1

Colouring graphs on surfaces

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Developments in graph colouring theory were motivated by the four-colour problem and Heawood's theorem. Both of these were originally formulated as map-colouring problems that can be expressed as colouring graphs embedded on surfaces. This chapter gives an overview of the abundance of results concerning the chromatic number of graphs that are embedded on surfaces.

1. Introduction

In 1852 Francis Guthrie asked whether the regions of every planar map can be coloured with four colours in such a way that no two regions of the map with common boundary receive the same colour. In effect, by duality Guthrie was asking whether every planar graph is 4-colourable. This easily stated problem became known as the famous four-colour problem. Attempts to solve it led to many important results in graph theory, but the problem itself remained unsolved for more than a century. It was finally answered in the positive by Appel and Haken [9], [10], [12]. More about its proof comes later in this chapter.

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For more than a century, the four-colour problem was one of the driving forces that led to developments in graph theory, and specifically graph colouring theory. Its generalization – the Heawood problem – was the main motivation for developments in topological graph theory (see [51] or [49]). Both of these were originally formulated as map-colouring problems, asking about colouring the faces of an embedded graph so that adjacent faces receive different colours. Every map-colouring problem can be expressed as a graph-colouring problem by considering the dual graph of the map. Its vertices correspond to the faces of the map, and two vertices are adjacent if the corresponding faces are adjacent on the surface.

In this chapter we discuss the colouring of graphs embedded on surfaces. We start by describing some of the ideas in the proof of the four-colour theorem and give a proof of the list-colouring version of the five-colour theorem that is due to Thomassen. Next, we consider a generalization to general surfaces, known as the Heawood problem.

What is the largest chromatic number of a graph that can be embedded in a given surface?

A solution was conjectured by Heawood [36] and eventually proved by Ringel and Youngs [52], [51]. Heawood's conjecture can be reduced to determining the genus and the non-orientable genus of complete graphs. The solution by Ringel and Youngs involves constructions of minimum genus embeddings of these graphs and introduces techniques that are of interest in topological graph theory. Finally, we outline colouring results and problems of a more specialized nature. There is an abundance of new results in this area and we will only give an overview of some of the most important achievements.

2. Planar graphs are 4-colourable and 5-choosable

A direct consequence of Euler's formula is that every planar graph has a vertex of degree 5 or less, from which it follows that every planar graph is 6-colourable. Heawood [36] proved that every planar graph is 5-colourable. In this section we present the extension of this result to list-colourings due to Thomassen [59].

Let G be a graph and suppose that, for each vertex v, some set $L(v)$ of natural numbers is given. We view $L(v)$ as a *list* of colours that can be used when colouring v. An *L-colouring* of G is an assignment of a colour to each vertex of G such that each vertex v receives a colour from its list $L(v)$ and adjacent vertices get different colours. Such a colouring is also called a *list-colouring*. The graph G is *k-choosable* if it admits an L-colouring for every list assignment in which each list $L(v)$ contains at least k colours. The *list-chromatic number* $\chi_l(G)$ is the minimum integer k for which G is k-choosable.

In 1975 Vizing raised the question of whether every planar graph is 5-choosable. Erdős, Rubin and Taylor [31] conjectured that every planar graph is 5-choosable, but

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not necessarily 4-choosable. In 1993 Voigt [65] proved the last part of this conjecture by exhibiting examples of planar graphs that are not 4-choosable. Subsequently, even 3-colourable non-4-choosable planar graphs have been found (see Voigt and Wirth [67] and Mirzakhani [46]). Thomassen [59] verified the first part of the conjecture, which implies, in particular, that every planar graph is 5-colourable.

Theorem 2.1 *Every planar graph is* 5*-choosable.*

In order to prove Theorem 2.1, it suffices to show that it holds for every planar triangulation (a plane graph with all faces of length 3). Thus, Theorem 2.1 follows immediately from Theorem 2.2 below. We recall that a plane graph G is a *near-triangulation* if it is 2-connected and if all faces, except possibly the outer face, are triangles. Thomassen [59] proved the following result.

Theorem 2.2 *Let G be a near-triangulation with outer cycle $C = v_1v_2 \dots v_pv_1$. Assume that $L(v_1) = \{1\}$ and $L(v_2) = \{2\}$ and that, for any other vertex v, the list $L(v)$ has at least three colours if $v \in V(C)\backslash\{v_1, v_2\}$ and at least five colours if $v \in V(G - C)$. Then G has an L-colouring.*

Proof (by induction on the number n of vertices in G) If $n = 3$, there is nothing to prove. So we proceed to the induction step.

If C has a chord, say v_iv_j, where $2 \le i \le j - 2 \le p - 1$ (and $v_{p+1} = v_1$), then we apply the induction hypothesis to the cycle $C' = v_1v_2 \dots v_iv_jv_{j+1} \dots v_1$ and its interior. We also apply induction to $C'' = v_jv_iv_{i+1} \dots v_{j-1}v_j$ and its interior, where the list of vertices v_i and v_j consists of the single colour that was obtained by using the induction hypothesis on C' and its interior.

Suppose now that C is a chordless cycle, and let $v_1, u_1, u_2, \dots, u_m, v_{p-1}$ be the neighbours of v_p in clockwise order around v_p. Since the interior of C is triangulated, G contains the path $P = v_1u_1u_2 \dots u_mv_{p-1}$. Since C is chordless, $C' = P \cup (C - v_p)$ is a cycle. Let x, y be two colours in $L(v_p)\backslash\{1\}$. Now define $L'(u_i) = L(u_i)\backslash\{x, y\}$, for $1 \le i \le m$, and $L'(v) = L(v)$ if $v \in V(G)\backslash\{u_1, \dots, u_m\}$. Now apply the induction hypothesis to C' and its interior and the new lists L'. We extend the obtained L'-colouring to an L-colouring of G by assigning x or y to v_p in such a way that v_p and v_{p-1} get distinct colours. ■

The four-colour problem was solved by Appel and Haken [9], [10], [12]. An updated version of their proof may be found in [11], and a simpler and cleaner proof was given by Robertson, Sanders, Seymour and Thomas [54]. An in-depth discussion of the four-colour theorem, comments about its proof and its consequences can be found in [68].

Four-colour theorem *Every planar graph is* 4*-colourable.*

The proof of Appel and Haken is lengthy, and part of it was done by a brute-force computer checking of the millions of possibilities that occurred in their proof.

