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Ruelle operator with weakly contractive iterated function systems

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Abstract. The Ruelle operator has been studied extensively both in dynamical systems and iterated function systems (IFSs). Given a weakly contractive IFS $(X, \{w_j\}_{j=1}^m)$ and an associated family of positive continuous potential functions $\{p_j\}_{j=1}^m$, a triple system $(X, \{w_j\}_{j=1}^m, \{p_j\}_{j=1}^m)$ is set up. In this paper we study Ruelle operators associated with the triple systems. The paper presents an easily verified condition. Under this condition, the Ruelle operator theorem holds provided that the potential functions are Dini continuous. Under the same condition, the Ruelle operator is quasi-compact, and the iterations sequence of the Ruelle operator converges with a specific geometric rate, if the potential functions are Lipschitz continuous.

1. *Introduction*

This paper is a follow-up and an improvement of paper [**14**]. We present here the development history of the Ruelle operator for the sake of completeness, especially for the convenience of new readers. The notations that appeared in [**14**] are adopted here as well. It seems that the style of this paper is similar to that of [**14**]. Ruelle introduced a convergence theorem to study the equilibrium state of an infinite one-dimensional lattice gas in his famous paper [**26**]. Bowen [**4**] further set up the theorem as the convergence of powers of a Ruelle operator on the space of continuous functions on a symbolic space. More precisely, let

$$\Sigma = \{1, \ldots, N\}^{\mathbb{N}} = \{\omega = i_0 i_1 \cdots i_{n-1} \cdots \mid i_{n-1} \in \{1, \ldots, N\}, n = 1, 2, \ldots\}$$

be the one-sided symbolic space and let

$$\sigma : \omega = i_0 i_1 \cdots i_{n-1} \cdots \to \sigma(\omega) = i_1 \cdots i_{n-1} \cdots$$

be the left shift of Σ. Then (Σ, σ) is called a symbolic system. Let ϕ be a Hölder continuous function on Σ (a potential). Let $C(\Sigma)$ be the space of all continuous functions on Σ. The *Ruelle operator* $\mathcal{T} : C(\Sigma) \to C(\Sigma)$ is defined as

$$\mathcal{T} f(x) = \sum_{y \in \sigma^{-1}(x)} e^{\phi(y)} f(y), \quad f \in C(\Sigma).$$

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Let ϱ be the *spectral radius* of the operator $\mathcal{T}$. It is known that ϱ is the unique positive simple maximal eigenvalue of $\mathcal{T}$ acting on the space of all Hölder continuous functions on Σ (see, for example, [**13**]). It was then proved that $\mathcal{T}$ has a unique positive eigenfunction $h \in C(\Sigma)$ and a unique probability eigenmeasure $\mu \in C^*(\Sigma)$ corresponding to the eigenvalue $\varrho > 0$ (see, for example, [**4**]). Moreover, for any $f \in C(\Sigma)$, $\varrho^{-n}\mathcal{T}^n(f)$ converges uniformly to a constant multiple of h. This is called the Ruelle operator theorem. In this theorem, $\sigma : \Sigma \to \Sigma$ is an expanding dynamical system. More general results about the Ruelle operator theorem for symbolic systems, expanding dynamical systems and contractive iterated function systems (IFSs) have also been obtained. We give a partial list in the literature [**6–9, 15, 16, 23, 29–31**].

Recently a parabolic system has aroused the attention of many people who are interested in the Ruelle operator theorem (refer to [**2, 19, 20, 25, 28, 32–34**]). However, in this case, it is known that the bounded eigenfunction of the spectral radius ϱ of T may not exist [**17**], and even if the eigenfunction exists, ϱ may not be an isolated point of the spectrum [**3**]. So far the results known are far from satisfactory, and a study of such a system remains a challenging problem. Lau and Ye studied the Ruelle operator theorem for a non-expansive system in a recent paper [**18**]. One requirement in [**18**] is that one of the iterations of the IFS must be strictly contractive. It is important to remove this requirement because many examples of IFSs will not satisfy this requirement. Jiang and the author [**14**] succeeded in removing the above mentioned requirement. However, the condition presented in [**14**] is a little difficult to check. Meanwhile, under the probability condition $\sum_{j=1}^{m} p_j(x) = 1$ and *average contractive* condition, Barnsley *et al* [**3**] showed that the Ruelle operator theorem holds. It is necessary to remove this probability condition, because in many place-dependent cases the sums of potential functions are not probability. We also note that, under the assumption that all potential functions are Lipschitz continuous, Hennion [**11**] discussed the quasi-compactness of the operator. In this paper, we continue to study the Ruelle operator associated with a weakly contractive IFS. This paper aims to extend the results of papers [**3, 11, 14**]. It will present an easily checked condition for the Ruelle operator theorem to hold. This improves the result of [**14**]. Furthermore, this paper will try to extend the result of [**11**]. Precisely, we will consider the quasi-compactness of the Ruelle operator and study the convergence speeds of the iterations sequence of the Ruelle operator, when the potential functions are Lipschitz continuous.

Let X be a non-empty convex compact subset of $\mathbb{R}^d$, let $\{w_j\}_{j=1}^m$ be a set of maps from X into X. Our IFS $\{w_j\}_{j=1}^m$ in this paper is *weakly contractive* as defined by

$$\alpha_{w_j}(t) := \sup_{|x-y|\le t} |w_j(x) - w_j(y)| < t \quad \text{for all } t > 0,\ 1 \le j \le m.$$

Throughout the paper, we always assume that the IFS $\{w_j\}_{j=1}^m$ is weakly contractive on a convex compact set $X(\subset \mathbb{R}^d)$. Under this (weak contractivity) assumption, there exists a non-empty compact set K such that $K = \bigcup_{j=1}^m w_j(K)$ (see [**10**]). We call K the invariant set of the IFS $(X, \{w_j\}_{j=1}^m)$.

With each w_j, we associate a positive continuous function p_j as a weight function (or potential function). We can set up the Ruelle operator as follows on the space $C(K)$ of real

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continuous functions on K:

$$T(f)(x) = \sum_{j=1}^{m} p_j(x) f(w_j(x)), \quad f \in C(K).$$

Let ϱ still be the spectral radius of the operator $T : C(K) \to C(K)$.

Jiang asked, in a private communication, if the Ruelle operator theorem holds for a system $(X, \{w_j\}_{j=1}^m, \{p_j\}_{j=1}^m)$ satisfying the condition

$$\sum_{j=1}^{m} p_j(x) \cdot |w_j'(x)| < \varrho \quad \text{for all } x \in X. \tag{1.1}$$

It seems that condition (1.1) is more natural than the condition given in [**14**]. To answer the above question, we present the following definition.

Definition 1.1. Let X be a non-empty convex compact subset of $\mathbb{R}^d$. We call $(X, \{w_j\}_{j=1}^m, \{p_j\}_{j=1}^m)$ a Dini (or Lipschitz) system, if all maps $w_j : X \to X$, $1 \le j \le m$, are both continuously differentiable and weakly contractive and all potentials $p_j : X \to \mathbb{R}^+$, $1 \le j \le m$, are positive Dini (or Lipschitz) continuous.

Throughout the paper, we always assume the maps $w_j : X \to X$, $1 \le j \le m$, are both weakly contractive and continuously differentiable. The assumption of continuous differentiability is reasonable, because many examples both in fractal geometry and smooth dynamical systems are continuously differentiable. We will present a positive answer to the above question. Our main results in this paper are the following Theorems 1.2 and 1.4.

THEOREM 1.2. *Let $(X, \{w_j\}_{j=1}^m, \{p_j\}_{j=1}^m)$ be a Dini system with $w_j'(x) \neq 0$ for all j. Suppose that condition (1.1) is satisfied. Then the Ruelle operator theorem holds for this Dini system.*

We will prove a more general result (Theorem 4.3) in §4. Actually, the above theorem is a special case of this more general result. This theorem generalizes partially the results of papers [**9, 14**] (see Corollary 4.4 and Remark 4.7). As a by-product, this theorem enables us to remove the probability requirement in [**3**] (see Corollary 4.5 and Remark 4.6). We would like to point out that the condition of Theorem 1.2 is easily verified (see Example 4.8). Moreover, its proof is simpler than the proof of the result in [**14**]. It is an important improvement.

In practice, it is difficult to calculate the spectral radius ϱ of T. However, a simple but useful lower bound of ϱ is

$$\min_{x \in X} \sum_{j=1}^{m} p_j(x) \le \varrho.$$

By using this, we have a simply checkable sufficient condition.

COROLLARY 1.3. *Let $(X, \{w_j\}_{j=1}^m, \{p_j\}_{j=1}^m)$ be a Dini system with $w_j'(x) \neq 0$ for all j. If*

$$\max_{x \in X} \sum_{j=1}^{m} p_j(x) \cdot |w_j'(x)| < \min_{x \in X} \sum_{j=1}^{m} p_j(x),$$

then the Ruelle operator theorem holds for this Dini system.

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We would like to point out that a general formula for the *essential spectral radius* $\varrho_e(T)$ for a general C^α IFS or Zygmund IFS can be found in [**2**]. Using this formula, one can check whether the essential spectral radius $\varrho_e(T)$ is strictly less than the spectral radius $\varrho(T)$, and then check the Ruelle operator theorem (see e.g. Proposition 3.4). However, these methods do not work for the Dini system (see Definition 1.1). The reason is that, in this case, $\varrho(T)$ is not an isolated point of the spectrum, and $\varrho(T) = \varrho_e(T)$ [**3, 22, 24, 27**]. Therefore, Theorem 1.2 provides a new method to check the Ruelle operator theorem for some Dini system.

For Lipschitz systems, we have the following Theorem 1.4.

THEOREM 1.4. *Let* $(X, \{w_j\}_{j=1}^m, \{p_j\}_{j=1}^m)$ *be a Lipschitz system. Suppose that condition (1.1) is satisfied. Then* $\varrho_e(T) < \varrho(T)$. *Moreover, for any Lipschitz continuous function* f *defined on* K, *the sequence* $\varrho^{-n}T^n f$ *converges with a specific geometric rate.*

We will prove Theorem 1.4 in §5 (see Theorems 5.2 and 5.3). Theorem 1.4 is a generalization of paper [**11**]. In fact, condition (1.1) is weaker than Hennion's condition [**11**] (see Remark 5.9). We would like to note that a probability measure μ on K is called *invariant measure* of T if $T^*\mu = \varrho\mu$ [**3, 9, 18**]. Since the IFS $(X, \{w_j\}_{j=1}^m)$ considered is weakly contractive, the system $(X, \{w_j\}_{j=1}^m, \{p_j\}_{j=1}^m)$ may not have the bounded distortion property [**32**]. This brings about difficulty in our study of the invariant measures. Having set up Theorems 1.2 and 1.4, we conclude that the invariant measure is unique. The above theorems can provide us some interesting information on the system, including the weak Gibbs property of the measure μ. It is helpful in our study of the multifractal of μ [**9, 34**].

