QUADRATIC INTEGRALS AND FACTORIZATION OF LINEAR OPERATORS

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Abstract

We consider operators on the matrix-valued disc algebra. We show that any bounded linear operator $T: A \overline{\otimes} \mathscr{K} \to X$ to a Banach space X of cotype 2 induces a bounded operator $G_v \to X$ defined on some weighted Bergman space G_v on the unit disc. We give sufficient conditions on the weight v for the formal inclusion $A \otimes \mathscr{K} \to G_v$ to be $2-C^*$-summing. Decompositions of T with respect to operator-valued measures are obtained.

1. *Introduction*

Let $\mathscr{K}$ be the space of compact operators on the separable Hilbert space H and let A be the disc algebra. In this paper we consider a bounded linear map $T: A \overline{\otimes} \mathscr{K} \to X$ taking values in a Banach space of cotype 2 and investigate the conditions under which T factors as a composition of linear maps $A \overline{\otimes} \mathscr{K} \to G_v \to X$. The space G_v is a Bergman space of operator-valued analytic functions on the disc with norm given by

$$\|f\|_{G_v}^2 = \frac{1}{2}\iint_D (1-r)\,\mathrm{trace}\,((f'(re^{i\theta})f'(re^{i\theta})^* + f'(re^{i\theta})^* f'(re^{i\theta}))\,v(\mathrm{d}r\,\mathrm{d}\theta),$$

where v is a trace-class valued measure.

In Section 2 we introduce some notation and state preliminary results. In Section 4 we show that if $T: \mathscr{A} \otimes \mathscr{K} \to X$ is a bounded linear map, there is a c^1-measure v on the unit disc such that T induces a bounded operator $G_v \to X$. This is deduced from the analogous, but stronger, result known for C^*-algebras by a localization argument given in Section 3. In Section 5 we consider how v is related to the Fourier representation of functions on the disc. The proofs depend upon an operator variant of the Littlewood–Paley identity for quadratic integrals.

In Section 6 we obtain sufficient conditions on v for the formal inclusion map $A \overline{\otimes} \mathscr{K} \to G_v$ to be $2-C^*$-summing. The results involve the duality between Carleson measures on D and H^1-functions. We also obtain conditions under which G_v contains the Hardy spaces $H^q\mathscr{K}$. Necessary conditions pertaining to discrete measures are considered in Section 7.

In Section 8 we show that if T factors through G_v, then there is a decomposition of T with respect to a finitely additive operator-valued measure on the disc. The argument was suggested by the abstract version of the Littlewood subordination principle considered in [**11**, p. 14].

2. *Notation and basic results*

Let $C(\mathbb{T})$ be the space of continuous functions on the circle $\mathbb{T}$, let A be the disc algebra of functions in $C(\mathbb{T})$ whose negative Fourier coefficients vanish. These are

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identified with their extensions to analytic functions on the unit disc D. The completion of $C(\mathbb{T})\otimes\mathscr{K}$ for the spatial tensor product norm is $C(\mathbb{T})\overline{\otimes}\mathscr{K}$. The matrix-valued disc algebra is $A\overline{\otimes}\mathscr{K}$, a closed linear subspace of $C(\mathbb{T})\overline{\otimes}\mathscr{K}$.

For any Banach space E we denote by L^pE the Bochner–Lebesgue space of E-valued strongly measurable functions f whose norms are pth power integrable. We denote by H^pE the Hardy space formed from the E-valued analytic functions on the unit disc for which the norm $\|f\|_{H^pE}=\sup_{0<r<1}\|f(re^{i\theta})\|_{L^p(d\theta,E)}$ is finite. We write A_0, H^p_0 for the corresponding spaces of functions f with $f(0)=0$.

For $1\leqslant p<\infty$ let c^p be the von Neumann Schatten ideal of operators on H whose singular numbers are pth power summable. It has norm $\|a\|_{c^p}=(\mathrm{trace}\,((aa^*)^{p/2}))^{1/p}$. In particular, c^1 is the space of trace class operators. The symmetric modulus of an operator is $|a|_s=((a^*a+aa^*)/2)^{1/2}$. The pairing of $\mathscr{K}$ with its dual c^1 will be denoted by $\langle a,v\rangle=\mathrm{trace}\,(av)$. The norm $\|\cdot\|$ will be the operator norm unless stated otherwise.

