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Conjugacy rigidity for non-positively curved graph manifolds

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Abstract. We show that the metric of non-positively curved graph manifolds is determined by its geodesic flow. More precisely, we show that if the geodesic flows of two non-positively curved graph manifolds are C^0 conjugate, then the spaces are isometric.

1. *Introduction*

Two complete Riemannian manifolds, N and N', are said to have C^0 conjugate geodesic flows if there is a continuous map $F : S^1N \to S^1N'$ such that $F \circ g_t = g'_t \circ F$ for all t, where g_t (respectively g'_t) represents the geodesic flow on the unit sphere bundle S^1N (respectively S^1N'). The conjugacy is called C^k if F is C^k.

Graph manifolds of non-positive curvature are compact three-dimensional manifolds whose geometry has been extensively studied in [**Sh**] and [**Le**]. Although there have been a number of definitions of 'graph manifold', we will use the rather general one used in [**Le**]: a graph manifold is a three-dimensional Haken manifold each of whose components under the minimal decomposition (in the sense of [**JS, Jo**]) is Seifert fibered. The decomposition is by cutting along embedded incompressible tori and Klein bottles. Not all such manifolds admit metrics of non-positive curvature (see [**Le**]), but this paper is only concerned with the non-positively curved Riemannian setting.

They are of interest for a number of reasons, but in particular they are the easiest non-trivial examples of non-positively curved manifolds that have Ballmann rank 1 but whose fundamental group is not hyperbolic. Here we show the following.

THEOREM 1.1. *If N and N' are compact non-positively curved graph manifolds whose geodesic flows are C^0 conjugate, then they must be isometric.*

The conjugacy rigidity question for compact Riemannian manifolds, 'When does a C^k conjugacy between the geodesic flows of two compact Riemannian manifolds N and N' imply that N and N' are isometric?', is a much studied but still largely open question.

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This question is closely related to other rigidity problems (such as the 'boundary rigidity' problem and, in negative curvature, the 'marked length spectrum' rigidity problem).

There are many examples where isometry does not hold (Zoll surfaces or more generally the examples in §6 of [**CK2**] for example), however none of these occur in the non-positive curvature setting. The only general conjugacy rigidity result known where there are no curvature assumptions made is that of [**CK2**], where one of the metrics is assumed to have a parallel vector field (hence, it is locally a Riemannian product where one factor is an interval). A natural setting for these questions is when at least one of the manifolds (say N) is assumed to have non-positive curvature. The two-dimensional case (when N has non-positive curvature) is completely taken care of by [**CFF**], which is based on [**O**] and [**Cr1**]. In higher dimensions, if N is a locally symmetric space of negative curvature (see [**BCG**]) or a flat manifold (see [**Cr2**]) and if the conjugacy is C^1 (or C^0 if $\mathrm{Vol}(N) = \mathrm{Vol}(N')$) then N' is isometric to N.

In this paper we consider the case where both manifolds N and N' have non-positive curvature and $n \geq 3$ (since the two-dimensional case was taken care of by [**CFF**]). In this setting, it is known that N' must be isometric to N if N has Ballmann rank ≥ 2 (see [**CEK**]).

In the above we restricted our attention only to rigidity theorems for a pair of compact manifolds without boundary. There are a number of results on compact manifolds with boundary or on finite-volume manifolds. Much work has also been done on the deformation question (i.e. can there be non-trivial differentiable one parameter families N_t with the 'same' geodesic flows; see, for example, [**GK**], [**CS**], [**DS**] and [**SU**]).

In a previous paper [**CK1**], motivated by a question of Gromov in [**Gr**], the present author along with Kleiner considered the space of geodesic rays on the universal cover of such non-positively curved graph manifolds (as well as other related spaces). The asymptotic properties of the geodesic rays are encoded in the geodesic boundary (equivalence classes of rays) and the induced action of the fundamental group on that boundary. We were able to give necessary and sufficient conditions on the metrics to guarantee that two such (necessarily homeomorphic) spaces have universal covers with equivalent boundaries (i.e. there is an equivariant, with respect to the action of the fundamental group, homeomorphism between the boundaries). In the process we saw that there were examples of non-isometric graph manifolds of non-positive curvature whose universal covers have equivalent boundaries. It thus became natural to ask if complete knowledge of the geodesic flow (i.e. knowledge up to C^0 conjugacy) is enough to completely determine the metric. (Such a conjugacy, of course, induces an equivalence between boundaries in a natural way.) Theorem 1.1 gives a positive answer to that.

Graph manifolds of non-positive curvature have a very restricted geometric form as studied in [**Sh**]. They are the union over totally geodesic boundaries of pieces (the 'geometric Seifert components') whose universal covers are isometric to a Riemannian product $X \times \mathbb{R}$, where X is a surface of non-positive curvature with geodesic boundary. The universal cover of the geometric Seifert components are minimal displacement sets (see §2 for a definition) of isometries. In fact, the minimal displacement set of an isometry of any simply connected space of non-positive curvature has the form $X \times \mathbb{R}$, where X is a convex subspace. They thus become amenable to some techniques developed in [**CK2**].