For a dynamical system $\sigma : K \to K$ defined on a compact Hausdorff metric space K satisfying a certain Markov property, more precisely, $K = \bigcup_{j=1}^m K_j$ is the union of finitely many pairwise disjoint compact subsets $\{K_j\}_{j=1}^m$ such that each $\sigma : K_j \to K$ is a homeomorphism, one can let w_j be the inverse of $\sigma : K_j \to K$ for each $1 \le j \le m$ and define $(K, \{w_j\}_{j=1}^m)$. It can be regarded as an IFS as well. We would like to note that our results in this paper are true for such a weakly contractive IFS $(K, \{w_j\}_{j=1}^m)$. Hence, the dynamical system (K, σ) admits a unique equilibrium state. This allows us to study the ergodic property of the dynamical system (K, σ) [**1**]. Furthermore, by making use of Theorems 1.2 and 1.4, we can study the rate of decay of its correlations as in paper [**32**].

The paper is organized as follows. In §2, we will present some elementary facts about the Ruelle operator and prove Proposition 2.1. In §3, we will introduce the quasi-compactness and the Ruelle operator theorem (or Perron–Frobenius property). We also set up the basic criteria for the assertion of the Ruelle operator theorem. In §4, when the potential functions are Dini continuous, we show the first main result of the paper. In §5, we present our second main result. Under the assumption that all potential functions are Lipschitz continuous, we discuss the quasi-compactness of the Ruelle operator and study the convergence speeds of the iterations sequence of the Ruelle operator.

2. *Preliminaries*

Consider the system

$$(X, \{w_j\}_{j=1}^m, \{p_j\}_{j=1}^m),$$

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where X is a non-empty convex compact subset of $\mathbb{R}^d$, $w_j : X \to X$, $1 \le j \le m$, are continuous maps and the $p_j(x)$, $1 \le j \le m$, are positive functions on X (they are called weights or potentials associated with w_j).

A map $w : X \to X$ is called *continuously differentiable* if $w'(x)$ is continuous on X.

We say that a map $w : X \to X$ is *contractive* if there exists a constant $0 < R < 1$ such that

$$|w(x) - w(y)| \le R|x - y| \quad \text{for all } x, y \in X$$

and *weakly contractive* if

$$\alpha_w(t) := \sup_{|x-y|\le t} |w(x) - w(y)| < t \quad \text{for all } t > 0.$$

It is clear that contractivity implies weak contractivity. A simple non-trivial example of a weakly contractive map is $w(x) = x/(1 + x)$ on $[0, 1]$. We call $(X, \{w_j\}_{j=1}^m)$ a weakly contractive IFS if all w_j, $1 \le j \le m$, are weakly contractive on X.

Throughout the paper, we always assume that all w_j, $1 \le j \le m$, are both continuously differentiable and weakly contractive on X. For any $n \in \mathbb{N}$, let

$$\mathcal{I}^n = \{J = (j_1 j_2 \cdots j_n) : 1 \le j_i \le m\}.$$

For any $J \in \mathcal{I}^n$, we have $|J| = n$. In the following, we often use $|J| = n$ to replace $J \in \mathcal{I}^n$ for simplicity. For any $J = (j_1 j_2 \cdots j_n) \in \mathcal{I}^n$, let

$$w_J(x) = w_{j_1} \circ w_{j_2} \circ \cdots \circ w_{j_n}(x).$$

Hata studied the invariant sets of the weakly contractive IFS on compact subset of $\mathbb{R}^d$ in [**10**]. By using the existence of fixed points for the weakly contractive maps, he showed the existence of a unique non-empty compact $K \subseteq X$ invariant under IFS $\{w_j\}_{j=1}^m$, i.e., $K = \bigcup_{j=1}^m w_j(K)$. Furthermore, from the weak contractivity of the IFS $(X, \{w_j\}_{j=1}^m)$, it follows that

$$\lim_{n\to\infty} \left(\max_{|J|=n} |w_J(X)| \right) = 0.$$

Then we can confirm that, for any $x \in K$,

$$K = \overline{\bigcup_{n=1}^{\infty} \bigcup_{|J|=n} w_J(x)}. \tag{2.1}$$

In the following, we always use K to denote the invariant set of the IFS $\{w_j\}_{j=1}^m$ considered.

The set K is fractal in general, and we cannot apply the mean value theorem on K. However, we have the following Proposition 2.1.

PROPOSITION 2.1. *Let each w_j $(1 \le j \le m)$ be both continuously differentiable and weakly contractive on X. If $\max_{x\in K} |w'_J(x)| < 1$ for some $J \in \bigcup_{n=1}^{\infty} \mathcal{I}^n$, then the map w_J is contractive on K.*

This proposition will play an important role in this paper (see the proofs of Corollary 3.7 and Theorem 3.8). To prove it, we need some preparation. For any $x \in \mathbb{R}^d$ and $\delta > 0$, we let

$$B_\delta(x) = \{y \in \mathbb{R}^d : |y - x| < \delta\},$$

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and

$$\overline{B_\delta(x)} = \{y \in \mathbb{R}^d : |y - x| \le \delta\}.$$

For any $x, y \in \mathbb{R}^d$ with $x \neq y$, we denote, for the sake of simplicity, by $[x, y]$ the line segment joining x and y, i.e.,

$$[x, y] = \{\lambda x + (1 - \lambda)y \in \mathbb{R}^d : 0 \le \lambda \le 1\}.$$

We call $[x, y]$ an interval. For any $\delta > 0$, we define

$$K_\delta = \{x \in X : |x - y| < \delta \text{ for some } y \in K\}.$$

LEMMA 2.2. *Let the IFS $(X, \{w_j\}_{j=1}^m)$ and the map w_J be as in Proposition 2.1. Then there exist $\delta > 0$ and $0 < r < 1$ such that*

$$|w_J(x) - w_J(y)| \le r|x - y| \quad \textit{for all } x, y \in K_\delta \textit{ with } |x - y| < \delta.$$

Proof. Let $r_0 = \max_{x \in K} |w'_J(x)|$. Choose $r_0 < r < 1$. From the continuity of $w'_j(\cdot)$ on the compact set X, we deduce that $|w'_J(\cdot)|$ is equicontinuous on X. From this, it follows that there exists a $\delta > 0$ such that, for any $x \in K$,

$$|w'_J(y)| < r \quad \text{for all } y \in X \cap B_{2\delta}(x).$$

For any $x, y \in K_\delta$ with $0 < |x - y| < \delta$, we can find a $z \in K$ such that $|x - z| < \delta$. From this, we get the implication that $x, y \in B_{2\delta}(z)$. From the convexity of the set X, it follows that there exists a $\xi \in [x, y]$ such that

$$\frac{|w_J(x) - w_J(y)|}{|x - y|} = |w'_J(\xi)|.$$

Note that $\xi \in B_{2\delta}(z)$. It follows that $|w'_J(\xi)| < r$. Hence, we conclude that

$$|w_J(x) - w_J(y)| \le r|x - y|. \qquad \square$$

Proof of Proposition 2.1. Let K be the invariant set of the IFS $(X, \{w_j\}_{j=1}^m)$, and let r and δ be as in Lemma 2.2. Let $d = |X|$, and let

$$R_0 = 1 - \frac{(1 - r)\delta}{d}.$$

Let $R = \max\{r, R_0\}$. Then $0 < R < 1$. We will prove that

$$|w_J(x) - w_J(y)| \le R|x - y| \quad \text{for all } x, y \in K.$$

Let $x, y \in K$ with $x \neq y$. From the convexity of the set X, we see that $[x, y] \subseteq X$. If $[x, y] \subseteq K_\delta$, then, by Lemma 2.2, we can deduce that

$$|w_J(x) - w_J(y)| \le r|x - y| \le R|x - y|.$$

If $[x, y] \not\subseteq K_\delta$, then $|x - y| \ge 2\delta$. We define first

$$[a_1, b_1] = [x, y] \cap \overline{B_{2^{-1}\delta}(x)} \quad \text{and} \quad [a_2, b_2] = [x, y] \cap \overline{B_{2^{-1}\delta}(y)}.$$

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It follows that $(a_1, b_1) \cap (a_2, b_2) = \emptyset$,

$$|b_1 - a_1| = |b_2 - a_2| = 2^{-1}\delta, \tag{2.2}$$

and

$$[a_i, b_i] \subset K_\delta, \quad i = 1, 2.$$

Along this way, we can divide the interval $[x, y]$ into two families of subintervals $[a_i, b_i]$, $1 \le i \le k$, and $[c_j, d_j]$, $1 \le j \le \ell$, such that

$$[a_i, b_i] \subset K_\delta \quad \text{and} \quad [c_j, d_j] \not\subseteq K_\delta.$$

Precisely, there exist two families of intervals $\{[a_i, b_i]\}_{i=1}^{k}$ $(k \ge 2)$ and $\{[c_j, d_j]\}_{j=1}^{\ell}$ $(\ell \ge 1)$ satisfying the following three conditions:

(i) $[x, y] = (\bigcup_{i=1}^{k}[a_i, b_i]) \cup (\bigcup_{j=1}^{\ell}[c_j, d_j])$;

(ii) $[a_i, b_i] \subset K_\delta$ for all i, and $[c_j, d_j] \not\subseteq K_\delta$ for all j;

(iii) $(a_i, b_i) \cap (a_{i'}, b_{i'}) = \emptyset$ for all $i \ne i'$, $(c_j, d_j) \cap (c_{j'}, d_{j'}) = \emptyset$ for all $j \ne j'$, and $(a_i, b_i) \cap (c_j, d_j) = \emptyset$ for all i, j.

Let $b = |x - y|$, $q = \sum_{i=1}^{k} |b_i - a_i|$ and $t = \sum_{j=1}^{\ell} |d_j - c_j|$. Then we can confirm that $b \le d$ and $q + t = b$. From the hypothesis that $[x, y] \not\subseteq K_\delta$, it follows that $0 < q, t < b$.

