101.25 A note on tilings and associated linear recurrences

Introduction

This Note is concerned with the enumeration of tilings of mathematical objects known as *n-bracelets* [1, 2]. It will be shown, in two different ways, how to derive a recurrence relation that allows us easily to carry out such enumerations. Before discussing n-bracelets, however, we need to describe briefly a related and somewhat more straightforward situation. The simplest non-trivial tilings are those associated with a $1 \times n$ rectangular board $\mathcal{R}_n$ for some $n \in \mathbb{N}$, where the board may be thought of as a $1 \times n$ grid comprising n square cells labelled consecutively from 1 to n. For example, in Figure 1 we see $\mathcal{R}_7$. It is well known [3, 4] that the number of ways of tiling $\mathcal{R}_n$ with uncoloured 1×1 squares and 1×2 dominos is given by F_{n+1}, the $(n+1)$th Fibonacci number.

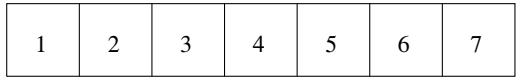


FIGURE 1: A 1×7 rectangular grid $\mathcal{R}_7$

This is easily generalised to tilings of $\mathcal{R}_n$ comprising, for a fixed positive integer $j \geqslant 2$, the set $\mathcal{S}(j) = \{(1 \times 1)_{\mathcal{S}}, (1 \times 2)_{\mathcal{S}}, \ldots, (1 \times j)_{\mathcal{S}}\}$ of uncoloured *straight* tiles. If we define $G_n(j)$ by the linear recurrence relation

$$G_n(j) = G_{n-1}(j) + G_{n-2}(j) + \ldots + G_{n-j}(j),$$

where $G_0(j) = G_1(j) = \ldots = G_{j-2}(j) = 0$ and $G_{j-1}(j) = 1$, then $G_{n+j-1}(j)$ enumerates the number of tilings of $\mathcal{R}_n$ using elements from $\mathcal{S}(j)$. Note that $(G_n(j))_{n \geqslant 0}$ is a generalised Fibonacci sequence.

An n-bracelet $\mathcal{B}_n$ may be constructed from $\mathcal{R}_n$ by stretching and bending $\mathcal{R}_n$ until it has been transformed into a circular grid. As an example, $\mathcal{B}_7$ can be seen in Figure 2. We refer to the kth cell of $\mathcal{B}_n$ as $\mathcal{K}_k$, noting that, in transforming $\mathcal{R}_n$ into $\mathcal{B}_n$, the labelling of the cells is maintained so that $\mathcal{K}_1$ and $\mathcal{K}_n$ are adjacent in $\mathcal{B}_n$.

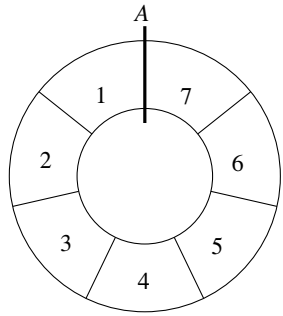


FIGURE 2: A 7-bracelet $\mathcal{B}_7$

We point out here that $\mathcal{B}_n$ is not a bracelet in the conventional mathematical sense. In the field of combinatorics, bracelets of length n are

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mathematical entities that may be regarded as equivalence classes of n-character strings over some alphabet of size k, in which rotations and reflections are taken as equivalent. Such bracelets represent, for example, structures comprising n beads of up to k different colours set out in a circle. However, the fact that the cells of $\mathcal{B}_n$ are numbered means that there are no such symmetry considerations in our scenario.

Let us now consider the enumeration of the tilings of $\mathcal{B}_n$ using uncoloured *curved* tiles from the set $\mathcal{C}(j) = \{(1 \times 1)_{\mathcal{C}}, (1 \times 2)_{\mathcal{C}}, \ldots, (1 \times j)_{\mathcal{C}}\}$. The element $(1 \times m)_{\mathcal{C}}$, where $1 \leqslant m \leqslant j$, is such that, when oriented correctly, it fits exactly onto any m consecutive cells of $\mathcal{B}_n$. Note that tiles are allowed to lie directly on top of the boundary A separating $\mathcal{X}_1$ and $\mathcal{X}_n$. Indeed, a particular tiling of $\mathcal{B}_n$ is termed *out-of-phase* if one of its tiles does in fact cover A, and *in-phase* if this is not the case [1, 2]. In the former situation we say that the tiling contains a 'straddling tile'.