By a (positive) c^1-measure on D we mean a finitely additive Borel set function μ with $\mu(S)\in c^1$ (with $S\subset D$) for which $S\mapsto\|\mu(S)\|_{c^1}$ is a bounded Radon measure and $\mu(S)\geqslant 0$ as an operator for each S. The total variation norm will be denoted by $\|\cdot\|_{\mathrm{var}}$.

We use P^θ_r to denote the operator given by convolution in the θ variable with the Poisson kernel $P_r(\theta-\phi)$. The gradient operator ∇ is defined on scalar functions on the disc by $\nabla: C^\infty\to C^\infty\oplus C^\infty: f\mapsto(\partial f/\partial x,\partial f/\partial y)$. The Laplace operator will be denoted by Δ. An unadorned C means a positive constant.

As a general reference on factorization of operators on Banach spaces we suggest [**8**].

DEFINITION 2.1. (i) A Banach space X is said to be of *cotype* 2 if there is a constant $C_X>0$ such that

$$\sum_n\|x_n\|_X^2\leqslant C_X^2\,\mathrm{Ave}_\pm\|\sum_n\pm x_n\|_X^2 \tag{2.1}$$

for all finite sequences of elements x_n of X. The optimal C_X is the *cotype* 2 *constant* of X.

(ii) Let $\mathscr{A}$ be a closed linear subspace of $B(H)$. A bounded linear operator $T:\mathscr{A}\to X$ is said to be $2-C^*$-*summing* if there is a $K>0$ for which

$$\sum_j\|T(a_j)\|_X^2\leqslant K^2\|\tfrac{1}{2}\sum_j(a_ja_j^*+a_j^*a_j)\|_{B(H)} \tag{2.2}$$

for all finite sequences of elements a_j of $\mathscr{A}$. The optimal constant K in (2.2) defines the $2-C^*$-*summing norm* of T, denoted by $\pi_2^*(T)$. We denote by $\Pi_2^*(\mathscr{A},X)$ the Banach space formed from the $2-C^*$-summing operators from $\mathscr{A}$ to X with this norm.

THEOREM 2.2 [**8**, p. 128]. *Let $\mathscr{A}$ be a C^*-algebra and $T:\mathscr{A}\to X$ a bounded linear operator where X has cotype* 2. *Then there is a constant K_X, depending only on the cotype* 2 *constant of X, and a positive linear functional ϕ on $\mathscr{A}$ with $\|\phi\|=1$ such that*

$$\|Ta\|_X^2\leqslant K_X^2\|T\|^2\phi(|a|_s^2)\quad(\textit{for } a\in\mathscr{A}). \tag{2.3}$$

REMARK 2.3. There is a state ϕ satisfying (2.3) if and only if T is $2-C^*$-summing. See [**8**, p. 123; **6**, Appendix].

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REMARK 2.4. Bourgain showed [**1**, Corollary 2.8] that if $T: A \to X$ is bounded, where X is of cotype 2, then T is $2-C^*$-summing. See also [**9**, Conjecture 8.3].

3. *Local extension problem*

It has been conjectured that Theorem 2.2 and 2.4 may be combined to hold for $A \overline{\otimes} \mathscr{K}$. Here we give a local version. Set $\Delta_j = \{n: 2^j \leqslant n < 2^{j+1}\}$ and let $D_j: C(\mathbb{T}) \to \operatorname{span}\{e^{in\theta}: n \in \Delta_j\}$ be the projection onto the dyadic block of characters given by convolution with a modified Dirichlet kernel. For simplicity we also denote $D_j \otimes I$: $C(\mathbb{T}) \otimes \mathscr{K} \to A \otimes \mathscr{K}$ by D_j. Since D_j has range in $A \overline{\otimes} \mathscr{K}$, we can form $TD_j P_r$.