From (2.2), we conclude that

$$q = \sum_{i=1}^{k} |b_i - a_i| \ge \sum_{i=1}^{2} |b_i - a_i| = \delta.$$

By making use of Lemma 2.2, we can deduce

$$|w_J(b_i) - w_J(a_i)| \le r|b_i - a_i| \quad \text{for all } 1 \le i \le k.$$

From the weak contractivity of the IFS $(X, \{w_j\}_{j=1}^{m})$, we get the implication that

$$|w_J(d_j) - w_J(c_j)| \le |d_j - c_j| \quad \text{for all } 1 \le j \le \ell.$$

Hence, it follows that

$$\begin{aligned} |w_J(x) - w_J(y)| &\le \sum_{i=1}^{k} |w_J(b_i) - w_J(a_i)| + \sum_{j=1}^{\ell} |w_J(d_j) - w_J(c_j)| \\ &\le r \sum_{i=1}^{k} |b_i - a_i| + \sum_{j=1}^{\ell} |d_j - c_j| = rq + t = rq + (b - q) \\ &= \left(1 - \frac{(1-r)q}{b}\right) b \le \left(1 - \frac{(1-r)\delta}{d}\right) b = R_0|x - y|. \end{aligned}$$

This implies that

$$|w_J(x) - w_J(y)| \le R|x - y|.$$

Hence, we finish proof. □

In the following, we assume, without loss of generality, that

$$|X| = \sup\{|x - y| : x, y \in X\} = 1.$$

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Let p be a function defined on X. Define

$$\alpha_p(t) = \sup_{|x-y|\le t} |p(x) - p(y)|.$$

The function p is called *Lipschitz continuous* if there exists a constant $C > 0$ such that $\alpha_p(t) \le Ct$ for any $t > 0$, i.e.,

$$|p(x) - p(y)| \le C|x - y| \quad \text{for all } x, y \in X.$$

The function p is called *Dini continuous* if

$$\int_0^1 \frac{\alpha_p(t)}{t}\, dt < \infty.$$

Definition 2.3. Let w_j, $1 \le j \le m$, be both continuously differentiable and weakly contractive on X, and let p_j, $1 \le j \le m$, be positive continuous on X. We call

$$(X, \{w_j\}_{j=1}^m, \{p_j\}_{j=1}^m)$$

a Dini (or Lipschitz) system, if each $p_j (1 \le j \le m)$ is Dini (or Lipschitz) continuous on X.

It is obvious that a Lipschitz system must be a Dini system. Let $C(K)$ be the space of all real continuous functions on K. For a Dini system $(K, \{w_j\}_{j=1}^m, \{p_j\}_{j=1}^m)$, we define an operator $T : C(K) \to C(K)$ by

$$Tf(x) = \sum_{j=1}^m p_j(x) f(w_j(x)). \tag{2.3}$$

We call T the *Ruelle operator* associated with the (triple) system

$$(K, \{w_j\}_{j=1}^m, \{p_j\}_{j=1}^m).$$

The dual operator T^* on the measure space $M(K)$ is given by

$$T^*\mu(E) = \sum_{j=1}^m \int_{w_j^{-1}(E)} p_j(x)\, d\mu(x) \quad \text{for any Borel set } E \subseteq K$$

(see e.g. [**3**]).

For $J = (j_1 j_2 \cdots j_n)$, $1 \le j_i \le m$, define

$$p_{wJ}(x) = p_{j_1}(w_{j_2} \circ w_{j_3} \circ \cdots \circ w_{j_n}(x)) \cdots p_{j_{n-1}}(w_{j_n}(x)) p_{j_n}(x).$$

Then

$$T^n f(x) = \sum_{|J|=n} p_{wJ}(x) f(w_J(x)).$$

Let $\varrho = \varrho(T)$ be the spectral radius of T. Since T is a positive operator, we have that $\|T^n 1\| = \|T^n\|$ and

$$\varrho = \lim_n \|T^n\|^{1/n} = \lim_n \|T^n 1\|^{1/n}.$$

PROPOSITION 2.4. [**14**] *Let $(X, \{w_j\}_{j=1}^m, \{p_j\}_{j=1}^m)$ be a Dini system, and let T be the Ruelle operator on $C(K)$ defined by (2.3). Then we have the following.*

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(i) $\min_{x\in K} \varrho^{-n}T^n 1(x) \le 1 \le \max_{x\in K} \varrho^{-n}T^n 1(x)$ *for all* $n > 0$.

(ii) *If there exist* $\lambda > 0$ *and* $0 < h \in C(K)$ *such that* $Th = \lambda h$, *then* $\lambda = \varrho$ *and there exist* $A, B > 0$ *such that, for any* $x \in K$,

$$A \le \varrho^{-n}T^n 1(x) \le B \quad \text{for all } n > 0.$$

Proof. The proposition is [**14**, Proposition 2.3], and we omit the proof. □

Let Y be a non-empty compact subset of $\mathbb{R}^d$. An operator $\mathcal{L} : C(Y) \to C(Y)$ is called *irreducible* [**18**], if for any non-trivial, non-negative $f \in C(Y)$ and for any $x \in Y$ there exists an integer $n > 0$ such that $\mathcal{L}^n f(x) > 0$.

PROPOSITION 2.5. *Let* $(X, \{w_j\}_{j=1}^m, \{p_j\}_{j=1}^m)$ *be a Dini system. Then the Ruelle operator* T *is irreducible, and if* $h \ge 0$ *is a non-trivial* ϱ*-eigenfunction of* T *then* $h > 0$.

Proof. The proof can be found in [**18**], and we omit it. □

3. *Quasi-compactness and Perron–Frobenius property*

Let $(\mathcal{B}, \|\cdot\|)$ be a complex Banach space, and let $\mathcal{L} : (\mathcal{B}, \|\cdot\|) \to (\mathcal{B}, \|\cdot\|)$ be a bounded linear operator. We denote by $\varrho(\mathcal{L})$ the spectral radius of $\mathcal{L}$, and we use $\varrho_e = \varrho_e(\mathcal{L})$ to denote the essential spectral radius of $\mathcal{L}$ (see e.g. [**1**]). Note that (see e.g. [**21**], [**1**] or [**2**])

$$\varrho_e(\mathcal{L}) = \lim_{n\to\infty} (\inf\{\|\mathcal{L}^n - Q\| : Q \text{ is compact on } \mathcal{B}\})^{1/n}.$$

Definition 3.1. The operator $\mathcal{L}$ is called quasi-compact, if there exist $0 \le r < \varrho(\mathcal{L})$ and two closed subspaces E and F of $(\mathcal{B}, \|\cdot\|)$ satisfying the following three conditions:

(i) $\dim F < \infty$ and $\mathcal{B} = E \oplus F$;

(ii) $\mathcal{L}E \subseteq E$ and $\mathcal{L}F \subseteq F$;

(iii) $\varrho(\mathcal{L}|_E) \le r < \varrho = \varrho(\mathcal{L}|_F)$.

It is important to note that the quasi-compactness of the operator $\mathcal{L}$ is equivalent to $\varrho_e(\mathcal{L}) < \varrho(\mathcal{L})$ (see e.g. [**1, 11**]).

Assume that there is another norm $|\cdot|$ on $\mathcal{B}$ such that operator $\mathcal{L} : (\mathcal{B}, \|\cdot\|) \to (\mathcal{B}, |\cdot|)$ is compact. Hennion [**11**] gives an elegant short proof of the following Proposition 3.2.

PROPOSITION 3.2. *Let the operator* $\mathcal{L} : (\mathcal{B}, \|\cdot\|) \to (\mathcal{B}, |\cdot|)$ *be compact. If there are sequences* $\{r_n\}$ *and* $\{\rho_n\}$ *of positive numbers, with*

$$r := \lim_{n\to\infty} \inf(r_n)^{1/n} < \varrho,$$

and, for all $n \in \mathbb{N}$,

$$\|\mathcal{L}^n f\| \le r_n \|f\| + \rho_n |f| \quad \text{for all } f \in \mathcal{B},$$

then the operator $\mathcal{L}$ *is quasi-compact, i.e.,* $\varrho_e < \varrho$.

Let each $p_j (1 \le j \le m)$ be positive Lipschitz continuous on X. We use $L(X)$ to denote the set of all real Lipschitz continuous defined on X. Let $\hat{L}(X) = L(X) + iL(X)$. Then we can define the Ruelle operator $P : \hat{L}(X) \to \hat{L}(X)$ by

$$P(f)(x) = \sum_{j=1}^{m} p_j(x) f(w_j(x)), \quad f \in \hat{L}(X).$$

As a consequence of Proposition 3.2, Hennion [**11**] showed the following Corollary 3.3.

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COROLLARY 3.3. *Let the Lipschitz system* $(X, \{w_j\}_{j=1}^m, \{p_j\}_{j=1}^m)$ *satisfy the condition*

$$\sup_{\substack{x,y\in X\\ y\neq x}} \sum_{j=1}^{m} p_j(x)\cdot \frac{|w_j(x)-w_j(y)|}{|x-y|} < \varrho. \tag{3.1}$$

Then the operator P *is quasi-compact, i.e.,* $\varrho_e(P) < \varrho(P)$.

The proofs of Proposition 3.2 and Corollary 3.3 can be found in [**11**], and we omit them.

Let us come back to consider the Dini system and discuss the Ruelle operator T defined as in (2.3). For this, we extend the space $C(K)$ to a complex one, and denote it by $\hat{C}(K)$, i.e.,

$$\hat{C}(K) = C(K) + iC(K).$$

For any $f \in \hat{C}(K)$, let $|f|_\infty = \max_{x\in K} |f(x)|$. Then $(\hat{C}(K), |\cdot|_\infty)$ is a complex Banach space. Define $\hat{T} : \hat{C}(K) \to \hat{C}(K)$ by

$$\hat{T}(f_1 + if_2) = Tf_1 + iTf_2, \quad f_1, f_2 \in C(K).$$

Then $\hat{T}$ is a natural extension of T. It is obvious that the operator $\hat{T}$ is positive, i.e., $\hat{T}f > 0$ whenever $f > 0$. From this, we confirm that $\varrho(\hat{T}) = \varrho(T)$. We use $\varrho_e(\hat{T})$ to denote the *essential spectral radius* of $\hat{T}$ (see e.g. [**1**]). In the following, if there is no confusion, we will call it the *essential spectral radius* of T, and denote it by $\varrho_e = \varrho_e(T)$, for simplicity.

PROPOSITION 3.4. *Let* $(X, \{w_j\}_{j=1}^m, \{p_j\}_{j=1}^m)$ *be a Dini system, and let the operator* T *be defined as in (2.3). If* $\varrho_e(T) < \varrho(T)$, *then there exists an* $0 < h \in C(K)$ *such that* $Th = \varrho h$.