We shall use $H_n(j)$ to denote the number of ways of tiling $\mathcal{B}_n$ employing elements from the set $\mathcal{C}(j)$. The aim of this Note is to demonstrate how a recurrence relation for $H_n(j)$ may be obtained via either an indirect or a direct approach. The indirect approach, shown in the following section, proceeds by way of the generalised Fibonacci numbers $G_n(j)$. We make the point here that we are not concerned so much with the sequences themselves, some of which are well-known, as we are with the derivation of the recurrence relations that generate these sequences.

An indirect approach

The evaluation of $H_n(j)$ may be carried out by first considering the case in which the tilings are in-phase, and then going on to look at the out-of-phase tilings. Note that the number of in-phase tilings of $\mathcal{B}_n$ using tiles from $\mathcal{C}(j)$ is equal to the number of tilings of $\mathcal{R}_n$ using tiles from $\mathcal{S}(j)$. The number of in-phase tilings is therefore given by $G_{n+j-1}(j)$. As for the out-of-phase tilings, $(1 \times 2)_{\mathcal{C}}$ can, for $n \geqslant 2$, straddle A in exactly one way, in which case the number of tilings is given by $G_{n+j-3}(j)$. Next, for $n \geqslant 3$, $(1 \times 3)_{\mathcal{C}}$ can straddle A in two distinct ways, leaving either $\mathcal{X}_1$, $\mathcal{X}_n$ and $\mathcal{X}_{n-1}$ covered or $\mathcal{X}_2$, $\mathcal{X}_1$ and $\mathcal{X}_n$ covered. In this case there are $2G_{n+j-4}(j)$ tilings. In general, for $n \geqslant m$ and $2 \leqslant m \leqslant j$, $(1 \times m)_{\mathcal{C}}$ is able to straddle A in $m - 1$ ways, giving rise to $(m - 1)\,G_{n+j-m-1}(j)$ tilings.

It follows that, for $j \geqslant 2$,

$$H_n(j) = G_{n+j-1}(j) + \sum_{k=1}^{j-1} kG_{n+j-k-2}(j), \tag{1}$$

for $n \geqslant 1$. Notice that, rather than giving a recurrence relation for $H_n(j)$ directly, this leaves us with a formula for $H_n(j)$ that may be expressed as a linear combination of the entries in $(G_n(j))_{n \geqslant 0}$.

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For the special case $j = 2$, (1) gives

$$\begin{aligned} H_n(2) &= G_{n+2-1}(2) + \sum_{k=1}^{2-1} kG_{n+2-k-2}(2) \\ &= G_{n+1}(2) + G_{n-1}(2) \\ &= F_{n+1} + F_{n-1} \\ &= L_n, \end{aligned}$$

which, incidentally, provides us with a combinatorial interpretation of the nth Lucas number L_n [2, 5, 6]; namely, L_n enumerates the tilings of an n-bracelet using the curved tiles $(1 \times 1)_{\mathfrak{C}}$ and $(1 \times 2)_{\mathfrak{C}}$.

To take a further example, set $j = 4$ in (1). In this case the sequence $(G_n(j))_{n \geqslant 0}$ specialises to the so-called 'tetranacci' sequence given by $(T_n)_{n \geqslant 0}$, for which the recurrence is given by

$$T_n = T_{n-1} + T_{n-2} + T_{n-3} + T_{n-4}$$

when $n \geqslant 4$, with initial conditions $T_0 = T_1 = T_2 = 0$ and $T_3 = 1$. Note that, for $n \geqslant 1$, T_{n+3} enumerates the tilings of $\mathcal{R}_n$ using tiles from $\mathcal{S}(4)$. From (1) we obtain

$$H_n(4) = T_{n+3} + T_{n+1} + 2T_n + 3T_{n-1}.$$

On using the above formula for $H_n(4)$ and the definition of T_n (momentarily dropping the argument $j = 4$ for economy of notation), we have, for $n \geqslant 4$,

$$\begin{aligned} H_n + H_{n-1} + H_{n-2} + H_{n-3} &= \sum_{k=0}^{3} \left(T_{n-k+3} + T_{n-k+1} + 2T_{n-k} + 3T_{n-k-1}\right) \\ &= T_{n+4} + T_{n+2} + 2T_{n+1} + 3T_n \\ &= H_{n+1}, \end{aligned}$$

giving us a recurrence relation for $H_n(4)$. In order to complete the definition of $H_n(4)$, we are just left to obtain the initial conditions, which are given by $H_1 = 1, H_2 = 3, H_3 = 7$ and $H_4 = 15$.

