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Pushout complements for partly total algebras[†]

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Let Σ be an arbitrary signature and Υ be a non-empty set of operation symbols within it. A (partial) Σ-algebra is Υ-*total* when all its operations in Υ are total: these are the *partly total algebras* in the title, and they include total algebras and attributed graphs. In this paper we establish a necessary and sufficient condition on a pair of homomorphisms of Υ-total Σ-algebras for the existence of a pushout complement of them. This solves the application problem for the double-pushout transformation of these kinds of structure.

1. Introduction

The double-pushout (DPO) approach to algebraic transformation was initially developed for total many-sorted unary algebras (Ehrig 1979), but it can be used in any category $\mathscr{C}$ provided some basic constructions within it are known. In this DPO approach, a *production rule* in a category $\mathscr{C}$ is a span of morphisms $P = (\mathbf{L} \xleftarrow{l} \mathbf{K} \xrightarrow{r} \mathbf{R})$ in $\mathscr{C}$, and it can be *applied* to an object $\mathbf{D}$ through a morphism $m : \mathbf{L} \to \mathbf{D}$ if there exists a diagram

$$\begin{array}{ccccc} \mathbf{L} & \xleftarrow{l} & \mathbf{K} & \xrightarrow{r} & \mathbf{R} \\ {\scriptstyle m}\downarrow & & {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g'} \\ \mathbf{D} & \xleftarrow[l']{} & \mathbf{B} & \xrightarrow[r']{} & \mathbf{H} \end{array}$$

such that both squares are pushout squares in $\mathscr{C}$. Should such a diagram exist, the object $\mathbf{H}$ is said to be *derived* from $\mathbf{D}$ by the application of rule P through m, and the object $\mathbf{B}$ is called the *context object* of this derivation.

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The first basic step in the development of the DPO transformation in a category $\mathscr{C}$ is the solution of the *application problem* in $\mathscr{C}$: when can a given production rule be applied through a given morphism? If $\mathscr{C}$ has all pushouts, solving this application problem amounts to finding what has been classically called a *gluing condition*: a necessary and sufficient condition on two morphisms $l : \mathbf{K} \to \mathbf{L}$ and $m : \mathbf{L} \to \mathbf{D}$ in $\mathscr{C}$ for the existence of a *pushout complement* of them, that is, an object $\mathbf{B}$ and two morphisms $g : \mathbf{K} \to \mathbf{B}$ and $l' : \mathbf{B} \to \mathbf{D}$ such that $\mathbf{D}$, together with m and l', is a pushout of l and g in $\mathscr{C}$.

Such a gluing condition for partial algebras over an arbitrary signature was established in Llabrés and Rosselló (2000). In the present paper we extend the main results obtained in that paper to establish a gluing condition for *partly total algebras*: partial algebras with some of their operations specified to be total. This covers, of course, the case of total algebras, but also the case of attributed graphs (see below).

The motivation for developing the DPO transformation of partial algebras over arbitrary signatures with some (possibly all) operations specified to be total stems from two broad sources. First, the interaction between ADT specification, which requires arbitrary signatures in a natural way, and algebraic transformation has generated an increasing interest in the transformation of arbitrary (partial, total or partly total) algebras. This has led to the recent development of single-pushout (SPO) approaches to the transformation of arbitrary algebras (see, for instance, Wagner and Gogolla (1996)), and of operationally described (but not categorically formalized) algebra rewriting systems (Große-Rhode 1999). Contrary to what happens for unary algebras, the DPO transformation of arbitrary algebras turns out to be independent of any sensible SPO transformation: there are non-trivial application conditions in both approaches and they are different, and even when a rule can be applied in both the DPO and SPO approaches through a given matching, the results may be different. On the other hand, the transformation formalism introduced in Große-Rhode (1999) requires no application condition, but, again, the result of the application of a given rule through a given matching is, in general, different in that approach and in the DPO approach. (See Llabrés (2001) for details on all the facts asserted so far in this paragraph.) Hence, the development of a DPO approach will strictly increase the toolkit available in this area.

Second, attributes are used in all graph grammar proposals for Software Engineering, since they allow us to enrich graph-like structures by means of data type information. Attributed (hyper)graphs are structures consisting of two components: the graphical part and the data part – the former describes the structural aspects, and the latter is devoted to data type aspects and usually consists of a fixed arbitrary algebra. Both parts are connected by 'labelling operators' that assign data type attributes to structural items. The SPO transformation of attributed graphs is well established (see, for instance, Löwe *et al.* (1993), where it is introduced, and Bardohl (2000), where it is used to formalize aspects of visual languages), but, so far, only one DPO approach to the transformation of attributed graphs has been introduced (Berthold *et al.* 2000). This approach does not allow the rewriting of attributes if nodes and arcs are directly labelled with them, which makes necessary the introduction of artificial attribute holder sorts. Under the DPO approach to the transformation of attributed graphs based on our model, the labelling operators

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can be understood as unary operations (as in the SPO approach) and easily rewritten as such (see, for instance, Example 38). Moreover, in general, the DPO transformation of attributed graphs is also independent of their SPO transformation in the sense explained in the previous paragraph.

The rest of the paper is organized as follows. Section 2 contains some preliminary material: in Section 2.1 we briefly summarize the main notions of the theory of partial algebras and fix some notation and conventions, and in Section 2.2 we recall the main results from Llabrés and Rosselló (2000) that are used in this paper and prove a new result on pushout complements for partial algebras (Proposition 4) that is also used later. In Section 3 we prove the main results on algebras with some fixed total operations, prior to the investigation of their gluing condition. Section 4 contains the main results of this paper: the gluing condition for algebras with some fixed total operations (Theorem 34 and Corollary 35), the existence – when the gluing condition is satisfied – and the uniqueness of pushout complements given by closed subalgebras (Proposition 27), and an explicit description for them (Theorem 34). More results were announced, for total algebras and without proofs, in the extended abstract (Burmeister *et al.* 2000) that the present paper develops. They can be easily generalized to algebras with some fixed total operations, but, unfortunately, space constraints prevent us from including them here: they will appear in Llabrés (2001).

2. Preliminaries

2.1. *Partial algebras*

We assume that the reader is familiar with the basic language of the theory of partial algebras, as presented for instance in Burmeister (1986, Part I). However, and to ease the task for the reader, in this section we briefly recall some of the main notions in this theory, and take the opportunity to fix some notation and conventions to be used throughout this paper. We refer the reader to Burmeister *et al.* (1997, Appendix A) for all notions, results and notation that are not introduced explicitly here or for which we do not give any specific reference.

Throughout this paper, $\Sigma = (S, \Omega, \eta)$ denotes a *signature* with set of *sorts* S, set of *operation symbols* Ω and *arity function* $\eta : \Omega \to S^* \times S$ with $\eta(\varphi) = (\omega(\varphi), \sigma(\varphi))$ for every $\varphi \in \Omega$ (S^* denotes the set of all words over S; the empty word is denoted by λ).

Let $\Omega^{(0)}$ denote the set of all *nullary* operation symbols in Σ, that is, those $\varphi \in \Omega$ such that $\omega(\varphi) = \lambda$. Set $\Omega^{(+)} = \Omega - \Omega^{(0)}$ and $\Sigma^{(+)} = (S, \Omega^{(+)}, \eta\big|_{\Omega^{(+)}})$.

A (*partial*) Σ-*algebra* is a structure $\mathbf{A} = (A, (\varphi^{\mathbf{A}})_{\varphi \in \Omega})$ where:

— $A = (A_s)_{s \in S}$ is an S-*set* (that is, an S-indexed family of sets), called the *carrier* of the algebra; for every $s \in S$, the set A_s is called the *carrier of sort* s of $\mathbf{A}$.

— For every $\varphi \in \Omega$, we have $\varphi^{\mathbf{A}} : A^{\omega(\varphi)} \to A_{\sigma(\varphi)}$ is a partial mapping, called, generically, an *operation* in $\mathbf{A}$ (where $A^{\lambda} = \{\varnothing\}$ and $A^{s_1 \dots s_p} = A_{s_1} \times \cdots \times A_{s_p}$ if $s_1 \dots s_p \neq \lambda$). We shall use $\operatorname{dom} \varphi^{\mathbf{A}} \subseteq A^{\omega(\varphi)}$ to denote the domain of the operation $\varphi^{\mathbf{A}}$.

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An operation $\varphi^{\mathbf{A}}$ is *total* when it is a totally defined mapping. When a nullary operation in $\mathbf{A}$ is total, we say that it is *defined* in $\mathbf{A}$, and we identify it with its image. A *total* Σ-*algebra* is a Σ-algebra where all operations are total.

If $\Sigma' = (S', \Omega', \eta')$ is a *subsignature* of Σ (that is, a signature with $S' \subseteq S$, $\Omega' \subseteq \Omega$ and $\eta' = \eta\big|_{\Omega'}$) and $\mathbf{A} = (A, (\varphi^{\mathbf{A}})_{\varphi\in\Omega})$, with $A = (A_s)_{s\in S}$, is a Σ-algebra, then the Σ'-*reduct* of $\mathbf{A}$ is the Σ'-algebra $((A_s)_{s\in S'}, (\varphi^{\mathbf{A}})_{\varphi\in\Omega'})$. If $\mathbf{A} = (A, (\varphi^{\mathbf{A}})_{\varphi\in\Omega})$ is a Σ-algebra, its $\Sigma^{(+)}$-reduct will always be denoted by $\mathbf{A}^{(+)}$.

The *relative subalgebra* of a Σ-algebra $\mathbf{A} = (A, (\varphi^{\mathbf{A}})_{\varphi\in\Omega})$ supported on a subset B of its carrier is the Σ-algebra $\mathbf{B} = (B, (\varphi^{\mathbf{B}})_{\varphi\in\Omega})$ obtained by restricting the structure of $\mathbf{A}$ to B, in the sense that for every $\varphi \in \Omega$,

$$\operatorname{dom}\varphi^{\mathbf{B}} = \{\underline{b} \in B^{\omega(\varphi)} \cap \operatorname{dom}\varphi^{\mathbf{A}} \mid \varphi^{\mathbf{A}}(\underline{b}) \in B_{\sigma(\varphi)}\}$$

and if $\underline{b} \in \operatorname{dom}\varphi^{\mathbf{B}}$, then $\varphi^{\mathbf{B}}(\underline{b}) = \varphi^{\mathbf{A}}(\underline{b})$.

Such a relative subalgebra is called a *closed subalgebra* of $\mathbf{A}$ when B is a closed subset of $\mathbf{A}$: that is, when for every $\varphi \in \Omega$, if $\underline{b} \in B^{\omega(\varphi)} \cap \operatorname{dom}\varphi^{\mathbf{A}}$, then $\varphi^{\mathbf{A}}(\underline{b}) \in B_{\sigma(\varphi)}$.

There always exists the least closed subset of a Σ-algebra $\mathbf{A}$ containing a given subset X of its carrier; it is denoted by $C_{\mathbf{A}}(X)$ and called the closed subset of $\mathbf{A}$ *generated* by X. We say that X *generates* the Σ-algebra $\mathbf{A}$ when it generates its carrier.

A subset Y of the carrier of a Σ-algebra $\mathbf{A}$ is an *initial segment* of $\mathbf{A}$ when for every $\varphi \in \Omega$, if $\varphi^{\mathbf{A}}(\underline{a}) \in Y_{\sigma(\varphi)}$, then $\underline{a} \in Y^{\omega(\varphi)}$. There always exists the least initial segment of a Σ-algebra $\mathbf{A}$ containing a given subset X of its carrier, and it is denoted $\downarrow_{\mathbf{A}} X$.

A *homomorphism* $f : \mathbf{A} \to \mathbf{B}$ between two Σ-algebras $\mathbf{A} = (A, (\varphi^{\mathbf{A}})_{\varphi\in\Omega})$ and $\mathbf{B} = (B, (\varphi^{\mathbf{B}})_{\varphi\in\Omega})$ is a mapping $f : A \to B$ that preserves the operations defined in the source algebra, in the sense that for every $\varphi \in \Omega$, if $\underline{a} \in \operatorname{dom}\varphi^{\mathbf{A}}$, then $f(\underline{a}) \in \operatorname{dom}\varphi^{\mathbf{B}}$ and $f(\varphi^{\mathbf{A}}(\underline{a})) = \varphi^{\mathbf{B}}(f(\underline{a}))$. Such a homomorphism is *closed* when, moreover, for every $\varphi \in \Omega$, if $f(\underline{a}) \in \operatorname{dom}\varphi^{\mathbf{B}}$, then $\underline{a} \in \operatorname{dom}\varphi^{\mathbf{A}}$. Any homomorphism between total algebras is closed.

Taking as objects all Σ-algebras (respectively, all total Σ-algebras) and their homomorphisms as morphisms, we obtain a category Alg_{Σ} (respectively, TAlg_{Σ}).

In Alg_{Σ}, the isomorphisms are the closed and bijective homomorphisms, the epimorphisms are those homomorphisms whose set-theoretical image generates the target algebra, and the monomorphisms are the injective homomorphisms (see Propositions 2.4.6, 3.6.1 and 2.10.3 in Burmeister (1986), respectively).

An equivalence $\theta = (\theta_s)_{s\in S}$ on the carrier of a Σ-algebra $\mathbf{A} = (A, (\varphi^{\mathbf{A}})_{\varphi\in\Omega})$ is a *congruence* on it when it is compatible with the operations in $\mathbf{A}$, in the sense that for each $\varphi \in \Omega^{(+)}$, say with $\eta(\varphi) = (s_1 \cdots s_n, s)$, if $\underline{a} = (a_1, \ldots, a_n), \underline{b} = (b_1, \ldots, b_n) \in \operatorname{dom}\varphi^{\mathbf{A}}$ are such that $(a_i, b_i) \in \theta_{s_i}$ for every $i = 1, \ldots, n$, then $(\varphi^{\mathbf{A}}(\underline{a}), \varphi^{\mathbf{A}}(\underline{b})) \in \theta_s$.

Given any relation X on the carrier of a Σ-algebra $\mathbf{A}$, there always exists the least congruence on $\mathbf{A}$ containing X.

Given a congruence θ on a Σ-algebra $\mathbf{A} = (A, (\varphi^{\mathbf{A}})_{\varphi\in\Omega})$, the *quotient* algebra

$$\mathbf{A}/\theta = (A/\theta, (\varphi^{\mathbf{A}/\theta})_{\varphi\in\Omega})$$

is the Σ-algebra with carrier set the quotient S-set A/θ and whose algebraic structure is the least one making the quotient mapping $nat_{\theta} : \mathbf{A} \to \mathbf{A}/\theta$ that maps each element of each A_s to its equivalence class modulo θ_s, a homomorphism of Σ-algebras.

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Let $X = (X_s)_{s \in S}$ be any S-set. The S-set $\mathrm{T}_\Sigma(X) = (\mathrm{T}_\Sigma(X)_s)_{s \in S}$ of *Σ-terms with variables in X* is defined by recurrence as follows: if $x \in X_s$, then $x \in \mathrm{T}_\Sigma(X)_s$; if $\varphi \in \Omega^{(0)}$, then $\varphi \in \mathrm{T}_\Sigma(X)_{\sigma(\varphi)}$; and if $\varphi \in \Omega^{(+)}$ with $\eta(\varphi) = (s_1 \dots s_p, s)$ and $t_i \in \mathrm{T}_\Sigma(X)_{s_i}$ for every $i = 1, \dots, p$, then $\varphi(t_1, \dots, t_p) \in \mathrm{T}_\Sigma(X)_s$.