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Initially, there was a lot of controversy about the legitimacy of this proof. It was considered by some mathematicians to be inadequate since it did not qualify as a conventional proof that could be checked by a human in a lifetime. The original proof also had some flaws, but all those that were discovered were subsequently corrected by the authors in their monograph [11].

The proof of Appel and Haken follows the outline proposed by the German mathematician Heinrich Heesch. First, it is shown that every planar graph contains a subgraph from a certain list $\mathcal{C}$ of configurations. The proof of this part is based on a discharging method, a very elaborate use of Euler's polyhedron formula. Secondly, it is shown that no configuration from $\mathcal{C}$ can occur in a minimal counter-example to the four-colour theorem. Having a particular configuration $X \in \mathcal{C}$ contained in a minimal counter-example G, the minimality would ensure that the rest of the graph, when X is removed from G, can be 4-coloured. There may be many possibilities for how the colouring would affect the neighbourhood of X. For some configurations considered by Appel and Haken, several hundred thousands of cases needed to be considered. It is here where the use of computers was necessary. For each such case, it was possible to argue that the colouring can be changed in such a way that it can be extended to X, yielding a contradiction to the assumption that G is a counter-example.

The proof method described above can be illustrated by the following example. A simple corollary of Euler's formula states that every planar graph contains a vertex with at most five neighbours, so the set $\mathcal{C}$ could consist of six configurations, each corresponding to a vertex of degree i, for $0 \leq i \leq 5$. If v is a vertex of degree at most 3, and if $G - v$ has a 4-colouring, then this colouring can be extended to G, since v can be coloured by a colour that does not appear among its neighbours. It can also be shown that a vertex of degree 4 cannot appear in a minimal counter-example, but there is no way to prove the same for vertices of degree 5. This case has to be replaced by hundreds of more complicated configurations.

In 1997 Robertson, Sanders, Seymour and Thomas [54] developed another proof, based on similar principles. The human-checkable part of the proof was made more transparent, and the computer-verified part was supported by a well-structured collection of data that enabled independent checking of the proofs. There may be doubts about the correctness of the compilers and the stability of the hardware used in the computer checking, but this is probably much less susceptible to error than human checking, especially because the latter proof has been independently checked on different platforms.

The complete proof of the four-colour theorem is available on-line with links from [57]. It includes the encoding of the unavoidable set of the 633 near-triangulations, the discharging rules, the computer programs, and other helpful information needed for the proof.

The four-colour theorem is interesting, in that it has many equivalent formulations, some of which do not seem to be related to map colourings at all. Two equivalent versions most frequently used in the literature are dual statements about nowhere-zero flows and edge-colourings, the topics considered in Chapters 5 and 9.

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Theorem 2.3 *Every* 2*-edge-connected planar graph has a nowhere-zero* 4*-flow.*

Theorem 2.4 *Every* 2*-connected cubic planar graph is* 3*-edge-colourable.*

An important side result of the proof of the four-colour theorem is a polynomially bounded algorithm for 4-colouring planar graphs (see [53]).

Theorem 2.5 *A* 4*-colouring of a given planar graph of order n can be found in* $O(n^2)$ *steps.*

In contrast to Theorem 2.5, deciding whether a planar graph is 3-colourable is much harder.

Theorem 2.6 *The decision problem as to whether a given planar graph is* 3*-colourable is NP-complete.*

An interesting (but rather easy) fact about 3-colourings of planar graphs is the following folklore result.

Theorem 2.7 *A planar graph is* 3*-colourable if and only if it is a spanning subgraph of a* 3*-colourable triangulation of the plane. A triangulation is* 3*-colourable if and only if all its vertices have even degree.*

Another folklore result that should be mentioned here concerns 2-colourable planar graphs. By König's theorem, a graph is bipartite if and only if all its cycles have even lengths. For a planar graph, the face-boundaries generate the cycle space, giving the following statement.

Theorem 2.8 *A planar graph is bipartite if and only if all its faces have even length.*

A list-colouring result for planar bipartite graphs was proved by Alon and Tarsi [7] who used the polynomial method that proved to be an indispensable tool for working with the list-chromatic number.

Theorem 2.9 *Every planar bipartite graph is* 3*-choosable.*

This result is somewhat surprising since, as noted earlier, there are 3-colourable planar graphs whose list-chromatic number is 5 (see Voigt and Wirth [67] and Mirzakhani [46]).

To conclude this section, we present some results about extensions of a partial colouring. Albertson [2] proved that every precolouring of vertices of a planar graph that are far apart from each other can be extended to a 5-colouring of the whole graph.

Theorem 2.10 *If S is a subset of the vertices of a planar graph G, and if the distance between any two vertices in S is at least* 4*, then any* 5*-colouring of S can be extended to a* 5*-colouring of G.*

Proof The proof is based on the four-colour theorem. We start with a 4-colouring of G using colours 1, 2, 3, 4. For each vertex $s \in S$, recolour s according to the given

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precolouring. If c is the colour given to s, then we recolour all the neighbours of s coloured c with the colour 5. It is easy to see that this gives a 5-colouring of G that extends the given precolouring. ■

Albertson asked whether this theorem can be extended to list-colourings. This problem, which became known as Albertson's conjecture, was solved recently by Dvořák, Lidický, Mohar and Postle [28].

Theorem 2.11 *If G is a planar graph with list-assignment L that gives a list of size* 1 *or* 5 *to each vertex, and if the distance between any pair of vertices with lists of size* 1 *is at least* 1984*, then G is L-colourable.*

Dvořák, Lidický and Mohar [27] obtained another result of a similar flavour.

Theorem 2.12 *Every graph drawn in the plane with some edges crossing each other is* 5*-choosable if all of the crossings are at graph distance* 15 *or more from each other.*

3. Heawood's formula

From now on we consider graphs embedded on arbitrary surfaces. Recall that every surface is homeomorphic to precisely one of the following surfaces:

- the *orientable surface S_g of genus g* homeomorphic to the sphere with g handles ($g \geq 0$)
- the *non-orientable surface N_h of genus h* homeomorphic to the sphere with h crosscaps ($h \geq 1$).

Since adding two crosscaps to a non-orientable surface is equivalent to adding one handle, it makes sense to unify the notion of the genus by introducing the *Euler genus* $\widehat{\mathbf{g}}(S)$ of a surface S by setting $\widehat{\mathbf{g}}(S) = 2g$ if S is homeomorphic to S_g, and $\widehat{\mathbf{g}}(S) = h$ if S is homeomorphic to N_h.