We would like to point out that Hennion [**11**] made use of the quasi-compactness and the irreducibility of T to prove Proposition 3.4. The proof can be found there, and we omit it.

Now we consider the case that $\varrho_e = \varrho$. We give a basic criterion for the existence of the eigenfunction corresponding to the spectral radius ϱ in this case. From Proposition 4.2 in [**14**], we get the following Proposition 3.5 immediately.

PROPOSITION 3.5. *Let* $(X, \{w_j\}_{j=1}^m, \{p_j\}_{j=1}^m)$ *be a Dini system. Suppose that we have the following.*

(i) $r := \min_{1\le j\le m} \sup_{\substack{x,y\in K\\ x\neq y}} (|w_j(x) - w_j(y)|/|x-y|) < 1.$

(ii) *There exist* $A, B > 0$ *such that* $A \le \varrho^{-n} T^n 1(x) \le B$ *for any* $x \in K$ *and* $n > 0$.

Then there exist a unique positive function $h \in C(K)$ *and a unique probability measure* $\mu \in M(K)$ *such that*

$$Th = \varrho h, \quad T^*\mu = \varrho\mu, \quad \langle \mu, h\rangle = 1.$$

Moreover, for every $f \in C(K)$, $\varrho^{-n}T^n f$ *converges to* $\langle \mu, f\rangle h$ *in the supremum norm, and, for every* $\xi \in M(K)$, $\varrho^{-n}T^{*n}\xi$ *converges weakly to* $\langle \xi, h\rangle \mu$.

Proof. Condition (i) here implies condition (i) of [**14**, Proposition 4.2] and the assertion follows. □

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Definition 3.6. Let $(X, \{w_j\}_{j=1}^m, \{p_j\}_{j=1}^m)$ be a Dini system. We say that the Ruelle operator theorem holds for this system, if there exist a unique positive function $h \in C(K)$ and a unique probability $\mu \in M(K)$ such that

$$Th = \varrho h, \quad T^*\mu = \varrho\mu, \quad \langle \mu, h\rangle = 1,$$

and, for every $f \in C(K)$, $\varrho^{-n}T^n f$ converges to $\langle \mu, f\rangle h$ in the supremum norm.

The Ruelle operator T is said to have the *Perron–Frobenius property* [**18**] if the Ruelle operator theorem holds for the system.

COROLLARY 3.7. *Let $(X, \{w_j\}_{j=1}^m, \{p_j\}_{j=1}^m)$ be a Dini system. Suppose that the following hold.*

(i) $\min_{1\le j\le m} \max_{x\in K} |w_j'(x)| < 1$.

(ii) *There exist constants $A, B > 0$ such that $A \le \varrho^{-n}T^n 1(x) \le B$ for any $x \in K$ and $n > 0$.*

Then the Ruelle operator theorem holds.

Proof. From condition (i), by applying Proposition 2.1, we conclude that the system $(X, \{w_j\}_{j=1}^m, \{p_j\}_{j=1}^m)$ satisfies condition (i) of Proposition 3.5. Hence, we get the assertion immediately. □

THEOREM 3.8. *Let $(X, \{w_j\}_{j=1}^m, \{p_j\}_{j=1}^m)$ be a Dini system. Suppose that the following hold.*

(i) *There exists a positive integer k such that*

$$\max_{x\in K} \sum_{|J|=k} p_{w_J}(x) \cdot |w_J'(x)| < \varrho^k.$$

(ii) *There exist constants $A, B > 0$ such that $A \le \varrho^{-n}T^n 1(x) \le B$ for any $x \in K$ and $n > 0$.*

Then the Ruelle operator theorem holds.

The above Theorem 3.8 is a fundamental theorem of this section. To prove it, we need to set up the following lemma first.

LEMMA 3.9. *Let $(X, \{w_j\}_{j=1}^m, \{p_j\}_{j=1}^m)$ be a Dini system. Then, for any $x \in K$,*

$$\sum_{|J|=n} p_{w_J}(x) \cdot |w_J'(x)| \le a^n \quad \text{for all } n > 0,$$

provided that

$$\sum_{j=1}^m p_j(x) \cdot |w_j'(x)| \le a. \tag{3.2}$$

Proof. From (3.2), it follows that

$$\begin{aligned}\sum_{|J|=2} p_{w_J}(x) \cdot |w_J'(x)| &= \sum_{i=1}^m \Bigg(\sum_{j=1}^m p_j(w_i(x)) \cdot |w_j'(w_i(x))|\Bigg) \cdot p_i(x)|w_i'(x)| \\ &\le \sum_{i=1}^m a \cdot p_i(x)|w_i'(x)| \le a^2 \quad \text{(by (3.2))}.\end{aligned}$$

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Inductively we can prove that for all $n \in \mathbb{N}$,

$$\sum_{|J|=n} p_{w_J}(x) \cdot |w'_J(x)| \le a^n. \qquad \square$$

Proof of Theorem 3.8. By condition (i), there exists an $0 < \eta < 1$ such that

$$\max_{x\in K} \sum_{|J|=k} p_{w_J}(x) \cdot |w'_J(x)| \le \eta \varrho^k.$$

From this, by applying Lemma 3.9, we conclude that, for any $x \in K$ and $\ell \in \mathbb{N}$,

$$\sum_{|J|=\ell k} p_{w_J}(x) \cdot |w'_J(x)| \le \eta^\ell \varrho^{\ell k}. \tag{3.3}$$

We claim that

$$\sup_{x\in K} \inf_{\ell\in\mathbb{N}} \min_{|J|=\ell k} |w'_J(x)| = 0.$$

Otherwise, we suppose that

$$\sup_{x\in K} \inf_{\ell\in\mathbb{N}} \min_{|J|=\ell k} |w'_J(x)| > 0.$$

Then there exist a $b_0 > 0$ and an $x_0 \in K$ such that

$$\inf_{\ell\in\mathbb{N}} \min_{|J|=\ell k} |w'_J(x_0)| \ge b_0.$$

This, combined with (3.3), implies that, for any $\ell \in \mathbb{N}$,

$$\begin{aligned}\eta^\ell &\ge \varrho^{-\ell k} \sum_{|J|=\ell k} p_{w_J}(x_0) \cdot |w'_J(x_0)| \ge b_0 \cdot \varrho^{-\ell k} \sum_{|J|=\ell k} p_{w_J}(x_0) \\ &= b_0 \cdot \varrho^{-\ell k} T^{\ell k} 1(x_0) \ge b_0 A \quad \text{(by condition (ii))}.\end{aligned}$$

This contradicts the choice of $0 < \eta < 1$. Then the claim follows. Thus there exist an $\ell_0 \in \mathbb{N}$ and a multi-index $J_0 \in \mathcal{I}^{\ell_0 k}$ such that $\max_{x\in K} |w'_{J_0}(x)| < 1$. Hence, by Corollary 3.7, the Ruelle operator theorem for $T^{\ell_0 k}$ holds. This implies that the Ruelle operator theorem for T holds. $\square$

In the next two sections, we will study the Perron–Frobenius property and quasi-compactness of the Ruelle operator for a Dini (or Lipschitz) system under the framework in Theorem 3.8 and Proposition 3.2, respectively.

4. *Some sufficient conditions for the Ruelle operator theorem*

Throughout this section we consider a Dini system $(X, \{w_j\}_{j=1}^m, \{p_j\}_{j=1}^m)$, and we always assume that $w'_j(x) \ne 0$ for all j. We mainly consider the case that $\varrho_e = \varrho$. We will give a basic criterion for the existence of the eigenfunction corresponding to the spectral radius ϱ in this case. We prove one of the main results (Theorem 4.3) by applying Theorem 3.8. The proof is a modification of the argument in [**14**]. In fact, having made full preparations, we only need to make some key changes, compared with the proof of Theorem 4.5 in [**14**]. For this, we still need to set up the following two lemmas first.

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Let $\alpha_0(t) = \max_{1\le j\le m}\{\alpha_{\log p_j}(t)\}$. Note that $|X| = 1$. We define

$$\alpha(t) = \begin{cases} \alpha_0(t) & \text{if } 0 \le t \le 1, \\ \alpha_0(1) & \text{if } t > 1. \end{cases}$$

Remember that an equivalent condition for a positive function $p(x)$ on X to be Dini continuous is

$$\sum_{n=0}^{\infty} \alpha_{\log p}(a\theta^n) < \infty$$

for any $0 < \theta < 1$ and constant $a > 0$.

For any $J \in \mathcal{I}^n$ and any $0 \le k < l \le n$, we let $r_J = \min_{x\in X} |w_J'(x)|$ and $R_J = \max_{x\in X} |w_J'(x)|$. We define $J|_l^k = (j_{n-l+1} j_{n-l+2} \cdots j_{n-k})$. In the following, for any $x \in X$, we let $w_{J|_0^0}(x) = x$ for convenience.

LEMMA 4.1. *Suppose that $(X, \{w_j\}_{j=1}^m, \{p_j\}_{j=1}^m)$ is a Dini system. Let $a \ge 1$ and $0 < \theta < 1$, and let $a_0 = \sum_{n=0}^{\infty} \alpha(a\theta^n)$. If $J = (j_1 \cdots j_n) \in \mathcal{I}^n$ satisfies the condition*

$$R_{J|_k^0} \le a\theta^k \quad \textit{for all } 1 \le k \le n,$$

then

$$p_{w_J}(x) \le e^{a_0} p_{w_J}(y) \quad \textit{for all } x, y \in K.$$

Proof. From the convexity of X, by applying the mean value theorem, we conclude that, for any $x, y \in K$,

$$|w_{J|_k^0}(x) - w_{J|_k^0}(y)| \le R_{J|_k^0} \quad \text{for all } 1 \le k \le n.$$

(We use $K \subseteq X$ and $|X| = 1$ here.) Then we have

$$|w_{J|_k^0}(x) - w_{J|_k^0}(y)| \le a\theta^k \quad \text{for all } 1 \le k \le n.$$

The inequality follows from the estimate that

$$\begin{aligned} \left|\log \frac{p_{w_J}(x)}{p_{w_J}(y)}\right| &\le \sum_{k=1}^{n} |\log p_{j_k}(w_{J|_{n-k}^0}(x)) - \log p_{j_k}(w_{J|_{n-k}^0}(y))| \\ &\le \sum_{k=1}^{n} \alpha(a\theta^{n-k}) \le a_0. \end{aligned}$$