LEMMA 3.1. *The operator $D_j: C(\mathbb{T}) \overline{\otimes} \mathscr{K} \to C(\mathbb{T}) \overline{\otimes} \mathscr{K}$ satisfies*
(i) $\|D_j\| \leqslant C(1+j)$,
(ii) $\|TD_j P_r\| \leqslant Cr^{2^j}(1+j)\|T\| \log(2/(1-r))$.

Proof. (i) Elementary estimates on the Dirichlet kernel give that

$$\|D_j\| \leqslant \| \sum_{n \in \Delta_j} e^{in\theta} \|_{L^1} \leqslant C(1+\log|\Delta_j|) \leqslant C(1+j).$$

(ii) Since D_j commutes with P_r, the operator norm of $TD_j P_r$ is at most

$$\|TD_j P_r\| \leqslant \|TD_j D_j P_r\| \leqslant \|TD_j\| \, \|P_r D_j\| \leqslant \|T\| \, \|D_j\| \, \| \sum_{n \in \Delta_j} r^n e^{in\theta} \|_{L^1(d\theta)}. \tag{3.1}$$

Using (i) and summing the geometric series, we get that (3.1) is at most

$$C2(1+j)\|T\| r^{2^j} \|(1-re^{i\theta})^{-1}\|_{L^1(d\theta)} \leqslant C(1+j)\, r^{2^j} \left(\log \frac{2}{1-r}\right) \|T\|, \tag{3.2}$$

where the final step follows from standard estimates on the Poisson kernel. See [**3**, p. 66].

PROPOSITION 3.2. *Let $T: A \overline{\otimes} \mathscr{K} \to X$ be a bounded linear operator where X is a Banach space of cotype 2. Then there is a constant K_X, depending only on the cotype 2 constant of X, such that for each $r < 1$ there is a c^1-measure v_r on $\mathbb{T}$ with $\|v_r\|_{\mathrm{var}} \leqslant K_X^2 (\log(2/(1-r)))^6 \|T\|^2$ for which*

$$\|TP_r f\|_X^2 \leqslant \int_{\mathbb{T}} \langle |f(e^{i\theta})|_s^2, v_r(d\theta) \rangle \quad (\textit{for } f \in A_0 \overline{\otimes} \mathscr{K}). \tag{3.3}$$

Proof. We can decompose TP_r as

$$TP_r = \sum_{j=0}^{\infty} TP_r D_j, \tag{3.4}$$

where each summand may be regarded as an operator on $C(\mathbb{T}) \overline{\otimes} \mathscr{K}$. By the Grothendieck–Pisier Theorem 2.2 for operators on the C^*-algebra $C(\mathbb{T}) \overline{\otimes} \mathscr{K}$ we have that

$$\pi_2^*(TP_r D_j) \leqslant K_X \|TP_r D_j\|. \tag{3.5}$$

Using the triangle inequality in the Banach space $\Pi_2^*(C(\mathbb{T}) \overline{\otimes} \mathscr{K}, X)$ we have that

$$\pi_2^*(TP_r) \leqslant \sum_{j=0}^{\infty} \pi_2^*(TP_r D_j) \leqslant K_X \sum_{j=0}^{\infty} \|TP_r D_j\|.$$

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By the lemma this is at most

$$CK_X\|T\|\sum_{j=0}^{\infty}(1+j)\,r^{2^j}\log\frac{2}{1-r}. \tag{3.6}$$

Now choose j_r so that $r^{2^{j_r}}\leqslant 2^{-1}$ and $jr^{2^j}\leqslant\log(2/(1-r))$ for $j>j_r$. One can take $j_r=O(\log(2/(1-r)))$. On splitting the sum we have

$$\begin{aligned}\sum_{j=1}^{\infty}jr^{2^j}&\leqslant\sum_{j=1}^{2j_r}j+\sum_{j=2j_r+1}^{\infty}jr^{2^{j-1}+2^{j-1}}\\&\leqslant 2j_r^2+\left(\log\frac{2}{1-r}\right)\sum_{j=1}^{\infty}(r^{2^{j_r}})^j=O\left(\left(\log\frac{2}{1-r}\right)^2\right).\end{aligned}$$

