambda) = q\}, \qquad q \geq 0.$$

We further assume that X is not the negative of a subordinator.

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Scale functions play a key role in analyzing spectrally negative Lévy processes. The scale function $W^{(p)}$ for X is a nonnegative, increasing function specified by the following Laplace transform. For any $q \geq 0$,

$$\int_0^\infty \mathrm{e}^{-\lambda x} W^{(q)}(x)\,\mathrm{d}x = \frac{1}{\psi(\lambda) - q}, \qquad \lambda > \Phi(q),$$

and $W^{(q)}(x) := 0$ for $x < 0$. We also need the scale functions

$$Z^{(p)}(x) := 1 + p \int_0^x W^{(p)}(y)\,\mathrm{d}y$$

and

$$Z^{(p)}(x,\theta) := \mathrm{e}^{\theta x}\left(1 + (p - \psi(\theta)) \int_0^x \mathrm{e}^{-\theta y} W^{(p)}(y)\,\mathrm{d}y\right)$$

with $Z^{(p)}(x) := 1$ and $Z^{(p)}(x,\theta) := \mathrm{e}^{\theta x}$ for $x < 0$, respectively. Write $W(x)$, $Z(x)$, and $Z(x,\theta)$ for $W^{(0)}(x)$, $Z^{(0)}(x)$, and $Z^{(0)}(x,\theta)$, respectively.

For all $p, q \geq 0$, process X has sample paths of unbounded variation if and only if $W^{(p)}(0) = 0$; in addition, if X is of bounded variation then $W^{(p)}(0) = W^{(q)}(0) > 0$, and if X is of unbounded variation then

$$\lim_{\varepsilon \to 0+} \frac{W^{(p)}(\varepsilon)}{W^{(q)}(\varepsilon)} = 1;$$

see [6].

For any $a \in \mathbb{R}$, define one-sided exit times

$$\tau_a^+ := \inf\{t > 0 \colon X_t > a\}, \qquad \tau_a^- := \inf\{t > 0 \colon X_t < a\}$$

with the convention $\inf \varnothing = \infty$.

In the following we collect some fluctuation identities for X concerning the solutions to the exit problems and potential measures, which can be found in [5] and [6]. For event A and random variable Y, write $\mathbb{E}[Y; A] = \mathbb{E}Y\mathbf{1}_A$, where $\mathbf{1}_A$ denotes the indicator function on A.

We first present solutions to the (first) exit problems. For any $0 < x < a$ and $u, v > 0$,

$$\mathbb{E}[\mathrm{e}^{-q\tau_a^+};\ \tau_a^+ < \infty] = \mathrm{e}^{-\Phi(q)a}, \qquad \mathbb{E}_x[\mathrm{e}^{-q\tau_0^-};\ \tau_0^- < \infty] = Z^{(q)}(x) - \frac{q}{\Phi(q)} W^{(q)}(x),$$

$$\begin{aligned}
&\mathbb{E}_x[\mathrm{e}^{-u\tau_0^- + vX_{\tau_0^-}};\ \tau_0^- < \infty] \\
&\quad = Z^{(u)}(x,v) - \frac{u - \psi(v)}{\Phi(u) - v} W^{(u)}(x) \\
&\quad = \mathrm{e}^{vx}\left(1 + (u - \psi(v)) \int_0^x \mathrm{e}^{-vy} W^{(u)}(y)\,\mathrm{d}y - \frac{u - \psi(v)}{\Phi(u) - v} \mathrm{e}^{-vx} W^{(u)}(x)\right)
\end{aligned}$$

and

$$\mathbb{E}_x[\mathrm{e}^{-u\tau_0^- + \Phi(u)X_{\tau_0^-}};\ \tau_0^- < \infty] = \mathrm{e}^{\Phi(u)x} - \frac{1}{\Phi'(u)} W^{(u)}(x),$$

$$\begin{aligned}
&\mathbb{E}_x[\mathrm{e}^{-u\tau_0^- + vX_{\tau_0^-}};\ \tau_0^- < \tau_a^+] \\
&\quad = Z^{(u)}(x,v) - \frac{u - \psi(v)}{\Phi(u) - v} W^{(u)}(x) - \frac{W^{(u)}(x)}{W^{(u)}(a)}\left(Z^{(u)}(a,v) - \frac{u - \psi(v)}{\Phi(u) - v} W^{(u)}(a)\right) \\
&\quad = Z^{(u)}(x,v) - \frac{W^{(u)}(x)}{W^{(u)}(a)} Z^{(u)}(a,v),
\end{aligned}$$

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where, for $\Phi(u) = v$, the ratio $(u - \psi(v))/(\Phi(u) - v)$ is understood in the sense of taking a limit.

We then present expressions for potential measures. For $a > 0$,

$$\int_0^\infty \mathrm{e}^{-qt}\mathbb{P}\{X_t \in \mathrm{d}y,\ t < \tau_a^+\}\,\mathrm{d}t = (\mathrm{e}^{-\Phi(q)a}W^{(q)}(a-y) - W^{(q)}(-y))\,\mathrm{d}y. \tag{2.1}$$

For $a \le 0$,

$$\int_0^\infty \mathrm{e}^{-qt}\mathbb{P}\{X_t \in \mathrm{d}y,\ t < \tau_a^-\}\,\mathrm{d}t = (\mathrm{e}^{-\Phi(q)(y-a)}W^{(q)}(-a) - W^{(q)}(-y))\,\mathrm{d}y. \tag{2.2}$$

For an independent exponential random variable e_r with rate r, let

$$T^+(r) := \sup\{0 < t \le e_r \colon X_t < 0\}, \qquad T^-(r) := \sup\{0 < t \le e_r \colon X_t > 0\}$$

with the convention $\sup \varnothing = 0$. Then $T^+(r)$ and $T^-(r)$ are, respectively, the last times of exiting the negative and positive part of the real line up to the time e_r. Write T^+ and T^- for $T^+(0)$ and $T^-(0)$, respectively.

In this paper, for constants $p, q, \theta, r > 0$, and $x \in \mathbb{R}$, we want to find expressions for

$$\mathbb{E}_x\left[\exp\left(-pT^+(r) - q\int_0^{T^+(r)} \mathbf{1}_{\{X_s \le 0\}}\,\mathrm{d}s + \theta X(T^+(r))\right);\ T^+(r) = e_r\right],$$

$$\mathbb{E}_x\left[\exp\left(-pT^+(r) - q\int_0^{T^+(r)} \mathbf{1}_{\{X_s \le 0\}}\,\mathrm{d}s\right);\ T^+(r) < e_r\right],$$

and

$$\mathbb{E}_x\left[\exp\left(-pT^-(r) + \theta X(T^-(r)) - q\int_0^{T^-(r)} \mathbf{1}_{\{X_s \ge 0\}}\,\mathrm{d}s\right);\ T^-(r) < e_r\right],$$

$$\mathbb{E}_x\left[\exp\left(-pT^-(r) - q\int_0^{T^-(r)} \mathbf{1}_{\{X_s \ge 0\}}\,\mathrm{d}s - \theta X(T^-(r))\right);\ T^-(r) = e_r\right].$$