Suppose that G is a graph with n vertices, m edges and f faces, that is embedded in a surface S. Then, by *Euler's formula*,

$$n - m + f \geq 2 - \widehat{\mathbf{g}}(S),$$

where equality holds if and only if the embedding is cellular (all faces are homeomorphic to open discs in the plane).

Heawood [36] proved the following analogue of the four-colour theorem for graphs embedded on arbitrary surfaces.

Theorem 3.1 *Let S be a surface with Euler genus $g > 0$ and let G be a graph embedded in S. Then*

$$\chi(G) \leq \left\lfloor \tfrac{1}{2}\left(7 + \sqrt{1 + 24g}\right) \right\rfloor. \tag{1}$$

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Proof Let $r = \chi(G)$. We may assume that each vertex of G has degree at least $r-1$, since if $\deg(v) < r-1$, then $\chi(G-v) = \chi(G)$, and so we may reduce the problem to $G - v$. Euler's formula implies that $m \leq 3n - 6 + 3g$. Since $2m \geq (r-1)n$,

$$(r-7)n + 12 - 6g \leq 0. \tag{2}$$

Since $g > 0$, the right-hand side of (1) is at least 6, and so we may assume that $r \geq 7$. Since $n \geq r$, (2) implies that $r^2 - 7r + 12 - 6g \leq 0$. This proves (1). ■

We note that, by the four-colour theorem, the bound (1) holds also for the sphere.

The proof of Theorem 3.1 shows that every graph embeddable on S has a vertex of degree not exceeding $\frac{1}{2}(5 + \sqrt{1+24g})$. Thus, the Heawood bound also holds for the list-chromatic number:

$$\chi_l(G) \leq \left\lfloor \tfrac{1}{2}\left(7 + \sqrt{1+24g}\right) \right\rfloor. \tag{3}$$

Heawood conjectured that equality holds in (1). This was eventually verified by Ringel and Youngs [51], [52] who proved that, for each surface except the Klein bottle, there exists a complete graph G for which equality holds in (1). Below we discuss this result in more detail in terms of colour-critical graphs.

A graph G is *k-colour-critical* (or *k-critical*) if $\chi(G) = k$ but $\chi(G') < k$ for every proper subgraph G' of G.

If v is a vertex of a k-critical graph G, then $G - v$ is $(k-1)$-colourable. Since its colouring cannot be extended to a $(k-1)$-colouring of G, all $k-1$ colours must be present on the neighbours of v. In particular, v has degree at least $k-1$. If G has n vertices and m edges, this implies that

$$2m \geq (k-1)n, \tag{4}$$

with equality for K_k. One of Dirac's results [23] on k-critical graphs is the following improvement of (4). If G is a k-critical graph with n vertices and m edges, and if $G \neq K_k$, then

$$2m \geq (k-1)n + k - 3. \tag{5}$$

Let $H(g) = \left\lfloor \frac{1}{2}(7 + \sqrt{1+24g}) \right\rfloor$. Heawood's theorem asserts that every graph G on a surface of Euler genus g has chromatic number at most $H(g)$. Dirac [21], [22] proved the following extension for all values of g except 1 and 3; these cases were proved later by Albertson and Hutchinson [4].

Theorem 3.2 *If G is a graph embeddable in the surface of Euler genus $g \geq 1$, then $\chi(G) < H(g)$ unless G contains the complete graph of order $H(g)$ as a subgraph.*

Proof We present Dirac's proof which works for $g \neq 1$ or 3. Let $h = \chi(G) = H(g)$, and let H be an h-critical subgraph of G. Assume that H has n vertices and m edges,

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and that $H \neq K_h$. By (5), we have $2m \geq (h-1)n + h - 3$. This inequality, combined with the aforementioned consequence of Euler's formula that $m \leq 3n - 6 + 3g$, gives

$$(h-1)n + h - 3 \leq 6n - 12 + 6g,$$

which implies that

$$(h-7)n \leq 6g - 9 - h. \tag{6}$$

Since $G \neq K_h$, and since there are no h-critical graphs on $h+1$ vertices, $n \geq h+2$. Since $g \geq 2$, $h \geq 7$, and thus (6) implies

$$(h-7)(h+2) \leq 6g - 9 - h. \tag{7}$$

It is easy to see that this inequality leads to a contradiction for $g = 2$ and when $4 \leq g \leq 11$, so we may assume that $g \geq 12$. From (7) we obtain $h^2 - 4h - 5 - 6g \leq 0$, implying that

$$h < 2 + \sqrt{6g+9}.$$

On the other hand, $h = H(g) > \frac{1}{2}(5 + \sqrt{24g+1})$. Hence

$$1 + \sqrt{24g+1} < \sqrt{24g+36}.$$

This implies that $2\sqrt{24g+1} < 34$ and therefore $g < 12$, a contradiction. ■

Since the complete graph K_7 cannot be embedded in the Klein bottle (see, for example, [49]), Theorem 3.2 implies in particular that the Heawood bound can be improved for the Klein bottle.

Theorem 3.3 *If G is a graph embeddable in the Klein bottle, then $\chi(G) \leq 6$.*

Škrekovski [55] considered $(H(g)-1)$-critical graphs on surfaces of given Euler genus g and proved that, for $g \geq 10$, the only such graphs are $K_{H(g)-1}$ and, in some exceptional cases, the join $K_{H(g)-4} + C_5$.

Theorem 3.2 was extended to list-colourings by Böhme, Mohar and Stiebitz [13], except for the case of Euler genus $g = 3$, which was solved later by Král' and Škrekovski [45].

Theorem 3.4 *If G is a graph embeddable in a surface of Euler genus $g \geq 1$, then $\chi_l(G) < H(g)$, unless G contains the complete graph of order $H(g)$ as a subgraph, in which case $\chi_l(G) = H(g)$.*

4. Colouring with few colours

For each surface and each natural number $k \geq 8$, there are only finitely many k-critical graphs embeddable on the surface (see Dirac [24] and Edwards [29]).

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Theorem 4.1 *Let S be a surface of Euler genus $g \geq 3$. Every 7-critical graph embeddable on S has at most $69(g-2)$ vertices, and for $k \geq 8$, every k-critical graph on S has at most $6(g-2)/(k-7)$ vertices.*

Since a graph is $(k-1)$-colourable if and only if it contains no k-critical subgraph, Theorem 4.1 implies the following result.