Substituting this estimate into (3.6) we obtain

$$\pi_2^*(TP_r)\leqslant CK_X\|T\|\left(\log\frac{2}{1-r}\right)^3\quad(\text{for } r<1). \tag{3.7}$$

By the noncommutative Pietsch representation theorem [**8**, p. 123; **6**, Appendix] there is a c^1 measure v_r with $\|v_r\|_{\mathrm{var}}\leqslant(\pi_2^*(TP_r))^2$ for which the integral inequality (3.3) is valid. When combined with the previous estimate (3.7) this completes the proof.

4. *Existence of weights*

THEOREM 4.1. *Let $T\colon A\overline{\otimes}\mathscr{H}\to X$ be a bounded linear operator where X has cotype 2. Then there is a bounded c^1-measure v on D for which*

$$\|Tf\|_X^2\leqslant\iint_D(1-r)\langle|f'(re^{i\theta})|_s^2,v(dr\,d\theta)\rangle\quad(\textit{for } f\in A_0\overline{\otimes}\mathscr{H}). \tag{4.1}$$

Proof. By Proposition 3.2 there is a c^1-measure v_r on $\mathbb{T}$ for which

$$\|T(P_{r^4}-P_{r^2})f)\|_X^2\leqslant\int_{\mathbb{T}}\langle|(P_{r^2}-I)\,P_rf(e^{i\theta})|_s^2,v_r(d\theta)\rangle. \tag{4.2}$$

Now using the semigroup property we obtain $P_{r^2}-I=(P_r-I)+P_r(P_r-I)$; so, by the Cauchy–Schwarz inequality, the right-hand side of (4.2) is at most

$$2\int_{\mathbb{T}}\langle|(P_r-I)\,P_rf(e^{i\theta})|_s^2,v_r(d\theta)\rangle+2\int_{\mathbb{T}}\langle|P_r(P_r-I)\,P_rf(e^{i\theta})|_s^2,v_r(d\theta)\rangle. \tag{4.3}$$

Using the Cauchy–Schwarz inequality for operators in the form $(P_rg)^*(P_rg)\leqslant P_r(g^*g)$, this last integral in (4.3) may be replaced by

$$\int_{\mathbb{T}}\langle|(P_r-I)\,P_rf(e^{i\theta})|_s^2,P_rv_r(d\theta)\rangle.$$

We introduce a 'dyadic' decomposition of the disc. We set $r_j=e^{-1/2^j}$ for each integer j. Using the estimates (4.2) and (4.3) with $r=r_{j+2}$ we get

$$\|T((P_{r_{j+1}}-P_{r_j})f)\|_X^2\leqslant 2\int_{\mathbb{T}}\left\langle\left|\int_{r_{j+1}}^{r_{j+2}}\frac{\partial}{\partial r}f(re^{i\theta})\,dr\right|_s^2,\mu_{r_{j+1}}(d\theta)\right\rangle, \tag{4.4}$$

where $\mu_{r_{j+1}}=v_{r_{j+2}}+P_{r_{j+2}}v_{r_{j+2}}$ is a c^1-measure on the circle. An application of the Cauchy–Schwarz inequality to the inner integral gives that (4.4) is at most

$$2(r_{j+2}-r_{j+1})\int_{r_{j+1}}^{r_{j+2}}\int_{\mathbb{T}}\langle|f'(re^{i\theta})|_s^2,\mu_{r_{j+1}}(d\theta)\rangle\,dr. \tag{4.5}$$

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We now consider the left-hand side of (4.1). Using the triangle inequality one shows that

$$\|Tf\|_X \leqslant \|TP_0 f\|_X + \sum_{j=-\infty}^{\infty} \|T((P_{r_{j+1}} - P_{r_j})f)\|_X.$$