3. Main results

We first proceed to find expressions for

$$\omega_1^+(x) := \mathbb{E}_x\left[\exp\left(-pT^+(r) - q\int_0^{T^+(r)} \mathbf{1}_{\{X_s \le 0\}}\,\mathrm{d}s\right);\ T^+(r) < e_r\right]$$

and

$$\omega_2^+(x) := \mathbb{E}_x\left[\exp\left(-pT^+(r) - q\int_0^{T^+(r)} \mathbf{1}_{\{X_s \le 0\}}\,\mathrm{d}s + \theta X(T^+(r))\right);\ T^+(r) = e_r\right].$$

Theorem 3.1. *For any $p, q, r, \theta > 0$, and $x \in \mathbb{R}$, we have*

$$\omega_1^+(x) = \frac{r(W^{(r)}(x) - W^{(p+r)}(x))}{\Phi(r)} + \frac{r(\Phi(p+q+r) - \Phi(p+r))}{q\Phi(r)} Z^{(p+r)}(x, \Phi(p+q+r)) - r\int_0^x W^{(r)}(y)\,\mathrm{d}y$$

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and

$$\omega_2^+(x) = \frac{r}{p+q+r-\psi(\theta)}\left(-\frac{p+r-\psi(\theta)}{q}\frac{\Phi(p+q+r)-\Phi(p+r)}{\Phi(p+r)}\right) \times Z^{(p+r)}(x, \Phi(p+q+r)) + \frac{r}{p+q+r-\psi(\theta)} Z^{(p+r)}(x,\theta).$$

Proof. Write e_q for an independent exponential random variable. Let $T_1 < T_2 < \cdots$ be the sequence of arrival times for an independent Poisson process with rate q.

For any $\epsilon > 0$, let

$$T_\varepsilon^+(r) := \inf\{T^+(r) \le s \le e_r : X_s > \varepsilon\}$$

with the convention $\inf \varnothing = e_r$. Write $T_\varepsilon^+ = T_\varepsilon^+(0)$. Then with either $T^+(r) < e_r$ or $T^+(r) = e_r$, we have $T_\varepsilon^+(r) \downarrow T^+(r)$ as $\varepsilon \to 0+$. Write

$$\omega_{1,\varepsilon}^+(x) := \mathbb{E}_x\left[\exp\left(-pT_\varepsilon^+(r) - q\int_0^{T_\varepsilon^+(r)} \mathbf{1}_{\{X_s \le 0\}}\,\mathrm{d}s\right);\ T_\varepsilon^+(r) < e_r\right]$$

and

$$\omega_{2,\varepsilon}^+(x) := \mathbb{E}_x\left[\exp\left(-pT_\varepsilon^+(r) - q\int_0^{T_\varepsilon^+(r)} \mathbf{1}_{\{X_s \le 0\}}\,\mathrm{d}s + \theta X(e_r)\right);\ T_\varepsilon^+(r) = e_r\right].$$

Then, for

$$A_\varepsilon^+ := \bigcap_{i\,:\,T_i < T_\varepsilon^+(r)} \{X_{T_i} > 0\},$$

we have

$$\omega_{1,\varepsilon}^+(x) = \mathbb{E}_x[\exp(-pT_\varepsilon^+(r));\ T_\varepsilon^+(r) < e_r, A_\varepsilon^+]$$

and

$$\omega_{2,\varepsilon}^+(x) = \mathbb{E}_x[\exp(-pT_\varepsilon^+(r) + \theta X(e_r));\ T_\varepsilon^+(r) = e_r, A_\varepsilon^+]$$

by the property of a Poisson process; see a similar argument in [9]. It follows that

$$\lim_{\varepsilon \to 0+} \omega_{i,\varepsilon}^+(x) = \omega_i^+(x), \qquad i = 1, 2.$$

Note that, by (2.1),

$$\mathbb{E}[\mathrm{e}^{-pT_\varepsilon^+(r)};\ T_\varepsilon^+(r) < e_r, T_1 < \tau_\varepsilon^+ \wedge e_r, A^+] \le \mathbb{P}\{T_1 < \tau_\varepsilon^+, 0 \le X_{T_1} \le \varepsilon\} = o(W(\varepsilon)),$$

where $o(W(\varepsilon))$ is simply $o(1)$ for X with bounded variation. Comparing the first Poisson arrival time T_1 with τ_ε^+ and e_r, we have

$$\begin{aligned}\omega_{1,\varepsilon}^+(0) = {} & \mathbb{E}[\mathrm{e}^{-p\tau_\varepsilon^+};\ \tau_\varepsilon^+ < e_q \wedge e_r] \\ & \times \left(\mathbb{P}_\varepsilon\{\tau_0^- > e_r\}\right. \\ & \quad \left. + \int_{-\infty}^0 \mathbb{E}_\varepsilon[\mathrm{e}^{-p\tau_0^-};\ \tau_0^- < e_r, X_{\tau_0^-} \in \mathrm{d}y]\mathbb{E}_y[\mathrm{e}^{-p\tau_0^+};\ \tau_0^+ < e_q \wedge e_r]\omega_{1,\varepsilon}^+(0)\right) \\ & + o(W(\varepsilon))\end{aligned}$$

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$$
\begin{aligned}
&= \mathrm{e}^{-\Phi(p+q+r)\varepsilon}\left\{1 - Z^{(r)}(\varepsilon) + \frac{r}{\Phi(r)}W^{(r)}(\varepsilon)\right.\\
&\qquad\left. + \mathbb{E}_\varepsilon \mathrm{e}^{-(p+r)\tau_0^- + \Phi(p+q+r)X(\tau_0^-)}\omega_{1,\varepsilon}^+(0)\right\} + o(W(\varepsilon))\\
&= \mathrm{e}^{-\Phi(p+q+r)\varepsilon}\frac{r}{\Phi(r)}W^{(r)}(\varepsilon) + o(W(\varepsilon))\\
&\quad + \left(1 - \frac{q}{\Phi(p+q+r) - \Phi(p+r)}\mathrm{e}^{-\Phi(p+q+r)\varepsilon}W^{(p+r)}(\varepsilon)\right)\omega_{1,\varepsilon}^+(0), \qquad (3.1)
\end{aligned}
$$

where we have used the fact that $\lim_{\varepsilon\to 0+} W^{(q)}(\varepsilon)/W(\varepsilon) = 1$. First solve (3.1) for $\omega_{1,\varepsilon}^+(0)$ and then take a limit. It follows that

$$
\begin{aligned}
\omega_1^+(0) &= \lim_{\varepsilon\to 0+}\omega_{1,\varepsilon}^+(0)\\
&= \lim_{\varepsilon\to 0+}\frac{\mathrm{e}^{-\Phi(p+q+r)\varepsilon}(r/\Phi(r))W^{(r)}(\varepsilon)}{(q/\Phi(p+q+r) - \Phi(p+r))\mathrm{e}^{-\Phi(p+q+r)\varepsilon}W^{(p+r)}(\varepsilon)}\\
&= \frac{r}{q\Phi(r)}(\Phi(p+q+r) - \Phi(p+r)).
\end{aligned}
$$