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In §§2 and 3 of the paper we thus consider C^0 geodesic conjugacies between arbitrary non-positively curved manifolds of dimension ≥ 3. In §3 we show that for corresponding minimal displacement sets the conjugacy induces affine maps between the corresponding spaces X (see Proposition 3.3).

In §4 we restrict our attention to graph manifolds and prove Theorem 1.1. Using the results in §3, it is straightforward to show that corresponding Seifert components are isometric. However, we note that it is easy to glue isometric Seifert components together so as to make non-isometric spaces. The main point of §4 is to show that the components must be glued together in an isometric manner.

2. *Notation and preliminaries*

A standard consequence of the flat strip theorem says that if ν is a geodesic in a simply connected non-positively curved manifold M, then the space $M_\nu = \{x \in M \mid x \text{ lies on a geodesic that stays a bounded distance from } \nu\}$ is a convex subset of M which is isometric to $X_\nu \times \mathbb{R}$, where X_ν is a simply connected manifold of non-positive curvature with convex boundary. The unit speed geodesics that stay a bounded distance from ν are precisely the 'vertical' curves $t \to (x_0, t + t_0) \in X_\nu \times \mathbb{R}$ for fixed (x_0, t_0). We let $\pi_\nu : M_\nu \to X_\nu$ (respectively $\mathbb{R}_\nu : M_\nu \to \mathbb{R}$) be the projection.

Let N be a Riemannian manifold of dimension ≥ 3 and of non-positive curvature and $\pi_N : M \to N$ be its universal cover. Since basepoint projection, $b : S^1N \to N$, induces an isomorphism between $\pi_1(S^1N)$ and $\pi_1(N) \equiv \Gamma$, we see that $\pi_{N*} : S^1M \to S^1N$ is also a universal cover. (For any tangent vector v to any space, we let $b(v)$ represent its base point.) After appropriate choices of base points, we get cocompact actions of Γ on M and S^1M with $M/\Gamma = N$ and $S^1M/\Gamma = S^1N$. For every $\gamma \in \Gamma$, we let $M_\gamma \subset M$ be the set of points of minimal γ-displacement, i.e. the set of m such that $d(m, \gamma(m))$ is a minimum. This is exactly the space M_ν, where ν is any axis of γ (i.e. $\gamma(\nu) = \nu$) and the axes of γ are precisely the vertical geodesics in M_γ. Thus M_γ is a convex subset of M which is isometric to $X_\gamma \times \mathbb{R}$, where γ acts by translation on the $\mathbb{R}$ factor and X_γ is a simply connected manifold of non-positive curvature with convex boundary. (We use $\pi_\gamma : M_\gamma \to X_\gamma$ and $\mathbb{R}_\gamma : M_\gamma \to \mathbb{R}$ to represent the projections.) The centralizer, $Z(\gamma) \subset \Gamma$, of γ in Γ will act on M_γ preserving the vertical directions.

Note that the geometric Seifert components of a non-positively curved graph manifold are compact subspaces of the form M_γ/Γ_γ, where Γ_γ is the subset of Γ that preserves M_γ. Thus $Z(\gamma) \subset \Gamma_\gamma$, but Γ_γ may also contain elements which reverse the vertical direction.

For every $m \in M_\gamma$, π_N takes the (vertical) geodesic segment $\overline{m\gamma(m)}$ to a closed geodesic in the free homotopy class of γ, and all such closed geodesics arise in this way.

3. *Geodesic conjugacies and the M_ν*

In this section we consider compact non-positively curved manifolds N and N' of dimension ≥ 3. We discuss the consequences that a C^0 conjugacy of the geodesic flows has on the spaces M_ν (and, in particular, on the M_γ).

The isomorphism $F_* : \pi_1(S^1N) \to \pi_1(S^1N')$ induces an isomorphism between $\pi_1(N)$ and $\pi_1(N')$. We let Γ represent the common abstract group. Thus (after choices of base

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points compatible with F for M, M', N and N') we can lift F to a uniformly continuous conjugacy $\tilde{F} : S^1M \to S^1M'$ and have cocompact actions of Γ on M and M' whose differentials commute with $\tilde{F}$. We let $D(F) < \infty$ be the maximum over $m \in M$ of the diameters of the sets $\tilde{F}(S^1_m M) \subset S^1M'$, where $S^1_m M \subset S^1M$ represents the unit vectors with basepoint m.

LEMMA 3.1. *For all $v, w \in S^1M$,*

$$-2D(F) \leq d_M(b(v), b(w)) - d_N(b(\tilde{F}(v)), b(\tilde{F}(w))) \leq 2D(F).$$

Proof. For $v, w \in SM$, let $\tau : [0, l] \to M$ be a minimizing geodesic from $b(v)$ to $b(w)$. Then both $d_N(b(\tilde{F}(v)), b(\tilde{F}(\tau'(0))))$ and $d_N(b(\tilde{F}(w)), b(\tilde{F}(\tau'(l))))$ are bounded by $D(F)$ and $d_N(b(\tilde{F}(\tau'(0))), b(\tilde{F}(\tau'(l)))) = d_N(b(\tilde{F}(\tau'(0))), b(g^l(\tilde{F}(\tau'(0))))) = l = d_M(b(v), b(w))$. Hence, the inequalities follow from the triangle inequality. □

For any geodesic ν of M, $\tilde{F}$ will take the unit tangent vector field of ν to the unit tangent field of a geodesic of M' which, by abuse of notation, we call $\tilde{F}(\nu)$.