The *free Σ-algebra* generated by X is the total Σ-algebra

$$\mathbf{T}_\Sigma(X) = (\mathrm{T}_\Sigma(X), (\varphi^{\mathbf{T}_\Sigma(X)})_{\varphi \in \Omega})$$

where, for every $\varphi \in \Omega$, the operation $\varphi^{\mathbf{T}_\Sigma(X)}$ is defined as follows:

— if $\varphi \in \Omega^{(0)}$, then $\varphi^{\mathbf{T}_\Sigma(X)} = \varphi \in \mathrm{T}_\Sigma(X)_{\sigma(\varphi)}$;

— if $\varphi \in \Omega^{(+)}$, say with $\omega(\varphi) = s_1 \dots s_p$, and $t_i \in \mathrm{T}_\Sigma(X)_{s_i}$ for every $i = 1, \dots, p$, then

$$\varphi^{\mathbf{T}_\Sigma(X)}(t_1, \dots, t_p) = \varphi(t_1, \dots, t_p) \in \mathrm{T}_\Sigma(X)_{\sigma(\varphi)}.$$

To close this subsection, there are two conventions used throughout this paper that we want to point out. Given an algebra denoted by a capital letter in boldface type (**A**, **B**, *etc.*), we shall always denote, unless otherwise stated, its carrier by the same capital letter, but in slanted type (A, B, *etc.*): in particular, given an algebra **A** and a subset of its carrier denoted by a capital letter in slanted type, we shall always denote the relative subalgebra of **A** supported on this subset by the same capital letter in boldface type. On the other hand, whenever there is no danger of confusion, and in order to lighten the notation, we shall skip all subscripts corresponding to sorts in the names of the carriers of the algebras, the components of the mappings or the congruences, *etc.*, and we shall denote things just as if we were dealing with homogeneous algebras.

2.2. *Pushout complements for partial algebras*

In this section we gather together some results on pushouts and pushout complements in Alg_Σ. We first recall from Burmeister (1986, Section 4.3) the construction of pushouts in Alg_Σ.

Theorem 1. Let **K**, **A** and **B** be three Σ-algebras and let $f : \mathbf{K} \to \mathbf{A}$ and $g : \mathbf{K} \to \mathbf{B}$ be two homomorphisms of Σ-algebras. Let

$$\mathbf{A}^{(+)} + \mathbf{B}^{(+)} = (A \sqcup B, (\varphi^{\mathbf{A}^{(+)} + \mathbf{B}^{(+)}})_{\varphi \in \Omega^{(+)}})$$

be the $\Sigma^{(+)}$-algebra with carrier the disjoint union[†] $A \sqcup B$ of A and B, and operations the disjoint unions of the corresponding operations on the $\Sigma^{(+)}$-reducts $\mathbf{A}^{(+)}$ and $\mathbf{B}^{(+)}$ of **A** and **B**. Let $\theta(f, g)$ be the least congruence on $\mathbf{A}^{(+)} + \mathbf{B}^{(+)}$ containing the relation

$$\{(f(x), g(x)) \mid x \in K\} \cup \{(\varphi_0^{\mathbf{A}}, \varphi_0^{\mathbf{B}}) \mid \varphi_0 \in \Omega^{(0)},\ \varphi_0^{\mathbf{A}}, \varphi_0^{\mathbf{B}} \text{ defined}\}.$$

Finally, let **D** be the Σ-algebra whose $\Sigma^{(+)}$-reduct is the quotient $(\mathbf{A}^{(+)} + \mathbf{B}^{(+)})/\theta(f, g)$ and where, for every $\varphi_0 \in \Omega^{(0)}$, $\varphi_0^{\mathbf{D}}$ is defined if and only if $\varphi_0^{\mathbf{A}}$ or $\varphi_0^{\mathbf{B}}$ are defined, in which case $\varphi_0^{\mathbf{D}}$ is its (or their) equivalence class modulo $\theta(f, g)$.

[†] Formally, the disjoint union $A \sqcup B$ of A and B is defined as $A \times \{1\} \cup B \times \{2\}$, but in order to simplify the notation we shall identify A and B with their images in $A \sqcup B$.

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Then $\mathbf{D}$, together with the homomorphisms $\mathbf{A} \to \mathbf{D}$ and $\mathbf{B} \to \mathbf{D}$ given by the restrictions to $\mathbf{A}$ and $\mathbf{B}$ of the quotient mapping $A \sqcup B \to (A \sqcup B)/\theta(f,g)$, is a pushout of f and g in Alg_Σ.

The next two results are, respectively, Theorem 10 and Proposition 11 in Llabrés and Rosselló (2000) and they deal with the existence of pushout complements in Alg_Σ.

Henceforth, we shall say that a pushout complement $(\mathbf{B}, g : \mathbf{K} \to \mathbf{B}, d : \mathbf{B} \hookrightarrow \mathbf{D})$ of two homomorphisms $f : \mathbf{K} \to \mathbf{A}$ and $m : \mathbf{A} \to \mathbf{D}$ is *natural* when $d : \mathbf{B} \hookrightarrow \mathbf{D}$ is the embedding of a closed subalgebra.

Theorem 2. Let $f : \mathbf{K} \to \mathbf{A}$ and $m : \mathbf{A} \to \mathbf{D}$ be two homomorphisms of Σ-algebras. Set $B_0 = C_{\mathbf{D}}((D - m(A)) \cup m(f(K)))$ and let $d : \mathbf{B}_0 \hookrightarrow \mathbf{D}$ be the corresponding closed embedding and $g : \mathbf{K} \to \mathbf{B}_0$, the composition $m \circ f$ understood with target algebra $\mathbf{B}_0$.

If f and m have a pushout complement in Alg_Σ, then $(\mathbf{B}_0, g : \mathbf{K} \to \mathbf{B}_0, d : \mathbf{B}_0 \hookrightarrow \mathbf{D})$ is their unique natural pushout complement and it is *terminal* in the following sense: if $(\mathbf{C}, m' : \mathbf{K} \to \mathbf{C}, f' : \mathbf{C} \to \mathbf{D})$ is another pushout complement of them, then there exists a unique homomorphism $h : \mathbf{C} \to \mathbf{B}_0$, which is, moreover, an epimorphism, such that $h \circ m' = g$ and $d \circ h = f'$.

Proposition 3. Let $f : \mathbf{K} \to \mathbf{A}$ and $m : \mathbf{A} \to \mathbf{D}$ be two homomorphisms of Σ-algebras. Set $B_0 = C_{\mathbf{D}}((D - m(A)) \cup m(f(K)))$ and let $\theta(f,m)$ be the least congruence on $\mathbf{A}^{(+)} + \mathbf{B}_0^{(+)}$ containing the relation

$$\{(f(k), mf(k)) \mid k \in K\} \cup \{(\varphi_0^{\mathbf{A}}, m(\varphi_0^{\mathbf{A}})) \mid \varphi_0 \in \Omega^{(0)},\ \varphi_0^{\mathbf{A}} \text{ defined}\}.$$

Then, f and m have a pushout complement in Alg_Σ if and only if they satisfy the following properties (jointly called the *gluing condition* in Alg_Σ):

GC1 $\ker m \subseteq \theta(f,m)$.
GC2 If $m(a) \in B_0$, then $(a, m(a)) \in \theta(f,m)$.
GC3 For every $\varphi \in \Omega$, if $\underline{d} \in \mathrm{dom}\,\varphi^{\mathbf{D}}$ and $\underline{d} \notin B_0^{\omega(\varphi)}$, then $\underline{d} = m(\underline{a})$ for some $\underline{a} \in \mathrm{dom}\,\varphi^{\mathbf{A}}$.

Finally, we have the following new result, which will be used several times in Section 4.

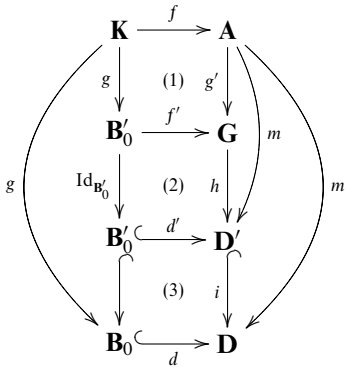
Proposition 4. Let $f : \mathbf{K} \to \mathbf{A}$ and $m : \mathbf{A} \to \mathbf{D}$ be two homomorphisms of Σ-algebras that have a pushout complement in Alg_Σ, and let $(\mathbf{B}_0, g : \mathbf{K} \to \mathbf{B}_0, d : \mathbf{B}_0 \hookrightarrow \mathbf{D})$ be their natural pushout complement. Let D' be any subset of the carrier of $\mathbf{D}$ containing $m(A)$. Then, $f : \mathbf{K} \to \mathbf{A}$ and $m : \mathbf{A} \to \mathbf{D}'$ also have a pushout complement in Alg_Σ.

Moreover, if we set $B_0' = B_0 \cap D'$, $d' : \mathbf{B}_0' \hookrightarrow \mathbf{D}'$ the corresponding embedding and $g : \mathbf{K} \to \mathbf{B}_0'$ the homomorphism $g : \mathbf{K} \to \mathbf{B}_0$ understood with target algebra $\mathbf{B}_0'$, then $(\mathbf{B}_0', g : \mathbf{K} \to \mathbf{B}_0', d' : \mathbf{B}_0' \hookrightarrow \mathbf{D}')$ is the natural pushout complement of $f : \mathbf{K} \to \mathbf{A}$ and $m : \mathbf{A} \to \mathbf{D}'$ in Alg_Σ.

Proof. The embedding $d' : \mathbf{B}_0' \hookrightarrow \mathbf{D}'$ is closed by Burmeister (1986, Proposition 3.1.9). Consider now the following commutative diagram, where (1) is a pushout square in Alg_Σ, $h : \mathbf{G} \to \mathbf{D}'$ is the unique homomorphism such that $h \circ g' = m : \mathbf{A} \to \mathbf{D}'$ and $h \circ f' = d' : \mathbf{B}_0' \to \mathbf{D}'$, and $i : \mathbf{D}' \hookrightarrow \mathbf{D}$ denotes the corresponding embedding. We shall

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prove that h is an isomorphism, that is, is bijective and closed.



To prove that h is surjective, let $x \in D'$ and consider it as an element of D. If $x \in B_0$, then $x \in D' \cap B_0 = B'_0$ and therefore $x = h(f'(x))$. If $x \notin B_0$, then by Llabrés and Rosselló (2000, Lemma 6.(i)) we have that $x \in m(A) = h(g'(A))$. So, in both cases $x \in h(G)$.

To prove that h is injective, notice that, since d' is closed and injective, f' is also closed (by Burmeister (1986, Proposition 2.4.5)) and injective. Now, since square (1) and the outer rectangle in the previous diagram are pushouts in Alg_Σ, the usual pushout decomposition property implies that (2)+(3) is a pushout square, too. Then, by Llabrés and Rosselló (2000, Corollary 2), $i \circ h$ is injective, and thus h is injective as well.

Finally, to prove that h is closed, let $\varphi \in \Omega$ be any operation symbol and let $\underline{x} \in G^{\omega(\varphi)}$ be such that $h(\underline{x}) \in \mathrm{dom}\,\varphi^{\mathbf{D}'}$. If $h(\underline{x}) \in B_0^{\omega(\varphi)}$, then $h(\underline{x}) \in \mathrm{dom}\,\varphi^{\mathbf{B}'_0}$, and hence $\underline{x} = f'(h(\underline{x})) \in \mathrm{dom}\,\varphi^{\mathbf{G}}$. If $h(\underline{x}) \notin B_0^{\omega(\varphi)}$, then property GC3 in Proposition 3 implies that $i(h(\underline{x})) = m(\underline{a}) = i(h(g'(\underline{a})))$ for some $\underline{a} \in \mathrm{dom}\,\varphi^{\mathbf{A}}$. But then, $i \circ h$ being injective, $\underline{x} = g'(\underline{a}) \in \mathrm{dom}\,\varphi^{\mathbf{G}}$. So, in both cases $\underline{x} \in \mathrm{dom}\,\varphi^{\mathbf{G}}$.

Thus, h is an isomorphism and therefore $(\mathbf{B}'_0, g : \mathbf{K} \to \mathbf{B}'_0, d' : \mathbf{B}'_0 \hookrightarrow \mathbf{D}')$ is the natural pushout complement of $f : \mathbf{K} \to \mathbf{A}$ and $m : \mathbf{A} \to \mathbf{D}'$. □

3. Υ-total Σ-algebras

Let $\Sigma = (S, \Omega, \eta)$ be an arbitrary signature and let $\Upsilon \subseteq \Omega$ be a non-empty set of operation symbols in Σ. We shall use Σ_Υ to denote the subsignature $\Sigma_\Upsilon = (S, \Upsilon, \eta|_\Upsilon)$ of Σ, and given any Σ-algebra $\mathbf{A}$, we shall always denote its Σ_Υ-reduct by $\mathbf{A}_\Upsilon$.

A Σ-algebra $\mathbf{A}$ is said to be Υ-*total* when $\varphi^{\mathbf{A}}$ is total for every $\varphi \in \Upsilon$, that is, when its Σ_Υ-reduct $\mathbf{A}_\Upsilon$ is total. The full subcategory of Alg_Σ with objects all Υ-total Σ-algebras will be denoted by $\mathrm{Alg}_{\Sigma,\Upsilon}$. Sometimes we shall use the expression *homomorphism of* Υ-*total* Σ-*algebras* when we want to speak about a homomorphism of Σ-algebras between two Υ-total Σ-algebras.

Example 5. If Υ is the set of all operation symbols in Σ, then the Υ-total Σ-algebras are exactly the total Σ-algebras, and hence $\mathrm{Alg}_{\Sigma,\Upsilon} = \mathrm{TAlg}_\Sigma$.

Example 6. Let $\Sigma_0 = (S_0, \Omega_0, \eta_0)$ and $\Sigma_1 = (S_1, \Omega_1, \eta_1)$ be two signatures with $S_0 \cap S_1 = \Omega_0 \cap \Omega_1 = \varnothing$. Let $\Sigma = (S, \Omega, \eta)$ be a signature with $S = S_0 \cup S_1$ and $\Omega = \Omega_0 \cup \Omega_1 \cup \mathscr{A}$,

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where $\mathscr{A} \cap (\Omega_0 \cup \Omega_1) = \varnothing$, and $\eta : \Omega \to S^* \times S$ such that $\eta\big|_{\Omega_0} = \eta_0$, $\eta\big|_{\Omega_1} = \eta_1$ and $\eta(\alpha) \in S_0 \times S_1$ for every $\alpha \in \mathscr{A}$.

The Ω_1-total Σ-algebras are then structures consisting of a Σ_0-algebra whose elements may have attributes (given by the possibly partial attribution mappings corresponding to the operation symbols in $\mathscr{A}$) in a total Σ_1-algebra. On the other hand, a $(\Omega_1 \cup \mathscr{A})$-total Σ-algebra would be a similar structure except that the attribution mappings must be total. These would be two different kinds of what could be called Σ_1-*attributed* Σ_0-*algebras*, generalizing the notion of attributed hypergraphs (*cf.* Löwe *et al.* (1993)).

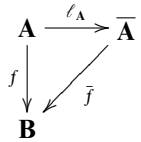
Of course, by modifying the set Υ we can also impose some operations in Σ_0 to be total as well as some operations in Σ_1 to be possibly partial.