Similarly, for $\theta > 0$ we have

$$
\begin{aligned}
\mathbb{E}[\mathrm{e}^{-pT_\varepsilon^+(r)+\theta X(T_\varepsilon^+(r))};\ T_\varepsilon^+(r) = e_r, T_1 < \tau_\varepsilon^+ \wedge e_r, A_\varepsilon^+] &\le \mathbb{P}\{T_1 < \tau_\varepsilon^+,\ 0 \le X_{T_1} \le \varepsilon\}\\
&= o(W(\varepsilon)).
\end{aligned}
$$

Then, for $\theta > p+q+r$,

$$
\begin{aligned}
\omega_{2,\varepsilon}^+(0) &= \mathbb{E}[\mathrm{e}^{-p\tau_\varepsilon^+};\ \tau_\varepsilon^+ < e_q \wedge e_r]\int_{-\infty}^0 \mathbb{E}_\varepsilon[\mathrm{e}^{-p\tau_0^-};\ \tau_0^- < e_r, X_{\tau_0^-} \in \mathrm{d}y]\\
&\quad \times (\mathbb{E}_y[\mathrm{e}^{-p\tau_0^+};\ \tau_0^+ < e_q \wedge e_r]\omega_{2,\varepsilon}^+(0) + \mathbb{E}_y[\mathrm{e}^{-pe_r+\theta X(e_r)};\ e_r < \tau_0^+ \wedge e_q])\\
&\quad + \mathbb{E}[\mathrm{e}^{-pe_r+\theta X(e_r)};\ e_r < \tau_\varepsilon^+ \wedge e_q, X_{e_r} < 0] + o(W(\varepsilon))\\
&= \mathrm{e}^{-\Phi(p+q+r)\varepsilon}\mathbb{E}_\varepsilon \mathrm{e}^{-(p+r)\tau_0^- + \Phi(p+q+r)X(\tau_0^-)}\omega_{2,\varepsilon}^+(0)\\
&\quad + \mathrm{e}^{-\Phi(p+q+r)\varepsilon}\frac{r}{p+q+r-\psi(\theta)}\\
&\qquad \times (\mathbb{E}_\varepsilon \mathrm{e}^{-(p+r)\tau_0^- + \theta X(\tau_0^-)} - \mathbb{E}_\varepsilon \mathrm{e}^{-(p+r)\tau_0^- + \Phi(p+q+r)X(\tau_0^-)})\\
&\quad + \frac{r}{p+q+r-\psi(\theta)}(1 - \mathrm{e}^{(\theta-\Phi(p+q+r))\varepsilon}) + o(W(\varepsilon))\\
&= \left(1 - \frac{q}{\Phi(p+q+r) - \Phi(p+r)}\mathrm{e}^{-\Phi(p+q+r)\varepsilon}W^{(p+r)}(\varepsilon)\right)\omega_{2,\varepsilon}^+(0) + o(W(\varepsilon))\\
&\quad + \frac{r}{p+q+r-\psi(\theta)}\left(\frac{q}{\Phi(p+q+r) - \Phi(p+r)}W^{(p+r)}(\varepsilon)\right.\\
&\qquad\left. - \mathrm{e}^{(\theta-\Phi(p+q+r))\varepsilon}\frac{p+r-\psi(\theta)}{\Phi(p+r) - \theta}W^{(p+r)}(\varepsilon)\right),
\end{aligned}
$$

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where, by (2.1),

$$\begin{aligned}
&\mathbb{E}[\mathrm{e}^{-pe_r+\theta X(e_r)};\ e_r < \tau_\varepsilon^+ \wedge e_q, X_{e_r} < 0] \\
&\quad = \mathbb{E}[\mathrm{e}^{-(p+q)e_r+\theta X(e_r)};\ e_r < \tau_\varepsilon^+] \\
&\quad = \int_{-\infty}^{0} \mathrm{e}^{\theta y} r(\mathrm{e}^{-\Phi(p+q+r)\varepsilon} W^{(p+q+r)}(\varepsilon - y) - W^{(p+q+r)}(-y))\,\mathrm{d}y \\
&\quad = r\mathrm{e}^{-\Phi(p+q+r)\varepsilon} \int_{-\infty}^{-\varepsilon} \mathrm{e}^{\theta(y+\varepsilon)} W^{(p+q+r)}(-y)\,\mathrm{d}y - r\int_{-\infty}^{0} \mathrm{e}^{\theta y} W^{(p+q+r)}(-y)\,\mathrm{d}y \\
&\quad = \mathrm{e}^{(\theta-\Phi(p+q+r))\varepsilon} \frac{r}{\psi(\theta) - p - q - r} - \frac{r}{\psi(\theta) - p - q - r} + o(W(\varepsilon)).
\end{aligned}$$

Then

$$\begin{aligned}
\omega_2^+(0) &= \lim_{\varepsilon\to 0+} \omega_{2,\varepsilon}^+(0) \\
&= \frac{r}{p+q+r-\psi(\theta)} \left(\frac{q}{\Phi(p+q+r) - \Phi(p+r)} - \mathrm{e}^{(\theta-\Phi(p+q+r))\varepsilon} \frac{p+r-\psi(\theta)}{\Phi(p+r)-\theta} \right) \\
&\quad \times \left[\frac{q}{\Phi(p+q+r) - \Phi(p+r)} \right]^{-1} \\
&= \frac{r}{p+q+r-\psi(\theta)} \left(1 - \frac{p+r-\psi(\theta)}{q} \frac{\Phi(p+q+r) - \Phi(p+r)}{\Phi(p+r)-\theta} \right).
\end{aligned}$$

For $x < 0$,

$$\begin{aligned}
\omega_1^+(x) &= \mathbb{E}_x[\mathrm{e}^{-(p+q)\tau_0^+};\ \tau_0^+ < e_r]\omega_1^+(0) \\
&= \mathrm{e}^{\Phi(p+q+r)x} \frac{r}{q\Phi(r)} (\Phi(p+q+r) - \Phi(p+r)).
\end{aligned}$$

Similarly,

$$\begin{aligned}
\omega_2^+(x) &= \mathbb{E}_x[\mathrm{e}^{-(p+q)e_r+\theta X(e_r)};\ e_r < \tau_0^+] + \mathbb{E}_x[\mathrm{e}^{-(p+q)\tau_0^+};\ \tau_0^+ < e_r]\omega_2^+(0) \\
&= \frac{r}{p+q+r-\psi(\theta)} \\
&\quad \times \left(\mathrm{e}^{\theta x} - \mathrm{e}^{\Phi(p+q+r)x} \frac{p+r-\psi(\theta)}{q} \frac{\Phi(p+q+r) - \Phi(p+r)}{\Phi(p+r)-\theta} \right). \qquad (3.2)
\end{aligned}$$

By analytical extension, (3.2) holds for all $\theta \geq 0$.