For ν a geodesic in M and $m \in M_\nu$, we let $V(m)$ be the vertical direction at m (i.e. points in the positive $\mathbb{R}$ direction of $M_\nu = X_\nu \times \mathbb{R}$). An immediate consequence of the above lemma is that if a geodesic ν_1 stays a bounded distance from ν, then $\tilde{F}(\nu_1)$ stays a bounded distance from $\tilde{F}(\nu)$. Thus $\tilde{F}(V(m)) = V'(m')$ for some $m' \in M'_{\tilde{F}(\nu)}$. This allows us to define a map $G_\nu : M_\nu \to M'_{\tilde{F}(\nu)}$ with $\tilde{F}(V(m)) = V'(G_\nu(m))$ and then, since we have $G_\nu(\tau_t m) = \tau'_t(G_\nu(m))$ (where τ_t and τ'_t represent vertical translation in $M_\nu = X_\nu \times \mathbb{R}$ and M'_ν, respectively), we get a well-defined function $GX_\nu : X_\nu \to X'_{\tilde{F}(\nu)}$.

For $\gamma \in \Gamma$ the above applies to M_γ. Since F must take closed geodesics in N in the free homotopy class of γ to corresponding closed geodesics (of the same length) in N', we see that $\tilde{F}$ takes M_γ to M'_γ and gives rise to maps G_γ and GX_γ as above. This also tells us that the translation by γ on M_γ and M'_γ is by the same amount. We also note that since $\tilde{F}$ commutes with the differentials of $Z(\gamma)$ and since $Z(\gamma)$ preserves M_γ and vertical directions, we have that G_γ commutes with $Z(\gamma)$.

A curve c in $M_\nu = X_\nu \times \mathbb{R}$ is said to have $\mathbb{R}$ growth rate A if

$$\lim_{t\to\infty} \left(\frac{\mathbb{R}_\nu(c(t))}{t} \right) = A.$$

LEMMA 3.2. *If a curve c in M_ν has growth rate A, then $G_\nu(c)$ also has $\mathbb{R}$ growth rate A.*

Proof. Let B_ν be the Busemann function of ν, i.e. $B_\nu(x) = \lim_{t\to\infty}(t - d(x, \nu(t)))$. Now Lemma 3.1 (with both v and w vertical vectors) implies that $|B_\nu(x) - B_{\tilde{F}(\nu)}(G_\nu(x))| \leq 2D(F)$. On the other hand, $B_\nu(x) = \mathbb{R}_\nu(x) + \text{constant}$ and hence $\mathbb{R}$ growth rates are the same as Busemann growth rates (also in $M'_{\tilde{F}(\nu)}$) and the lemma follows. □

PROPOSITION 3.3. *For any geodesic ν in M, GX_ν takes each unit speed geodesic segment in X_ν to a constant speed geodesic segment in $X'_{\tilde{F}(\nu)}$. Further, if the segment is a subset of a ray in X_ν, then the image is unit speed.*

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Proof. Let $\sigma(t)$ for $0 \leq t \leq L$ be a unit speed geodesic segment in X_v and for each i let $\sigma_i(t) = (\sigma(t/\sqrt{1+i^2}), it/\sqrt{1+i^2}) \subset M_v$ for $0 \leq t \leq L\sqrt{1+i^2}$ be a unit speed geodesic in M_v. Now $F(\sigma_i'(t))$ will be the tangent vector field $\tau'(t)$ to a unit speed geodesic $\tau_i(t)$ in M'. Let $p_i \in M'_{\tilde{F}(v)}$ (respectively $q_i \in M'_{\tilde{F}(v)}$) be the unique closest point in $M'_{\tilde{F}(v)}$ to $\tau_i(0)$ (respectively $\tau_i(L\sqrt{1+i^2})$). The uniform continuity of F guarantees $d(F(\sigma_i'(t)), F(V(\sigma_i(t))))$ converges to zero (d computed in the tangent bundle) and hence that $\lim_{i\to\infty} d(p_i, \tau_i(0)) = 0$ and $\lim_{i\to\infty} d(q_i, \tau_i(L\sqrt{1+i^2})) = 0$. Hence, if $\bar{\tau}_t(t)$ for $0 \leq t \leq L\sqrt{1+i^2}$ is the constant speed geodesic from p_i to q_i, we see (by the non-positivity of the curvature) that for every t, $d(\tau_i'(t), \bar{\tau}_i'(t))$ and, hence, $d(\bar{\tau}_i'(t), F(V(\sigma_i(t))))$ converges to zero. In particular, for every t we have that $d(\pi'_{\tilde{F}(v)}(\bar{\tau}_i'(t)), GX_v(\sigma(t/\sqrt{1+i^2})))$ converges to zero. However, each $\pi'_{\tilde{F}(v)}(\bar{\tau}_i'(t))$ is a constant speed geodesic in $X'_{\tilde{F}(v)}$ and hence so is $GX_v(\sigma(t))$.