The class of all Υ-total Σ-algebras, for any pair (Σ, Υ), is an *E-variety* in the sense of Burmeister (1986, Section 6.1), that is, it is the class of all models of some set of existence equations: these existence equations are those requiring that the operations in Υ must be total. We then have the following result, which is a special case of Burmeister (1986, Proposition 6.2.1.(i)) and can be proved directly in a straightforward way.

Lemma 7.

(i) Every closed subalgebra of a Υ-total Σ-algebra is again Υ-total.
(ii) If $\mathbf{A}$ is a Υ-total Σ-algebra and $f : \mathbf{A} \to \mathbf{B}$ is a surjective homomorphism of Σ-algebras, then $\mathbf{B}$ is also Υ-total. In particular, every quotient of a Υ-total Σ-algebra and every Σ-algebra isomorphic to a Υ-total Σ-algebra are again Υ-total.

A *free* Υ-*completion* of a Σ-algebra $\mathbf{A}$ is a Υ-total Σ-algebra $\overline{\mathbf{A}}$ together with an epimorphism $\ell_{\mathbf{A}} : \mathbf{A} \to \overline{\mathbf{A}}$ of Σ-algebras such that for every homomorphism $f : \mathbf{A} \to \mathbf{B}$ of Σ-algebras with $\mathbf{B}$ Υ-total, there exists one, and only one, homomorphism $\bar{f} : \overline{\mathbf{A}} \to \mathbf{B}$ such that $\bar{f} \circ \ell_{\mathbf{A}} = f$.



When $\Omega = \Upsilon$, that is, when $\mathrm{Alg}_{\Sigma,\Upsilon} = \mathrm{TAlg}_{\Sigma}$, the free Υ-completion of a Σ-algebra is its *free completion* in the sense of Burmeister (1986, Section 5.3). In general, Burmeister (1986, Proposition 6.2.1) guarantees that free Υ-completions always exist (in the language of Burmeister (1986), they would be called $\mathrm{Alg}_{\Sigma,\Upsilon}$-*universal* $\mathrm{Alg}_{\Sigma,\Upsilon}$-*solutions*). We describe next how they can be constructed explicitly.

Let $\mathbf{A} = (A, (\varphi^{\mathbf{A}})_{\varphi \in \Omega})$ be a Σ-algebra. Let $\mathbf{T}_{\Sigma_\Upsilon}(A)$ be the free Σ_Υ-algebra generated by A, with carrier the S-set $\mathrm{T}_{\Sigma_\Upsilon}(A)$ of Σ_Υ-terms with variables in A. Let $\mathbf{A}^* = (\mathrm{T}_{\Sigma_\Upsilon}(A), (\varphi^{\mathbf{A}^*})_{\varphi \in \Upsilon})$ be the total Σ_Υ-algebra such that for every $\varphi \in \Upsilon$ and for every $\underline{t} \in \mathrm{T}_{\Sigma_\Upsilon}(A)^{\omega(\varphi)}$,

$$\varphi^{\mathbf{A}^*}(\underline{t}) = \begin{cases} \varphi^{\mathbf{A}}(\underline{t}) & \text{if } \underline{t} \in \mathrm{dom}\, \varphi^{\mathbf{A}} \\ \varphi^{\mathbf{T}_{\Sigma_\Upsilon}(A)}(\underline{t}) & \text{otherwise.} \end{cases}$$

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Set $\overline{A} = C_{\mathbf{A}^*}(A)$ and let $\overline{\mathbf{A}}_\Upsilon$ be the closed subalgebra of $\mathbf{A}^*$ supported on $\overline{A}$. The Σ_Υ-reduct $\mathbf{A}_\Upsilon$ of $\mathbf{A}$ is a relative subalgebra of $\overline{\mathbf{A}}_\Upsilon$ and, by Burmeister and Schmidt (1967), the embedding $\mathbf{A}_\Upsilon \hookrightarrow \overline{\mathbf{A}}_\Upsilon$ is a free completion of Σ_Υ-algebras.

Finally, let $\overline{\mathbf{A}}$ be the Σ-algebra whose Σ_Υ-reduct is $\overline{\mathbf{A}}_\Upsilon$ and where each operation $\varphi^{\overline{\mathbf{A}}}$ with $\varphi \in \Omega - \Upsilon$ is defined to be equal to the corresponding operation $\varphi^{\mathbf{A}}$. It is clear that $\mathbf{A}$ is still a relative subalgebra of $\overline{\mathbf{A}}$.

Proposition 8. With the previous notation, the embedding $\mathbf{A} \hookrightarrow \overline{\mathbf{A}}$ is a free Υ-completion.

Proof. By construction, $\overline{\mathbf{A}}$ is a Υ-total Σ-algebra generated by A, and, as we have mentioned, the embedding $\mathbf{A}_\Upsilon \hookrightarrow \overline{\mathbf{A}}_\Upsilon$ is a free completion. Now let $f : \mathbf{A} \to \mathbf{C}$ be a homomorphism of Σ-algebras with $\mathbf{C}$ Υ-total. Then, there exists a unique homomorphism of total Σ_Υ-algebras $\bar{f} : \overline{\mathbf{A}}_\Upsilon \to \mathbf{C}_\Upsilon$ whose restriction to $\mathbf{A}$ is equal to f, and the definition of $\overline{\mathbf{A}}$ entails that $\bar{f}$ is also a homomorphism of Σ-algebras $\bar{f} : \overline{\mathbf{A}} \to \mathbf{C}$. □

By Burmeister (1986, Proposition 5.1.3.(i)), free Υ-completions are unique up to isomorphism, in the sense that if $\ell_{\mathbf{A}} : \mathbf{A} \to \overline{\mathbf{A}}$ and $\widetilde{\ell_{\mathbf{A}}} : \mathbf{A} \to \widetilde{\mathbf{A}}$ are two free Υ-completions of $\mathbf{A}$, there exists an isomorphism $\bar{\ell} : \overline{\mathbf{A}} \to \widetilde{\mathbf{A}}$ such that $\widetilde{\ell_{\mathbf{A}}} = \bar{\ell} \circ \ell_{\mathbf{A}}$. This fact, together with the fact that the free Υ-completion explicitly described above is the embedding of a relative subalgebra, allows us to assume henceforth that *free Υ-completions are always given by embeddings of relative subalgebras*. In particular, whenever we say that a Υ-total Σ-algebra $\mathbf{B}$ is a free Υ-completion of a relative subalgebra $\mathbf{A}$, we shall mean that the corresponding embedding $\mathbf{A} \hookrightarrow \mathbf{B}$ is a free Υ-completion.

Moreover, the uniqueness up to isomorphism of free Υ-completions and the previous explicit construction of one of them also entail that if $\ell_{\mathbf{A}} : \mathbf{A} \to \overline{\mathbf{A}}$ is a free Υ-completion, then $\ell_{\mathbf{A}} : \mathbf{A}_\Upsilon \to \overline{\mathbf{A}}_\Upsilon$ is a free completion of Σ_Υ-algebras.

If $\mathbf{A}$ and $\mathbf{B}$ are two Σ-algebras, $\ell_{\mathbf{A}} : \mathbf{A} \to \overline{\mathbf{A}}$ and $\ell_{\mathbf{B}} : \mathbf{B} \to \overline{\mathbf{B}}$ are free Υ-completions, and $f : \mathbf{A} \to \mathbf{B}$ is a homomorphism of Σ-algebras, then f extends in a unique way to a homomorphism $\hat{f} : \overline{\mathbf{A}} \to \overline{\mathbf{B}}$: namely, $\hat{f}$ is the unique homomorphism extending $\ell_{\mathbf{B}} \circ f : \mathbf{A} \to \overline{\mathbf{B}}$.

The following result provides an 'internal' axiomatic characterization of free Υ-completions, and it can be proved along the lines of the characterization of free completions given in Burmeister (1986, Section 5.3).

Proposition 9. Let $\Sigma = (S, \Omega, \eta)$ be a signature and let $\Upsilon \subseteq \Omega$. Let $\overline{\mathbf{A}}$ be a Υ-total Σ-algebra and $\mathbf{A}$ be a relative subalgebra of it. Then, $\overline{\mathbf{A}}$ is a free Υ-completion of $\mathbf{A}$ if and only if the following properties are satisfied:

FAC1 A is an initial segment of $\overline{\mathbf{A}}$.
FAC2 For every $\varphi \in \Omega - \Upsilon$, $\operatorname{dom} \varphi^{\overline{\mathbf{A}}} = \operatorname{dom} \varphi^{\mathbf{A}}$.
FAC3 For every $\varphi, \psi \in \Upsilon$, if $\varphi^{\overline{\mathbf{A}}}(\underline{a}) = \psi^{\overline{\mathbf{A}}}(\underline{a}') \notin A$, then $\varphi = \psi$ and $\underline{a} = \underline{a}'$.
FAC4 A generates $\overline{\mathbf{A}}$.

Corollary 10. Let $\mathbf{B}$ be a Σ-algebra, let $\mathbf{A}$ be a closed subalgebra of it, let $\ell_{\mathbf{B}} : \mathbf{B} \hookrightarrow \overline{\mathbf{B}}$ be a free Υ-completion of $\mathbf{B}$, and let $\overline{\mathbf{A}}$ be the closed subalgebra of $\overline{\mathbf{B}}$ supported on $C_{\overline{\mathbf{B}}}(A)$. Then, the embedding $\mathbf{A} \hookrightarrow \overline{\mathbf{A}}$ is a free Υ-completion.

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Proof. It is straightforward to prove that the embedding $\mathbf{A} \hookrightarrow \overline{\mathbf{A}}$ satisfies properties (FAC1) to (FAC4) in the previous proposition using the fact that $\mathbf{B} \hookrightarrow \overline{\mathbf{B}}$ satisfies them and that A is a closed subset of $\mathbf{B}$ that generates $\overline{\mathbf{A}}$. □

Next we introduce the notion of the Υ-freeness core of a partial Σ-algebra, which plays a key role in the next section, and we prove some of its main properties.

Let $\mathbf{D} = (D, (\varphi^{\mathbf{D}})_{\varphi\in\Omega})$ be a Σ-algebra with $D = (D_s)_{s\in S}$. We use $\prec^+_{\mathbf{D}}$ to denote the transitive closure of the relation $\prec_{\mathbf{D}}$ on $\bigcup_{s\in S} D_s$ defined as follows:

$d' \prec_{\mathbf{D}} d$ iff $d = \varphi^{\mathbf{D}}(d_1, \ldots, d', \ldots, d_m)$ for some operation symbol $\varphi \in \Omega$ and some tuple $(d_1, \ldots, d', \ldots, d_m) \in \operatorname{dom} \varphi^{\mathbf{D}}$.

We want to point out that $\prec^+_{\mathbf{D}}$ need not be reflexive: in fact, $d \prec^+_{\mathbf{D}} d$ for some $d \in D$ if and only if there exists a finite chain

$$d = d_1 \prec_{\mathbf{D}} \cdots \prec_{\mathbf{D}} d_m = d.$$

If M is any subset of D, then

$$\downarrow_{\mathbf{D}} M = M \cup \{\, d \in D \mid \text{there is some } m \in M \text{ such that } d \prec^+_{\mathbf{D}} m \,\}$$

is the least initial segment of $\mathbf{D}$ containing M, and $\downarrow_{\mathbf{D}} (M \cup N) = \downarrow_{\mathbf{D}} M \cup \downarrow_{\mathbf{D}} N$ for every $M, N \subseteq D$.

Definition 11. The Υ-*freeness core* $K_\Upsilon(\mathbf{D})$ of a Σ-algebra $\mathbf{D} = (D, (\varphi^{\mathbf{D}})_{\varphi\in\Omega})$ is the initial segment $\downarrow_{\mathbf{D}} K_0(\mathbf{D})$, where

$$\begin{aligned} K_0(\mathbf{D}) = \quad & (D - \textstyle\bigcup_{\varphi\in\Omega} \varphi^{\mathbf{D}}(\operatorname{dom}\varphi^{\mathbf{D}})) \cup (\bigcup_{\varphi\in\Omega-\Upsilon} \varphi^{\mathbf{D}}(\operatorname{dom}\varphi^{\mathbf{D}})) \cup \{d \in D \mid d \prec^+_{\mathbf{D}} d\} \\ & \cup \{d \in D \mid \text{there exist } \varphi, \psi \in \Upsilon, \underline{b} \in \operatorname{dom}\varphi^{\mathbf{D}}, \underline{c} \in \operatorname{dom}\psi^{\mathbf{D}} \\ & \qquad \text{with } \varphi^{\mathbf{D}}(\underline{b}) = d = \psi^{\mathbf{D}}(\underline{c}) \text{ and } (\varphi \neq \psi \text{ or } \underline{b} \neq \underline{c})\} \end{aligned}$$

Example 12. Let $\Sigma = (S, \Omega, \eta)$ be the signature with $S = \{s, t\}$, $\Omega = \{\varphi\}$ and $\eta(\varphi) = (st, t)$. Let $\mathbf{D}$ be the total Σ-algebra whose carrier set $D = (D_s, D_t)$ is given by $D_s = \{a\}$ and $D_t = \{b_k\}_{k\in\mathbb{Z}}$ and whose operation $\varphi^{\mathbf{D}}$ is defined by $\varphi^{\mathbf{D}}(a, b_i) = b_{i+1}$ for every $i \in \mathbb{Z}$. Then $K_\Omega(\mathbf{D}) = \{a\}$, that is, $K_\Omega(\mathbf{D})_s = \{a\}$ and $K_\Omega(\mathbf{D})_t = \varnothing$.

The next result enlightens the role that Υ-freeness cores play in free Υ-completions: namely, $K_\Upsilon(\mathbf{D})$ is the set of those elements of $\mathbf{D}$ that cannot be generated by any Υ-free completion $\mathbf{D}' \hookrightarrow \mathbf{D}$.

Proposition 13. Let $\mathbf{D}$ be a Υ-total Σ-algebra and let $\mathbf{B}$ be a relative subalgebra of it. Then, $\mathbf{D}$ is a free Υ-completion of $\mathbf{B}$ if and only if B is an initial segment of $\mathbf{D}$ that generates it and contains $K_\Upsilon(\mathbf{D})$.

Proof. It is clear that if B is an initial segment of $\mathbf{D}$ that generates it and contains $K_\Upsilon(\mathbf{D})$, then $\mathbf{D}$ satisfies properties (FAC1) to (FAC4) in Proposition 9 with respect to B, and therefore $\mathbf{B} \hookrightarrow \mathbf{D}$ is a free Υ-completion.

Conversely, let $\mathbf{B} \hookrightarrow \mathbf{D}$ be a free Υ-completion. Since B is an initial segment of $\mathbf{D}$, to prove that $K_\Upsilon(\mathbf{D}) \subseteq B$, it is enough to prove that $K_0(\mathbf{D}) \subseteq B$. Now, again by Proposition 9, B contains all elements of D that are values of operations in $\mathbf{D}$ not belonging to Υ as well as all elements of D that are values of operations in $\mathbf{D}$ belonging to Υ in at least two

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different ways. Moreover, since B generates $\mathbf{D}$, it also contains all elements of D that are not the image of any operation in $\mathbf{D}$. Thus, to prove that $K_0(\mathbf{D}) \subseteq B$, it remains to prove that B contains all those $d \in D$ such that $d \prec^+_{\mathbf{D}} d$.