For $x > 0$,

$$\begin{aligned}
\omega_1^+(x) &= \mathbb{P}_x\{\tau_0^- > e_r\} \\
&\quad + \int_{-\infty}^{0} \mathbb{E}_x[\mathrm{e}^{-p\tau_0^-};\ \tau_0^- < e_r, X_{\tau_0^-} \in \mathrm{d}y]\mathbb{E}_y[\mathrm{e}^{-(p+q)\tau_0^+};\ \tau_0^+ < e_r]\omega_1^+(0) \\
&= \frac{r}{\Phi(r)} W^{(r)}(x) - r\int_0^x W^{(r)}(y)\,\mathrm{d}y \\
&\quad + \frac{r(\Phi(p+q+r) - \Phi(p+r))}{q\Phi(r)} \\
&\quad \times \left\{ Z^{(p+r)}(x, \Phi(p+q+r)) - \frac{q}{\Phi(p+q+r) - \Phi(p+r)} W^{(p+r)}(x) \right\}
\end{aligned}$$

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$$= \frac{r(W^{(r)}(x) - W^{(p+r)}(x))}{\Phi(r)} - r\int_0^x W^{(r)}(y)\,\mathrm{d}y + \frac{r(\Phi(p+q+r) - \Phi(p+r))}{q\Phi(r)} Z^{(p+r)}(x, \Phi(p+q+r))$$

and

$$\begin{aligned}
\omega_2^+(x) &= \int_{-\infty}^0 \mathbb{E}_x[\mathrm{e}^{-p\tau_0^-};\ \tau_0^- < e_r, X_{\tau_0^-} \in \mathrm{d}y] \\
&\quad \times (\mathbb{E}_y[\mathrm{e}^{-(p+q)\tau_0^+};\ \tau_0^+ < e_r]\omega_2^+(0) + \mathbb{E}_y[\mathrm{e}^{-(p+q)e_r + \theta X(e_r)};\ e_r < \tau_0^+]) \\
&= \mathbb{E}_x \mathrm{e}^{-(p+r)\tau_0^- + \Phi(p+q+r)X(\tau_0^-)}\omega_2^+(0) \\
&\quad + \frac{r}{p+q+r-\psi(\theta)}(\mathbb{E}_x \mathrm{e}^{-(p+r)\tau_0^- + \theta X(\tau_0^-)} - \mathbb{E}_x \mathrm{e}^{-(p+r)\tau_0^- + \Phi(p+q+r)X(\tau_0^-)}) \\
&= \frac{r}{p+q+r-\psi(\theta)}\left(-\frac{p+r-\psi(\theta)}{q}\frac{\Phi(p+q+r) - \Phi(p+r)}{\Phi(p+r) - \theta}\right) \\
&\quad \times Z^{(p+r)}(x, \Phi(p+q+r)) \\
&\quad + \frac{r}{p+q+r-\psi(\theta)} Z^{(p+r)}(x, \theta).
\end{aligned}$$

The desired results then follow. □

Remark 3.1. For $x < 0$,

$$\begin{aligned}
\mathbb{E}_x \mathrm{e}^{-pT^+(r)} &= \lim_{q,\theta \to 0+} (\omega_1^+(x) + \omega_2^+(x)) \\
&= \frac{r}{p+r} + r\mathrm{e}^{\Phi(p+r)x}\Phi'(p+r)\left(\frac{1}{\Phi(r)} - \frac{1}{\Phi(p+r)}\right).
\end{aligned}$$

For $x > 0$,

$$\begin{aligned}
\mathbb{E}_x \mathrm{e}^{-pT^+(r)} &= \frac{r(W^{(r)}(x) - W^{(p+r)}(x))}{\Phi(r)} + r\mathrm{e}^{\Phi(p+r)x}\Phi'(p+r)\left(\frac{1}{\Phi(r)} - \frac{1}{\Phi(p+r)}\right) \\
&\quad + 1 - Z^{(r)}(x) + \frac{r}{p+r} Z^{(p+r)}(x),
\end{aligned}$$

which recovers Theorem 2 in [2].

Remark 3.2. For $\psi'(0+) > 0$, letting $r, q \to 0+$, we have

$$\mathbb{E}_x \mathrm{e}^{-pT^+} = \psi'(0+)(W(x) - W^{(p)}(x)) + \mathrm{e}^{\Phi(p)x}\psi'(0+)\Phi'(p),$$

which recovers a result in [3]; letting $r, p \to 0+$ for $\psi'(0+) > 0$, we have

$$\mathbb{E}_x \exp\left(-q\int_0^\infty \mathbf{1}_{\{X_s < 0\}}\,\mathrm{d}s\right) = \frac{\psi'(0+)}{q}\Phi(q)Z(x, \Phi(q)),$$

which recovers Corollary 3(ii) of [10].

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We also want to find

$$\omega_1^-(x) := \mathbb{E}_x\left[\exp\left(-pT^-(r) + \theta X_{T^-(r)} - q\int_0^{T^-(r)} \mathbf{1}_{\{X_s\geq 0\}}\,\mathrm{d}s\right);\ T^-(r) < e_r\right]$$

and

$$\omega_2^-(x) := \mathbb{E}_x\left[\exp\left(-pT^-(r) - q\int_0^{T^-(r)} \mathbf{1}_{\{X_s\geq 0\}}\,\mathrm{d}s - \theta X(T^-(r))\right);\ T^-(r) = e_r\right].$$

Theorem 3.2. *For any $p, q, r, \theta > 0$, and $x \in \mathbb{R}$, we have*

$$\begin{aligned}\omega_1^-(x) = {}& \frac{1}{q}(\Phi(p+q+r) - \Phi(p+r))Z^{(p+q+r)}(x, \Phi(p+r))\\ &\times\left(\frac{p+q+r-\psi(\theta+\Phi(r))}{\Phi(p+q+r)-\theta-\Phi(r)} - \frac{p+q+r-\psi(\theta)}{\Phi(p+q+r)-\theta}\right)\\ &+ Z^{(p+q+r)}(x,\theta) - Z^{(p+q+r)}(x,\theta+\Phi(r))\end{aligned} \tag{3.3}$$

and

$$\begin{aligned}\omega_2^-(x) = {}& \frac{r}{q}\frac{\Phi(p+q+r)-\Phi(p+r)}{\theta+\Phi(p+q+r)}Z^{(p+q+r)}(x,\Phi(p+r))\\ &- r\int_0^x \mathrm{e}^{-\theta y}W^{(p+q+r)}(x-y)\,\mathrm{d}y.\end{aligned}$$