□

LEMMA 4.2. *Let $(X, \{w_j\}_{j=1}^m)$ be a weakly contractive IFS. Suppose that each w_j $(1 \le j \le m)$ is continuously differentiable with $w_j'(x) \ne 0$. Then for any $\epsilon > 0$ there exists a constant $a \ge 1$ such that, for any integer $n > 0$,*

$$R_J \le a e^{n\epsilon} r_J \quad \textit{for all } J \in \mathcal{I}^n.$$

Proof. Let $d_n = \max_{|J|=n} |w_J(X)|$, $\delta(t) = \max_{1\le j\le m} \alpha_{w_j'}(t)$. Since $|w_j'(x)|$ is continuous and positive on the compact set X, there exists a constant $c_1 > 0$ such that

$$\frac{|w_j'(x)|}{|w_j'(y)|} \le 1 + c_1\delta(|x - y|) \quad \text{for all } x, y \in X.$$

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Hence, for any $J = (j_1 j_2 \cdots j_n) \in \mathcal{I}^n$

$$\frac{|w'_J(x)|}{|w'_J(y)|} = \prod_{k=1}^{n} \frac{|w'_{j_k}(w_{j_{k+1}} \circ w_{j_{k+2}} \circ \cdots \circ w_{j_n}(x))|}{|w'_{j_k}(w_{j_{k+1}} \circ w_{j_{k+2}} \circ \cdots \circ w_{j_n}(y))|}$$
$$\leq \prod_{k=0}^{n-1}(1 + c_1\delta(|w_{J|_k^0}(X)|)) \leq \prod_{k=0}^{n-1}(1 + c_1\delta(d_{n-k})).$$

Since the IFS $(X, \{w_j\}_{j=1}^m)$ is weakly contractive, we have $\lim_{n\to\infty} d_n = 0$. From the continuity of w'_j on the compact set X, we deduce that the function $\delta(t)$ is continuous and $\delta(0) = 0$. Then, for any $\epsilon > 0$, there exists $k_0 > 0$ such that $\delta(d_k) < {c_1}^{-1}\epsilon$ whenever $k \geq k_0$. Hence there exists a constant $a \geq 1$ such that, for any integer $n > 0$ and any $x, y \in X$,

$$\frac{|w'_J(x)|}{|w'_J(y)|} \leq a(1 + \epsilon)^n \leq ae^{n\epsilon}.$$

From this, we deduce that, for any integer $n > 0$,

$$R_J \leq ae^{n\epsilon} r_J \quad \text{for all } J \in \mathcal{I}^n. \qquad \square$$

THEOREM 4.3. *Let $(X, \{w_j\}_{j=1}^m, \{p_j\}_{j=1}^m)$ be a Dini system with $w'_j(x) \neq 0$ for all j. If there exists a positive integer k such that*

$$\max_{x\in K} \sum_{|J|=k} p_{w_J}(x) \cdot |w'_J(x)| < \varrho^k, \tag{4.1}$$

then the Ruelle operator theorem holds.

Proof. Since the Ruelle operator theorem for T^k implies the Ruelle operator theorem for T, we may assume $k = 1$ in the hypothesis, so that (4.1) is reduced to

$$\max_{x\in K} \sum_{j=1}^{m} p_j(x) \cdot |w'_j(x)| < \varrho. \tag{4.2}$$

This means that condition (i) of Theorem 3.8 is satisfied. Hence, we only need to show that condition (ii) of Theorem 3.8 is also satisfied, i.e., there exist $A, B > 0$ such that, for any $x \in K$,

$$A \leq \varrho^{-n} \sum_{|J|=n} p_{w_J}(x) \leq B \quad \text{for all } n.$$

By (4.2), we can find $0 < \eta < 1$ such that

$$\max_{x\in K} \sum_{j=1}^{m} p_j(x) \cdot |w'_j(x)| \leq \eta\varrho. \tag{4.3}$$

Choose θ such that $0 < \eta < \theta < 1$. For any fixed $x \in K$ and any $J \in \mathcal{I}^n$, let n_J be the largest integer such that

$$|w'_{J|_{n_J}^0}(x)| \geq \theta^{n_J}.$$

Then, by the chain rule, we have

$$|w'_{J|_i^{n_J}}(w_{J|_{n_J}^0}(x))| < \theta^{i-n_J} \quad \text{for all } n_J < i \leq n. \tag{4.4}$$

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For any $1 \le k \le n$, let

$$\Omega(n, k) = \{J \in \mathcal{I}^n : n_J = k\},$$

and let $\Omega(n, 0) = \{J \in \mathcal{I}^n : n_J = 0\}$, i.e.,

$$\Omega(n, 0) = \{J \in \mathcal{I}^n : |w'_{J|_i^0}(x)| < \theta^i \text{ for all } 0 < i \le n\}.$$

Then

$$\mathcal{I}^n = \bigcup_{k=0}^{n} \Omega(n, k).$$

From (4.3), by Lemma 3.9, we can confirm that

$$\sum_{|J|=n} p_{w_J}(x) \cdot |w'_J(x)| \le (\eta\varrho)^n \quad \text{for all } n > 0.$$

Then we have

$$\sum_{J\in\Omega(n,n)} p_{w_J}(x) \cdot \theta^n \le \sum_{|J|=n} p_{w_J}(x) \cdot |w'_J(x)| \le (\eta\varrho)^n.$$

Thus, it follows that

$$\varrho^{-n} \sum_{J\in\Omega(n,n)} p_{w_J}(x) \le \left(\frac{\eta}{\theta}\right)^n. \tag{4.5}$$

Choose $\epsilon > 0$ such that $\theta_0 := e^\epsilon\theta < 1$. Let $a \ge 1$ be the number, corresponding to ϵ, determined by Lemma 4.2. Let $a_0 = \sum_{k=0}^{\infty} \alpha(a\theta_0^k)$. Then a_0 is finite because all of the p_j are Dini continuous functions on X. For any $n > 0$, we can make use of Proposition 2.4(i) to find $x_n \in K$ such that

$$\varrho^{-n} \sum_{|J|=n} p_{w_J}(x_n) \le 1. \tag{4.6}$$

For any $J = (j_1 j_2 \cdots j_n) \in \Omega(n, k)$, we have $J|_k^0 \in \Omega(k, k)$. From Lemma 4.2, we conclude that

$$R_{J|_i^k} \le ae^{(i-k)\epsilon} r_{J|_i^k} \quad \text{for all } k < i \le n.$$

This, combined with (4.4), implies that

$$R_{J|_i^k} \le ae^{(i-k)\epsilon} r_{J|_i^k} \le ae^{(i-k)\epsilon}\theta^{i-k} = a\theta_0^{i-k} \quad \text{for all } k < i \le n. \tag{4.7}$$

From this, together with Lemma 4.1, we deduce that

$$p_{w_{J|_n^k}}(w_{J|_k^0}x) \le e^{a_0} p_{w_{J|_n^k}}(y) \quad \text{for all } y \in K.$$

(We use $|X| = 1$ here.) Hence

$$p_{w_J}(x) = p_{w_{J|_n^k}}(w_{J|_k^0}x) p_{w_{J|_k^0}}(x) \le e^{a_0} p_{w_{J|_n^k}}(x_{n-k}) p_{w_{J|_k^0}}(x). \tag{4.8}$$

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It follows that

$$\begin{aligned}
\varrho^{-n}\sum_{|J|=n} p_{w_J}(x) &= \varrho^{-n}\sum_{k=0}^{n}\sum_{J\in\Omega(n,k)} p_{w_J}(x)\\
&\le \varrho^{-n}\sum_{k=0}^{n}\sum_{J\in\Omega(n,k)} e^{a_0}\, p_{w_{J|_n^k}}(x_{n-k})\, p_{w_{J|_k^0}}(x) \quad \text{(by (4.8))}\\
&\le e^{a_0}\sum_{k=0}^{n}\left(\varrho^{-n+k}\sum_{|J'|=n-k} p_{w_{J'}}(x_{n-k})\right)\left(\varrho^{-k}\sum_{J''\in\Omega(k,k)} p_{w_{J''}}(x)\right)\\
&\le e^{a_0}\sum_{k=0}^{n} 1\cdot\left(\frac{\eta}{\theta}\right)^k \quad \text{(by (4.5), (4.6))}\\
&\le e^{a_0}\sum_{k=0}^{\infty}\left(\frac{\eta}{\theta}\right)^k := B. \qquad (4.9)
\end{aligned}$$

This concludes the upper bound estimate.

For the lower bound estimation, similarly to the proof of [**14**, Theorem 4.5], we can prove, by making use of Proposition 2.4(i) and (4.9), that there exists an $A > 0$ such that

$$\varrho^{-n}\sum_{|J|=n} p_{w_J}(x) \ge A.$$

This completes the proof. □

COROLLARY 4.4. *Let $(X, \{w_j\}_{j=1}^m, \{p_j\}_{j=1}^m)$ be a Dini system with $w_j'(x) \neq 0$ for all j. If the IFS $\{w_j\}_{j=1}^m$ is contractive, then the Ruelle operator theorem holds for this Dini system.*

Proof. Since the IFS $\{w_j\}_{j=1}^m$ is contractive, we have

$$R := \max_{1\le j\le m}\max_{x\in X} |w_j'(x)| < 1.$$

Note that, for any $n \in \mathbb{N}$,

$$|w_J'(x)| \le R^n \quad \text{for any } x \in X \text{ and any } J \in \mathcal{I}^n.$$

From this, together with $K \subseteq X$, we deduce that

$$\limsup_{n\to\infty}\left(\max_{x\in K}\sum_{|J|=n} p_{w_J}(x)\cdot|w_J'(x)|\right)^{1/n} \le R\cdot\lim_{n\to\infty}\|T^n\|^{1/n} = R\varrho < \varrho.$$

This implies that condition (4.1) in Theorem 4.3 is satisfied. The assertion follows. □

COROLLARY 4.5. *Let $(X, \{w_j\}_{j=1}^m, \{p_j\}_{j=1}^m)$ be a Dini system with $w_j'(x) \neq 0$ for all j. If*

$$\sup_{y\neq x}\sum_{j=1}^{m} p_j(x)\cdot\frac{|w_j(x)-w_j(y)|}{|x-y|} < \varrho, \qquad (4.10)$$

then the Ruelle operator theorem holds for this Dini system.