Corollary 4.2 *Let S be a surface and let $k \geq 7$. Then there are only finitely many k-critical graphs on S, and there exists a polynomially bounded algorithm for testing whether an arbitrary S-embeddable graph is $(k-1)$-colourable.*

The algorithm in Corollary 4.2 for testing $(k-1)$-colourability can be extended to a linear-time algorithm which also returns a $(k-1)$-colouring if one exists. Given a graph G on S, remove successively its vertices of degree less than $k-1$ until no such vertex exists. If $k = 7$, then also remove a vertex of degree 6 if all neighbours have degree 6 and they induce a cycle. It turns out that the order of the remaining graph H is bounded above by a constant that depends on S only, and every $(k-1)$-colouring of H can be extended (in linear time) to G. Hence G can be $(k-1)$-coloured if and only if H has a $(k-1)$-colouring (which can be discovered in constant time by checking all possibilities). A similar linear-time algorithm was proposed by Edwards [29] who also discussed in detail how to perform the removal of vertices in linear time.

Theorem 4.1 says that a graph on a fixed surface is 6-colourable unless the graph contains a small obstruction. Such an obstruction must contain a non-contractible cycle of length $O(g)$, where g is the Euler genus of the surface. Graphs that do not contain short non-contractible cycles are said to be *locally planar*, since a large neighbourhood of each vertex is planar. To quantify local planarity, we define the notion of a width of the embedded graph as follows (see [49] for more details). The *edge-width* $\mathrm{ew}(G)$ of a cellular embedded graph G is the shortest length of a cycle in G that is not contractible on the surface of this embedding. A related parameter is the *surface non-separating edge-width* $\mathrm{ew}'(G)$ which is the minimum length of a surface non-separating (homologically non-trivial) cycle in G.

We now describe a 6-colour theorem by Fisk and Mohar [33].

Theorem 4.3 *There is a universal constant c for which every graph G embedded in a surface of Euler genus $g > 0$ with $\mathrm{ew}'(G) > c \log g$ is 6-colourable.*

Let us now consider 5-colourings.

If G is k-critical and H is l-critical, then their join $G+H$ is $(k+l)$-critical. Therefore the graphs $C_3 + C_5$, $K_2 + H_7$ (where H_7 is the graph of Fig. 1) are 6-critical. Figure 2 shows that they are toroidal (where the grey vertices of the second graph correspond to K_2). Thomassen [60] characterized those toroidal graphs that are 5-colourable.

Theorem 4.4 *There are precisely four 6-critical toroidal graphs: K_6, $C_3 + C_5$, $K_2 + H_7$ (where H_7 is the graph of Fig. 1), and the graph T_{11} obtained from the 11-cycle $x_0x_1 \ldots x_{10}x_0$ by adding all chords x_ix_{i+2} and x_ix_{i+3} for $i = 0, 1, \ldots, 10$ (see Fig. 3).*

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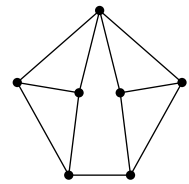


Fig. 1. The 4-critical graph H_7

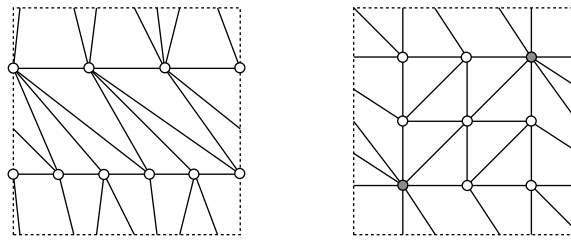


Fig. 2. The graphs $C_3 + C_5$ and $K_2 + H_7$ on the torus

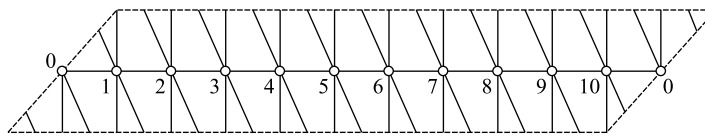


Fig. 3. The graph T_{11} on the torus

All 6-critical graphs in Theorem 4.4 contain non-contractible triangles in all their embeddings on the torus. This implies the following result.

Corollary 4.5 *If a graph G is embedded in the torus such that* $\mathrm{ew}(G) \geq 4$*, then G is* 5*-colourable.*

Thomassen [58] extended this result to all orientable surfaces.

Theorem 4.6 *If G is embeddable in an orientable surface of genus g so that the edge-width is at least* 2^{14g+6}*, then G is* 5*-colourable.*

The proof of Theorem 4.6 uses the fact that, for embedded graphs of sufficiently large edge-width, the surface can be cut in such a way that we obtain a planar graph after the cutting, and pairs of vertices that are duplicated by the cutting procedure are far from each other. An extension of the 5-colour theorem for planar graphs is then used, where we take care of colouring the duplicate pairs with the same colour.

Thomassen asked whether his Theorem 4.6 can be extended to list-colourings. His question was answered in the affirmative by DeVos, Kawarabayashi and Mohar [20].

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Theorem 4.7 *For each surface S there exists a constant M for which every graph that is embedded in S with edge-width at least M is 5-choosable.*

Another extension of Theorem 4.6 was proved by Thomassen [62].

Theorem 4.8 *For each surface S, there are only finitely many 6-critical S-embeddable graphs.*

The complete list of 6-critical graphs is known for the sphere (where there are none), the projective plane (where K_6 is the only one, by Theorem 3.2), for the torus (Theorem 4.4), and for the Klein bottle [18], [39]. A generalization of Theorem 4.8 to 5-list-colourings has been obtained by Kawarabayashi and Mohar [40].

Corollary 4.9 *For each fixed surface S, there is a polynomial-time algorithm for deciding whether a graph on S can be 5-coloured. The same holds for list-colourings: it can be decided whether a given graph G embedded in S is 5-choosable and, given G and a list-assignment L with all lists of size at least 5, whether G is L-colourable.*

As mentioned earlier, the problem of 3-colouring graphs on the sphere (and hence any surface) is NP-complete. Corollary 4.9 thus suggests the following problem. Let S be a fixed surface. Does there exist a polynomially bounded algorithm for deciding whether a graph on S can be 4-coloured?

The following result of Fisk [32] implies that there is no 4-colour analogue of Theorem 4.8.