Proof. For $\varepsilon > 0$ and $x \in \mathbb{R}$, let

$$T_\varepsilon^-(r) := \inf\{T^-(r) \leq s \leq e_r\colon X_s < -\varepsilon\}$$

with the convention $\inf\varnothing = e_r$,

$$\omega_{1,\varepsilon}^-(x) := \mathbb{E}_x\left[\exp\left(-pT_\varepsilon^-(r) + \theta X_{T_\varepsilon^-(r)} - q\int_0^{T_\varepsilon^-(r)} \mathbf{1}_{\{X_s\geq 0\}}\,\mathrm{d}s\right);\ T_\varepsilon^-(r) < e_q\right]$$

and

$$\omega_{2,\varepsilon}^-(x) := \mathbb{E}_x\left[\exp\left(-pT_\varepsilon^-(r) - q\int_0^{T_\varepsilon^-(r)} \mathbf{1}_{\{X_s\geq 0\}}\,\mathrm{d}s - \theta X(e_r)\right);\ T_\varepsilon^-(r) = e_q\right].$$

Then, for

$$A_\varepsilon^- := \bigcap_{i\colon T_i < T_\varepsilon^-(r)} \{X_{T_i} < 0\},$$

we have

$$\omega_{1,\varepsilon}^-(x) = \mathbb{E}_x\left[\exp\left(-pT_\varepsilon^-(r) + \theta X_{T_\varepsilon^-(r)} - q\int_0^{T_\varepsilon^-(r)} \mathbf{1}_{\{X_s\geq 0\}}\,\mathrm{d}s\right);\ T_\varepsilon^-(r) < e_r, A_\varepsilon^-\right].$$

By (2.2), we have

$$\begin{aligned}\mathbb{E}[\mathrm{e}^{-pT_\varepsilon^-(r)+\theta X(T_\varepsilon^-(r))};\ T_1 < e_r\wedge\tau_{-\varepsilon}^-, T_\varepsilon^-(r) < e_r, A_\varepsilon^-] &\leq \mathbb{P}\{T_1 < \tau_{-\varepsilon}^-,\ -\varepsilon \leq X_{T_1} \leq 0\}\\ &= o(W(\varepsilon)).\end{aligned}$$

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Then

$$\begin{aligned}\omega_{1,\varepsilon}^{-}(0) &= \int_{-\infty}^{-\varepsilon} \mathbb{E}[\mathrm{e}^{-p\tau_{-\varepsilon}^{-}+\theta y};\ \tau_{-\varepsilon}^{-} < e_q \wedge e_r, X_{\tau_{-\varepsilon}^{-}} \in \mathrm{d}y]\mathbb{P}_y\{\tau_0^{+} > e_r\} \\ &\quad + \int_{-\infty}^{-\varepsilon} \mathbb{E}[\mathrm{e}^{-p\tau_{-\varepsilon}^{-}};\ \tau_{-\varepsilon}^{-} < e_q \wedge \mathrm{e}_r, X_{\tau_{-\varepsilon}^{-}} \in \mathrm{d}y]\mathbb{E}_y[\mathrm{e}^{-p\tau_0^{+}};\ \tau_0^{+} < e_r]\omega_{1,\varepsilon}^{-}(0) \\ &\quad + o(W(\varepsilon)) \\ &= \int_{-\infty}^{-\varepsilon} \mathbb{E}[\mathrm{e}^{-(p+q+r)\tau_{-\varepsilon}^{-}+\theta y};\ X_{\tau_{-\varepsilon}^{-}} \in \mathrm{d}y](1-\mathrm{e}^{\Phi(r)y}) \\ &\quad + \int_{-\infty}^{-\varepsilon} \mathbb{E}[\mathrm{e}^{-(p+q+r)\tau_{-\varepsilon}^{-}};\ X_{\tau_{-\varepsilon}^{-}} \in \mathrm{d}y]\mathrm{e}^{\Phi(p+r)y}\omega_{1,\varepsilon}^{-}(0) + o(W(\varepsilon)) \\ &= 1 + o(W(\varepsilon)) - \frac{p+q+r-\psi(\theta)}{\Phi(p+q+r)-\theta}\mathrm{e}^{-\theta\varepsilon}W^{(p+q+r)}(\varepsilon) \\ &\quad - 1 + o(W(\varepsilon)) + \frac{p+q+r-\psi(\theta+\Phi(r))}{\Phi(p+q+r)-\theta-\Phi(r)}\mathrm{e}^{-(\theta+\Phi(r))\varepsilon}W^{(p+q+r)}(\varepsilon) \\ &\quad + \left(1 - \frac{q}{\Phi(p+q+r)-\Phi(p+r)}\mathrm{e}^{-\Phi(p+r)\varepsilon}W^{(p+q+r)}(\varepsilon)\right)\omega_{1,\varepsilon}^{-}(0).\end{aligned}$$

It follows that

$$\begin{aligned}\omega_1^{-}(0) &= \lim_{\varepsilon\to 0+}\omega_{1,\varepsilon}^{-}(0) \\ &= \frac{1}{q}(\Phi(p+q+r)-\Phi(p+r)) \\ &\quad \times \left(\frac{p+q+r-\psi(\theta+\Phi(r))}{\Phi(p+q+r)-\theta-\Phi(r)} - \frac{p+q+r-\psi(\theta)}{\Phi(p+q+r)-\theta}\right).\end{aligned}$$

Similarly,

$$\begin{aligned}\omega_{2,\varepsilon}^{-}(0) &= \int_{-\infty}^{-\varepsilon} \mathbb{E}[\mathrm{e}^{-p\tau_{-\varepsilon}^{-}};\ \tau_{-\varepsilon}^{-} < e_q \wedge e_r, X_{\tau_{-\varepsilon}^{-}} \in \mathrm{d}y]\mathbb{E}_y[\mathrm{e}^{-p\tau_0^{+}};\ \tau_0^{+} < e_r]\omega_{2,\varepsilon}^{-}(0) \\ &\quad + \mathbb{E}[\mathrm{e}^{-pe_r-\theta X(e_r)};\ e_r < \tau_{-\varepsilon}^{-} \wedge e_q] + o(W(\varepsilon)) \\ &= \left(1 - \frac{q}{\Phi(p+q+r)-\Phi(p+r)}\mathrm{e}^{-\Phi(p+r)\varepsilon}W^{(p+q+r)}(\varepsilon)\right)\omega_{2,\varepsilon}^{-}(0) \\ &\quad + \frac{r}{\theta+\Phi(p+q+r)}W^{(p+q+r)}(\varepsilon) + o(W(\varepsilon)),\end{aligned}$$

where

$$\begin{aligned}&\mathbb{E}[\mathrm{e}^{-pe_r-\theta X(e_r)}; e_r < \tau_{-\varepsilon}^{-} \wedge e_q] \\ &\quad = r\int_{-\varepsilon}^{\infty} \mathrm{e}^{-\theta y}(\mathrm{e}^{-\Phi(p+q+r)(y+\varepsilon)}W^{(p+q+r)}(\varepsilon) - W^{(p+q+r)}(-y))\,\mathrm{d}y \\ &\quad = \frac{r}{\theta+\Phi(p+q+r)}W^{(p+q+r)}(\varepsilon) + o(W(\varepsilon)).\end{aligned}$$

Then

$$\omega_2^{-}(0) = \frac{r}{q}\frac{\Phi(p+q+r)-\Phi(p+r)}{\theta+\Phi(p+q+r)}.$$