Now let $r; [0,\infty) \to X_v$ be a unit speed ray and assume that $GX_v(r)$ is a ray with constant speed a. We need to show that $a = 1$. Using the construction above for segments, we first define (by letting L go to infinity) for each i the unit speed ray $\tau_i(t)$ whose distance to $M'_{\tilde{F}(v)}$ goes uniformly (since $\tilde{F}$ is uniform) to zero as i goes to infinity and hence we can define (by a standard limiting argument) a ray $\bar{\tau}_i(t) \subset M'_{\tilde{F}(v)}$ such that as i goes to infinity both $d(\bar{\tau}_i'(t), \tau_i'(t))$ and $d(\pi'_{\tilde{F}(v)}(\bar{\tau}_i'(t)), GX_v(r(t/\sqrt{1+i^2})))$ converge uniformly to zero. Note that $d(\bar{\tau}_i(t), G_v(\sigma_i(t)))$ is uniformly bounded since both curves are at a uniformly bounded distance (depending on D) from $\tau_i(t)$ and, hence, by Lemma 3.2, $\bar{\tau}_i$ has the same $\mathbb{R}$ growth rate as σ_i (i.e. $i/\sqrt{1+i^2}$). However, since $GX_v(r)$ has speed a (and hence $GX_v(r(t/\sqrt{1+i^2}))$ has speed $a/\sqrt{1+i^2}$) and $\bar{\tau}_i$ has speed approaching one (the speed of τ_i), we see that $\bar{\tau}_i$ can have the same $\mathbb{R}$ growth as σ_i for all i if and only if $a = 1$. □

Maps GX_v that satisfy the conclusion of Proposition 3.3 are called affine maps. We now recall some of the standard properties of such maps (see, for example, ch. VI of [**KN**]). For $V \in S^1X_v$, let $s(V)$ represent the speed of the GX_v image of the geodesic determined by V. Since s is continuous, it is locally bounded above and hence GX_v is locally Lipschitz (in fact, globally in the cocompact case). Thus GX_v is differentiable almost everywhere and at every $x \in X_v$ where GX_v is differentiable, we see that $GX_v = \exp_{GX_v(x)} \circ D(GX_v)_x \circ \exp_x^{-1}$ (this is always true locally, but in our case $\exp_x^{-1}$ is globally defined). In particular, GX_v is C^∞ and $GX_v = \exp_{GX_v(x)} \circ D(GX_v)_x \circ \exp_x^{-1}$ holds for all x. This implies that GX_v preserves the respective connections and hence that s is invariant under holonomy. Since X_v is convex (with boundary) and simply connected, invariant subspaces of the holonomy action induce deRham splittings as usual. Thus we obtain the following.

COROLLARY 3.4. *For any geodesic v in M, GX_v is C^∞, takes deRham factors to deRham factors, acts homothetically on each non-Euclidean deRham factor (and hence is an isometry on each non-compact one) and acts as an affine transformation on the flat factor. In particular, if X_v is irreducible, not flat and non-compact (e.g. in the case of M_γ when $Z(\gamma)$ is not cyclic), then GX_v is an isometry.*

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Remark 3.5. It is certainly possible that in the above GX_v is always an isometry. In fact, it is highly likely if the conjugacy rigidity conjecture for non-positively curved manifolds is true since in this case X_v and $X'_{\tilde{F}(v)}$ would be isometric.

COROLLARY 3.6. *When GX_v is an isometry, $G_v : M_v \to M'_v$ is also an isometry.*

Proof. We know that $G_v : X_v \times \mathbb{R} \to X'_v \times \mathbb{R}$ is of the form $G_v(x, t) = (GX_v(x), t + f(x))$ for some real valued function $f : X_v \to \mathbb{R}$. We need only see that f is constant. Along a geodesic segment σ (using notations from the proof of Proposition 3.3), we see that $G_v(\sigma_i(t))$ gets uniformly close to the geodesic $\bar{\tau}_i(t)$ as i gets large. This shows that f must be linear along σ, but in fact it must be constant, since when i gets large, $\bar{\tau}_i(t)$ becomes unit speed. □

Remark 3.7. Consider the case that N is three dimensional and, for some $\gamma \in \Gamma$, X_γ is non-compact. Combining Proposition 3.3 and the corollaries we see that either X_γ is a flat strip or G_γ is an isometry that commutes with $Z(\gamma)$ and $M_\gamma / Z(\gamma)$ is isometric to $M'_\gamma / Z(\gamma)$.