Theorem 4.10 *If G is a triangulation of some surface, and if G has exactly two vertices of odd degree and they are adjacent, then G is not 4-colourable.*

The proof of Theorem 4.10 actually works for every graph in which each edge is in precisely two triangles.

Corollary 4.11 *If S is a surface other than the sphere, and if $k = 3, 4$ or 5, then there are infinitely many k-critical S-embeddable graphs.*

The following problem was proposed by Albertson [1]. Let S be any surface. Does there exist a natural number $q = q(S)$ for which any graph G embedded on S contains a set A of at most q vertices such that $G - A$ is 4-colourable? This problem is open even for the torus where possibly $q = 3$ will do, as conjectured by Albertson [1].

5. Grötzsch's theorem and its generalizations

In this section we discuss the chromatic number of graphs on a fixed surface when conditions on the girth are imposed. We start with a classical result of Grötzsch [35].

Theorem 5.1 *Every planar graph with girth at least 4 is 3-colourable.*

Concerning a list-colouring version of Grötzsch's theorem, Voigt [66] gave an example of a planar graph of girth 4 that is not 3-choosable. Of course, planar graphs

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of girth 4 are 4-choosable since they are 3-degenerate. On the other hand, Thomassen [61] proved 3-choosability under a stronger condition.

Theorem 5.2 *Every planar graph of girth at least* 5 *is* 3*-choosable.*

Grötzsch's theorem can be strengthened by allowing triangles, as long as these are far from each other. This was conjectured in 1969 by Havel and proved by Dvořák, Král' and Thomas [26].

Theorem 5.3 *There exists a constant M for which every planar graph whose triangles are at distance at least M from each other is* 3*-colourable.*

The constant M in [26] is extremely large. Dvořák recently succeeded in extending Theorem 5.2 to 3-list-colourings, under the assumption that cycles of length 3 or 4 are far apart.

Inspired by Theorems 5.1 and 4.6, Hutchinson [37] proved the following.

Theorem 5.4 *For each positive integer g there exists a number $f(g)$ for which the following holds. If G is embedded in the orientable surface $\mathbb{S}_g$ so that* $\mathrm{ew}(G) > f(g)$ *and all facial walks have even length, then G is* 3*-colourable.*

Dvořák, Král' and Thomas [25] extended Theorem 5.4 to arbitrary triangle-free graphs on any fixed surface S. As a corollary, they showed that 3-colourability of a triangle-free graph embeddable in S can be tested in polynomial time. In particular, this gives a linear-time algorithm to compute the chromatic number of such graphs.

While the 5-colour result in Theorem 4.6 holds both for orientable and non-orientable surfaces, the 3-colour result of Hutchinson does not extend to non-orientable surfaces because of the following counter-intuitive result of Youngs [69].

Theorem 5.5 *If G is embeddable in the projective plane such that every face is of length* 4*, then G has chromatic number* 2 *or* 4*. Thus, if G is not bipartite, then G is not* 3*-colourable.*

Theorem 5.5 does not extend to toroidal graphs. This follows from Theorem 5.4, or by taking the Cartesian product of two cycles. An easy modification of this example shows also that Theorem 5.5 does not hold for graphs on the Klein bottle. However, Klavžar and Mohar [44] proved that twisted Cartesian products of two cycles of arbitrarily large edge-width on the Klein bottle sometimes need four colours. Mohar and Seymour [47] obtained a counterpart of Theorem 5.4 for non-orientable surfaces, by proving that every non-3-colourable graph of large edge-width with all faces of even length contains a quadrangulation that is not 3-colourable, and then characterizing completely the non-3-colourable quadrangulations. A generalization and full explanation of these phenomena were demystified by DeVos *et al.* [19] and are presented in the section on the duality of colourings and flows.

Gimbel and Thomassen [34] extended Theorem 5.5 to a complete characterization of those graphs in the projective plane which have girth at least 4 and are not 3-colourable.

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Theorem 5.6 *Let G be a graph of girth at least 4 embedded in the projective plane. Then G is 3-colourable if and only if G does not contain a quadrangulation of the projective plane that is not bipartite.*

Gimbel and Thomassen [34] also gave an upper bound for the chromatic numbers of triangle-free graphs on general surfaces, analogous to Theorem 3.1 except that the bound is not sharp. For simplicity we formulate it for orientable surfaces only.

Theorem 5.7 *There exist positive constants c_1 and c_2 for which the following statements hold.*

- *Every triangle-free graph of genus g has chromatic number at most $c_1\sqrt[3]{g/\log g}$.*
- *For each $g \geq 1$, there exists a triangle-free graph of genus g and chromatic number at least $c_2\sqrt[3]{g}/\log g$.*

The following result was proved independently by Fisk and Mohar [33] and by Gimbel and Thomassen [34].

Theorem 5.8 *Let S be a surface.*

- *For each $k \geq 5$, there are only finitely many k-critical triangle-free graphs embeddable on S.*
- *There are only finitely many 4-critical graphs with girth at least 6 on S.*

Theorem 5.8 implies that, for each positive integer g, there exists an integer $w(g)$ for which any triangle-free graph embedded on a surface of Euler genus g with edge-width exceeding $w(g)$ can be 4-coloured. Fisk and Mohar [33] proved that $w(g) < c \log g$, where c is an absolute constant. An analogous result holds for 3-colourings of graphs with girth at least 6.

Theorem 5.8 also shows that, for each surface S, the chromatic number of a graph of girth 6 embeddable on S can be found in polynomial time. It can also be decided in polynomial time whether a triangle-free graph on S can be 4-coloured. The same applies for 3-colourings if the girth is at least 5 (see Thomassen [63]).

Theorem 5.9 *For each surface S, there are only finitely many 4-critical graphs of girth at least 5 that can be embedded in S.*

It is known that there are no such graphs on the torus or the projective plane.

Dvořák, Král' and Thomas have announced a structural description of 4-critical triangle-free graphs embedded in a fixed orientable surface, based on which one could obtain a linear-time algorithm for deciding whether they are 3-colourable.

6. Colouring–flow duality

Tutte introduced the notion of a nowhere-zero flow (see Chapter 9) as a notion dual to a colouring. In this section we state the colouring–flow duality for planar graphs,

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and show that there is an important extension to locally planar graphs embedded on other surfaces.

For a graph G without cut-edges, let $\varphi(G)$ denote its *flow index*, the smallest k for which G admits a nowhere-zero k-flow. Tutte [64] has proved the following result.