So, assume there exists some $d \in D - B$ such that $d \prec^+_{\mathbf{D}} d$, say $d = d_1 \prec_{\mathbf{D}} \cdots \prec_{\mathbf{D}} d_m = d$. No element d_i in this chain belongs to B, because it is an initial segment of $\mathbf{D}$ and $d \notin B$. Moreover, any operation making that $d_i \prec_{\mathbf{D}} d_{i+1}$ for some $i = 1, \ldots, m-1$ must belong to Υ, because otherwise the corresponding d_{i+1} would belong to B. Therefore, $d \prec^+_{\mathbf{D}_\Upsilon} d$, and since $\mathbf{B}_\Upsilon \hookrightarrow \mathbf{D}_\Upsilon$ is a free completion of Σ_Υ-algebras, Burmeister (1986, Proposition 5.5.2) shows that this can only happen if $d \in B$. □

Corollary 14. Let $\mathbf{D}$ be a Υ-total Σ-algebra and X, an initial segment of it. Set

$$D_X = C_{\mathbf{D}}(K_\Upsilon(\mathbf{D}) \cup X)$$

and let $\mathbf{D}_X$ be the closed subalgebra of $\mathbf{D}$ supported on D_X.

(i) D_X is an initial segment of $\mathbf{D}$.
(ii) $\mathbf{D}_X$ is a free Υ-completion of its relative subalgebra supported on $K_\Upsilon(\mathbf{D}) \cup X$.

Proof.

(i) Let $\varphi \in \Omega$ be any operation symbol and let $\underline{x} \in \operatorname{dom} \varphi^{\mathbf{D}}$ be such that $\varphi^{\mathbf{D}}(\underline{x}) \in D_X$: we must prove that $\underline{x} \in D_X^{\omega(\varphi)}$. To do that, we distinguish two cases.
If $\varphi^{\mathbf{D}}(\underline{x}) \in K_\Upsilon(\mathbf{D}) \cup X$, which is an initial segment, then $\underline{x} \in (K_\Upsilon(\mathbf{D}) \cup X)^{\omega(\varphi)} \subseteq D_X^{\omega(\varphi)}$.
Otherwise, if $\varphi^{\mathbf{D}}(\underline{x}) \in D_X - (K_\Upsilon(\mathbf{D}) \cup X)$, then $\varphi \in \Upsilon$, and there exist some $\psi \in \Upsilon$ and $\underline{x}' \in D_X^{\omega(\psi)}$ such that $\varphi^{\mathbf{D}}(\underline{x}) = \psi^{\mathbf{D}}(\underline{x}')$. Since $\varphi^{\mathbf{D}}(\underline{x}) \notin K_\Upsilon(\mathbf{D})$ by assumption, this implies $\varphi = \psi$ and $\underline{x} = \underline{x}' \in D_X^{\omega(\varphi)}$.
(ii) $K_\Upsilon(\mathbf{D}) \cup X$ is an initial segment of $\mathbf{D}_X$ that generates it, and it is straightforward to check that it contains $K_\Upsilon(\mathbf{D}_X)$. By Proposition 13, this implies that $\mathbf{D}_X$ is a free Υ-completion of its relative subalgebra supported on $K_\Upsilon(\mathbf{D}) \cup X$. □

Corollary 15. Let $\mathbf{D}$ be a Υ-total Σ-algebra, and let $\mathbf{K}$ be its relative subalgebra supported on its Υ-freeness core $K_\Upsilon(\mathbf{D})$. Then, $\mathbf{D}$ is a free Υ-completion of $\mathbf{K}$ if and only if $K_\Upsilon(\mathbf{D})$ generates $\mathbf{D}$.

In the next section we shall see that to determine whether or not two homomorphisms of Υ-total Σ-algebras $f : \mathbf{K} \to \mathbf{A}$ and $m : \mathbf{A} \to \mathbf{D}$ have a pushout complement in $\mathrm{Alg}_{\Sigma,\Upsilon}$ is easier when we can guarantee that $\mathbf{D}$ is generated by $K_\Upsilon(\mathbf{D})$. To close this section, we show two general cases in which this happens.

Proposition 16. Every finitely generated Υ-total Σ-algebra is generated by its Υ-freeness core.

Proof. Let $\mathbf{D}$ be a Υ-total Σ-algebra generated by a finite set E of elements: without any loss of generality we may assume that no $e \in E$ is such that $e \in C_{\mathbf{D}}(E - \{e\})$. If there exists some $e \in E$ such that $e \notin K_\Upsilon(\mathbf{D})$, then there exist some $\varphi \in \Upsilon$ and $(d_1, \ldots, d_n) \in \operatorname{dom} \varphi^{\mathbf{D}}$ such that $e = \varphi^{\mathbf{D}}(d_1, \ldots, d_n)$, and $e \notin \downarrow_{\mathbf{D}} \{d_1, \ldots, d_n\}$, because otherwise we would have $e \prec^+_{\mathbf{D}} e$. But then $e \in C_{\mathbf{D}}(E - \{e\})$, which is contrary to the assumption. This proves that $E \subseteq K_\Upsilon(\mathbf{D})$. □

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Definition 17. A signature $\Sigma = (S, \Omega, \eta)$ is *hierarchical* when there does not exist any infinite sequence of operations $(\varphi_n)_{n\in\mathbb{N}}$ in Ω such that the target sort of φ_{n+1} is a source sort of φ_n, for every $n \geqslant 0$.

Proposition 18. If $\Sigma = (S, \Omega, \eta)$ and $\Upsilon \subseteq \Omega$ are such that Σ_Υ is hierarchical, then every Υ-total Σ-algebra is generated by its Υ-freeness core.

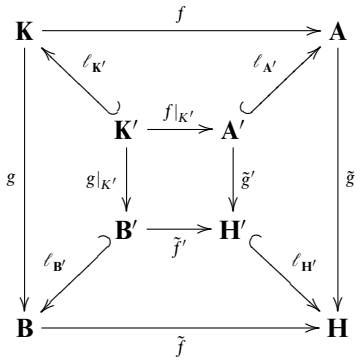
Proof. Assume that there exists some Υ-total Σ-algebra $\mathbf{D}$ not generated by its Υ-freeness core $K_\Upsilon(\mathbf{D})$. Let $d^{(0)}$ be an element of $D - C_\mathbf{D}(K_\Upsilon(\mathbf{D}))$. Since $K_\Upsilon(\mathbf{D})$ contains all elements of D that are not the value of any operation in $\mathbf{D}$ as well as all elements that are the value of some operation in $\Omega - \Upsilon$, there exist $\varphi \in \Upsilon$ and $\underline{d} = (d_1, \ldots, d_p) \in \operatorname{dom} \varphi^\mathbf{D}$ such that $d^{(0)} = \varphi^\mathbf{D}(\underline{d})$. By assumption, not every d_k in $\underline{d}$ belongs to $C_\mathbf{D}(K_\Upsilon(\mathbf{D}))$. Hence, there exists some $d^{(1)} \in D - C_\mathbf{D}(K_\Upsilon(\mathbf{D}))$ such that $d^{(1)} \prec_{\mathbf{D}_\Upsilon} d^{(0)}$. Repeating the argument with $d^{(1)}$, *etc.*, we get a sequence of elements $(d^{(n)})_{n\in\mathbb{N}}$ in $D - C_\mathbf{D}(K_\Upsilon(\mathbf{D}))$ such that $d^{(n)} \succ_{\mathbf{D}_\Upsilon} d^{(n+1)}$ for every $n \geqslant 0$, whose existence is prevented by the definition of a hierarchical signature. □

4. A gluing condition for Υ-total Σ-algebras

Let $\Sigma = (S, \Omega, \eta)$ and $\Upsilon \subseteq \Omega$ be throughout this section, and unless otherwise stated, an arbitrary signature and a non-empty subset of its set of operation symbols. The goal of this section is to find a *gluing condition* for homomorphisms of Υ-total Σ-algebras, that is, a necessary and sufficient condition for the existence of a pushout complement in $\mathrm{Alg}_{\Sigma,\Upsilon}$ of two given homomorphisms $f : \mathbf{K} \to \mathbf{A}$ and $m : \mathbf{A} \to \mathbf{D}$ of Υ-total Σ-algebras.

First, we describe the construction of a pushout of two homomorphisms in $\mathrm{Alg}_{\Sigma,\Upsilon}$.

Proposition 19. Let $f : \mathbf{K} \to \mathbf{A}$ and $g : \mathbf{K} \to \mathbf{B}$ be two homomorphisms of Υ-total Σ-algebras. Let K' be a subset of $\mathbf{K}$ such that $\mathbf{K}$ is a free Υ-completion of $\mathbf{K}'$, let A' be a subset of $\mathbf{A}$ containing $f(K')$ and such that $\mathbf{A}$ is a free Υ-completion of $\mathbf{A}'$, and let B' be a subset of $\mathbf{B}$ containing $g(K')$ and such that $\mathbf{B}$ is a free Υ-completion of $\mathbf{B}'$. Let $\ell_{\mathbf{K}'} : \mathbf{K}' \hookrightarrow \mathbf{K}$, $\ell_{\mathbf{A}'} : \mathbf{A}' \hookrightarrow \mathbf{A}$ and $\ell_{\mathbf{B}'} : \mathbf{B}' \hookrightarrow \mathbf{B}$ denote the embeddings of the corresponding relative subalgebras. Let $(\mathbf{H}', \tilde{g}' : \mathbf{A}' \to \mathbf{H}', \tilde{f}' : \mathbf{B}' \to \mathbf{H}')$ be a pushout of the restrictions $f|_{K'} : \mathbf{K}' \to \mathbf{A}'$ and $g|_{K'} : \mathbf{K}' \to \mathbf{B}'$ in Alg_Σ, let $\ell_{\mathbf{H}'} : \mathbf{H}' \hookrightarrow \mathbf{H}$ be a free Υ-completion of $\mathbf{H}'$, and let $\tilde{g} : \mathbf{A} \to \mathbf{H}$ and $\tilde{f} : \mathbf{B} \to \mathbf{H}$ be, respectively, the unique homomorphisms of Υ-total Σ-algebras extending $\tilde{g}' : \mathbf{A}' \to \mathbf{H}'$ and $\tilde{f}' : \mathbf{B}' \to \mathbf{H}'$.



Then $(\mathbf{H}, \tilde{g} : \mathbf{A} \to \mathbf{H}, \tilde{f} : \mathbf{B} \to \mathbf{H})$ is a pushout of $f : \mathbf{K} \to \mathbf{A}$ and $g : \mathbf{K} \to \mathbf{B}$ in $\mathrm{Alg}_{\Sigma,\Upsilon}$.

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Proof. The mapping that associates to every Σ-algebra $\mathbf{A}$ a free Υ-completion $\overline{\mathbf{A}}$ of it and to every homomorphism $f : \mathbf{A} \to \mathbf{B}$ of Σ-algebras its unique extension $\hat{f} : \overline{\mathbf{A}} \to \overline{\mathbf{B}}$ to these free Υ-completions, is a functor $\mathrm{Alg}_\Sigma \to \mathrm{Alg}_{\Sigma,\Upsilon}$ left-adjoint to the full embedding $\mathrm{Alg}_{\Sigma,\Upsilon} \hookrightarrow \mathrm{Alg}_\Sigma$, and therefore it preserves pushouts (see, for instance, the dual of Borceux (1994, Proposition 3.2.2)). □

Remark 20. It is easy to prove directly, without using any category theory, that in the previous result it is enough to assume that K' generates $\mathbf{K}$ to guarantee that its thesis holds. We shall not use this generalization here.

Taking the particular case of this proposition with $\mathbf{K}' = \mathbf{K}$, $\mathbf{A}' = \mathbf{A}$ and $\mathbf{B}' = \mathbf{B}$, we obtain the following result.

Corollary 21. Let $f : \mathbf{K} \to \mathbf{A}$ and $g : \mathbf{K} \to \mathbf{B}$ be two homomorphisms of Υ-total Σ-algebras, let $(\mathbf{H}, g' : \mathbf{A} \to \mathbf{H}, f' : \mathbf{B} \to \mathbf{H})$ be a pushout of f and g in Alg_Σ, and let $\ell_{\mathbf{H}} : \mathbf{H} \hookrightarrow \overline{\mathbf{H}}$ be a free Υ-completion of $\mathbf{H}$. Then $(\overline{\mathbf{H}}, \ell_{\mathbf{H}} \circ g' : \mathbf{A} \to \overline{\mathbf{H}}, \ell_{\mathbf{H}} \circ f' : \mathbf{B} \to \overline{\mathbf{H}})$ is a pushout of f and g in $\mathrm{Alg}_{\Sigma,\Upsilon}$.

Since we have already furnished explicit descriptions of pushouts in Alg_Σ (Proposition 1) and free Υ-completions (Section 3), this result provides an explicit description of pushouts in $\mathrm{Alg}_{\Sigma,\Upsilon}$.

Example 22. Let Σ be a one-sorted signature with a single, binary operation symbol $\varphi \in \Omega$. Let $\mathbf{K}$ be the empty algebra, let $\mathbf{A}$ be a total Σ-algebra with carrier set $A = \{a\}$ and operation $\varphi^{\mathbf{A}}(a, a) = a$ and let $\mathbf{B}$ be any non-empty total Σ-algebra. Let f and g be the empty homomorphisms from $\mathbf{K}$ to $\mathbf{A}$ and $\mathbf{B}$, respectively.

The pushout of f and g in Alg_Σ is given by the disjoint union $\mathbf{A} + \mathbf{B}$ of $\mathbf{A}$ and $\mathbf{B}$, and their pushout in TAlg_Σ is then given by a free completion of $\mathbf{A} + \mathbf{B}$.

Corollary 23. Let

$$\begin{array}{ccc} \mathbf{K} & \xrightarrow{f} & \mathbf{A} \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle \tilde{g}} \\ \mathbf{B} & \xrightarrow[\tilde{f}]{} & \mathbf{D} \end{array}$$

be a pushout square in $\mathrm{Alg}_{\Sigma,\Upsilon}$. If f is closed and injective and g is injective, then $\tilde{f}$ is closed and injective and $\tilde{g}$ is injective.

Proof. Let $(\mathbf{D}_0, g_0 : \mathbf{A} \to \mathbf{D}_0, f_0 : \mathbf{B} \to \mathbf{D}_0)$ be a pushout of f and g in Alg_Σ. By Llabrés and Rosselló (2000, Corollary 2), f_0 is closed and injective and g_0 is injective and, by Corollary 21, if $\ell_{\mathbf{D}_0} : \mathbf{D}_0 \hookrightarrow \overline{\mathbf{D}}_0$ is a free Υ-completion of $\mathbf{D}_0$, then

$$(\overline{\mathbf{D}}_0, \ell_{\mathbf{D}_0} \circ g_0 : \mathbf{A} \to \overline{\mathbf{D}}_0, \ell_{\mathbf{D}_0} \circ f_0 : \mathbf{B} \to \overline{\mathbf{D}}_0)$$

is also a pushout of $f : \mathbf{K} \to \mathbf{A}$ and $g : \mathbf{K} \to \mathbf{B}$ in $\mathrm{Alg}_{\Sigma,\Upsilon}$. In this pushout, the homomorphisms $\ell_{\mathbf{D}_0} \circ f_0$ and $\ell_{\mathbf{D}_0} \circ g_0$ are clearly injective, and it is straightforward to check that $\ell_{\mathbf{D}_0} \circ f_0$ is closed using the fact that $\varphi^{\mathbf{B}}$ is total for every $\varphi \in \Upsilon$, that f_0 is closed, and that $\psi^{\mathbf{D}_0} = \psi^{\overline{\mathbf{D}}_0}$ for every $\psi \in \Omega - \Upsilon$. Then, the uniqueness up to isomorphism

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of pushouts entails that the homomorphisms $\tilde{f}$ and $\tilde{g}$ in the statement are also closed and injective, and injective, respectively. □

We establish now a result that relates the existence of a pushout complement of two homomorphisms in $\mathrm{Alg}_{\Sigma,\Upsilon}$ to the existence of a suitable pushout complement in Alg_{Σ}.