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Proof. Note that, for any $x \in K$,

$$\lim_{y\to x} \sum_{j=1}^{m} p_j(x) \cdot \frac{|w_j(x) - w_j(y)|}{|x - y|} = \sum_{j=1}^{m} p_j(x) \cdot |w_j'(x)|.$$

This implies that

$$\sum_{j=1}^{m} p_j(x) \cdot |w_j'(x)| \le \sup_{y\neq x} \sum_{j=1}^{m} p_j(x) \cdot \frac{|w_j(x) - w_j(y)|}{|x - y|}.$$

From this, together with Theorem 4.3, we conclude the assertion. □

Remark 4.6. We would like to point out that Corollary 4.5 extends partially the result in [**3**]. In fact, under the probability condition $\sum_{j=1}^{m} p_j(x) = 1$, we conclude immediately that $\varrho = 1$. In this case, condition (4.10) is reduced to the *average contractive* condition:

$$\sup_{y\neq x} \sum_{j=1}^{m} p_j(x) \cdot \frac{|w_j(x) - w_j(y)|}{|x - y|} < 1.$$

Hence, we succeed in removing the probability condition in [**3**].

Remark 4.7. By noting that, for any $x \in K$,

$$\sum_{j=1}^{m} p_j(x) \cdot |w_j'(x)| \le \sum_{j=1}^{m} p_j(x) \cdot \sup_{y\neq x} \frac{|w_j(x) - w_j(y)|}{|x - y|},$$

we see that Theorem 4.3 generalizes partially the main results of [**14**] as well.

The following example reveals that the condition of Theorem 1.2 (or Theorem 4.3) in this paper is more easily checked than the condition in [**14**] (compared with [**14**, Example 4.7]).

Example 4.8. Let the system $(X, \{w_j\}_{j=1}^2, \{p_j\}_{j=1}^2)$ be as in [**14**, Example 4.7]. That is, let $X = [0, 1]$, and let $w_1(x) = x - (x^2/2)$, $w_2(x) = (1/2) + (x^2/2)$. Also let p_1 be any positive Dini (not Lipschitz) continuous function on X with the inequalities $0 < p_1(\cdot) < 1$. Let

$$\delta = \tfrac{1}{5} \cdot \min_{x\in X}\{p_1(x), 1 - p_1(x)\} > 0,$$

and let $g : X \to X$ be defined by

$$g(x) = \begin{cases} \delta - 2^{-1} + x & \text{if } 2^{-1} - \delta < x \le 2^{-1}, \\ \delta + 2^{-1} - x & \text{if } 2^{-1} < x < 2^{-1} + \delta, \\ 0 & \text{otherwise.} \end{cases}$$

It is obvious that $0 \le g(x) \le \delta$ for all $x \in X$. Define

$$p_2(x) = 1 - p_1(x) + g(x) \quad \text{for all } x \in X.$$

Then p_j, $1 \le j \le 2$, are Dini functions, and

$$1 \le \sum_{j=1}^{2} p_j(x) = 1 + g(x) \le 1 + \delta.$$

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Let T be the Ruelle operator defined by the system $(X, \{w_j\}_{j=1}^2, \{p_j\}_{j=1}^2)$. Then

$$1 \le \varrho \le 1 + \delta. \tag{4.11}$$

By the choice of δ, we have

$$5\delta \le p_1(x) \le 1 - 5\delta \quad \text{for all } x \in X. \tag{4.12}$$

Note that $w_1'(x) = 1 - x$, $w_2'(x) = x$. Then we have

$$\begin{aligned}\sum_{j=1}^{2} p_j(x) \cdot |w_j'(x)| &= p_1(x) \cdot (1 - x) + x(1 - p_1(x) + g(x)) \\ &\le (1 - 5\delta) \cdot (1 - x) + x(1 - 5\delta + \delta) \quad \text{(by (4.12))} \\ &= (1 - 5\delta) + x\delta \le 1 - 4\delta < 1.\end{aligned}$$

This, together with (4.11), implies that

$$\sup_{x \in X} \sum_{j=1}^{2} p_j(x) \cdot |w_j'(x)| < \varrho.$$

Then Theorem 4.3 implies that the Ruelle operator theorem holds for this Dini system.

5. *Quasi-compact Ruelle operator*

Throughout this section, we always assume that each potential function $p_j (1 \le j \le m)$ is Lipschitz continuous on X. As before, we denote by K the invariant set of the IFS $(X, \{w_j\}_{j=1}^m)$, i.e., $K = \bigcup_{j=1}^m w_j(K)$.

Now let $Y = X$ or $Y = K$. We use $L(Y)$ to denote the set of all real Lipschitz continuous functions defined on Y. Let $\hat{L}(Y) = L(Y) + iL(Y)$.

For any $f \in \hat{L}(Y)$, we let $|f|_\infty = \max_{x \in Y} |f(x)|$, and let

$$|f|_{\mathrm{Lip}} = \sup_{\substack{x, y \in Y \\ x \ne y}} \frac{|f(x) - f(y)|}{|x - y|}.$$

Define

$$\|f\| = |f|_{\mathrm{Lip}} + |f|_\infty.$$

Then $(\hat{L}(Y), |\cdot|_\infty)$ is a complex normed linear space, and $(\hat{L}(Y), \|\cdot\|)$ is a complex Banach space. It is obvious that $(L(Y), \|\cdot\|)$, as a closed subspace of $(\hat{L}(Y), \|\cdot\|)$, is a Banach space.

We define Ruelle operator $P : \hat{L}(X) \to \hat{L}(X)$ by

$$P(f)(x) = \sum_{j=1}^{m} p_j(x) f(w_j(x)), \quad f \in \hat{L}(X),$$

and we let $T : L(K) \to L(K)$ be defined by

$$T(f)(x) = \sum_{j=1}^{m} p_j(x) f(w_j(x)), \quad f \in L(K).$$

We naturally extend the operator T to $\hat{L}(K)$, and denote it by $\hat{T}$, i.e.,

$$\hat{T}(f + ig) = Tf + iTg \quad \text{for all } f, g \in L(K).$$

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LEMMA 5.1. *Let* $(X, \{w_j\}_{j=1}^m, \{p_j\}_{j=1}^m)$ *be a Lipschitz system, and let the operators* P, T *and* $\hat{T}$ *be defined as above. Then* $\varrho(P) = \varrho(T) = \varrho(\hat{T})$.

Proof. Since both T and $\hat{T}$ are positive, and $\hat{T}$ is the extension of T, it is obvious that $\varrho(\hat{T}) = \varrho(T)$.

We will prove that $\varrho(P) = \varrho(T)$. It is easy to see that $\varrho(T) \le \varrho(P)$. On the other hand, we let $d_n = \max_{|J|=n} |w_J(X)|$. From the weak contractivity of the IFS $(X, \{w_j\}_{j=1}^m)$, it follows that $\lim_{n\to\infty} d_n = 0$. Let

$$\alpha(t) = \max_{1\le j\le m} \{\alpha_{\log p_j}(t)\}.$$

From the positiveness and Lipschitz continuity of the p_j on the compact set X, we can deduce that there exists a constant $c > 0$ such that $\alpha(t) \le ct$ and $\alpha(0) = 0$.

For any $J \in \mathcal{I}^n$, and for any $x \in X$ and any $y \in K$, we have

$$p_{w_J}(x) \le p_{w_J}(y) \cdot e^{\sum_{k=0}^{n-1} \alpha(d_k)} \le p_{w_J}(y) \cdot e^{\sum_{k=0}^{n-1} cd_k}.$$

This implies that

$$P^n 1(x) \le T^n 1(y) \cdot e^{c \sum_{k=0}^{n-1} d_k}.$$

This, combined with the positiveness of P and T, implies that

$$\|P^n\|^{1/n} \le \|T^n\|^{1/n} \cdot e^{c/n \sum_{k=0}^{n-1} d_k}.$$

From this, together with $\lim_{k\to\infty} d_k = 0$, we can deduce that $\varrho(P) \le \varrho(T)$. Therefore $\varrho(P) = \varrho(T)$. □

Having proved Lemma 5.1, in the following we can use ϱ to denote any one of $\varrho(T)$, $\varrho(\hat{T})$ and $\varrho(P)$, i.e., $\varrho = \varrho(T) = \varrho(\hat{T}) = \varrho(P)$.

THEOREM 5.2. *Let* $(X, \{w_j\}_{j=1}^m, \{p_j\}_{j=1}^m)$ *be a Lipschitz system. If*

$$\max_{x\in X} \sum_{j=1}^m p_j(x) \cdot |w_j'(x)| < \varrho, \tag{5.1}$$

then there exist a unique $0 < h \in L(K)$, *a unique probability measure* $\mu \in M(K)$ *and a constant* $0 < b < 1$ *such that* $Th = \varrho h$, $T^*\mu = \varrho\mu$, $\langle \mu, h\rangle = 1$ *and*

$$|\varrho^{-n} T^n f - \langle \mu, f\rangle h|_\infty = O(b^n) \quad \text{for all } f \in L(K).$$

To prove Theorem 5.2, we need to prove the following Theorem 5.3 first.

THEOREM 5.3. *Let the Lipschitz system* $(X, \{w_j\}_{j=1}^m, \{p_j\}_{j=1}^m)$ *satisfy condition (5.1). Then* $\varrho_e(T) < \varrho(T)$, *and there exists an* $0 < h \in L(K)$ *such that* $Th = \varrho h$.

From Theorem 5.3, we have $\varrho_e(T) < \varrho(T)$. This means that the operator $\hat{T}$ is quasi-compact. In this case, we sometimes say that the operator T is quasi-compact for simplicity, if there is no confusion. To prove Theorem 5.3, we need some preparation.