Theorem 6.1 *If G is a connected plane graph and if G^* is its dual graph, then $\chi(G) = \varphi(G^*)$.*

This duality extends to the circular version of flows and colourings. Let S_r be a circle in $\mathbb{R}^2$ with circumference $r > 0$. Given a graph G, a mapping $c : V(G) \to S_r$ is called a *circular r-colouring* if, for each edge vw in G, the distance between $c(v)$ and $c(w)$ on the circle S_r is at least 1. The *circular chromatic number* $\chi_c(G)$ is the smallest number r for which G admits a circular r-colouring. It is a fact that the minimum r in the definition is attained, and it turns out that $\chi(G) - 1 < \chi_c(G) \leq \chi(G)$. It is easy to see that $\chi_c(C_{2k+1}) = 2 + (1/k)$, so this coincides with intuition that long odd cycles are close to being 2-colourable.

There is similarly the corresponding notion of a circular flow. Let G be a graph and let us orient the edges of G arbitrarily. A map $\varphi : E(G) \to \mathbb{R}$ is a *flow* if, at each vertex, the sum of the values on the incoming edges is equal to the sum on the outgoing edges. We say that φ is an *α-flow* if $1 \leq |\varphi(e)| \leq \alpha - 1$, for each $e \in E(G)$. Using this notion, we define the *circular flow index* of G as

$$\varphi_c(G) = \inf\{\alpha \in \mathbb{R} : G \text{ admits an } \alpha\text{-flow}\}.$$

As with the circular chromatic number, it turns out that for every graph,

$$\varphi(G) - 1 < \varphi_c(G) \leq \varphi(G).$$

Note that the above notions are independent of the chosen orientation of the edges of G. For example, if we reverse the orientation of an edge e, then by replacing $\varphi(e)$ by $-\varphi(e)$ we preserve the property that φ is an α-flow.

Theorem 6.1 can be extended to circular colourings and flows.

Theorem 6.2 *If G is a connected plane graph and if G^* is its dual graph, then $\chi_c(G) = \varphi_c(G^*)$.*

This duality does not extend beyond planar graphs. It is known that the flow index of every graph is at most 6 (and it is conjectured that it is at most 5), but the dual graph may have arbitrarily large chromatic number. Nevertheless, Theorem 4.6 eliminates such an objection when the graph is locally planar and thus has sufficiently large edge-width. Indeed, DeVos, Goddyn, Mohar, Vertigan and Zhu [19] have generalized Theorem 6.2 to graphs on arbitrary surfaces, by proving the following approximate duality.

Theorem 6.3 *Let S be an orientable surface, and let $\varepsilon > 0$. Then there is an integer M such that, for every graph G embedded in S with edge-width at least M,*

$$\varphi_c(G^*) \leq \chi_c(G) < \varphi_c(G^*) + \varepsilon,$$

where G^* is the dual graph of G in S.

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In order to obtain the corresponding duality result for non-orientable surfaces, we have to extend the definition of a flow in the dual graph. This extension gives us an alternative way of defining the circular chromatic number, and we include both definitions for comparison.

Let G be a graph embedded in a surface S, and let us orient the edges of G arbitrarily. A map $\varphi : E(G) \to \mathbb{R}$ is a *tension* if, for each cycle C in G, the sum of the values of φ on the forward edges of C is equal to the corresponding sum on the backward edges of C. This condition is equivalent to a stronger one that, for every closed walk W, the sum of the values of φ on the forward edges of W minus the sum on the backward edges of W is 0. If this condition is satisfied only for every closed walk W in G that defines a contractible curve in the surface, then φ is said to be a *local tension*. If $1 \leq |\varphi(e)| \leq \alpha - 1$ holds for each $e \in E(G)$, we say that φ is a (*local*) α*-tension*. Using these notions, we have the following:

$$\chi_c(G) = \inf\{\alpha \in \mathbb{R} : G \text{ admits an } \alpha\text{-tension}\}.$$

Analogously to this, we define a new invariant, the *local circular chromatic number*:

$$\chi_{loc}(G) = \inf\{\alpha \in \mathbb{R} : G \text{ admits a local } \alpha\text{-tension}\}.$$

Note that $\chi_{loc}(G) \leq \chi_c(G)$, and that strict inequality is possible. The concepts of tension and local tension involve an auxiliary orientation of the edges, but the values $\chi_c(G)$ and $\chi_{loc}(G)$ are independent of the chosen orientation.

If the surface is orientable, then $\chi_{loc}(G^*) = \varphi_c(G)$ and $\lceil \chi_{loc}(G^*) \rceil = \varphi(G)$, the flow index of G. If the surface is non-orientable, then $\lceil \chi_{loc}(G^*) \rceil$ is the *biflow index* of Bouchet. Thus, χ_{loc} both unifies and refines these two indices for embedded graphs. It can be used to extend Theorem 6.3 to include non-orientable surfaces. The following result is due to DeVos, Goddyn, Mohar, Vertigan and Zhu [19].

Theorem 6.4 *Let S be a surface, and let $\varepsilon > 0$. Then there is an integer M such that, for each graph G embedded in S with edge-width at least M,*

$$\chi_{loc}(G) \leq \chi_c(G) < \chi_{loc}(G) + \varepsilon,$$

where G^* is the dual graph of G in S.

DeVos *et al.* [19] also applied this result to two families of embedded graphs for which χ_{loc} exhibits interesting behaviour. An embedded graph G is *even-faced* if all of its faces have even length. Note that even-faced graphs in the plane are 2-colourable (see Theorem 2.8). It turns out that even-faced embedded graphs come in two types, odd and even, and their chromatic number depends on their type. Another class of graphs whose chromatic number is well understood are planar triangulations (see Theorem 2.7). On surfaces of higher genus, triangulations can be classified as being of even or odd type. In both cases, the *type* of G depends on the 'signature parity' of certain closed walks in the dual G^* (although in strikingly different ways). This distinction is reflected in the value of $\chi_{loc}(G)$ (see [19]).

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Theorem 6.5 (Bimodality of χ_{loc}) *Let G be an embedded graph.*

- *If G is even-faced with maximum face-length $2r$, then*

$$\chi_{loc}(G) = 2 \text{ (even type)} \quad \text{or} \quad \chi_{loc}(G) \geq \frac{2r}{r-1} \text{ (odd type)}.$$

- *If G is a triangulation, then*

$$\chi_{loc}(G) = 3 \text{ (even type)} \quad \text{or} \quad \chi_{loc}(G) \geq 4 \text{ (odd type)}.$$

As a corollary, DeVos *et al.* [19] proved the following.