When $f(0) = 0$ the first term vanishes and the Cauchy–Schwarz inequality gives

$$\|Tf\|_X^2 \leqslant C \sum_{j=-\infty}^{\infty} (1+|j|)^2 \|T(P_{r_{j+1}} - P_{r_j})f\|_X^2. \tag{4.6}$$

We obtain the stated result (4.1) by estimating each summand in (4.6) using (4.5), and summing over j. The factor $(1-r)$ arises in (4.1) since $r_{j+1} - r_j \leqslant 1 - r_j \leqslant 2(1-r_{j+1})$ for each j. The required c^1-measure v is defined by the pairing

$$\langle g, v\rangle = C \sum_{j=-\infty}^{\infty} \int_{r_{j+1}}^{r_{j+2}} \int_{\mathbb{T}} (1+|j|)^2 \langle g(r,\theta), \mu_{r_{j+1}}(\mathrm{d}\theta)\rangle \,\mathrm{d}r \quad (\text{for } g \in C(\bar{D}) \mathbin{\overline{\otimes}} \mathscr{K}). \tag{4.7}$$

Since $\|\mu_{r_{j+1}}\|_{\mathrm{var}} = O((\log(1/(1-r_{j+1}))^6) = O(j^6)$ as $j \to \infty$, the measure v is bounded with variation norm

$$\begin{aligned} \|v\|_{\mathrm{var}} &\leqslant C \sum_{j=-\infty}^{\infty} (1+|j|)^2 (r_{j+2} - r_{j+1}) \|\mu_{r_{j+1}}\|_{\mathrm{var}} \\ &\leqslant C \sum_{j=-\infty}^{\infty} (1+|j|)^8 \, 2^{-|j|} K_X \|T\|^2 \leqslant C K_X \|T\|^2. \end{aligned} \tag{4.8}$$

5. *Fourier coefficients and moments*

THEOREM 5.1. *Let $T: A_0 \mathbin{\overline{\otimes}} \mathscr{K} \to X$ be a bounded linear operator, where X has cotype 2. Then there is a constant K_X, depending only upon the cotype 2 constant of X, and a c^1-measure w on $[0,1]$ whose moments $w_n = \int_0^1 u^n w(\mathrm{d}u)$ have the following properties: the w_n are positive trace-class operators with*

$$\|w_n\|_{c^1} \leqslant K_X n^{-2} (1 + (\log n)^8) \|T\|^2$$

for which

$$\sum_{n=1}^{\infty} \|T(\mathrm{e}^{\mathrm{i}n\theta} \otimes \hat{f}(n))\|_X^2 \leqslant \sum_{n=1}^{\infty} n^2 \langle |\hat{f}(n)|_s^2, w_n\rangle \quad (\textit{for } f \in A_0 \otimes \mathscr{K}). \tag{5.1}$$

Proof. For notational convenience we suppose that $\|T\| \leqslant 1$. Let v be the c^1-measure for T given by (4.7). Applying Theorem 4.1 to the functions $\mathrm{e}^{\mathrm{i}n\theta} \otimes \hat{f}(n)$ and summing over n we have

$$\sum_{n=1}^{\infty} \|T(\mathrm{e}^{\mathrm{i}n\theta} \otimes \hat{f}(n))\|_X^2 \leqslant \sum_{n=1}^{\infty} \iint_D (1-r) r^{2(n-1)} n^2 \langle |\hat{f}(n)|_s^2, v(\mathrm{d}r\,\mathrm{d}\theta)\rangle \tag{5.2}$$

for all $\mathscr{K}$-valued polynomials f. Consequently, (5.1) will be satisfied if we make the choice

$$w_n = \iint_D (1-r) r^{2(n-1)} v(\mathrm{d}r\,\mathrm{d}\theta) \quad (\text{for } n \geqslant 1). \tag{5.3}$$

On the annulus $\{r\mathrm{e}^{\mathrm{i}\theta} : r \in [r_j, r_{j+1})\}$ we have $v(\mathrm{d}r\,\mathrm{d}\theta) = v_r(\mathrm{d}\theta) \otimes \mathrm{d}r$ where, changing