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LEMMA 5.4. *Let the system* $(X, \{w_j\}_{j=1}^m, \{p_j\}_{j=1}^m)$ *be as in Theorem 5.3. Then there exist* $0 < \eta < 1$ *and* $\delta > 0$ *such that*

$$\sup_{\substack{x,y\in X\\ 0<|x-y|<\delta}} \sum_{j=1}^m p_j(x) \cdot \frac{|w_j(x) - w_j(y)|}{|x-y|} \le \eta\varrho.$$

Proof. Since $p_j(x)$ and $w_j'(x)$, $1 \le j \le m$, are continuous on the compact set X, it follows that both $p_j(x)$ and $|w_j'(x)|$, $1 \le j \le m$, are equicontinuous on X. From this, together with (5.1), we deduce that there exists an $0 < \eta < 1$ such that

$$\max_{x\in X} \sum_{j=1}^m p_j(x) \cdot |w_j'(x)| < \eta\varrho. \tag{5.2}$$

For any $x, y \in X$ with $x \ne y$, from the convexity of the set X, by the mean value theorem, it follows that, for any $1 \le j \le m$, there exists a ξ_j located in the line segment joining x and y such that

$$\frac{|w_j(x) - w_j(y)|}{|x-y|} = |w_j'(\xi_j)|.$$

Hence, from (5.2), together with the equicontinuity of the $p_j(x)$ and the $|w_j'(x)|$, we can deduce that there exists a $\delta > 0$ such that, for any $z \in X$,

$$\sup_{\substack{x,y\in B_\delta(z)\\ x\ne y}} \sum_{j=1}^m p_j(x) \cdot \frac{|w_j(y) - w_j(x)|}{|y-x|} \le \eta\varrho. \tag{5.3}$$

Now for any $x, y \in X$ with $0 < |x-y| < \delta$, we have $x, y \in B_\delta(x)$. Hence, from (5.3), it follows that

$$\sum_{j=1}^m p_j(x) \cdot \frac{|w_j(x) - w_j(y)|}{|x-y|} \le \eta\varrho.$$

From this, we deduce that

$$\sup_{\substack{x,y\in X\\ 0<|x-y|<\delta}} \sum_{j=1}^m p_j(x) \cdot \frac{|w_j(x) - w_j(y)|}{|x-y|} \le \eta\varrho. \qquad \square$$

PROPOSITION 5.5. *Let* $\eta > 0$ *and* $\delta > 0$ *be given as in Lemma 5.4. Then for any* $n \in \mathbb{N}$,

$$\sup_{\substack{x,y\in X\\ 0<|x-y|<\delta}} \sum_{|J|=n} p_{w_J}(x) \cdot \frac{|w_J(x) - w_J(y)|}{|x-y|} \le (\eta\varrho)^n.$$

Proof. For any $x, y \in X$ with $0 < |x-y| < \delta$, from the weak contractivity of the IFS $(X, \{w_j\}_{j=1}^m)$, it follows that

$$|w_j(w_i(x)) - w_j(w_i(y))| \le |w_i(x) - w_i(y)| < \delta \quad \text{for all } 1 \le i, j \le m.$$

From this, together with Lemma 5.4, we conclude that, for any $1 \le i \le m$,

$$\sum_{j=1}^m p_j(w_i(x)) \cdot |w_j(w_i(x)) - w_j(w_i(y))| \le \eta\varrho \cdot |w_i(x) - w_i(y)|.$$

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Then we have

$$\begin{aligned}&\sum_{|J|=2} p_{w_J}(x)\cdot|w_J(x)-w_J(y)|\\&=\sum_{i=1}^{m} p_i(x)\Bigg(\sum_{j=1}^{m} p_j(w_i(x))\cdot|w_j(w_i(x))-w_j(w_i(y))|\Bigg)\\&\le \eta\varrho\sum_{i=1}^{m} p_i(x)\cdot|w_i(x)-w_i(y)|\le(\eta\varrho)^2|x-y|.\end{aligned}$$

It follows that

$$\sum_{|J|=2} p_{w_J}(x)\cdot\frac{|w_J(x)-w_J(y)|}{|x-y|}\le(\eta\varrho)^2.$$

This implies that

$$\sup_{\substack{x,y\in X\\0<|x-y|<\delta}}\sum_{|J|=2} p_{w_J}(x)\cdot\frac{|w_J(x)-w_J(y)|}{|x-y|}\le(\eta\varrho)^2.$$

The proof follows by induction. □

PROPOSITION 5.6. *Let the Lipschitz system* $(X,\{w_j\}_{j=1}^m,\{p_j\}_{j=1}^m)$ *satisfy condition (5.1). Then the Ruelle operator* P *is quasi-compact, i.e., there exist* $0<r<\varrho=\varrho(P)$ *and two closed subspaces* G *and* H *of* $\hat{L}(X)$ *satisfying the following conditions:*

$$\begin{cases}\text{(i)} & \dim H<\infty \text{ and } \hat{L}(X)=G\oplus H;\\ \text{(ii)} & PG\subseteq G \text{ and } PH\subseteq H;\\ \text{(iii)} & \varrho(P|_G)\le r<\varrho=\varrho(P|_H).\end{cases}\tag{5.4}$$

Proof. Let $\delta>0$ be given by Lemma 5.4. Let $\delta_0=2^{-1}\delta$. Since $\{B_{\delta_0}(x):x\in X\}$ is an open covering of X, from the compactness of X, it follows that the covering family $\{B_{\delta_0}(x):x\in X\}$ contains a finite sub-covering, i.e., there exists a finite set $\{x_k\}_{k=1}^q\subseteq X$ such that $X\subseteq\bigcup_{k=1}^q B_{\delta_0}(x_k)$.

For any $f\in\hat{L}(X)$ and any $1\le k\le q$, we let $f_k=f|_{B_{\delta_0}(x_k)}$, and define

$$|f_k|_{\mathrm{Lip}}=\sup_{\substack{x,y\in B_{\delta_0}(x_k)\\x\ne y}}\frac{|f(x)-f(y)|}{|x-y|}.$$

Then, from the fact that $X\subseteq\bigcup_{k=1}^q B_{\delta_0}(x_k)$, together with the convexity of X, we can deduce that, for any $1\le k\le q$,

$$|f_k|_{\mathrm{Lip}}\le|f|_{\mathrm{Lip}}\le q\cdot\max_{1\le k\le q}|f_k|_{\mathrm{Lip}}.\tag{5.5}$$

For any $x,y\in X$ with $x\ne y$, we have

$$\begin{aligned}\frac{|P^nf(x)-P^nf(y)|}{|x-y|}&\le\Bigg(\sum_{|J|=n}p_{w_J}(x)\frac{|w_J(x)-w_J(y)|}{|x-y|}\Bigg)|f|_{\mathrm{Lip}}\\&\quad+\Bigg(\sum_{|J|=n}\frac{|p_{w_J}(x)-p_{w_J}(y)|}{|x-y|}\Bigg)|f|_\infty\\&\le\Bigg(\sum_{|J|=n}p_{w_J}(x)\frac{|w_J(x)-w_J(y)|}{|x-y|}\Bigg)|f|_{\mathrm{Lip}}+R_n\cdot|f|_\infty,\end{aligned}\tag{5.6}$$

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where

$$R_n := \sup_{x\neq y}\sum_{|J|=n}\frac{|p_{w_J}(x)-p_{w_J}(y)|}{|x-y|} < \infty,$$

because all the p_j are Lipschitz continuous on the compact set X.

For any $1 \le k \le q$ and any $x \neq y \in B_{\delta_0}(x_k)$, we have $0 < |x-y| < 2\delta_0 = \delta$. By applying Proposition 5.5, we conclude that

$$\sum_{|J|=n} p_{w_J}(x)\frac{|w_J(x)-w_J(y)|}{|x-y|} \le (\eta\varrho)^n.$$

From this, together with (5.6), we deduce that

$$|(P^n f)_k|_{\mathrm{Lip}} = \sup_{\substack{x,y\in B_{\delta_0}(x_k)\\ x\neq y}}\frac{|P^n f(x)-P^n f(y)|}{|x-y|} \le (\eta\varrho)^n\cdot|f|_{\mathrm{Lip}} + R_n\cdot|f|_\infty.$$

This, together with (5.5), implies that

$$|P^n f|_{\mathrm{Lip}} \le q\cdot\max_{1\le k\le q}|(P^n f)_k|_{\mathrm{Lip}} \le q(\eta\varrho)^n\cdot|f|_{\mathrm{Lip}} + qR_n\cdot|f|_\infty.$$

Hence,

$$\begin{aligned}\|P^n f\| &= |P^n f|_{\mathrm{Lip}} + |P^n f|_\infty\\ &\le q(\eta\varrho)^n\cdot|f|_{\mathrm{Lip}} + qR_n\cdot|f|_\infty + |P^n 1|_\infty\cdot|f|_\infty\\ &\le q(\eta\varrho)^n\cdot\|f\| + (qR_n + |P^n 1|_\infty)\cdot|f|_\infty.\end{aligned}$$

It is obvious that

$$0 < \lim_{n\to\infty}(q(\eta\varrho)^n)^{1/n} = \eta\varrho < \varrho.$$

Note that P is compact as an operator from $(\hat{L}(X), \|\cdot\|)$ into $(\hat{L}(X), |\cdot|_\infty)$. From this, together with Proposition 3.2, we deduce that the operator P is quasi-compact. Hence, there exist $0 < r < \varrho = \varrho(P)$ and two closed subspaces G and H of $\hat{L}(X)$ satisfying conditions (5.4). □

Proof of Theorem 5.3. Let $r > 0$ and the subspaces G and H be given by Proposition 5.6. Define

$$E = \{f|_K : f \in G\} \quad \text{and} \quad F = \{g|_K : g \in H\}.$$

Then we can confirm that both E and F are closed subspaces of $\hat{L}(K)$ with $\hat{T}(E) \subseteq E$ and $\hat{T}(F) \subseteq F$. It is obvious that $\dim F < \infty$ and $\hat{L}(K) = E \oplus F$.