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For $x < 0$,

$$\begin{aligned}\omega_1^-(x) &= \mathrm{e}^{\theta x}\mathbb{P}_x\{e_r < \tau_0^+\} + \mathbb{E}_x[\mathrm{e}^{-p\tau_0^+};\ \tau_0^+ < e_r]\omega_1^-(0)\\ &= \mathrm{e}^{\theta x}(1-\mathrm{e}^{\Phi(r)x}) + \frac{\mathrm{e}^{\Phi(p+r)x}}{q}(\Phi(p+q+r)-\Phi(p+r))\\ &\qquad\times\left(\frac{p+q+r-\psi(\theta+\Phi(r))}{\Phi(p+q+r)-\theta-\Phi(r)} - \frac{p+q+r-\psi(\theta)}{\Phi(p+q+r)-\theta}\right)\end{aligned}$$

and

$$\omega_2^-(x) = \mathbb{E}_x[\mathrm{e}^{-p\tau_0^+};\ \tau_0^+ < e_r]\omega_2^-(0) = \frac{r\mathrm{e}^{\Phi(p+r)x}}{q}\frac{\Phi(p+q+r)-\Phi(p+r)}{\theta+\Phi(p+q+r)}.$$

For $x > 0$,

$$\begin{aligned}\omega_1^-(x) &= \int_{-\infty}^0 \mathbb{E}_x[\mathrm{e}^{-(p+q)\tau_0^-};\ \tau_0^- < e_r, X_{\tau_0^-} \in \mathrm{d}y]\\ &\quad\times(\mathbb{E}_y[\mathrm{e}^{-p\tau_0^+};\ \tau_0^+ < e_r]\omega_1^-(0) + \mathrm{e}^{\theta y}\mathbb{P}_y\{e_r < \tau_0^+\})\\ &= \mathbb{E}_x\mathrm{e}^{-(p+q+r)\tau_0^- + \Phi(p+r)X(\tau_0^-)}\omega_1^-(0) + \mathbb{E}_x\mathrm{e}^{-(p+q+r)\tau_0^- + \theta X(\tau_0^-)}\\ &\quad - \mathbb{E}_x\mathrm{e}^{-(p+q+r)\tau_0^- + (\theta+\Phi(r))X(\tau_0^-)}\\ &= \left\{Z^{(p+q+r)}(x,\Phi(p+r)) - \frac{q}{\Phi(p+q+r)-\Phi(p+r)}W^{(p+q+r)}(x)\right\}\\ &\quad\times\frac{1}{q}(\Phi(p+q+r)-\Phi(p+r))\\ &\quad\times\left(\frac{p+q+r-\psi(\theta+\Phi(r))}{\Phi(p+q+r)-\theta-\Phi(r)} - \frac{p+q+r-\psi(\theta)}{\Phi(p+q+r)-\theta}\right)\\ &\quad + Z^{(p+q+r)}(x,\theta) - \frac{p+q+r-\psi(\theta)}{\Phi(p+q+r)-\theta}W^{(p+q+r)}(x)\\ &\quad - Z^{(p+q+r)}(x,\theta+\Phi(r)) + \frac{p+q+r-\psi(\theta+\Phi(r))}{\Phi(p+q+r)-\theta-\Phi(r)}W^{(p+q+r)}(x)\\ &= Z^{(p+q+r)}(x,\Phi(p+r))\omega_1^-(0) + Z^{(p+q+r)}(x,\theta) - Z^{(p+q+r)}(x,\theta+\Phi(r)).\end{aligned}$$

Similarly,

$$\begin{aligned}\omega_2^-(x) &= \int_{-\infty}^0 \mathbb{E}_x[\mathrm{e}^{-(p+q)\tau_0^-};\ \tau_0^- < e_r, X_{\tau_0^-} \in \mathrm{d}y]\mathbb{E}_y[\mathrm{e}^{-p\tau_0^+};\ \tau_0^+ < e_r]\omega_2^-(0)\\ &\quad + \mathbb{E}_x[\mathrm{e}^{-(p+q)e_r - \theta X(e_r)}; e_r < \tau_0^-]\\ &= \frac{r}{q}\frac{\Phi(p+q+r)-\Phi(p+r)}{\theta+\Phi(p+q+r)}\\ &\quad\times\left\{Z^{(p+q+r)}(x,\Phi(p+r)) - \frac{q}{\Phi(p+q+r)-\Phi(p+r)}W^{(p+q+r)}(x)\right\}\\ &\quad + \frac{r}{\theta+\Phi(p+q+r)}W^{(p+q+r)}(x) - r\int_0^x \mathrm{e}^{-\theta y}W^{(p+q+r)}(x-y)\,\mathrm{d}y\end{aligned}$$

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$$= \frac{r}{q}\frac{\Phi(p+q+r)-\Phi(p+r)}{\theta+\Phi(p+q+r)}Z^{(p+q+r)}(x,\Phi(p+r)) - r\int_0^x \mathrm{e}^{-\theta y}W^{(p+q+r)}(x-y)\,\mathrm{d}y,$$

where

$$\begin{aligned}&\mathbb{E}_x[\mathrm{e}^{-(p+q)e_r-\theta X(e_r)};\, e_r<\tau_0^-]\\ &\quad= r\int_0^\infty \mathrm{e}^{-\theta y}(\mathrm{e}^{-\Phi(p+q+r)y}W^{(p+q+r)}(x)-W^{(p+q+r)}(x-y))\,\mathrm{d}y\\ &\quad= \frac{r}{\theta+\Phi(p+q+r)}W^{(p+q+r)}(x)-r\int_0^x \mathrm{e}^{-\theta y}W^{(p+q+r)}(x-y)\,\mathrm{d}y.\end{aligned}$$

We thus finish the proof. □

Remark 3.3. Using Theorem 3.2 we can find the probability that X hits 0 at time $T^-(r)$. It shows that $\mathbb{P}_x\{X_{T^-(r)}=0\}>0$ if and only if process X has a nontrivial Brownian component, which is similar to what happens at the first exit time.