Corollary 6.6 (Bimodality of χ_c) *For any surface S and any $\varepsilon > 0$, there exists an integer M such that, for every S-embedded graph G with* $\mathrm{ew}(G) \geq M$,

- *if G is even-faced with maximum face-length $2r$, then*

$$\chi_c(G) \in [2, 2+\varepsilon] \cup [2r/(r-1), 4]$$

- *if G is a triangulation, then*

$$\chi_c(G) \in [3, 3+\varepsilon] \cup [4, 5].$$

DeVos *et al.* [19] also showed that certain Eulerian triangulations of large edge-width on non-orientable surfaces have circular chromatic number 5, and that there are quadrangulations of non-orientable surfaces with arbitrarily large edge-width, yet with circular chromatic number 4. These results explain Theorem 5.5, that projective plane quadrangulations never have chromatic number 3, and they completely generalize and explain similar phenomena on general surfaces that were observed in the past.

Note that a little more can be said in some special cases.

If S is the plane or projective plane, then the circular chromatic number $\chi_c(G)$ is relatively well behaved. For example, set $\varepsilon = 0$ in Theorem 6.4 and in Corollary 6.6. If G is even-faced, then there is an exact formula for $\chi_c(G)$ (see DeVos *et al.* [19]). In particular, if G has an even-faced embedding in the projective plane, then $\chi_c(G) = 2$ or $\chi_c(G) = 2 + (2/k)$, for some non-negative integer k. Moreover, if a quadrangulation of the projective plane has $\chi_c(G) > 2$, then $\chi_c(G) = 4$; this strengthens Theorem 5.5.

If the girth of G is at least 6, then the upper bound of 4 can be replaced by 3 in the first part of Corollary 6.6 (see [19]). More generally, it may be true that we can replace 4 by $2 + 2/(g-1)$, if the girth is at least $2g$.

The interval $[4, 5]$ in the second part of Corollary 6.6 can be replaced by $\{4\}$ if S is orientable and G is Eulerian. This is implied by a result of Hutchinson *et al.* [38], which states that all triangulations of orientable surfaces of sufficiently large edge-width satisfy $\chi(G) \leq 4$.

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7. The acyclic chromatic number

A colouring (or a list-colouring) of a graph G is *acyclic* if every cycle in the graph receives at least three different colours. Equivalently, if the colour classes of the colouring are $V_1, V_2, \ldots, V_k$, then the induced subgraph on $V_i \cup V_j$ is a forest for all $i \neq j$. We define the *acyclic chromatic number* $\chi_a(G)$ of a graph G as the minimum number of colours for which there exists an acyclic colouring of G. This notion was introduced in 1973 by Branko Grünbaum, who proved that every planar graph admits an acyclic 9-colouring. After various improvements of Grünbaum's result, Borodin [14] proved the ultimate version about the acyclic chromatic number of planar graphs.

Theorem 7.1 *Every planar graph is acyclically* 5*-colourable.*

The bound of 5 is best possible since there are planar graphs whose acyclic chromatic number is 5; an example is depicted in Fig. 4. If we have a 4-colouring of this graph, then the two vertices labelled 1 and 2 must be coloured differently or we obtain a bicoloured 4-cycle, since two of their common neighbours necessarily have the same colour. Thus, any acyclic 4-colouring is as shown in the figure. However, no available colour is left for the vertex z. If we replace each thick edge in this graph by four paths of length 2 joining the same pair of vertices, we obtain an even more striking example, a bipartite 2-degenerate planar graph whose acyclic chromatic number is 5.

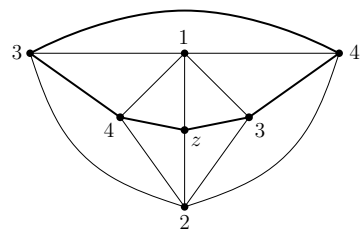


Fig. 4. A planar graph with acyclic chromatic number 5

The notion of acyclic colourings can be extended to list-colourings. We say that G is *acyclically k-choosable* if G admits an acyclic L-colouring for every list-assignment L for which each vertex has k colours in its list. The *acyclic list-chromatic number* $\chi_{a,l}(G)$ is the minimum k such that G is acyclically k-choosable.

There are many other results on acyclic list colourings of planar graphs. Borodin, Fon-der-Flaass, Kostochka, Raspaud and Sopena [16] conjectured that every planar graph is acyclically 5-choosable, and proved the following weaker statement.

Theorem 7.2 *Every planar graph is acyclically* 7*-choosable.*

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If the aforementioned conjecture is true, then it may even be true that the acyclic chromatic number and the acyclic list-chromatic number are equal for every planar graph.

Borodin and Ivanova [17] proved that every planar graph without 4-cycles is acyclically 5-choosable, and Borodin, Chen, Ivanova and Raspaud [15] conjectured that every planar graph with girth at least 5 is acyclically 3-choosable. The following weaker bound was proved by Montassier [50].

Theorem 7.3 *Every planar graph of girth at least* 5 *is acyclically* 4*-choosable.*

Concerning graphs on arbitrary surfaces, Borodin conjectured that, apart from the sphere, the maximum acyclic chromatic number for graphs embeddable on a surface equals the Heawood bound for the usual chromatic number. This conjecture was disproved for all surfaces of large genus by Alon, Mohar and Sanders [6], who proved the following result.

Theorem 7.4 *If G is a graph embeddable on a surface of Euler genus g, then* $\chi_a(G) \leq 100g^{4/7} + 10\,000$.

The corresponding bounds for the acyclic list-chromatic number have not appeared in the literature, but the proof in [6] can be adapted to give the same bounds for the list-chromatic version.

Theorem 7.5 *For each sufficiently large integer g, there is a graph G embeddable on any surface of Euler genus g that satisfies* $\chi_a(G) \geq \frac{1}{10}\, g^{4/7}/(\log g)^{1/7}$.

Examples for Theorem 7.5 were obtained by making small modifications to random graphs (with edge-probability $p = 3(\log n/n)^{1/4}$). Alon, Mohar and Sanders [6] also found Klein bottle graphs whose acyclic chromatic number is 7. For the projective plane, they provided an upper bound given in the next theorem. It is not known whether the bound can be improved to 6.