Note that $\varrho(\hat{T}|_E) \le \varrho(P|_G)$. This, together with Lemma 5.1, implies that

$$\varrho(\hat{T}|_E) \le \varrho(P|_G) \le r < \varrho(P) = \varrho(\hat{T}).$$

We claim that $\varrho(\hat{T}|_F) = \varrho(\hat{T})$. Otherwise, suppose that $\varrho(\hat{T}|_F) < \varrho(\hat{T})$. From $1 \in \hat{L}(K) = E \oplus F$, it follows that there exist $f_1 \in E$ and $f_2 \in F$ such that $1 = f_1 + f_2$. This implies that

$$\begin{aligned}\|\hat{T}^n\| &= |\hat{T}^n 1|_\infty \le |\hat{T}^n f_1|_\infty + |\hat{T}^n f_2|_\infty\\ &\le \|(\hat{T}|_E)^n\|\cdot|f_1|_\infty + \|(\hat{T}|_F)^n\|\cdot|f_2|_\infty\\ &\le (\|(\hat{T}|_E)^n\| + \|(\hat{T}|_F)^n\|)\cdot\max\{|f_1|_\infty,\ |f_2|_\infty\}.\end{aligned}$$

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From this, we can deduce that

$$\varrho(\hat{T}) = \lim_{n\to\infty} \|\hat{T}^n\|^{1/n} \le \max\{\varrho(\hat{T}|_E),\ \varrho(\hat{T}|_F)\} \le \max\{r, \varrho(\hat{T}|_F)\} < \varrho(\hat{T}).$$

This is a contradiction, and the claim follows. This implies that

$$\varrho(\hat{T}|_E) \le r < \varrho(\hat{T}|_F) = \varrho(\hat{T}).$$

Hence, the operator $\hat{T}$ is quasi-compact. This implies that $\varrho_e(\hat{T}) < \varrho(\hat{T})$. Thus, by Lemma 5.1, we have $\varrho_e(T) < \varrho(T)$. Note that T is irreducible. Proposition 3.4 yields an $0 < h \in L(K)$ such that $Th = \varrho h$. □

PROPOSITION 5.7. *Let the system* $(X, \{w_j\}_{j=1}^m, \{p_j\}_{j=1}^m)$ *be as in Theorem 5.3. Then the Ruelle operator theorem holds.*

Proof. By condition (5.1), we see that condition (i) of Theorem 3.8 is satisfied. To prove the assertion, we only need to show that condition (ii) of Theorem 3.8 is also satisfied. Indeed, by Theorem 5.3, there exists an $0 < h \in L(K)$ such that $Th = \varrho h$. Then, by Proposition 2.4(ii), there exist constants $A, B > 0$ such that

$$A \le T^n 1(x) \le B \quad \text{for all } x \in K \text{ and } n \in \mathbb{N}.$$

This means that condition (ii) of Theorem 3.8 is satisfied. Hence, we have completed the proof. □

Let $\mathbb{C}$ denote the set of complex numbers. We use Id to denote the identical operator from $\hat{L}(K)$ to itself. We denote by $\mathrm{Res}(\hat{T})$ the resolvent set of the operator $\hat{T}$, i.e.,

$$\mathrm{Res}(\hat{T}) = \{z \in \mathbb{C} : (\hat{T} - z\mathrm{Id}) \text{ has a bounded inverse operator}\}.$$

We denote by $\mathrm{Sp}(\hat{T})$ the spectrum of the operator $\hat{T}$, i.e., $\mathrm{Sp}(\hat{T}) = \mathbb{C}\backslash\mathrm{Res}(\hat{T})$. Then we have $\varrho(T) = \sup\{|z| : z \in \mathrm{Sp}(\hat{T})\}$.

PROPOSITION 5.8. *Let the system* $(X, \{w_j\}_{j=1}^m, \{p_j\}_{j=1}^m)$ *be as in Theorem 5.2. Then*

$$\tau := \sup\{|\lambda| : \lambda \in \mathrm{Sp}(\hat{T}) \text{ and } \lambda \ne \varrho\} < \varrho.$$

Proof. The operator $\hat{T}$ is, by Theorem 5.3, quasi-compact. Then there exist two closed subspaces E and F as in Definition 3.1. We denote by $\mathrm{Sp}(\hat{T}|_E)$ and $\mathrm{Sp}(\hat{T}|_F)$ the spectrum of the operators $\hat{T}|_E$ and $\hat{T}|_F$, respectively. Then we can confirm that

$$\mathrm{Sp}(\hat{T}) = \mathrm{Sp}(\hat{T}|_E) \cup \mathrm{Sp}(\hat{T}|_F). \tag{5.7}$$

Note that, for any λ with $|\lambda| > \varrho(\hat{T}|_E)$, the operator $(\hat{T}|_E - \lambda \cdot \mathrm{Id}|_E)$ has a bounded inverse operator. It follows that

$$\sup\{|\lambda| : \lambda \in \mathrm{Sp}(\hat{T}|_E)\} = \varrho(\hat{T}|_E) < \varrho = \varrho(T). \tag{5.8}$$

We denote by $\mathrm{Eg}(\hat{T}|_F)$ the set of eigenvalues of the operator $\hat{T}|_F$, i.e., let

$$\mathrm{Eg}(\hat{T}|_F) = \{z \in \mathbb{C} :\ \text{there exists } 0 \ne f \in F \text{ such that } \hat{T}f = zf\}.$$

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From the fact that $\dim F < \infty$, it follows that $\mathrm{Eg}(\hat{T}|_F)$ is a finite set, and $\mathrm{Sp}(\hat{T}|_F) = \mathrm{Eg}(\hat{T}|_F)$. This, together with (5.7) and (5.8), implies that we only need to show that

$$\{\lambda \in \mathrm{Eg}(\hat{T}|_F) : |\lambda| = \varrho\} = \{\varrho\}. \tag{5.9}$$

Without loss of generality, we assume that $\varrho = 1$ (otherwise, we may consider the operator $\varrho^{-1}\hat{T}$). Let $0 < h \in L(K)$ and $\mu \in M(K)$ be given by Proposition 5.7 (refer to Definition 3.6). If there exists a $\lambda \in \mathrm{Eg}(\hat{T}|_F)$ with $|\lambda| = 1$ and $0 \neq g \in \hat{L}(K)$ such that $\hat{T}g = \lambda g$, we claim that $\lambda = \varrho$. Otherwise, we suppose that $\lambda \neq \varrho$. Then it follows immediately that g and h are linearly independent.

By $\varrho = 1$, we have $Th = h$. By Proposition 2.4(ii), there exist constants $A, B > 0$ such that, for any $n > 0$,

$$A \leq T^n 1(x) \leq B \quad \text{for all } x \in K.$$

From this, we conclude that, for any $f \in \hat{L}(K)$, the sequence $n^{-1}\hat{T}^n f$ converges to 0 in the supremum norm. Hence, the operators sequence $n^{-1}\hat{T}^n$ converges to zero in the weak topology. Meanwhile, from Theorem 5.3, we see that the operator $\hat{T}$ is quasi-compact, i.e., $\varrho_e < \varrho$. This, combined with [**5**, Lemma 5 of VIII.8], implies that there is an integer $k > 0$ such that $\lambda^k = 1$. This, together with $\hat{T}g = \lambda g$, implies that $\hat{T}^k g = \lambda^k g = g$. Let $g = g_1 + ig_2$, where $g_1, g_2 \in L(K)$. Then for any integer $n > 0$, we have $\hat{T}^n g = T^n g_1 + iT^n g_2$. From this, together with Proposition 5.7, we deduce that

$$\lim_{n\to\infty} \hat{T}^n g = \lim_{n\to\infty} T^n g_1 + i \cdot \lim_{n\to\infty} T^n g_2 = \langle \mu, g_1\rangle h + i \cdot \langle \mu, g_2\rangle h = \langle \mu, g\rangle h.$$

On the other hand, we have $\hat{T}^{k\ell} g = g$ for all $\ell \in \mathbb{N}$. Hence, from the above argument, we conclude that $g = bh$ for some constant b. This is a contradiction. Then (5.9) follows. Hence, we have completed the proof. □

Proof of Theorem 5.2. Let $0 < h \in L(K)$ and $\mu \in M(K)$ be given by Proposition 5.7. Suppose that there exists another $0 \neq g \in L(K)$ such that $Tg = \varrho g$. We claim that g and h are linearly dependent. In fact, we let

$$c = \inf\left\{\frac{g(x)}{h(x)} : x \in K\right\}.$$

From the continuity of g and h, we conclude that there exists an $x_0 \in K$ such that $g(x_0)/h(x_0) = c$. It follows that $g(x_0) = c \cdot h(x_0)$, and $h_0 := g - ch$ is a continuous non-negative function with $Th_0 = \varrho h_0$. From this, together with the irreducibility of T, we deduce that $h_0 = 0$, and the claim follows. We suppose again that there exists a $0 \neq g_1 + ig_2 \in \hat{L}(K)$ such that $\hat{T}(g_1 + ig_2) = \varrho(g_1 + ig_2)$ and $g_1, g_2 \in L(K)$. It follows easily that $Tg_k = \varrho g_k$, $k = 1, 2$. From this, together with the above claim, we conclude that g_k, $k = 1, 2$, and h are linearly dependent.

Now we let

$$F_0 = \{g \in \hat{L}(K) : \hat{T}g = \varrho g\}.$$

From the above argument, we conclude that $\dim F_0 = 1$. Then $F_0 = \{ch : c \in \mathbb{C}\}$. From this, together with the quasi-compactness of $\hat{T}$ and Proposition 5.8, we can deduce that there exists a closed subspace E_0 of $\hat{L}(K)$ such that $\hat{L}(K) = E_0 \oplus F_0$ and

$$\varrho(\hat{T}|_{E_0}) < \varrho(\hat{T}|_{F_0}) = \varrho(\hat{T}). \tag{5.10}$$

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For any $f \in L(K)$, there exist $f_1 \in E_0$ and $f_2 \in F_0$ such that $f = f_1 + f_2$. Note that $f_2 = c_f h$ for some $c_f \in \mathbb{C}$. From this, it follows that

$$\varrho^{-n} T^n f = \varrho^{-n} \hat{T}^n f_1 + \varrho^{-n} \hat{T}^n (c_f h) = \varrho^{-n} (\hat{T}|_{E_0})^n f_1 + c_f h. \tag{5.11}$$

Let $b_0 = \varrho^{-1} \cdot \varrho(\hat{T}|_{E_0})$. From (5.10), we confirm that $0 < b_0 < 1$. From (5.11), it follows that

$$\begin{aligned} \limsup_{n\to\infty} (|\varrho^{-n} T^n f - c_f h|_\infty)^{1/n} &= \varrho^{-1} \cdot \limsup_{n\to\infty} (|(\hat{T}|_{E_0})^n f_1|_\infty)^{1/n} \\ &\le \varrho^{-1} \cdot \limsup_{n\to\infty} (\|(\hat{T}|_{E_0})^n\| \cdot |f_1|_\infty)^{1/n} \\ &\le \varrho^{-1} \cdot \lim_{n\to\infty} \|(\hat{T}|_{E_0})^n\|^{1/n} = b_0. \end{aligned}$$

Now we choose $b_0 < b < 1$. From the above argument, we can deduce that

$$|\varrho^{-n} T^n f - c_f h|_\infty = O(b^n) \quad \text{for all } f \in L(K).$$

This, together with Proposition 5.7, implies that $c_f = \langle \mu, f \rangle \in \mathbb{R}$, and the assertion follows. □

Remark 5.9. Note that, for any $x \in K$,

$$\sum_{j=1}^{m} p_j(x) \cdot |w_j'(x)| \le \sup_{y \ne x} \sum_{j=1}^{m} p_j(x) \cdot \frac{|w_j(x) - w_j(y)|}{|x - y|}.$$

From this, we conclude that condition (3.1) implies condition (5.1). Hence, Theorem 5.3, together with Theorem 5.2, generalizes partially the result in [**11**] (referred to in Corollary 3.3 and Proposition 3.4).

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