Proposition 24. Two homomorphisms of Υ-total Σ-algebras $f : \mathbf{K} \to \mathbf{A}$ and $m : \mathbf{A} \to \mathbf{D}$ have a pushout complement in $\mathrm{Alg}_{\Sigma,\Upsilon}$ if and only if there exists a relative subalgebra $\mathbf{D}'$ of $\mathbf{D}$ such that its carrier contains $m(A)$, $\mathbf{D}' \hookrightarrow \mathbf{D}$ is a free Υ-completion, and $f : \mathbf{K} \to \mathbf{A}$ and $m : \mathbf{A} \to \mathbf{D}'$ have a pushout complement in Alg_{Σ}.

Proof. Assume first that $f : \mathbf{K} \to \mathbf{A}$ and $m : \mathbf{A} \to \mathbf{D}$ have a pushout complement

$$(\mathbf{B}, g : \mathbf{K} \to \mathbf{B}, \tilde{f} : \mathbf{B} \to \mathbf{D})$$

in $\mathrm{Alg}_{\Sigma,\Upsilon}$. Let $(\mathbf{D}_0, g_0 : \mathbf{A} \to \mathbf{D}_0, f_0 : \mathbf{B} \to \mathbf{D}_0)$ be a pushout of $f : \mathbf{K} \to \mathbf{A}$ and $g : \mathbf{K} \to \mathbf{B}$ in Alg_{Σ} and let $\ell : \mathbf{D}_0 \hookrightarrow \overline{\mathbf{D}}_0$ be a free Υ-completion of $\mathbf{D}_0$. By Corollary 21, $(\overline{\mathbf{D}}_0, \ell \circ g_0 : \mathbf{A} \to \overline{\mathbf{D}}_0, \ell \circ f_0 : \mathbf{B} \to \overline{\mathbf{D}}_0)$ is a pushout of $f : \mathbf{K} \to \mathbf{A}$ and $g : \mathbf{K} \to \mathbf{B}$ in $\mathrm{Alg}_{\Sigma,\Upsilon}$, and thus, by the uniqueness up to isomorphism of pushouts, there exists an isomorphism $h : \overline{\mathbf{D}}_0 \to \mathbf{D}$ such that $h \circ \ell \circ g_0 = m$ and $h \circ \ell \circ f_0 = \tilde{f}$. Then the relative subalgebra of $\mathbf{D}$ supported on $h(D_0)$ satisfies the properties in the statement.

Conversely, let $\mathbf{D}'$ be a relative subalgebra of $\mathbf{D}$ that contains $m(A)$ and such that the embedding $\mathbf{D}' \hookrightarrow \mathbf{D}$ is a free Υ-completion and $f : \mathbf{K} \to \mathbf{A}$ and $m : \mathbf{A} \to \mathbf{D}'$ have a pushout complement $(\mathbf{B}', g' : \mathbf{K} \to \mathbf{B}', d' : \mathbf{B}' \to \mathbf{D}')$ in Alg_{Σ}. Let $\ell_{\mathbf{B}'} : \mathbf{B}' \hookrightarrow \overline{\mathbf{B}'}$ be a free Υ-completion of $\mathbf{B}'$, and let $\hat{d} : \overline{\mathbf{B}'} \to \mathbf{D}$ be the unique homomorphism of Σ-algebras that extends $d' : \mathbf{B}' \to \mathbf{D}$. Then, by Proposition 19, $(\overline{\mathbf{B}'}, \ell_{\mathbf{B}'} \circ g' : \mathbf{K} \to \overline{\mathbf{B}'}, \hat{d} : \overline{\mathbf{B}'} \to \mathbf{D})$ is a pushout complement of $f : \mathbf{K} \to \mathbf{A}$ and $m : \mathbf{A} \to \mathbf{D}$ in $\mathrm{Alg}_{\Sigma,\Upsilon}$. □

Corollary 25. If two homomorphisms of Υ-total Σ-algebras have a pushout complement in Alg_{Σ}, they also have a pushout complement in $\mathrm{Alg}_{\Sigma,\Upsilon}$.

Two homomorphisms of Υ-total Σ-algebras having a pushout complement in $\mathrm{Alg}_{\Sigma,\Upsilon}$ need not have a pushout complement in Alg_{Σ}, as the following example shows.

Example 26. Consider the situation in Example 22. Let $\mathbf{D} = (D, \varphi^{\mathbf{D}})$ denote a free completion of $\mathbf{A} + \mathbf{B}$, that yields the pushout of f and g in TAlg_{Σ}, and let $m : \mathbf{A} \to \mathbf{D}$ be the corresponding embedding. Then $f : \mathbf{K} \to \mathbf{A}$ and $m : \mathbf{A} \to \mathbf{D}$ have a pushout complement in TAlg_{Σ}: namely, $\mathbf{B}$ together with the empty mapping $g : \mathbf{K} \to \mathbf{B}$ and the embedding $\mathbf{B} \hookrightarrow \mathbf{D}$. But they do not have any pushout complement in Alg_{Σ}, since they do not satisfy condition GC3 of Proposition 3.

Returning to Proposition 24, the relative subalgebra $\mathbf{D}'$ of $\mathbf{D}$ satisfying the conditions in its statement need not be unique because, by Proposition 4, any other relative subalgebra of $\mathbf{D}$ between $m(A)$ and $\mathbf{D}'$ such that $\mathbf{D}$ is its free completion would also work. However, despite the possible lack of uniqueness of such a relative subalgebra $\mathbf{D}'$, the natural pushout complement of two homomorphisms in $\mathrm{Alg}_{\Sigma,\Upsilon}$ is, when it exists, unique, as the following result shows.

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Proposition 27. Let $f : \mathbf{K} \to \mathbf{A}$ and $m : \mathbf{A} \to \mathbf{D}$ be two homomorphisms of Υ-total Σ-algebras that have a pushout complement in $\mathrm{Alg}_{\Sigma,\Upsilon}$. Then, they have one and only one natural pushout complement, and it is terminal in the sense of Theorem 2.

Proof. Let us prove first that $f : \mathbf{K} \to \mathbf{A}$ and $m : \mathbf{A} \to \mathbf{D}$ have a natural pushout complement in $\mathrm{Alg}_{\Sigma,\Upsilon}$. Let $\mathbf{D}'$ be a relative subalgebra of $\mathbf{D}$ that contains $m(A)$ and such that the embedding $i : \mathbf{D}' \hookrightarrow \mathbf{D}$ is a free Υ-completion and $f : \mathbf{K} \to \mathbf{A}$ and $m : \mathbf{A} \to \mathbf{D}'$ have a pushout complement in Alg_{Σ}, which exists by Proposition 24. Let $(\mathbf{B}, g : \mathbf{K} \to \mathbf{B}, d : \mathbf{B} \hookrightarrow \mathbf{D}')$ be the natural pushout complement of $f : \mathbf{K} \to \mathbf{A}$ and $m : \mathbf{A} \to \mathbf{D}'$ in Alg_{Σ}, and let $\mathbf{B}_0$ be the subalgebra of $\mathbf{D}$ generated by the carrier of $\mathbf{B}$: by Corollary 10, the embedding $j : \mathbf{B} \hookrightarrow \mathbf{B}_0$ is a free Υ-completion. If we use $d_0 : \mathbf{B}_0 \hookrightarrow \mathbf{D}$ to denote the embedding of $\mathbf{B}_0$ into $\mathbf{D}$, then by Proposition 19,

$$(\mathbf{B}_0, j \circ g : \mathbf{K} \to \mathbf{B}_0, d_0 : \mathbf{B}_0 \hookrightarrow \mathbf{D})$$

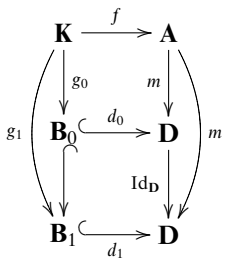
is a natural pushout complement of $f : \mathbf{K} \to \mathbf{A}$ and $m : \mathbf{A} \to \mathbf{D}$ in $\mathrm{Alg}_{\Sigma,\Upsilon}$.

As far as the uniqueness of such a natural pushout complement goes, let

$$(\mathbf{B}_0, g_0 : \mathbf{K} \to \mathbf{B}_0, d_0 : \mathbf{B}_0 \hookrightarrow \mathbf{D}), \qquad (\mathbf{B}_1, g_1 : \mathbf{K} \to \mathbf{B}_1, d_1 : \mathbf{B}_1 \hookrightarrow \mathbf{D})$$

be two of them; notice that each $g_k : \mathbf{K} \to \mathbf{B}_k$ is the composition $m \circ f$ understood with target algebra $\mathbf{B}_k$. We must prove that $\mathbf{B}_0 = \mathbf{B}_1$, since it will imply that $d_0 = d_1$ and $g_0 = g_1$. To do so, we distinguish two cases, depending on the relative position of $\mathbf{B}_0$ and $\mathbf{B}_1$ within $\mathbf{D}$.

Assume first that one of them contains the other, say $\mathbf{B}_0 \subseteq \mathbf{B}_1$. Consider the following commutative diagram:



Since the upper and outer squares in it are pushout squares in $\mathrm{Alg}_{\Sigma,\Upsilon}$, by the usual pushout decomposition property, the lower square is also a pushout square in $\mathrm{Alg}_{\Sigma,\Upsilon}$. Let us prove that this implies $\mathbf{B}_0 = \mathbf{B}_1$.

Let $C = D \times \{0\} \cup (D - B_0) \times \{1\}$, and let $\pi : C \to D$ denote the first projection: $\pi(b, i) = b$ for every $(b, i) \in C$. Let $\mathbf{C} = (C, (\varphi^{\mathbf{C}})_{\varphi \in \Omega})$ be the Υ-total Σ-algebra with carrier set C and with its operations defined as follows: for every $\varphi \in \Omega$

$$\mathrm{dom}\, \varphi^{\mathbf{C}} = \{\underline{c} \in C^{\omega(\varphi)} \mid \pi(\underline{c}) \in \mathrm{dom}\, \varphi^{\mathbf{D}}\}$$

and for every $\underline{c} \in \mathrm{dom}\, \varphi^{\mathbf{C}}$

$$\varphi^{\mathbf{C}}(\underline{c}) = \begin{cases} (\varphi^{\mathbf{D}}(\pi(\underline{c})), 0) & \text{if } \underline{c} \in (D \times \{0\})^{\omega(\varphi)} \text{ or } \varphi^{\mathbf{D}}(\pi(\underline{c})) \in B_0 \\ (\varphi^{\mathbf{D}}(\pi(\underline{c})), 1) & \text{if } \underline{c} \notin (D \times \{0\})^{\omega(\varphi)} \text{ and } \varphi^{\mathbf{D}}(\pi(\underline{c})) \in D - B_0. \end{cases}$$

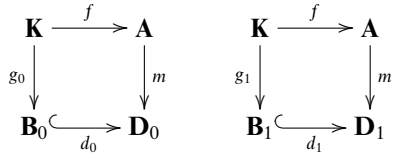
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Consider the homomorphisms of Σ-algebras $g, h : \mathbf{D} \to \mathbf{C}$ defined by

$$g : \begin{array}{ccc} D & \to & C \\ d & \mapsto & (d,0) \end{array} \qquad h : \begin{array}{ccc} D & \to & C \\ d & \mapsto & \begin{cases} (d,0) & \text{if } d \in B_0 \\ (d,1) & \text{if } d \in D - B_0. \end{cases} \end{array}$$

The universal property of pushouts applied to the lower square of the previous diagram and the homomorphisms $g : \mathbf{D} \to \mathbf{C}$, $h|_{\mathbf{B}_1} : \mathbf{B}_1 \to \mathbf{C}$ implies that $g|_{\mathbf{B}_1} = h|_{\mathbf{B}_1}$, and therefore $\mathbf{B}_1 = \mathbf{B}_0$, as we wanted to prove.

Consider now the general case. Let



be pushout squares in Alg_Σ. Using the proof of the 'only if' implication in Proposition 24, we see that each $\mathbf{D}_k$ may be assumed to be a relative subalgebra of $\mathbf{D}$ whose carrier contains $m(A)$ and such that the embedding $i_k : \mathbf{D}_k \hookrightarrow \mathbf{D}$ is a free Υ-completion and the corresponding homomorphisms $\mathbf{B}_k \to \mathbf{D}_k$ and $\mathbf{A} \to \mathbf{D}_k$ are, respectively, $d_k : \mathbf{B}_k \to \mathbf{D}$ and $m : \mathbf{A} \to \mathbf{D}$ understood with target algebra $\mathbf{D}_k$. Notice, in particular, that each $d_k : \mathbf{B}_k \hookrightarrow \mathbf{D}_k$ is the embedding of a closed subalgebra, and hence $(\mathbf{B}_k, g_k : \mathbf{K} \to \mathbf{B}_k, d_k : \mathbf{B}_k \hookrightarrow \mathbf{D}_k)$ is the natural pushout complement of $f : \mathbf{K} \to \mathbf{A}$ and $m : \mathbf{A} \to \mathbf{D}_k$ in Alg_Σ.

Applying Proposition 4 to these pushout squares and the relative subalgebra $\mathbf{D}_0 \cap \mathbf{D}_1$ of $\mathbf{D}_0$ and $\mathbf{D}_1$, we get that both $\mathbf{B}_0 \cap \mathbf{D}_1$ and $\mathbf{B}_1 \cap \mathbf{D}_0$ yield natural pushout complements of $f : \mathbf{K} \to \mathbf{A}$ and $m : \mathbf{A} \to \mathbf{D}_0 \cap \mathbf{D}_1$ in Alg_Σ. By the uniqueness of such a natural pushout complement, this entails that $\mathbf{B}_0 \cap \mathbf{D}_1 = \mathbf{B}_1 \cap \mathbf{D}_0$, and then an easy set-theoretical argument shows that this algebra is nothing but $\mathbf{B}_0 \cap \mathbf{B}_1$. Since $\mathbf{B}_0$ and $\mathbf{B}_1$ are closed subalgebras of $\mathbf{D}$, their intersection is also a closed subalgebra of it.

Now, we have $\mathbf{D}_0 \cap \mathbf{D}_1 \hookrightarrow \mathbf{D}$ is a free Υ-completion. Indeed, the satisfaction of properties (FAC1), (FAC2) and (FAC3) in Proposition 9 by this embedding is a direct consequence of their satisfaction by the embeddings $\mathbf{D}_0 \hookrightarrow \mathbf{D}$ and $\mathbf{D}_1 \hookrightarrow \mathbf{D}$. As far as (FAC4) goes, since D_1 is an initial segment of $\mathbf{D}$ and D_0 generates $\mathbf{D}$, we have that $D_0 \cap D_1$ generates $\mathbf{D}_1$, and, therefore, since D_1 also generates $\mathbf{D}$, we finally have that $D_0 \cap D_1$ generates $\mathbf{D}$.

Then, by Corollary 21, $\mathbf{B}_0 \cap \mathbf{B}_1$ yields a natural pushout complement of $f : \mathbf{K} \to \mathbf{A}$ and $m : \mathbf{A} \to \mathbf{D}$ in $\mathrm{Alg}_{\Sigma,\Upsilon}$, and it is contained in $\mathbf{B}_0$ and in $\mathbf{B}_1$. From the previous case we conclude that $\mathbf{B}_0 = \mathbf{B} = \mathbf{B}_1$.