Let $O \subset X_\gamma$ be a flat strip neighborhood of some boundary geodesic τ and let $\bar{O} \subset M_\gamma$ be $\pi_\gamma^{-1}(O)$, which has the metric of a flat plane cross an interval. Let $\alpha \in Z(\gamma)$, $\alpha \neq \gamma$ be such that $\pi_\gamma^{-1}(\tau) \subset M_\gamma \cap M_\alpha$, then the geometry of $\bar{O}$ forces $\bar{O} \subset M_\gamma \cap M_\alpha$.

4. *Graph manifolds*

We now consider the case where $N = M/\Gamma$ is a compact three-dimensional graph manifold with a non-positively curved Riemannian metric with fundamental group Γ and universal cover M.

By [**Sh**] and [**Le**], N has the following structure. There is a finite connected graph $\mathcal{G}$ with vertex set $\mathcal{V}$ and oriented edge set $\mathcal{E}$ (i.e. each edge shows up twice in $\mathcal{E}$). For each $v \in V$, there is a compact non-positively curved 3-manifold N_v with non-empty totally geodesic boundary (the geometric Seifert components of N) and Seifert fibrations $p_v : N_v \to X_v$, where the metric on N_v has local product structure compatible with the fibration p_v and the X_v are non-positively curved orbifolds; N is the union of the N_v. Each N_v is $M_{\gamma_v}/\Gamma_{\gamma_v}$ for some $\gamma_v \in \Gamma$. There is one boundary component T_e of N_v for each $e \in \mathcal{E}$ such that v is the initial vertex of e. (Note that a non-oriented edge from v to v will correspond to two boundary components of N_v.) The N_v are glued together as described by $\mathcal{G}$. That is, for each edge $e \in \mathcal{E}$, say from v_1 to v_2, we glue N_{v_1} to N_{v_2} along T_e and T_{-e}, respectively. (Note that, in the case where the metric on N_{v_1} near T_e is a product of an interval with T_e, then Remark 3.7 states that the picture looks the same on N_{v_2} near T_{-e} and the glueing identifies the whole product neighborhoods.) We will distinguish N_v of 'oriented type', where $\Gamma_{\gamma_v} = Z(\gamma_v)$, from the alternative 'non-oriented type'.

We note that these glueings are only unique up to translations in the vertical directions of N_v of oriented type. In fact different choices of glueings lead to non-isometric manifolds (we explain more later).

It is easier to explain what is going on in the universal covering space. The universal covering space is built by glueing copies of the M_{γ_v}. The glueing is described by a tree $\bar{\mathcal{G}}$ (the Basse–Serre tree of the fundamental group, see §2.5 of [**CK1**]) with vertex set $\bar{\mathcal{V}}$ and

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edge set $\bar{\mathcal{E}}$. Γ acts on $\bar{\mathcal{G}}$ with $\pi_{\mathcal{G}} : \bar{\mathcal{G}} \to \bar{\mathcal{G}}/\Gamma = \mathcal{G}$. We can define the vertex and edge spaces $M_{\bar{v}}$, $R^2_{\bar{e}}$ in the universal cover, where $M_{\bar{v}}$ is isometric to M_{γ_v} if $\pi_{\mathcal{G}}(\bar{v}) = v$. For every $s \in \mathbb{R}$ there is an isometry $\tau(\bar{v}, s) : M_{\bar{v}} \to M_{\bar{v}}$ which translates by s in the vertical direction. (The notation $\tau(\bar{v}, s)$ is somewhat unusual, rather than $\tau^s_{\bar{v}}$ for example, but it makes things easier to see later.) It is clear that $\tau(\bar{v}, s)$ commutes with elements of $Z(\gamma_v)$; however, in the non-oriented case (unless $s = 0$) it will not commute with those isometries that reverse the vertical orientation.

Now let N and N' be non-positively curved graph manifolds built from the same graph $\mathcal{G}$ and isometric pieces N_v and N'_v, and let I_v be a chosen isometry respecting the vertical directions. We can lift the I_v equivariantly to isometries $I_{\bar{v}} : M_{\bar{v}} \to M'_{\bar{v}}$. For each $e \in \bar{\mathcal{E}}$ from $\bar{v_1}$ to $\bar{v_2}$, we have that $I_{\bar{v_1}}$ and $I_{\bar{v_2}}$ define two isometries from $R^2_{\bar{e}}$ to $R^{2'}_{\bar{e}}$ which differ by a Euclidean translation. We can thus (after choosing upwards vertical directions in the N_v) uniquely define real numbers $s_{\bar{e}}$ and $r_{\bar{e}}$ such that on $R^2_{\bar{e}}$ we have

$$\tau(\bar{v_1}, s_{\bar{e}}) \circ I_{\bar{v_1}} = \tau(\bar{v_2}, r_{\bar{e}}) \circ I_{\bar{v_2}}.$$

We note that $r_{\bar{e}} = s_{-\bar{e}}$ and that in the oriented case the equivariance of the lifting states that if $\pi_{\mathcal{G}}(\bar{e_1}) = \pi_{\mathcal{G}}(\bar{e_2})$ then $s_{\bar{e_1}} = s_{\bar{e_2}}$ so we can define s_e for $e \in \mathcal{E}$. In the non-oriented case one cannot choose the upwards vertical direction equivariantly, so s_e is only defined up to sign and if $\bar{e_1}$ and $\bar{e_2}$ are edges from v and are related by an isometry that reverses the vertical direction of N_v, then $s_{\bar{e_1}} = -s_{\bar{e_2}}$ (independent of the choice of upwards).