Theorem 7.6 *Every projective planar graph is acyclically* 7*-colourable.*

For locally planar graphs, the following result of Kawarabayashi and Mohar [41] gives an analogue of Theorem 4.6.

Theorem 7.7 *For each $g \geq 0$, there exists an integer M such that every graph G embeddable on a surface of Euler genus g with edge-width at least M has* $\chi_a(G) \leq 7$.

8. Degenerate colourings

The notion of a degenerate colouring is a strengthening of the notion of an acyclic colouring. A graph G is *k-degenerate* if every subgraph has a vertex of degree at most k. Note that the requirement for the acyclic chromatic number says that the union of any two colour classes induces a forest, which is equivalent to the statement

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that it induces a 1-degenerate graph. One can impose an even stronger condition on a k-colouring, that the union of t colour classes always induces a $(t-1)$-degenerate subgraph, for every t. Such a colouring is said to be *degenerate*, and one can speak of the *degenerate chromatic number* $\chi_d(G)$ and the *degenerate list-chromatic number* $\chi_{d,l}(G)$. Note that the condition for $t = 1$ is just that of a proper colouring, and the condition for $t = 2$ is the same as that for acyclic colourings. Borodin [14] conjectured that Theorem 7.1 can be strengthened.

Conjecture A *Every planar graph admits a degenerate* 5*-colouring.*

The strongest result in this direction has been obtained by Kierstead, Mohar, Špacapan, Yang and Zhu [43].

Theorem 8.1 *Every planar graph has a degenerate* 9*-colouring. Moreover, the degenerate list-chromatic number of every planar graph is at most* 9*.*

9. The star chromatic number

A proper colouring of G with no 2-coloured path on four vertices is called a *star colouring*. This is equivalent to saying that the union of any two colour classes induces a star forest – that is, a subgraph in which each component is a star $K_{1,t}$, for some $t \geq 0$. The least number n for which G admits a star colouring with n colours is called the *star chromatic number* of G, denoted by $\chi_s(G)$. Clearly, $\chi_s(G) \geq \chi_a(G)$. If a colouring is both degenerate and star, then we speak of a *degenerate star colouring*. The corresponding chromatic number is denoted by χ_{sd}.

Albertson, Chappell, Kierstead, Kündgen and Ramamurthi [3] proved that every planar graph admits a star colouring with 20 colours.

Theorem 9.1 *If G is a planar graph, then $\chi_s(G) \leq 20$.*

The largest known value of the star chromatic number of a planar graph is 10, so Theorem 9.1 leaves some room for improvement.

Kierstead, Kündgen, and Timmons [42] proved that every bipartite planar graph G satisfies $\chi_s(G) \leq \chi_{s,l}(G) \leq 14$, and they provided examples with $\chi_s(G) \geq 8$. Under a different assumption that the girth is at least 5, Albertson *et al.* [3] proved that $\chi_s(G) \leq 16$. Planar graphs with girth 5 and star chromatic number 6 were found by Timmons [56].

For general surfaces, Mohar and Špacapan [48] improved the linear bound in [3] to a sublinear upper bound.

Theorem 9.2 *If G is a graph embeddable on a surface of Euler genus g, then*

$$\chi_{sd}(G) \leq \lceil 1000g^{3/5} + 100\,000 \rceil.$$

The bound in Theorem 9.2 is essentially best possible; this is certified by the following result of Mohar and Špacapan [48].

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Theorem 9.3 *For each sufficiently large integer g, there is a graph G embeddable in an orientable (or non-orientable) surface of genus g, for which*

$$\chi_{sd}(G) \geq \chi_s(G) \geq \tfrac{1}{32}\, g^{3/5}/(\log g)^{1/5}.$$

The corresponding result for locally planar graphs was proved by Kawarabayashi and Mohar [41].

Theorem 9.4 *For each positive integer g, there is a constant M such that every graph G embeddable on an orientable (or non-orientable) surface of genus g with edge-width at least M satisfies* $\chi_s(G) \leq 2s_0^* + 3 \leq 43$, *where* s_0^* (≤ 20) *is the maximum star chromatic number for the class of all planar graphs.*

10. Summary

The known results on the chromatic number of planar and locally planar graphs are collected in the following table. The entries with $\leq$ indicate that improvement is possible. The entries with $\geq$ indicate that there are known examples with the stated chromatic number.

Table 1.

	planar				locally planar
	all	even-faced	triangle-free	girth ≥ 5	ew $\geq M(g)$
χ	4 (4CT)	2 (Th. 2.8)	3 (Th. 5.1)	3 (Th. 5.1)	5 (Th. 4.6)
χ_l	5 (Th. 2.1)	3 (Th. 2.9)	4 (3-degen.)	3 (Th. 5.2)	5 (Th. 4.7)
χ_a	5 (Th. 7.1)	5	5	-	≤ 7 (Th. 7.7)
$\chi_{a,l}$	≤ 7 (Th. 7.2)	-	-	≤ 4 (Th. 7.3)	-
χ_d	≤ 9 (Th. 8.1)	-	-	-	-
χ_s	$10 \leq s_0^* \leq 20$ (Th. 9.1)	$\leq 14,\ \geq 8$ [42]	-	$\leq 16,\ \geq 6$ [3],[56]	$\leq 2s_0^* + 3$ (Th. 9.4)

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The dependence of the chromatic number of arbitrary graphs on the genus is summarized in the table below.

Table 2.

	upper bound (all graphs)	lower bound (all graphs)	upper bound (triangle-free)	lower bound (triangle-free)
χ, χ_l	$O(g^{1/2})$ (Th. 3.1)	$\Omega(g^{1/2})$ (Th. 3.2 & 3.4)	$O((g/\log g)^{1/3})$ (Th. 5.7)	$\Omega(g^{1/3}/\log g)$ (Th. 5.7)
χ_a, $\chi_{a,l}$	$O(g^{4/7})$ (Th. 7.4)	$\Omega(g^{4/7}/\log(g)^{1/7})$ (Th. 7.5)	-	-
χ_d	$O(g^{3/5})$ (Th. 9.2)	$\Omega(g^{4/7}/\log(g)^{1/7})$ (bound from χ_a)	-	-
χ_s, χ_{sd}	$O(g^{3/5})$ (Th. 9.2)	$\Omega(g^{3/5}/\log(g)^{1/5})$ (Th. 9.3)	-	-

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