This finishes the proof of the uniqueness of the natural pushout complement of $f : \mathbf{K} \to \mathbf{A}$ and $m : \mathbf{A} \to \mathbf{D}$. To prove that it is terminal among all their pushout complements, let $(\mathbf{C}, g_0 : \mathbf{K} \to \mathbf{C}, f' : \mathbf{C} \to \mathbf{D})$ be any pushout complement of them in $\mathrm{Alg}_{\Sigma,\Upsilon}$ and let $(\mathbf{D}', m' : \mathbf{A} \to \mathbf{D}', f'' : \mathbf{C} \to \mathbf{D}')$ be a pushout of $f : \mathbf{K} \to \mathbf{A}$ and $g_0 : \mathbf{K} \to \mathbf{C}$ in Alg_Σ such that $\mathbf{D}'$ is a relative subalgebra of $\mathbf{D}$, the embedding $\mathbf{D}' \hookrightarrow \mathbf{D}$ is a Υ-free completion, and m' and f'' are m and f' understood with target algebra $\mathbf{D}'$ (*cf.* the first part of the proof of Proposition 24). Now let $(\mathbf{B}', g_0' : \mathbf{K} \to \mathbf{B}', d' : \mathbf{B}' \hookrightarrow \mathbf{D}')$ be the natural pushout complement of $f : \mathbf{K} \to \mathbf{A}$ and $m' : \mathbf{A} \to \mathbf{D}'$ in Alg_Σ. By the terminality of natural

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pushout complements in Alg_Σ (Theorem 2), there exists a unique epimorphism $e : \mathbf{C} \to \mathbf{B}'$ such that $e \circ g_0 = g_0'$ and $d' \circ e = f''$. Since the natural pushout complement in $\mathrm{Alg}_{\Sigma,\Upsilon}$ of $f : \mathbf{K} \to \mathbf{A}$ and $m : \mathbf{A} \to \mathbf{D}$ is given by the closed subalgebra $\mathbf{B}_0$ of $\mathbf{D}$ generated by $\mathbf{B}'$, the composition $e : \mathbf{C} \to \mathbf{B}' \hookrightarrow \mathbf{B}_0$ is an epimorphism and satisfies the desired properties, and it is unique because the embedding of $\mathbf{B}_0$ into $\mathbf{D}$ is a monomorphism. □

We now have the following result, which takes us one step closer to the gluing condition in $\mathrm{Alg}_{\Sigma,\Upsilon}$.

Theorem 28. Let $f : \mathbf{K} \to \mathbf{A}$ and $m : \mathbf{A} \to \mathbf{D}$ be two homomorphisms of Υ-total Σ-algebras. Set $D_m = K_\Upsilon(\mathbf{D}) \cup {\downarrow_{\mathbf{D}}} m(A)$ and $D_0 = C_{\mathbf{D}}(D_m)$.

Then, $f : \mathbf{K} \to \mathbf{A}$ and $m : \mathbf{A} \to \mathbf{D}$ have a pushout complement in $\mathrm{Alg}_{\Sigma,\Upsilon}$ if and only if

(i) $f : \mathbf{K} \to \mathbf{A}$ and $m : \mathbf{A} \to \mathbf{D}_m$ have a pushout complement in Alg_Σ; and
(ii) if $(\mathbf{B}_m, g : \mathbf{K} \to \mathbf{B}_m, d_m : \mathbf{B}_m \hookrightarrow \mathbf{D}_m)$ is the natural pushout complement of $f : \mathbf{K} \to \mathbf{A}$ and $m : \mathbf{A} \to \mathbf{D}_m$ in Alg_Σ, and if $\mathbf{B}_0$ is the subalgebra of $\mathbf{D}$ generated by B_m, then the embeddings $\mathbf{B}_0 \hookrightarrow \mathbf{D}_0$ and $\mathbf{D}_0 \hookrightarrow \mathbf{D}$ have a pushout complement in $\mathrm{Alg}_{\Sigma,\Upsilon}$.

Proof. Assume first that $f : \mathbf{K} \to \mathbf{A}$ and $m : \mathbf{A} \to \mathbf{D}$ have a pushout complement in $\mathrm{Alg}_{\Sigma,\Upsilon}$. By Proposition 24, there exists a relative subalgebra $\mathbf{D}'$ of $\mathbf{D}$ whose carrier set contains $m(A)$, and such that $\mathbf{D}' \hookrightarrow \mathbf{D}$ is a free Υ-completion and $f : \mathbf{K} \to \mathbf{A}$ and $m : \mathbf{A} \to \mathbf{D}'$ have a pushout complement in Alg_Σ. Since $\mathbf{D}$ is a free Υ-completion of $\mathbf{D}'$, by Proposition 13 we have that $K_\Upsilon(\mathbf{D}) \subseteq D'$, and since $m(A) \subseteq D'$ and D' is an initial segment of $\mathbf{D}$, we finally have that $D_m \subseteq D'$. But then Proposition 4 entails that $f : \mathbf{K} \to \mathbf{A}$ and $m : \mathbf{A} \to \mathbf{D}_m$ have a pushout complement in Alg_Σ, which proves (i).

In order to prove (ii), let $(\mathbf{B}_m, g_m : \mathbf{K} \to \mathbf{B}_m, d_m : \mathbf{B}_m \hookrightarrow \mathbf{D}_m)$ be the natural pushout complement of $f : \mathbf{K} \to \mathbf{A}$ and $m : \mathbf{A} \to \mathbf{D}_m$ in Alg_Σ, and let $\mathbf{B}_0$ be the subalgebra of $\mathbf{D}$ supported on $B_0 = C_{\mathbf{D}}(B_m) = C_{\mathbf{D}_0}(B_m)$. By Corollary 14, $\mathbf{D}_m \hookrightarrow \mathbf{D}_0$ is a free Υ-completion and then the first part of the proof of Proposition 27 shows that $(\mathbf{B}_0, g_m : \mathbf{K} \to \mathbf{B}_0, d_0 : \mathbf{B}_0 \hookrightarrow \mathbf{D}_0)$ is the natural pushout complement of $f : \mathbf{K} \to \mathbf{A}$ and $m : \mathbf{A} \to \mathbf{D}_0$ in $\mathrm{Alg}_{\Sigma,\Upsilon}$, where $d_0 : \mathbf{B}_0 \hookrightarrow \mathbf{D}_0$ stands for the corresponding embedding.

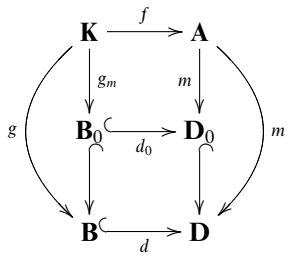
On the other hand, let $(\mathbf{B}, g : \mathbf{K} \to \mathbf{B}, d : \mathbf{B} \hookrightarrow \mathbf{D})$ be the natural pushout complement of $f : \mathbf{K} \to \mathbf{A}$ and $m : \mathbf{A} \to \mathbf{D}$ in $\mathrm{Alg}_{\Sigma,\Upsilon}$. It turns out that $\mathbf{B}_0 \subseteq \mathbf{B}$. Indeed, let

$$\begin{array}{ccc} \mathbf{K} & \xrightarrow{f} & \mathbf{A} \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle m} \\ \mathbf{B} & \xrightarrow[d_0]{} & \mathbf{D}'' \end{array}$$

be a pushout of $f : \mathbf{K} \to \mathbf{A}$ and $g : \mathbf{K} \to \mathbf{B}$ in Alg_Σ such that $\mathbf{D}''$ is a relative subalgebra of $\mathbf{D}$ whose carrier D'' contains $m(A) \cup B$ and the embedding $\mathbf{D}'' \hookrightarrow \mathbf{D}$ is a free Υ-completion: since $\mathbf{B}$ is a closed subalgebra of $\mathbf{D}$, it is also a closed subalgebra of $\mathbf{D}''$. Then, since D'' is an initial segment that contains $K_\Upsilon(\mathbf{D})$ and $m(A)$, it contains D_m, and therefore, by Proposition 4, $\mathbf{B} \cap \mathbf{D}_m$ yields the natural pushout complement of $f : \mathbf{K} \to \mathbf{A}$ and $m : \mathbf{A} \to \mathbf{D}_m$ in Alg_Σ. Hence, $\mathbf{B}_m = \mathbf{B} \cap \mathbf{D}_m$ and, in particular, $\mathbf{B}_m \subseteq \mathbf{B}$. Finally, since $\mathbf{B}$ is a closed subalgebra of $\mathbf{D}$, its carrier must contain $B_0 = C_{\mathbf{D}}(B_m)$, as we claimed.

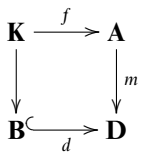
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Consider now the following commutative diagram:



The upper and outer squares in this diagram are pushout squares in $\mathrm{Alg}_{\Sigma,\Upsilon}$, so, by the pushout decomposition property, the lower square is also a pushout square in $\mathrm{Alg}_{\Sigma,\Upsilon}$, which proves assertion (ii) in the statement.

Conversely, if condition (i) is satisfied, we have, by Proposition 19 and Corollary 10, $(\mathbf{D}_0, m : \mathbf{A} \to \mathbf{D}_0, d_0 : \mathbf{B}_0 \hookrightarrow \mathbf{D}_0)$ is a pushout of $f : \mathbf{K} \to \mathbf{A}$ and $g : \mathbf{K} \to \mathbf{B}_0$ in $\mathrm{Alg}_{\Sigma,\Upsilon}$ (see the proof of the converse implication), and if (ii) is satisfied, there exists a pushout complement $(\mathbf{B}, \mathbf{B}_0 \to \mathbf{B}, \mathbf{B} \hookrightarrow \mathbf{D})$ of $d_0 : \mathbf{B}_0 \hookrightarrow \mathbf{D}_0$ and $\mathbf{D}_0 \hookrightarrow \mathbf{D}$. Combining the corresponding two pushout squares as in the last diagram, we obtain a pushout square in $\mathrm{Alg}_{\Sigma,\Upsilon}$



and hence $f : \mathbf{K} \to \mathbf{A}$ and $m : \mathbf{A} \to \mathbf{D}$ have a pushout complement in $\mathrm{Alg}_{\Sigma,\Upsilon}$. □

Notice that, with the notation of this theorem, if D_m generates $\mathbf{D}$, then $f : \mathbf{K} \to \mathbf{A}$ and $m : \mathbf{A} \to \mathbf{D}$ have a pushout complement in $\mathrm{Alg}_{\Sigma,\Upsilon}$ if and only if $f : \mathbf{K} \to \mathbf{A}$ and $m : \mathbf{A} \to \mathbf{D}_m$ have a pushout complement in Alg_{Σ}, that is, if and only if $f : \mathbf{K} \to \mathbf{A}$ and $m : \mathbf{A} \to \mathbf{D}_m$ satisfy conditions GC1 to GC3 in Proposition 3. One important case when we can guarantee that D_m generates $\mathbf{D}$, independently of m, is when $K_\Upsilon(\mathbf{D})$ generates $\mathbf{D}$. In the last part of Section 3 we gave some cases when this happens: for instance, when $\mathbf{D}$ is finitely generated.

Unfortunately, there are cases when D_m does not generate $\mathbf{D}$, and in these cases the satisfaction of the gluing condition in Alg_{Σ} by $f : \mathbf{K} \to \mathbf{A}$ and $m : \mathbf{A} \to \mathbf{D}_m$ need not be sufficient to guarantee the existence of a pushout complement of $f : \mathbf{K} \to \mathbf{A}$ and $m : \mathbf{A} \to \mathbf{D}$ in $\mathrm{Alg}_{\Sigma,\Upsilon}$, as the following example shows.

Example 29. Let Σ and $\mathbf{D}$ be as in Example 12. Let $\mathbf{K}$ be the empty Σ-algebra, let $\mathbf{A}$ be the closed subalgebra of $\mathbf{D}$ supported on $\{a\}$, let $f : \mathbf{K} \to \mathbf{A}$ be the empty mapping and $m : \mathbf{A} \to \mathbf{D}$ be the corresponding closed embedding.

Then $D_m = K_\Omega(\mathbf{D}) = \{a\}$ does not generate $\mathbf{D}$ and, although $f : \mathbf{K} \to \mathbf{A}$ and $m : \mathbf{A} \to \mathbf{D}_m$ trivially have a pushout complement in Alg_{Σ}, we shall see in Example 36 that $f : \mathbf{K} \to \mathbf{A}$ and $m : \mathbf{A} \to \mathbf{D}$ do not have a pushout complement in TAlg_{Σ}.

So, to establish a gluing condition in $\mathrm{Alg}_{\Sigma,\Upsilon}$, it remains to find a necessary and sufficient

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condition on two closed embeddings $d_0 : \mathbf{B}_0 \hookrightarrow \mathbf{D}_0$ and $i_0 : \mathbf{D}_0 \hookrightarrow \mathbf{D}$ of Υ-total Σ-algebras, with $\mathbf{D}_0$ supported on an initial segment of $\mathbf{D}$ containing its Υ-freeness core $K_\Upsilon(\mathbf{D})$, for the existence of a pushout complement of them in $\mathrm{Alg}_{\Sigma,\Upsilon}$. Finding it will be easier once we have proved, in Lemma 33 below, that there always exists the greatest initial segment H (the set $H_{\mathbf{D}}(D_0, B_0)$ in the next definition) of $\mathbf{D}$ such that $\mathbf{B}_0 \hookrightarrow \mathbf{D}_0$ and $\mathbf{D}_0 \hookrightarrow \mathbf{H}$ have a pushout complement in Alg_Σ: it will turn out that $\mathbf{B}_0 \hookrightarrow \mathbf{D}_0$ and $\mathbf{D}_0 \hookrightarrow \mathbf{D}$ have a pushout complement in $\mathrm{Alg}_{\Sigma,\Upsilon}$ if and only if H generates $\mathbf{D}$.

Definition 30. Let $\mathbf{D}$ be a Σ-algebra, let D_0 be a closed subset of $\mathbf{D}$ that is also an initial segment of it, and let B_0 be a closed subset of $\mathbf{D}_0$. Then

$$H_{\mathbf{D}}(D_0, B_0) = D_0 \cup \{x \in D - D_0 \mid \begin{array}{l}\text{every chain with respect to } \prec_{\mathbf{D}} \text{ going from} \\ \text{some element } d_0 \in D_0 - B_0 \text{ to } x \text{ involves} \\ \text{some element of } B_0\}.\end{array}$$

Example 31. Let Σ and $\mathbf{D}$ be as in Example 12. Let $\mathbf{B}_0 = \varnothing$, and let $\mathbf{D}_0$ be the closed subalgebra of $\mathbf{D}$ supported on $D_0 = \{a\}$. Then $H_{\mathbf{D}}(\mathbf{B}_0, \mathbf{D}_0) = D_0$ because for every element b_k of $D - D_0$ there exists a chain with respect to $\prec_{\mathbf{D}}$ going from a to b_k, and this chain does not involve any element of B_0, because it is empty.

Example 32. Let Σ be a signature with all its operations nullary or unary. Let $\mathbf{D}$ be a Σ-algebra, D_0 be a closed subset of $\mathbf{D}$ that is also an initial segment of it, and B_0 be a closed subset of $\mathbf{D}_0$. Then $H_{\mathbf{D}}(D_0, B_0) = D$ because in this case there is no chain with respect to $\prec_{\mathbf{D}}$ going from an element of D_0 to an element of $D - D_0$.