Corollary 3.1. *For any $p,q,r>0$, and $x\in\mathbb{R}$, we have*

$$\begin{aligned}&\mathbb{E}_x\left[\exp\left(-pT^-(r)-q\int_0^{T^-(r)}\mathbf{1}_{\{X_s\geq 0\}}\,\mathrm{d}s\right);\, T^-(r)<e_r,\, X_{T^-(r)}=0\right]\\ &\quad=\frac{\sigma^2}{2}\Phi(r)Z^{(p+q+r)}(x,\Phi(p+r))\frac{1}{q}(\Phi(p+q+r)-\Phi(p+r))\\ &\qquad-\frac{\sigma^2}{2}\Phi(r)W^{(p+q+r)}(x).\end{aligned}\tag{3.4}$$

In particular, for $\psi'(0+)<0$,

$$\mathbb{E}_x[\mathrm{e}^{-pT^-};\, X_{T^-}=0]=\tfrac{1}{2}\sigma^2\Phi(0)(\mathrm{e}^{\Phi(p)x}\Phi'(p)-W^{(p)}(x)).$$

Proof. Note that

$$\begin{aligned}&\lim_{\theta\to\infty}\left(\frac{p+q+r-\psi(\theta+\Phi(r))}{\Phi(p+q+r)-\theta-\Phi(r)}-\frac{p+q+r-\psi(\theta)}{\Phi(p+q+r)-\theta}\right)\\ &\quad=\lim_{\theta\to\infty}\frac{-\psi(\theta+\Phi(r))(\Phi(p+q+r)-\theta)+\psi(\theta)(\Phi(p+q+r)-\theta)+\Phi(r)(p+q+r-\psi(\theta))}{(\Phi(p+q+r)-\theta-\Phi(r))(\Phi(p+q+r)-\theta)}\\ &\quad=-\tfrac{1}{2}\sigma^2\Phi(p+q+r)+\sigma^2\Phi(r)+\tfrac{1}{2}\sigma^2(\Phi(p+q+r)-\Phi(r))\\ &\quad=\tfrac{1}{2}\sigma^2\Phi(r);\end{aligned}$$

see [5, Section 2.4].

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In addition,

$$\begin{aligned}
&\lim_{\theta\to\infty}(Z^{(p+q+r)}(x,\theta)-Z^{(p+q+r)}(x,\theta+\Phi(r)))\\
&\quad=\lim_{\theta\to\infty}\Bigg(\mathbb{E}_x\mathrm{e}^{-(p+q+r)\tau_0^-+\theta X(\tau_0^-)}\mathbf{1}_{\{\tau_0^-<\infty\}}+\frac{p+q+r-\psi(\theta)}{\Phi(p+q+r)-\theta}W^{(p+q+r)}(x)\\
&\qquad\quad-\mathbb{E}_x\mathrm{e}^{-(p+q+r)\tau_0^-+(\theta+\Phi(r))X(\tau_0^-)}\mathbf{1}_{\{\tau_0^-<\infty\}}\\
&\qquad\quad-\frac{p+q+r-\psi(\theta+\Phi(r))}{\Phi(p+q+r)-\theta-\Phi(r)}W^{(p+q+r)}(x)\Bigg)\\
&\quad=-\tfrac{1}{2}\sigma^2\Phi(r)W^{(p+q+r)}(x).
\end{aligned}$$

Then (3.4) follows from (3.3). □

Corollary 3.2. *For $\psi'(0+)<0$ and $x\in\mathbb{R}$,*

$$\begin{aligned}
&\mathbb{E}_x\exp\left(-pT^-+\theta X_{T^-}-q\int_0^{T^-}\mathbf{1}_{\{X_s\geq 0\}}\,\mathrm{d}s\right)\\
&\quad=Z^{(p+q)}(x,\Phi(p))\frac{1}{q}(\Phi(p+q)-\Phi(p))\left(\frac{p+q-\psi(\theta+\Phi(0))}{\Phi(p+q)-\theta-\Phi(0)}-\frac{p+q-\psi(\theta)}{\Phi(p+q)-\theta}\right)\\
&\qquad+Z^{(p+q)}(x,\theta)-Z^{(p+q)}(x,\theta+\Phi(0)).
\end{aligned}$$

Remark 3.4. For $x<0$,

$$\begin{aligned}
\mathbb{E}_x\mathrm{e}^{-pT^-(r)}&=\lim_{q,\theta\to 0+}(\omega_1^-(x)+\omega_2^-(x))\\
&=1-\mathrm{e}^{\Phi(r)x}+\mathrm{e}^{\Phi(p+r)x}\frac{\Phi'(p+r)}{\Phi(p+r)}\frac{p\Phi(r)}{\Phi(p+r)-\Phi(r)}.
\end{aligned}$$

For $x>0$,

$$\begin{aligned}
\mathbb{E}_x\mathrm{e}^{-pT^-(r)}&=Z^{(p+r)}(x,\Phi(p+r))\frac{\Phi'(p+r)}{\Phi(p+r)}\frac{(p+r)\Phi(r)-r\Phi(p+r)}{\Phi(p+r)-\Phi(r)}\\
&\quad+Z^{(p+r)}(x)-Z^{(p+r)}(x,\Phi(r))+\frac{r\Phi'(p+r)}{\Phi(p+r)}Z^{(p+r)}(x,\Phi(p+r))\\
&\quad-r\int_0^x W^{(p+r)}(x-y)\,\mathrm{d}y\\
&=\mathrm{e}^{\Phi(p+r)x}\frac{\Phi'(p+r)}{\Phi(p+r)}\frac{p\Phi(r)}{\Phi(p+r)-\Phi(r)}+\frac{r}{p+r}+\frac{p}{p+r}Z^{(p+r)}(x)\\
&\quad-Z^{(p+r)}(x,\Phi(r)).
\end{aligned}$$

We thus recover Theorem 2 of [2].

Remark 3.5. For $\psi'(0+)<0$ and $x\in\mathbb{R}$, letting $q,\theta,r\to 0+$, we have

$$\mathbb{E}_x\mathrm{e}^{-pT^-}=Z^{(p)}(x,\Phi(p))\frac{p\Phi(0)\Phi'(p)}{\Phi(p)(\Phi(p)-\Phi(0))}+Z^{(p)}(x)-Z^{(p)}(x,\Phi(0));$$

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letting $p, q, r \to 0+$, we have

$$\mathbb{E}_x e^{\theta X_{T^-}} = e^{\Phi(0)x}\Phi'(0)\left(\frac{\psi(\theta+\Phi(0))}{\theta} + \frac{\psi(\theta)}{\Phi(0)-\theta}\right) + Z(x,\theta) - Z(x,\theta+\Phi(0));$$

letting $p, \theta, r \to 0+$, we have

$$\begin{aligned}\mathbb{E}_x \exp\left(-q\int_0^\infty \mathbf{1}_{\{X_s>0\}}\,ds\right) &= Z^{(q)}(x,\Phi(0))\frac{\Phi(0)}{\Phi(q)} + Z^{(q)}(x,0) - Z^{(q)}(x,\Phi(0)) \\ &= Z^{(q)}(x,\Phi(0))\left(\frac{\Phi(0)}{\Phi(q)} - 1\right) + Z^{(q)}(x),\end{aligned}$$

which recovers Corollary 3(iv) of [10].

Remark 3.6. Using another approach for the proofs of Theorems 3.1 and 3.2, one can first prove the desired results for process X with bounded variation, and then apply a strong approximation of a spectrally negative Lévy process by a spectrally negative compound Poisson process to obtain the results for process X with unbounded variation. We refer the reader to [10] and [11] for details of technical issues concerning this alternative approach.