We call the relative glueings *consistent* if for every $v \in \mathcal{G}$ there is a $s_v \in \mathbb{R}$ such that for every $e \in \mathcal{E}$ which has v as initial point we have $s_v = s_e$ and, further, if for N_v of non-oriented type, $s_v = 0$. If the glueings are consistent, then we define an equivariant isometry $I : M \to M'$ by $I(m) = \tau(\bar{v}, s_{\bar{v}}) \circ I_{\bar{v}}(m)$ for $m \in M_{\bar{v}}$. Thus we have the following.

LEMMA 4.1. *If N and N' are non-positively curved graph manifolds built from the same graph $\mathcal{G}$ and isometric pieces N_v and with consistent relative glueings, then N and N' are isometric.*

Remark 4.2. If the relative glueings are not consistent, then it is not hard to see there will be no isometry which takes N_v to N'_v. Thus we can make many non-isometric spaces with the same $\mathcal{G}$ and isometric pieces.

Two such non-isometric graph manifolds with the same $\mathcal{G}$ and isometric N_v all of orientable type but different glueings will have the same 'geometric data' in the sense of [**CK1**] and hence give a new class of examples of non-isometric graph manifolds whose geometric boundaries are equivariantly homeomorphic.

In order to make sure that in the non-oriented case our isometries $G_{\gamma_v} : M_{\gamma_v} \to M'_{\gamma_v}$ descend to N_v, we need to make sure that G_{γ_v} commutes with the full group Γ_{γ_v} not just the subgroup $Z(\gamma_v)$. In fact, this will not *a priori* be the case. However, we will see in the next lemma that for each such v there is a number a_v such that this holds if we replace F by the conjugacy $g'_{a_v} \circ F$. This will allow us to define a Γ invariant (though *a priori* not continuous) isometry from M to M'.

LEMMA 4.3. *For each v of non-oriented type there is an a_v such that if we define G_{γ_v} via the conjugacy $g'_{a_v} \circ F$, then G_{γ_v} commutes with the full group Γ_{γ_v}.*

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Proof. If instead of using the 'upward' vertical direction of M_{γ_v} we used the 'downward' vertical direction, we would arrive at isometries $\bar{G}_v : M_{\gamma_v} \to M'_{\gamma_v}$ and $\bar{G}X_v : X_v \to X'_v$. When X_v is not a flat strip, the geometric structure of X_v guarantees that the identity is the only self isometry of X_v that takes each boundary component to itself. Hence we see that $\bar{G}X_v = GX_v$ and thus that $\bar{G}_v$ differs from G_v by at most a fixed vertical translation. Thus there is a (unique) a_v such that for vertical vectors V we have $g'_{a_v} \circ F(-V) = -g'_{a_v} \circ F(v)$. This fact, along with the fact that the conjugacy $g'_{a_v} \circ F$ is Γ equivariant (here Γ acts on S^1M and S^1M' via its differentials), easily yields the result, since if $\nu \in \Gamma_{\gamma_v}$ reverses the orientation of the vertical then $G \circ \nu = \nu \circ \bar{G} = \nu \circ G$.

We are assuming that the Seifert components are not products with an interval and, hence, the only case left to consider is when $X_v = [0, L] \times \mathbb{R}$ is a flat strip and there is an element $\nu \in \Gamma_{\gamma_v}$ such that ν reverses the vertical direction in M_v and switches boundary components. Thus the action of ν on $M'_v = [0, L] \times \mathbb{R} \times \mathbb{R}$ (using the fact that there are no fixed points) is of the form $\nu(x, s, t) = (L - x, a + s, b - t)$ for some numbers $a \neq 0$ and b. A similar argument to the above yields an a_v such that using $g'_{a_v} \circ F$ instead of F, we have $\bar{G}X = T \circ GX$, where $T : [0, L] \times \mathbb{R} \to [0, L] \times \mathbb{R}$ is of the form $T(x, s) = (x, s_0 \pm s)$ and $\bar{G} = (T \times \mathrm{id}) \circ G$. We first note that the equivariance of the conjugacy F implies $G \circ \nu = \nu \circ \bar{G} = \nu \circ (T \times \mathrm{id}) \circ G$. Since both $\bar{G}$ and G are $Z(\gamma_v)$ equivariant we see that $T \times \mathrm{id}$ commutes with the $Z(\gamma_v)$ action on M'_v which forces T to be of the form $T(x, s) = (x, s_0 + s)$. Now since ν^2 preserves the orientation of the fibers we see that ν^2 commutes with G and hence $\nu^2 = \nu \circ (T \times \mathrm{id}) \circ \nu \circ (T \times \mathrm{id})$. Thus $(x, 2a + s, t) = (x, 2a + 2s_0 + s, t)$ and so $s_0 = 0$ and $T = \mathrm{id}$. Thus $\bar{G} = G$ and the result follows as in the other case. □