Lemma 33. Let $\mathbf{D}_0$ be a closed subalgebra of a Σ-algebra $\mathbf{D}$ supported on an initial segment of it, and let $\mathbf{B}_0$ be a closed subalgebra of $\mathbf{D}_0$. Let $\mathbf{H}$ be the relative subalgebra of $\mathbf{D}$ supported on the set $H_{\mathbf{D}}(D_0, B_0)$. Then:

(a) $H_{\mathbf{D}}(D_0, B_0)$ is an initial segment of $\mathbf{D}$.
(b) The embeddings $\mathbf{B}_0 \hookrightarrow \mathbf{D}_0$ and $\mathbf{D}_0 \hookrightarrow \mathbf{H}$ have a pushout complement in Alg_Σ.
(c) If $\mathbf{D}'$ is a relative subalgebra of $\mathbf{D}$ supported on an initial segment of it containing D_0 such that the embeddings $\mathbf{B}_0 \hookrightarrow \mathbf{D}_0$ and $\mathbf{D}_0 \hookrightarrow \mathbf{D}'$ have a pushout complement in Alg_Σ, then $\mathbf{D}' \subseteq \mathbf{H}$.

Proof. To simplify the notation, we shall use H to denote $H_{\mathbf{D}}(D_0, B_0)$.

(a) Let $\varphi \in \Omega$ and $\underline{x} = (x_1, \ldots, x_m) \in \mathrm{dom}\, \varphi^{\mathbf{D}}$ be such that $\varphi^{\mathbf{D}}(\underline{x}) \in H$.
If $\varphi^{\mathbf{D}}(\underline{x}) \in D_0$, then $\underline{x} \in D_0^{\omega(\varphi)} \subseteq H^{\omega(\varphi)}$, because D_0 is an initial segment of $\mathbf{D}$. If $\varphi^{\mathbf{D}}(\underline{x}) \in H \cap (D - D_0)$, so that every chain with respect to $\prec_{\mathbf{D}}$ going from some $d_0 \in D_0 - B_0$ to $\varphi^{\mathbf{D}}(\underline{x})$ involves some element of B_0, then every x_i in $\underline{x}$ that does not belong to D_0 must also satisfy this same property, which entails that $\underline{x} \in H^{\omega(\varphi)}$ also.
(b) We shall prove that the embeddings $\mathbf{B}_0 \hookrightarrow \mathbf{D}_0$ and $\mathbf{D}_0 \hookrightarrow \mathbf{H}$ satisfy the conditions GC1 to GC3 in Proposition 3. Notice that the candidate for a natural pushout complement in Alg_Σ of these two embeddings is given by the closed subalgebra $\mathbf{B}$ of $\mathbf{H}$ supported on $B = C_{\mathbf{H}}((H - D_0) \cup B_0)$.
Since $\mathbf{D}_0 \hookrightarrow \mathbf{H}$ is injective, condition GC1 is clearly satisfied. To check condition GC2,

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it is enough to prove $B \cap D_0 \subseteq B_0$. And this inclusion holds, because

$$B \cap D_0 = C_{\mathbf{H}}((H - D_0) \cup B_0) \cap D_0 = C_{\mathbf{D}_0}(((H - D_0) \cup B_0) \cap D_0) = C_{\mathbf{D}_0}(B_0) = B_0$$

where the second equality holds because D_0 is an initial segment of $\mathbf{D}$.
As far as condition GC3 goes, let $\underline{x} = (x_1, \ldots, x_m) \in \operatorname{dom} \varphi^{\mathbf{H}}$ be such that $\underline{x} \notin B^{\omega(\varphi)}$; we must prove that $\underline{x} \in \operatorname{dom} \varphi^{\mathbf{D}_0}$. If $\varphi^{\mathbf{H}}(\underline{x}) \in D_0$, then $\underline{x} \in \operatorname{dom} \varphi^{\mathbf{D}_0}$ because D_0 is a closed initial segment of $\mathbf{D}$. Assume now that $\varphi^{\mathbf{H}}(\underline{x}) \notin D_0$. Since $\underline{x} \notin B^{\omega(\varphi)}$, there exists some x_k, $k = 1, \ldots, m$, such that $x_k \notin B = C_{\mathbf{H}}((H - D_0) \cup B_0)$. In particular, $x_k \in D_0 - B_0$, and then $x_k \prec_{\mathbf{D}} \varphi^{\mathbf{H}}(\underline{x})$ is a chain with respect to $\prec_{\mathbf{D}}$ going from an element of $D_0 - B_0$ to $\varphi^{\mathbf{H}}(\underline{x})$ without crossing B_0, which contradicts the fact that $\varphi^{\mathbf{H}}(\underline{x}) \in H$. Then, the case $\varphi^{\mathbf{H}}(\underline{x}) \notin D_0$ cannot happen.

(c) Let $\mathbf{D}'$ be a relative subalgebra of $\mathbf{D}$ supported on an initial segment of it containing $\mathbf{D}_0$ and such that the embeddings $\mathbf{B}_0 \hookrightarrow \mathbf{D}_0$ and $\mathbf{D}_0 \hookrightarrow \mathbf{D}'$ have a pushout complement in Alg_{Σ}. Let $B' = C_{\mathbf{D}'}((D' - D_0) \cup B_0)$, so that the subalgebra $\mathbf{B}'$ of $\mathbf{D}'$ yields a pushout complement of $\mathbf{B}_0 \hookrightarrow \mathbf{D}_0$ and $\mathbf{D}_0 \hookrightarrow \mathbf{D}'$.
Assume that there exists some element $x \in D'$ that does not belong to H. Then $x \notin D_0$ and there exists some chain of elements of D $x^0 \prec_{\mathbf{D}} x^1 \prec_{\mathbf{D}} \ldots \prec_{\mathbf{D}} x^n = x$ such that $x^0 \in D_0 - B_0$ and every $x^i \notin B_0$. Assume that this chain has the shortest length among all chains going from an element of $D_0 - B_0$ to x without crossing B_0.
Since D' is by assumption an initial segment of $\mathbf{D}$ and $x \in D'$, we have that $x^i \in D'$, for every $i = 0, \ldots, n-1$, and $x^0 \prec_{\mathbf{D}'} x^1 \prec_{\mathbf{D}'} \ldots \prec_{\mathbf{D}'} x^n = x$.
If $x^0 \in B'$, then, since $x^0 \in D_0$, we would have

$$x^0 \in C_{\mathbf{D}'}((D' - D_0) \cup B_0) \cap D_0 = C_{\mathbf{D}_0}(((D' - D_0) \cup B_0) \cap D_0) = C_{\mathbf{D}_0}(B_0) = B_0$$

which is contrary to the assumption that $x^0 \in D_0 - B_0$. Therefore $x^0 \notin B'$.
Now, $x^0 \prec_{\mathbf{D}'} x^1$ implies that $x^1 = \varphi^{\mathbf{D}'}(\underline{x})$ for some operation symbol φ and some tuple $\underline{x} \in \operatorname{dom} \varphi^{\mathbf{D}'}$ containing x^0. Since $x^0 \notin B'$, we have that $\underline{x} \notin (B')^{\omega(\varphi)}$, so, by condition GC3 applied to $\mathbf{B}_0 \hookrightarrow \mathbf{D}_0$ and $\mathbf{D}_0 \hookrightarrow \mathbf{D}'$, we have that $\underline{x} \in \operatorname{dom} \varphi^{\mathbf{D}_0}$, which implies that $x^1 \in D_0 - B_0$. But then $x^1 \prec_{\mathbf{D}} \ldots \prec_{\mathbf{D}} x^n = x$ is again a chain going from an element of $D_0 - B_0$ to x that does not cross B_0, and its length is strictly shorter than the previous one, which yields a contradiction. □

We have, at last, the following result, which gives a gluing condition in $\mathrm{Alg}_{\Sigma,\Upsilon}$.

Theorem 34. Let $f : \mathbf{K} \to \mathbf{A}$ and $m : \mathbf{A} \to \mathbf{D}$ be two homomorphisms of Υ-total Σ-algebras. Let

$$D_m = K_{\Upsilon}(\mathbf{D}) \cup \downarrow_{\mathbf{D}} m(A), \quad D_0 = C_{\mathbf{D}}(D_m),$$
$$B_0 = C_{\mathbf{D}}((D_m - m(A)) \cup m(f(K))), \quad H = H_{\mathbf{D}}(D_0, B_0).$$

Then, $f : \mathbf{K} \to \mathbf{A}$ and $m : \mathbf{A} \to \mathbf{D}$ have a pushout complement in $\mathrm{Alg}_{\Sigma,\Upsilon}$ if and only if

(i) $f : \mathbf{K} \to \mathbf{A}$ and $m : \mathbf{A} \to \mathbf{D}_m$ satisfy the gluing condition in Alg_{Σ}; and
(ii) H generates $\mathbf{D}$.

And if f and m satisfy conditions (i) and (ii), the closed subalgebra of $\mathbf{D}$ supported on $C_{\mathbf{D}}((H - D_0) \cup B_0)$ yields the natural pushout complement of $f : \mathbf{K} \to \mathbf{A}$ and $m : \mathbf{A} \to \mathbf{D}$ in $\mathrm{Alg}_{\Sigma,\Upsilon}$.

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Proof. Assume first that f and m have a pushout complement in $\mathrm{Alg}_{\Sigma,\Upsilon}$. Then, by Theorem 28, $f : \mathbf{K} \to \mathbf{A}$ and $m : \mathbf{A} \to \mathbf{D}_m$ have a pushout complement in Alg_{Σ} (and therefore they satisfy the gluing condition in Alg_{Σ}, as point (i) in the statement claims), and the embeddings $\mathbf{B}_0 \hookrightarrow \mathbf{D}_0$ and $\mathbf{D}_0 \hookrightarrow \mathbf{D}$ have a pushout complement in $\mathrm{Alg}_{\Sigma,\Upsilon}$. This last fact implies that there exists a relative subalgebra $\mathbf{D}'$ of $\mathbf{D}$ that contains $\mathbf{D}_0$ and such that $\mathbf{D}$ is its free Υ-completion and the embeddings $\mathbf{B}_0 \hookrightarrow \mathbf{D}_0$ and $\mathbf{D}_0 \hookrightarrow \mathbf{D}'$ have a pushout complement in Alg_{Σ}. But if $\mathbf{D}' \hookrightarrow \mathbf{D}$ is a free Υ-completion, then D' is an initial segment of $\mathbf{D}$, so, by Lemma 33.(c), $D' \subseteq H$. Since D' generates $\mathbf{D}$, this clearly implies that H generates $\mathbf{D}$, which proves point (ii) in the statement.

Conversely, assume that $f : \mathbf{K} \to \mathbf{A}$ and $m : \mathbf{A} \to \mathbf{D}$ are two homomorphisms of Υ-total Σ-algebras satisfying conditions (i) and (ii) in the statement. In this case H is an initial segment of $\mathbf{D}$ that contains $K_\Upsilon(\mathbf{D})$, because it contains D_0, and by assumption it generates $\mathbf{D}$. Therefore, by Proposition 13, $\mathbf{H} \hookrightarrow \mathbf{D}$ is a free Υ-completion. By Proposition 24 and Lemma 33.(b), this implies that $\mathbf{B}_0 \hookrightarrow \mathbf{D}_0$ and $\mathbf{D}_0 \hookrightarrow \mathbf{D}$ have a pushout complement in $\mathrm{Alg}_{\Sigma,\Upsilon}$. On the other hand, condition (i) in the statement says that $f : \mathbf{K} \to \mathbf{A}$ and $m : \mathbf{A} \to \mathbf{D}_m$ have a pushout complement in Alg_{Σ}. Then, by Theorem 28, $f : \mathbf{K} \to \mathbf{A}$ and $m : \mathbf{A} \to \mathbf{D}$ have a pushout complement in $\mathrm{Alg}_{\Sigma,\Upsilon}$.

Finally, the proof of the 'if' implication in Theorem 28 shows that the natural pushout complement in $\mathrm{Alg}_{\Sigma,\Upsilon}$ of $\mathbf{B}_0 \hookrightarrow \mathbf{D}_0$ and $\mathbf{D}_0 \hookrightarrow \mathbf{D}$ yields also the natural pushout complement in $\mathrm{Alg}_{\Sigma,\Upsilon}$ of $f : \mathbf{K} \to \mathbf{A}$ and $m : \mathbf{A} \to \mathbf{D}$. If H generates $\mathbf{D}$, then $\mathbf{H} \hookrightarrow \mathbf{D}$ is a free completion, and hence, by Proposition 19 and Corollary 10, the natural pushout complement in $\mathrm{Alg}_{\Sigma,\Upsilon}$ of $\mathbf{B}_0 \hookrightarrow \mathbf{D}_0$ and $\mathbf{D}_0 \hookrightarrow \mathbf{D}$ is given by the closure in $\mathbf{D}$ of the natural pushout complement in Alg_{Σ} of $\mathbf{B}_0 \hookrightarrow \mathbf{D}_0$ and $\mathbf{D}_0 \hookrightarrow \mathbf{H}$. Since the latter is the closed subalgebra of $\mathbf{H}$ generated by $(H - D_0) \cup B_0$, its closure in $\mathbf{D}$ is $C_{\mathbf{D}}((H - D_0) \cup B_0)$, as claimed in the statement. □

Now we can translate conditions (i) and (ii) in this theorem into a set-theoretical *gluing condition* for homomorphisms of Υ-total Σ-algebras using Proposition 3.

Corollary 35. Let $f : \mathbf{K} \to \mathbf{A}$ and $m : \mathbf{A} \to \mathbf{D}$ be two homomorphisms of Υ-total Σ-algebras. Let

$$D_m = K_\Upsilon(\mathbf{D}) \cup \downarrow_{\mathbf{D}} m(A), \quad B_m = C_{\mathbf{D}_m}((D_m - m(A)) \cup m(f(K)))$$
$$D_0 = C_{\mathbf{D}}(D_m), \quad B_0 = C_{\mathbf{D}}(B_m), \quad H = H_{\mathbf{D}}(D_0, B_0).$$

Let $\theta(f, m)$ be the congruence on $\mathbf{A}^{(+)} + \mathbf{B}_m^{(+)}$ generated by

$$\varepsilon(f, m) = \{(a, m(a)) \mid a \in f(K)\} \cup \{(\varphi_0^{\mathbf{A}}, \varphi_0^{\mathbf{D}}) \mid \varphi_0 \in \Omega^{(0)},\ \varphi_0^{\mathbf{A}} \text{ defined}\}.$$

Then, f and m have a pushout complement in $\mathrm{Alg}_{\Sigma,\Upsilon}$ if and only if they satisfy the following properties:

GC1 $\ker m \subseteq \theta(f, m)$.
GC2 If $m(a) \in B_m$, then $(a, m(a)) \in \theta(f, m)$.
GC3 For every $\varphi \in \Omega$, if $\underline{d} \in \mathrm{dom}\, \varphi^{\mathbf{D}_m}$ and $\underline{d} \notin B_m^{\omega(\varphi)}$, then there exists some $\underline{a} \in \mathrm{dom}\, \varphi^{\mathbf{A}}$ such that $\underline{d} = m(\underline{a})$.
GC4 $C_{\mathbf{D}}(H) = D$.

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From this corollary, a gluing condition for total algebras can be easily derived. We leave the details to the reader.