4. Examples

In this section we consider some examples for more explicit results. Let $X_t = -\mu t + B_t$ for constant $\mu > 0$ and standard Brownian motion B. Then

$$\psi(\lambda) = -\mu\lambda + \tfrac{1}{2}\lambda^2, \qquad \Phi(q) = \sqrt{\mu^2+2q} + \mu,$$
$$W^{(q)}(x) = \frac{1}{\sqrt{\mu^2+2q}}(\exp((\sqrt{\mu^2+2q}+\mu)x) - \exp(-(\sqrt{\mu^2+2q}-\mu)x))\mathbf{1}_{\{x\geq 0\}},$$

and

$$W(x) = \frac{1}{\mu}(e^{2\mu x} - 1)\mathbf{1}_{\{x\geq 0\}}.$$

Observe that $T^- < \infty$ almost surely since $\psi'(0) < 0$.

By Corollary 3.1,

$$\begin{aligned}&\mathbb{E}_x[\exp(-pT^-);\ X_{T^-}=0] \\ &\quad= \tfrac{1}{2}\Phi(0)(\exp(\Phi(p)x)\Phi'(p) - W^{(p)}(x)) \\ &\quad= \mu\left(\frac{1}{\sqrt{\mu^2+2p}}\exp((\sqrt{\mu^2+2p}+\mu)x)\right. \\ &\qquad\left. - \frac{1}{\sqrt{\mu^2+2p}}(\exp((\sqrt{\mu^2+2p}+\mu)x) - \exp(-(\sqrt{\mu^2+2p}-\mu)x))\mathbf{1}_{\{x\geq 0\}}\right) \\ &\quad= \frac{\mu}{\sqrt{\mu^2+2p}}(\exp((\sqrt{\mu^2+2p}+\mu)x)\mathbf{1}_{\{x<0\}} + \exp(-(\sqrt{\mu^2+2p}-\mu)x)\mathbf{1}_{\{x\geq 0\}}).\end{aligned}$$

It follows that

$$\mathbb{P}_x\{X_{T^-}=0\} = e^{2\mu x}\mathbf{1}_{\{x<0\}} + \mathbf{1}_{\{x\geq 0\}} = \mathbb{P}_x\{\tau_0^+ < \infty\}\mathbf{1}_{\{x<0\}} + \mathbf{1}_{\{x\geq 0\}},$$

which is expected intuitively.

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Let X be a spectrally negative α-stable process with a negative drift whose Laplace exponent is given by

$$\psi(\lambda) = -\mu\lambda + \lambda^\alpha \quad \text{for } \mu > 0,\ 1 < \alpha < 2.$$

Then $\Phi(0) = \mu^{1/(\alpha-1)}$ and by Remark 3.5,

$$\begin{aligned}\mathbb{E}e^{\theta X_{T^-}} &= \frac{1}{\psi'(\Phi(0))}\left(\frac{\psi(\theta+\Phi(0))}{\theta} + \frac{\psi(\theta)}{\Phi(0)-\theta}\right)\\ &= \frac{1}{(\alpha-1)\mu}\left(\frac{-\mu\theta - \mu^{\alpha/(\alpha-1)} + (\theta+\mu^{1/(\alpha-1)})^\alpha}{\theta} + \frac{-\mu\theta+\theta^\alpha}{\mu^{1/(\alpha-1)}-\theta}\right).\end{aligned}$$

It is not difficult to see that

$$\lim_{\theta\to\infty} \mathbb{E}e^{\theta X_{T^-}} = 0,$$

i.e. $\mathbb{P}\{X_{T^-} = 0\} = 0$ which agrees with Corollary 3.1.

Let X be a compound Poisson process with a positive drift μ and negative exponential jumps with common rate ρ arriving according to a Poisson process with rate a. Then

$$\psi(\lambda) = \mu\lambda - \frac{a\lambda}{\rho+\lambda}, \qquad \Phi(0) = 0 \iff \mu - \frac{a}{\rho} \geq 0.$$

We further assume that $\mu - a/\rho < 0$ so that the last exit time is finite. Then

$$\Phi(0) = \frac{a}{\mu} - \rho, \qquad \psi'(\Phi(0)) = \mu - \frac{a\rho}{(\rho + a/\mu - \rho)^2} = \mu - \frac{\rho\mu^2}{a},$$

and

$$\psi(\theta+\Phi(0)) = \mu\left(\theta + \frac{a}{\mu} - \rho\right) - \frac{a(\theta + a/\mu - \rho)}{\rho + \theta + a/\mu - \rho} = \left(\theta + \frac{a}{\mu} - \rho\right)\frac{\mu\theta}{\theta + a/\mu}.$$

It follows that

$$\begin{aligned}\mathbb{E}e^{\theta X_{T^-}} &= \frac{1}{\psi'(\Phi(0))}\left(\frac{\psi(\theta+\Phi(0))}{\theta} + \frac{\psi(\theta)}{\Phi(0)-\theta}\right)\\ &= \frac{1}{\mu - \rho\mu^2/a}\left(\left(\theta + \frac{a}{\mu} - \rho\right)\frac{\mu}{\theta + a/\mu} + \frac{\mu\theta - a\theta/(\rho+\theta)}{a/\mu - \rho - \theta}\right)\\ &= \frac{a\rho}{(\rho+\theta)(\mu\theta + a)}\\ &= \frac{a}{a-\mu\rho}\frac{\rho}{\rho+\theta} - \frac{\mu\rho}{a-\mu\rho}\frac{a}{a+\mu\theta}.\end{aligned}$$

Therefore, under $\mathbb{P}$, X_{T^-} has a density

$$\left(\frac{a}{a-\mu\rho}\rho e^{\rho x} - \frac{\mu\rho}{a-\mu\rho}\frac{a}{\mu}e^{ax/\mu}\right)\mathbf{1}_{\{x<0\}} = \frac{a\rho}{a-\mu\rho}(e^{\rho x} - e^{ax/\mu})\mathbf{1}_{\{x<0\}}. \tag{4.1}$$

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There is a more direct way to find the density f of X_{T^-}. Conditioning on τ_0^- and using the fact that $-X(\tau_0^-)$ follows the exponential distribution we have, for $x < 0$,

$$\begin{aligned} f(x) &= \rho \mathrm{e}^{\rho x} \mathbb{P}_x\{\tau_0^+ = \infty\} + f(x) \int_{-\infty}^0 \rho \mathrm{e}^{\rho y} \mathbb{P}_y\{\tau_0^+ < \infty\} \, \mathrm{d}y \\ &= \rho \mathrm{e}^{\rho x}(1 - \mathrm{e}^{(a/\mu - \rho)x}) + f(x) \int_{-\infty}^0 \rho \mathrm{e}^{\rho y} \mathrm{e}^{(a/\mu-\rho)y} \, \mathrm{d}y \\ &= \rho(\mathrm{e}^{\rho x} - \mathrm{e}^{ax/\mu}) + \frac{\mu\rho}{a} f(x). \end{aligned}$$

Solving the above equation for $f(x)$ we can recover expression (4.1).

It is remarkable that $-X_{T^-}$ does not follow an exponential distribution, which is in contrast with what happens at the first exit time τ_0^- when the overshoot $-X_{\tau_0^-}$ follows the exponential distribution of e_ρ. In addition, compared with the exponential random variable e_ρ, $-X_{T^-}$ has a higher chance of taking large values.

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