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notation, we write $v_r(\mathrm{d}\theta) = Cj^2\mu_{r_j}(\mathrm{d}\theta)$. Setting $u = r^2$ in (5.3) we see that the w_n are the moments of the c^1-measure on $[0,1]$ defined by

$$w(\mathrm{d}u) = 2^{-1}(1-u^{1/2})\,u^{-3/2}\bar{v}(u^{1/2})\,\mathrm{d}u, \tag{5.4}$$

where

$$\bar{v}(r) = \int_{\mathbb{T}} v_r(\mathrm{d}\theta) \quad (\text{for } 0 < r < 1).$$

The required estimates on w_n are obtained as in (4.8). For $j > 0$, integrating in the angle variable gives

$$\begin{aligned}\|\bar{v}(r)\|_{c^1} \leqslant Cj^2\|\mu_{r_j}\|_{\mathrm{var}} &= j^2 O\left(\log\frac{2}{1-r_{j+1}}\right)^6 \\ &= O\left(\log\frac{2}{1-r_{j+1}}\right)^8 = O\left(\log\frac{4}{1-r}\right)^8 \quad (\text{for } r_j \leqslant r < r_{j+1}).\end{aligned} \tag{5.5}$$

This bound also holds for $j \leqslant 0$. We substitute (5.5) in (5.3) to obtain

$$n^2\|w_n\|_{c^1} \leqslant K\int_0^1 n^2(1-r)\,r^{2(n-1)}\left(\log\frac{4}{1-r}\right)^8\mathrm{d}r. \tag{5.6}$$

Splitting the integral in (5.6) gives

$$n^2\|w_n\|_{c^1} \leqslant K\left(\log\frac{4}{\varepsilon}\right)^8\int_0^{1-\varepsilon} n^2r^{2(n-1)}(1-r)\,\mathrm{d}r + K\int_{1-\varepsilon}^1 n^2r^{2(n-1)}\mathrm{d}r, \tag{5.7}$$

since $(1-r)(\log(4/(1-r)))^8$ is bounded for $r\in(0,1)$. The choice of $\varepsilon = n^{-2}$ gives the bound $\|w_n\|_{c^1} = O(n^{-2}(\log n)^8)$ as $n\to\infty$. The implied constants may be chosen to depend only upon the cotype 2 constant of X.

REMARK 5.2. When X is a Hilbert space one can improve the bounds on $\|w_n\|_{c^1}$ to

$$\|w_n\|_{c^1} \leqslant Kn^{-2}(1+(\log n)^6)\|T\|^2 \quad (\text{for } n > 1), \tag{5.8}$$

where K is an absolute constant. The proof uses the Littlewood–Paley formula [**4**, p. 236] to improve upon (4.6). Expressions such as (4.6) are considered in [**9**, Section 7].

6. *The space G and Hardy spaces*

Let v be a c^1-measure on the disc. We introduce a Hilbert space G_v by forming the completion of the polynomials with coefficients in $\mathscr{K}$ and zero constant term for the inner product

$$\langle f,g\rangle_{G_v} = \frac{1}{2}\iint_D (1-r)\,\langle f'(r\mathrm{e}^{\mathrm{i}\theta})\,(g'(r\mathrm{e}^{\mathrm{i}\theta}))^* + (g'(r\mathrm{e}^{\mathrm{i}\theta}))^*f'(r\mathrm{e}^{\mathrm{i}\theta}), v(\mathrm{d}r\,\mathrm{d}\theta)\rangle. \tag{6.1}$$

The space G_v is a weighted vector Bergman space. In this section we shall present various sufficient conditions on v for G_v to contain $A_0\,\overline{\otimes}\,\mathscr{K}$ or, more generally, the Hardy space $H_0^p\,\mathscr{K}$. We let $\tilde{G}_v$ be the completion of $H_0^\infty\,\mathscr{K}$ for the norm (6.1) where this is well defined. For the time being let us suppose that $v(dr\,\mathrm{d}\theta)$ has a density with respect to planar Lebesgue measure and may be represented as $v(r,\theta)r\,\mathrm{d}r\,\mathrm{d}\theta$.