Example 36. Let Σ, $f : \mathbf{K} \to \mathbf{A}$ and $m : \mathbf{A} \to \mathbf{D}$ be given as in Example 29. Then, with the notation of the previous results, $H_{\mathbf{D}}(\mathbf{B}_0, \mathbf{D}_0) = \{a\}$ (see Example 31) does not generate $\mathbf{D}$. Therefore, $f : \mathbf{K} \to \mathbf{A}$ and $m : \mathbf{A} \to \mathbf{D}$ do not have a pushout complement in $\mathrm{Alg}_{\Sigma,Y}$.

Example 37. Let Σ and $f : \mathbf{K} \to \mathbf{A}$ be as in Example 22 and let $m : \mathbf{A} \to \mathbf{D}$ be any homomorphism of total Σ-algebras. When do f and m have a pushout complement in TAlg_Σ? We can use Corollary 35 to answer this question.

To begin with, we have that $D_m = K_\Omega(\mathbf{D})$, because $\varphi^{\mathbf{D}}(m(a), m(a)) = m(a)$ implies that $m(a) \in K_\Omega(\mathbf{D})$, and $B_m = C_{\mathbf{D}_m}(D_m - \{m(a)\})$. Since $\mathbf{K}$ is empty and there is no nullary operation in the signature, the relation $\theta(f, m)$ is the diagonal $\Delta_{A \sqcup B_m}$.

Let us check conditions GC1 to GC3 in Corollary 35. Since $m : \mathbf{A} \to \mathbf{D}$ is injective (A is a singleton), condition GC1 is clearly satisfied. On the other hand, since $\theta(f, m) = \Delta_{A \sqcup B_m}$, condition GC2 is equivalent to the fact that $m(a) \notin B_m$, that is, to the equality $B_m = D_m - \{m(a)\}$. Furthermore, since D_m is an initial segment of $\mathbf{D}$, this is clearly equivalent to the fact that $D - \{m(a)\}$ is closed in $\mathbf{D}$. Finally, since A is a singleton, and assuming condition GC2 already, condition GC3 says in this case that if $d \in D_m - \{m(a)\}$, then $(d, m(a)), (m(a), d) \notin \mathrm{dom}\, \varphi^{\mathbf{D}_m}$. This condition turns out to be equivalent (again, because D_m is an initial segment of $\mathbf{D}$) to the fact that if $d \in D - \{m(a)\}$, then $\varphi^{\mathbf{D}}(d, m(a)), \varphi^{\mathbf{D}}(m(a), d) \notin D_m$.

Thus, to sum up, f and m satisfy conditions GC1 to GC3 if and only if

(1) $\{m(a)\}$ is an initial segment of $\mathbf{D}$; and
(2) $\varphi^{\mathbf{D}}(d, m(a)), \varphi^{\mathbf{D}}(m(a), d) \notin K_\Omega(\mathbf{D})$ for every $d \neq m(a)$.

Now, $B_0 = C_{\mathbf{D}}(K_\Omega(\mathbf{D}) - \{m(a)\}) \subseteq D - \{m(a)\}$ and $D_0 = C_{\mathbf{D}}(K_\Omega(\mathbf{D}))$. Thus $m(a) \in {\downarrow_{\mathbf{D}_0}} y$ for every $y \in D_0 - B_0$. It is then easy to prove that if $b_0 \in B_0$, then $m(a) \notin {\downarrow_{\mathbf{D}}} b_0$. Using this fact, one can check that $H_{\mathbf{D}}(D_0, B_0) = D_0 \cup \{x \in D - D_0 \mid m(a) \notin {\downarrow_{\mathbf{D}}} x\}$. Therefore, since D_0 is generated by $K_\Omega(\mathbf{D})$, condition (GC4) now says

(3) $\mathbf{D}$ is generated by $H = K_\Omega(\mathbf{D}) \cup \{x \in D - C_{\mathbf{D}}(K_\Omega(\mathbf{D})) \mid m(a) \notin {\downarrow_{\mathbf{D}}} x\}$.

So, f and m have a pushout complement in TAlg_Σ if and only if they satisfy conditions (1), (2) and (3) above. And when they satisfy these conditions, their natural pushout complement (which, in this case, can be proved to be their unique pushout complement, up to isomorphisms) is given by the closed subalgebra of $\mathbf{D}$ supported on $B = \{x \in D \mid m(a) \notin {\downarrow_{\mathbf{D}}} x\}$.

Actually, it can be easily seen *a posteriori* that conditions (1) to (3) are equivalent to the fact that $\mathbf{D}$ is the coproduct in TAlg_Σ of its closed subalgebras supported on $\{m(a)\}$ and on the free completion of $K_\Omega(\mathbf{D}) - \{m(a)\}$, respectively.

So, conditions (1) to (3) characterize when a DPO rule in TAlg_Σ ($\mathbf{A} \xleftarrow{f} \mathbf{K} \xrightarrow{r} \mathbf{R}$) can be applied to a total Σ-algebra $\mathbf{D}$ through a homomorphism $m : \mathbf{A} \to \mathbf{D}$. The context algebra of such an application would be obtained by removing from $\mathbf{D}$ all elements that can be reached from $m(a)$ through a sequence of operations, and then the derived object would be the free completion of the direct sum of this context algebra and the right-hand

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side algebra **R** of the rule: the global effect is, then, the replacement of $\{m(a)\}$ by **R** within **D**, as if we blew it out into **R**.

Finally, notice that if all operations in Υ are nullary or unary, the argument used in Example 32 shows that, using the notation of Theorem 34, $H = D$, and thus condition (ii) in Theorem 34 and condition GC4 in Corollary 35 are automatically satisfied. It is not difficult to prove then that if all operations in Σ are unary and $\Omega = \Upsilon$, condition GC3 is always satisfied and conditions GC1 and GC2 are equivalent to the usual identification and dangling condition for unary total algebras (Ehrig 1979).

5. Conclusion

A Υ-total Σ-algebra, for some signature Σ and some non-empty set Υ of operation symbols within it, is a partial Σ-algebra all of whose operations belonging to Υ are total. In this paper we have established a gluing condition for Υ-total Σ-algebras, and we have distinguished and explicitly described a natural pushout complement of any pair of homomorphisms satisfying it. These results generalize the corresponding well-known results for unary total algebras (Ehrig 1979), and they shape the first basic step in the development of the DPO transformation of Υ-total Σ-algebras.

There are some results on the DPO transformation of Υ-total Σ-algebras that, due to the lack of space, have been left out of this paper and will be proved elsewhere (Llabrés 2001):

— The *uniqueness condition* (a necessary and sufficient condition on a morphism $l : \mathbf{K} \to \mathbf{L}$ for the uniqueness up to isomorphism of the pushout complement of f and each morphism $m : \mathbf{L} \to \mathbf{D}$ that satisfies the gluing condition with respect to f) for Υ-total Σ-algebras is exactly the same as the uniqueness condition for partial Σ-algebras established in Llabrés and Rosselló (2000, Proposition 15).

— In general, the DPO transformation in the category $\mathrm{Alg}_{\Sigma,\Upsilon}$ of Υ-total Σ-algebras strictly subsumes the procedure consisting of carrying out the DPO transformation of Υ-total Σ-algebras in Alg_{Σ} and then computing the Υ-free completion of the derived Σ-algebra. For instance, blowing out a closed subalgebra supported on one point of a total binary algebra **D** into a fixed algebra (see Example 37) needs, in Alg_{Σ}, essentially a rule for every algebra **D** (see Example 26).

— The DPO transformation in the category $\mathrm{Alg}_{\Sigma,\Upsilon}$ is independent of the *SPO* (*single-pushout*) transformation of Υ-total Σ-algebras using different sensible types of partial homomorphisms, in the same sense as for arbitrary partial algebras (Llabrés and Rosselló 2000, Section 6).

We will finish this paper with an easy example of DPO transformation of attributed graphs in our framework.

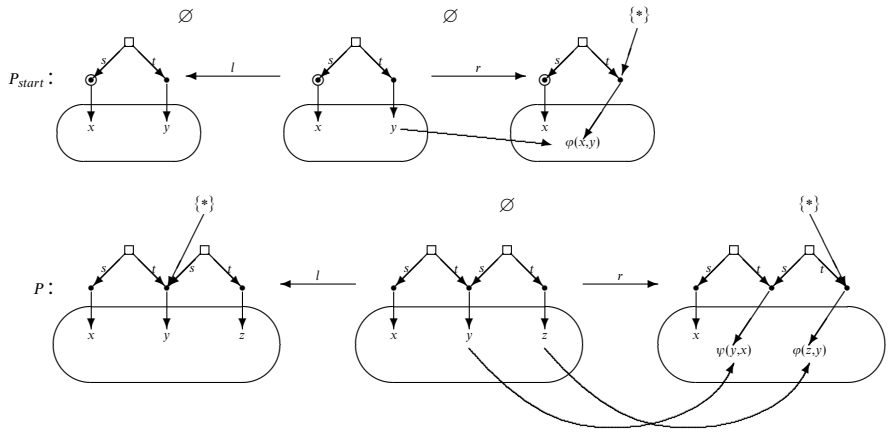
Example 38. A *weighted graph* is a directed graph in which all nodes have an integer weight. We want to move a pointer along the nodes of a path in such a weighted graph, starting from a distinguished node, and add up the weights of the nodes visited. To use DPO transformations to perform such an operation, we first describe weighted graphs

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(with a distinguished node and, possibly, a pointer) as partly total algebras. So, let Σ_{wg} be a signature with set of sorts $\{e, v, z, m\}$ and operation symbols

$$s, t : e \to v, \quad \rho : \lambda \to v, \quad \mu : m \to v, \quad \alpha : v \to z, \quad \varphi, \psi : zz \to z$$

(where, as is usual in ADT specification, $\phi : \omega(\phi) \to \sigma(\phi)$ denotes the fact that the arity of the operation symbol ϕ is $(\omega(\phi), \sigma(\phi))$). If we understand a weighted graph as a structure consisting of a directed graph (with arcs of sort e, nodes of sort v, and the operations s and t sending every arc to its source and target nodes, respectively), the algebra $\mathbb{N}$ of natural numbers (of sort z) with addition (corresponding to φ) and partial subtraction (corresponding to ψ) as data part, and the total attribution mapping α sending every node of the graph to its weight in $\mathbb{N}$, then a weighted graph is a special case of an $\{s, t, \alpha, \mu\}$-total Σ_{wg}-algebra (with, furthermore, empty carrier of sort m and the nullary operation ρ undefined). To add a pointer to such a weighted graph, we only have to make its carrier of sort m be a singleton $\{*\}$, and then the pointer corresponds to the image of $*$ under the operation μ. Finally, to distinguish a node, we simply define the nullary operation ρ to be equal to that node.



Consider the DPO production rules P_{start} and P in $\mathrm{Alg}_{\Sigma_{wg},\{s,t,\alpha,\mu\}}$ described in the figure above[†]. An application of rule P_{start} to a weighted graph without a pointer and with a distinguished node adds a pointer to a node v that is the target of some arc whose source node is the distinguished node, and it also adds the weight of the distinguished node to

[†] In the figure, small squares correspond to elements of sort e and small bullets to elements of sort v. The carrier of sort m is always empty or a singleton $\{*\}$. The carrier of sort z in the left-hand side and central objects in rule P_{start} is $\{x, y\}$ and the operations φ, ψ are discrete on it, while that of its right-hand side object is $\{x, y, \varphi(x, y)\}$, the operation φ is defined on it by $(x, y) \mapsto \varphi(x, y)$, and the operation ψ by $(\varphi(x, y), x) \mapsto y$. The carrier of sort z in the left-hand side and central objects in rule P is $\{x, y, z\}$ and the operations φ, ψ are discrete on it, while that of its right-hand side object is $\{x, y, z, \varphi(y, z), \psi(y, x)\}$, the operation φ is defined on it by $(y, z) \mapsto \varphi(y, z)$, and the operation ψ by $(y, x) \mapsto \psi(y, x)$ and $(\varphi(y, z), y) \mapsto z$. In all cases it is denoted by an oval. The source and target operations are represented by arrows going from a square to a bullet and labelled s and t, respectively. The operation ρ is denoted by a circle. The attribution and pointer operations are also represented by arrows, and we do not draw the operations φ, ψ. The homomorphisms l and r are those that respect the configuration of the representation. Some extra hint of the definition of these homomorphisms is represented, when needed, by a dashed arrow.

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the weight of v. Then, successive applications of rule P make the pointer advance through a path, restoring the value of the weight of the node it visited last, and changing the weight of the pointed node to the total weight of the path covered so far.

If we apply these rules to graphs whose nodes are labelled with words, we move a pointer along a path starting on the distinguished node, and we label the pointed node with the concatenation of the labels of the nodes visited so far (leaving the other labels unchanged). These rules can even be used to generate, with the labels of the nodes of a path in the graph, a partial $\{\varphi, \psi\}$-algebra satisfying the equation $\psi(\varphi(x, y), x) \approx y$.

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References

Bardohl, R. (2000) *Visual Definition of Visual Languages based on Algebraic Graph Transformation*, Ph. D. Thesis, Technical University of Berlin, Kovak-Verlag.

Berthold, M., Fischer, I. and Koch, M. (2000) Attributed Graph Transformation with Partial Attribution and its Application to Neural Networks. *Proc. APPLIGRAPH/GETGRATS Workshop on Graph Transformation Systems, Berlin* 171–178.

Borceux, F. (1994) Handbook of Categorical Algebra I: Basic Category Theory. *Encyclopedia of Mathematics and its Applications* **50**, Cambridge University Press.

Burmeister, P. (1986) A model theoretic oriented approach to partial algebras. *Mathematical Research* **32**, Akademie-Verlag.

Burmeister, P., Llabrés, M. and Rosselló, F. (2000) Pushout Complements for Arbitrary Total Algebras (Extended Abstract). *Proc. APPLIGRAPH/GETGRATS Workshop on Graph Transformation Systems, Berlin* 163–170.

Burmeister, P., Rosselló, F., Torrens, J. and Valiente, G. (1997) Algebraic transformation of unary partial algebras I: Double-pushout approach. *Theoretical Computer Science* **184** 145–193.

Burmeister, P. and Schmidt, J. (1967) On the completion of partial algebras. *Colloquium Mathematicum* **17** 235–245.

Ehrig, H. (1979) Introduction to the Algebraic Theory of Graph Grammars. Proceedings 1st Workshop on Graph Grammars and their Application to Computer Science and Biology. *Springer-Verlag Lecture Notes in Computer Science* **73** 1–69.

Große-Rhode, M. (1999) Specification of State Based Systems by Algebra Rewrite Systems and Refinements. Technical Report 99-04, TU Berlin.

Llabrés, M. (2001) *Double-pushout transformation of algebras*, Ph. D. Thesis, Univ. Illes Balears.

Llabrés, M. and Rosselló, F. (2000) Pushout complements for arbitrary partial algebras. Proceedings 6th International Workshop on Theory and Application of Graph Transformation TAGT'98. *Springer-Verlag Lecture Notes in Computer Science* **1764** 131–144.

Löwe, M., Korff, M. and Wagner, A. (1993) An algebraic framework for the transformation of attributed graphs. In: *Term Graph Rewriting: Theory and Practice*, John Wiley & Sons Ltd. 185–199.

Wagner, A. and Gogolla, M. (1996) Defining operational behavior of object specifications by attributed graph transformations. *Fundamenta Informaticæ* **26** 407–431.

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