Proof of Theorem 1.1. To prove Theorem 1.1, we use Remark 3.7 and Lemma 4.3 to define our isometries $I_{\bar{v}} : M_{\bar{v}} \to M'_{\bar{v}}$ and assume (using Lemma 4.1) that N and N' do not have consistent glueings. We then find a closed geodesic $c \in N$ and a curve $\xi \in N'$ representing the same (as identified via the conjugacy F) free homotopy class such that $L(\xi) < L(c)$. This is the desired contradiction since any closed geodesic minimizes the lengths of curves freely homotopic to it and since ξ is freely homotopic to $F(c)$, hence $L(c) = L(F(c)) \leq L(\xi)$. We do this by choosing c so that it intersects appropriate boundary tori nearly tangentially and such that the discontinuous image of c when we reglue the Seifert components to get N' can be shortened to get a continuous curve in the appropriate homotopy class. We actually work in the universal covers.

We first consider the case where v is of oriented type, and we assume that there are edges $e_1, e_2 \in \mathcal{E}$ and a vertex v such that v is the final point of e_1 and the initial point of e_2 with $s_{-e_1} \neq s_{e_2}$. We let v_1 be the initial point of e_1 and v_2 the final point of e_2.

We first consider the case where N_v is of oriented type. The first step is to find an appropriate closed geodesic c in $N_{v_1} \cup N_v \cup N_{v_2}$ such that c intersects each of T_{e_1} and T_{e_2} in exactly two points and intersects no other T_e. To choose such a c, we first choose closed geodesics $a_i \subset N_{v_i}$ and $b \subset N_v$. We also choose minimizing geodesics τ_i from the base point of a_i to the base point of b. They are chosen so that b is in the homotopy class of γ_v and the classes corresponding to a_i (or more precisely $-\tau_i \cup a_i \cup \tau_i$) do not commute

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with γ_v. For each $n \in \mathbb{Z}$, we consider the curves

$$c_n = a_1 \cup \tau_1 \cup b^n \cup -\tau_2 \cup a_2 \cup \tau_2 \cup b^{-n} \cup -\tau_1.$$

We choose c to be the closed geodesic in the free homotopy class of some c_n (for some n large in absolute value). Since c can be obtained from c_n by curve shortening, the convexity of the Seifert components can be used to make sure that each curve in the shortening process, and hence the final curve c, satisfies the required conditions.

Thus, we can choose a unit speed parameter $c : [0, L] \to N$ so that there are $t_i \in R$, $0 = t_0 < t_1 < t_2 < t_3 < t_4 < t_5 = L$ with $c([0, t_1]) \subset N_{v_1}$, $c(t_1) \in T_{e_1}$, $c([t_1, t_2]) \subset N_v$, $c(t_2) \in T_{e_2}$, $c([t_2, t_3]) \subset N_{v_2}$, $c(t_3) \in T_{e_2}$, $c([t_3, t_4]) \subset N_v$, $c(t_4) \in T_{e_1}$ and $c([t_4, L]) \subset N_{v_1}$.

For such a c, we let $\bar{c}[0, l] \to M$ be a continuous lift of c. There are thus $\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{v}_4, \bar{v}_5 \in \bar{\mathcal{V}}$ and $\bar{e}_1, \bar{e}_2, \bar{e}_3, \bar{e}_4 \in \bar{\mathcal{E}}$ such that for

$$\begin{gathered} i = 1, 2, 3, 4, 5, \quad \bar{c}_i \equiv \bar{c}([t_{i-1}, t_i]) \subset M_{\bar{v}_i}, \\ i = 1, 2, 3, 4, \quad \bar{c}(t_i) \in \mathbb{R}^2_{\bar{e}_i}, \\ \pi_{\mathcal{G}}(\bar{v}_1) = \pi_{\mathcal{G}}(\bar{v}_5) = v_1, \\ \pi_{\mathcal{G}}(\bar{v}_2) = \pi_{\mathcal{G}}(\bar{v}_4) = v, \\ \pi_{\mathcal{G}}(\bar{v}_3) = v_2, \\ \pi_{\mathcal{G}}(\bar{e}_1) = -\pi_{\mathcal{G}}(\bar{e}_4) = e_1, \end{gathered}$$

and

$$\pi_{\mathcal{G}}(\bar{e}_2) = -\pi_{\mathcal{G}}(\bar{e}_3) = e_2.$$

We now determine which n to use. Let $\mathbb{R}_{\bar{v}_i} : M_{\bar{v}_i} \to \mathbb{R}$ represent the projection onto the vertical. Choose $|n|$ large enough so that $|\mathbb{R}_{\bar{v}_2}(\bar{c}(t_2)) - \mathbb{R}_{\bar{v}_2}(\bar{c}(t_1))| > |s_{\bar{e}_2} - s_{-\bar{e}_1}|$ (respectively $|\mathbb{R}_{\bar{v}_4}(\bar{c}(t_4)) - \mathbb{R}_{\bar{v}_4}(\bar{c}(t_3))| > |s_{\bar{e}_2} - s_{-\bar{e}_1}| = |s_{\bar{e}_4} - s_{-\bar{e}_3}|$). Choose the sign of n (hence the signs of $\mathbb{R}_{\bar{v}_2}(\bar{c}(t_2)) - \mathbb{R}_{\bar{v}_2}(\bar{c}(t_1))$ and $\mathbb{R}_{\bar{v}_4}(\bar{c}(t_4)) - \mathbb{R}_{\bar{v}_4}(\bar{c}(t_3))$ for large enough n) so that