More generally, we may use (1) to obtain the following:

$$\begin{aligned} \sum_{i=0}^{j-1} H_{n-1}(j) &= \sum_{i=0}^{j-1} \left(G_{n+j-i-1}(j) + \sum_{k=1}^{j-1} kG_{n+j-k-i-2}(j) \right) \\ &= \sum_{i=0}^{j-1} G_{n+j-i-1}(j) + \sum_{k=1}^{j-1} k \sum_{i=0}^{j-1} G_{n+j-k-i-2}(j) \\ &= G_{n+j}(j) + \sum_{k=1}^{j-1} kG_{n+j-k-1}(j) \\ &= H_{n+1}(j). \end{aligned}$$

Alternatively, this recurrence relation may be written as

$$H_n(j) = \sum_{i=1}^{j} H_{n-i}(j), \tag{2}$$

for which the initial conditions are given by $H_n(j) = 2^n - 1, 1 \leqslant n \leqslant j$.

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A direct approach

Our recurrence relation (2) may also be derived by establishing a bijection between the set of tilings of $\mathcal{B}_{n-j}$, $\mathcal{B}_{n-j+1}$, … , $\mathcal{B}_{n-1}$ and the set of tilings of $\mathcal{B}_n$. The mapping we employ for this task augments each tiling $\mathcal{T}_{n-m}$ of $\mathcal{B}_{n-m}$ to some tiling $\mathcal{T}_n$ of $\mathcal{B}_n$ by inserting the tile $(1 \times m)_{\mathscr{C}}$ between a particular pair of adjacent cells in $\mathcal{T}_{n-m}$. This will be elucidated further in the following two paragraphs.

Let n and j be such that $n \geqslant j+1$. We define the *first tile* of a tiling $\mathcal{T}_n$ of $\mathcal{B}_n$ to be the non-straddling tile covering the lowest-numbered cell possible. By way of an example, let us consider tilings of $\mathcal{B}_7$, where $j = 3$. If $(1 \times 2)_{\mathscr{C}}$ covers $\mathcal{X}_1$ and $\mathcal{X}_2$, $(1 \times 2)_{\mathscr{C}}$ covers $\mathcal{X}_3$ and $\mathcal{X}_4$, and $(1 \times 3)_{\mathscr{C}}$ covers $\mathcal{X}_5$ to $\mathcal{X}_7$, the first tile is the one covering $\mathcal{X}_1$ and $\mathcal{X}_2$. However, if $(1 \times 2)_{\mathscr{C}}$ covers $\mathcal{X}_2$ and $\mathcal{X}_3$, $(1 \times 2)_{\mathscr{C}}$ covers $\mathcal{X}_4$ and $\mathcal{X}_5$, and $(1 \times 3)_{\mathscr{C}}$ is a straddling tile covering $\mathcal{X}_1$, $\mathcal{X}_7$ and $\mathcal{X}_6$, the first tile is the one covering $\mathcal{X}_2$ and $\mathcal{X}_3$. Note that the condition $n \geqslant j+1$ ensures that each tiling $\mathcal{T}_n$ of $\mathcal{B}_n$ using elements from $\mathscr{C}(j)$ will possess at least one tile that does not straddle A.

Let us now consider a tiling $\mathcal{T}_{n-m}$ of $\mathcal{B}_{n-m}$ using elements from $\mathscr{C}(j)$, where m is any integer such that $1 \leqslant m \leqslant j$. We are going to map this to a tiling $\mathcal{T}_n$ of $\mathcal{B}_n$, as follows. If $\mathcal{T}_{n-m}$ is an in-phase tiling of $\mathcal{B}_{n-m}$, then its image, $\mathcal{T}_n$, is obtained by covering cells $\mathcal{X}_1$ to $\mathcal{X}_m$ of $\mathcal{B}_n$ with $(1 \times m)_{\mathscr{C}}$, the first tile of $\mathcal{T}_n$, and then covering the remaining cells of $\mathcal{B}_n$, $\mathcal{X}_{m+1}$ to $\mathcal{X}_n$, with a tiling corresponding exactly to that of the cells $\mathcal{X}_1$ to $\mathcal{X}_{n-m}$ in $\mathcal{B}_{n-m}$. Suppose, on the other hand, that $\mathcal{T}_{n-m}$ is an out-of-phase tiling of $\mathcal{B}_{n-m}$. Its straddling tile is given by $(1 \times (a+b))_{\mathscr{C}}$ for some $a, b \geqslant 1$ such that $a + b \leqslant j$, where, to the left of boundary A in Figure 2, this tile occupies $\mathcal{X}_1$ to $\mathcal{X}_a$, and to the right of A it occupies $\mathcal{X}_{n-m-b+1}$ to $\mathcal{X}_{n-m}$. We obtain $\mathcal{T}_n$, the image of $\mathcal{T}_{n-m}$, by covering $\mathcal{X}_{a+1}$ to $\mathcal{X}_{a+m}$ of $\mathcal{B}_n$ with $(1 \times m)_{\mathscr{C}}$, the first tile of $\mathcal{T}_n$, then covering both $\mathcal{X}_1$ to $\mathcal{X}_a$ and $\mathcal{X}_{n-b+1}$ to $\mathcal{X}_n$ with the straddling tile $(1 \times (a+b))_{\mathscr{C}}$, and finally covering $\mathcal{X}_{a+m+1}$ to $\mathcal{X}_{n-b}$ with a tiling corresponding exactly to that of the cells $\mathcal{X}_{a+1}$ to $\mathcal{X}_{n-m}$ in $\mathcal{B}_{n-m}$.

Now let $m_1, m_2 \in \mathbb{N}$, where $1 \leqslant m_1 \leqslant j$ and $1 \leqslant m_2 \leqslant j$. Whether or not $m_1 = m_2$, it is clear, from the definition of the above mapping, that for any two distinct tilings, $\mathcal{T}_{n-m_1}$ and $\mathcal{T}'_{n-m_2}$, of $\mathcal{B}_{n-m_1}$ and $\mathcal{B}_{n-m_2}$, respectively, their images will be distinct. It is thus the case that the given mapping from the set of tilings of $\mathcal{B}_{n-j}$, $\mathcal{B}_{n-j+1}$, … , $\mathcal{B}_{n-1}$ to the set of tilings of $\mathcal{B}_n$ is injective.

Finally, since the condition $n \geqslant j+1$ guarantees that each tiling $\mathcal{T}_n$ of $\mathcal{B}_n$ using elements from $\mathscr{C}(j)$ will possess at least one tile that does not straddle A, it follows that for each m with $1 \leqslant m \leqslant j$, every tiling $\mathcal{T}_n$ of $\mathcal{B}_n$ with a first tile given by $(1 \times m)_{\mathscr{C}}$ has some pre-image $\mathcal{T}_{n-m}$ of $\mathcal{B}_{n-m}$. This tells us that the mapping from the set of tilings of $\mathcal{B}_{n-j}$, $\mathcal{B}_{n-j+1}$, … , $\mathcal{B}_{n-1}$ to the set of tilings of $\mathcal{B}_n$ is surjective. From this we see that the mapping is a bijection, showing that (2) is true. As in the previous section, in order to complete the definition of $H_n(j)$, we simply need the initial conditions $H_n(j) = 2^n - 1, 1 \leqslant n \leqslant j$.

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MARTIN GRIFFITHS

Dept. of Mathematical Sciences, University of Essex, Colchester CO4 3SQ

Nemo, continued from page 68.

3. Thus how might you express an accumulation containing the integers b, a^1, a^2, s, v ?

$$b = s - v \overset{\times}{+} a^1$$

or $a^2 + v + a^1 \times s = b$?

4. Algebra as far as the quadratic equation and the use of logarithms are often of value in ordinary cases: but all beyond these is but a luxury; a delicious luxury indeed; but not be in indulged in by one who is to have a profession to follow for his subsistence.
5. This is a peculiar kind of arithmetick, which takes the quantity sought, whether it be a number or a line, or any other quantity, as if it were granted, and by means of one or more quantities given, proceeds by consequence, till the quantity at first only supposed to be known, or at least some power thereof, is found to be equal to some quantity or quantities which are known, and consequently itself is known.
6. "$x^2 - y^2$," S − was saying, " $= (x + y)(x - y)$. And the equation holds good whatever the values of x and y…".

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