$$|\mathbb{R}_{\bar{v}_2}(\bar{c}(t_2)) - \mathbb{R}_{\bar{v}_2}(\bar{c}(t_1)) + s_{\bar{e}_2} - s_{-\bar{e}_1}| < |\mathbb{R}_{\bar{v}_2}(\bar{c}(t_2)) - \mathbb{R}_{\bar{v}_2}(\bar{c}(t_1))|$$

and

$$|\mathbb{R}_{\bar{v}_4}(\bar{c}(t_4)) - \mathbb{R}_{\bar{v}_4}(\bar{c}(t_3)) + s_{\bar{e}_4} - s_{-\bar{e}_3}| < |\mathbb{R}_{\bar{v}_4}(\bar{c}(t_4)) - \mathbb{R}_{\bar{v}_4}(\bar{c}(t_3))|.$$

Having chosen c, we now proceed with the rest of the argument. We let $\gamma \in \Gamma$ be such that $\gamma(\bar{c}(0)) = \bar{c}(L)$ and hence, in particular,

$$\gamma\tau(\bar{v}_1, s_{\bar{e}_1})(\bar{c}(0)) = \tau(\bar{v}_5, s_{\bar{e}_1})(\bar{c}(L)) = \tau(\bar{v}_5, s_{-\bar{e}_4})(\bar{c}(L)).$$

Consider the following continuous curve, ξ in M' between $\tau(\bar{v}_1, s_{\bar{e}_1})(\bar{c}(0))$ and $\tau(\bar{v}_5, s_{-\bar{e}_4})(\bar{c}(L))$:

- for $t_0 \le t \le t_1$, $\xi(t) = \tau(\bar{v}_1, s_{\bar{e}_1})(\bar{c}_1(t))$;
- for $t_1 \le t \le t_2$, $\xi(t) = \tau(\bar{v}_2, (t_2 - t)/(t_2 - t_1)s_{-\bar{e}_1} + (t - t_1)/(t_2 - t_1)s_{\bar{e}_2})(\bar{c}_2(t))$;
- for $t_2 \le t \le t_3$, $\xi(t) = \tau(\bar{v}_3, s_{\bar{e}_3})(\bar{c}_3(t))$ (note that $s_{\bar{e}_3} = s_{-\bar{e}_2}$);
- for $t_3 \le t \le t_4$, $\xi(t) = \tau(\bar{v}_4, (t_2 - t)/(t_2 - t_1)s_{-\bar{e}_3} + (t - t_1)/(t_2 - t_1)s_{\bar{e}_4})(\bar{c}_4(t))$;
- for $t_4 \le t \le t_5$, $\xi(t) = \tau(\bar{v}_5, s_{-\bar{e}_4})(\bar{c}_5(t))$.

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We now only need to see that $L(\xi) < L = L(c)$. The five pieces ξ_i of ξ correspond to the pieces c_i of c. It is easy to see that $L(c_i) = L(\xi_i)$ for $i = 1, 3, 5$, since one comes from the other by an isometry composed with a vertical shift (also an isometry). For $i = 2$, we note that $\pi_{X_{v_2}}(\bar{c}_2)$ is the same as $\pi_{X_{v_2}}(\xi_2)$ and hence $L(\xi_2) < L(c_2)$ since

$$\begin{aligned}|\mathbb{R}_{\bar{v}_2}(\xi_2(t_2)) - \mathbb{R}_{\bar{v}_2}(\xi_2(t_1))| &= |\mathbb{R}_{\bar{v}_2}(\bar{c}_2(t_2)) - \mathbb{R}_{\bar{v}_2}(\bar{c}_2(t_1)) + s_{\bar{e}_2} - s_{-\bar{e}_1}| \\ &< |\mathbb{R}_{\bar{v}_2}(\bar{c}_2(t_2)) - \mathbb{R}_{\bar{v}_2}(\bar{c}_2(t_1))|.\end{aligned}$$

The case $i = 4$ is similar.

We need now consider the case where N_v is of non-oriented type and e is an edge with initial vertex v. This case is similar to, but easier than, the previous case. We can choose our closed geodesic c to lie in only two Seifert components of N crossing the boundary corresponding to e exactly twice and nearly vertical (in either the up or down sense). Since N_v is of non-oriented type, we can choose c so that in the universal cover, the corresponding edges $\bar{e_1}$ and $\bar{e_1}$ are related by an isometry that reverses the vertical direction and hence $s_{\bar{e_1}} = -s_{\bar{e_2}}$. An argument similar to above then forces $s_e = 0$.

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