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THEOREM 6.1. (i) *Suppose that there is q with $1 \leqslant q < \infty$ for which*

$$\iint_D (1-r)^{-1-2/q} \| v(r,\theta) \|_{c^1} \, \mathrm{d}r \, \mathrm{d}\theta < \infty. \tag{6.2}$$

Then $H^q_0 \mathscr{K}$ is boundedly contained in G_v.

(ii) *If* (6.2) *is satisfied with $q = \infty$, then $A_0 \mathbin{\overline{\otimes}} \mathscr{K}$ is boundedly contained in G_v and the inclusion map is $2-C^*$-summing.*

Proof. In each case it suffices to compare the norms of an f belonging to the dense linear subspace $A \otimes \mathscr{K}$. To prove (ii), we use the Cauchy integral formula

$$f'(r\mathrm{e}^{\mathrm{i}\theta}) = \frac{1}{2\pi} \int_0^{2\pi} \frac{\mathrm{e}^{\mathrm{i}\phi}}{(\mathrm{e}^{\mathrm{i}\phi} - r\mathrm{e}^{\mathrm{i}\theta})^2} f(\mathrm{e}^{\mathrm{i}\phi}) \, \mathrm{d}\phi \quad (\text{for } f \in A \mathbin{\overline{\otimes}} \mathscr{K}).$$

Hence by the Cauchy–Schwarz inequality

$$\langle |f'(r\mathrm{e}^{\mathrm{i}\theta})|_s^2, v(r,\theta) \rangle \leqslant \| (\mathrm{e}^{\mathrm{i}\phi} - r\mathrm{e}^{\mathrm{i}\theta})^{-2} \|_{L^1(\mathrm{d}\phi)} \int_0^{2\pi} \frac{\langle |f(\mathrm{e}^{\mathrm{i}\phi})|_s^2, v(r,\theta) \rangle}{|\mathrm{e}^{\mathrm{i}\phi} - r\mathrm{e}^{\mathrm{i}\theta}|^2} \frac{\mathrm{d}\phi}{2\pi}. \tag{6.3}$$

The first factor is $(1-r^2)^{-1}$. We integrate (6.3) over the disc with respect to $(1-r)\, r \, \mathrm{d}r \, \mathrm{d}\theta$ to get

$$\| f \|^2_{G_v} = \iint_D (1-r) \langle |f'(r\mathrm{e}^{\mathrm{i}\theta})|_s^2, v(r,\theta) \rangle r \, \mathrm{d}r \, \mathrm{d}\theta \leqslant \int_0^{2\pi} \langle |f(\mathrm{e}^{\mathrm{i}\phi})|_s^2, \omega(\phi) \rangle \, \mathrm{d}\phi, \tag{6.4}$$

where ω is the $L^1 c^1$ function

$$\omega(\phi) = \iint_D \frac{r v(r,\theta)}{|\mathrm{e}^{\mathrm{i}\phi} - r\mathrm{e}^{\mathrm{i}\theta}|^2} \, \mathrm{d}r \, \mathrm{d}\theta.$$

Since the functions take positive values we can conclude from Fubini's Theorem that

$$\| \omega(\phi) \|_{L^1 c^1} \leqslant \iint_D (1-r)^{-1} \| v(r,\theta) \|_{c^1} \, \mathrm{d}r \, \mathrm{d}\theta,$$

which is finite by (6.2). We can deduce from (6.4) that the inclusion map $A_0 \mathbin{\overline{\otimes}} \mathscr{K} \to G_v$ is $2-C^*$-summing.

To prove (i) we let p be the conjugate exponent of q. By the Cauchy Integral Formula and Hölder's Inequality we have, for fixed r, that

$$\| f'(r\mathrm{e}^{\mathrm{i}\theta}) \|_{L^\infty \mathscr{K}} \leqslant \| (1 - r\mathrm{e}^{\mathrm{i}\theta})^{-2} \|_{L^p} \| f(\mathrm{e}^{\mathrm{i}\theta}) \|_{L^q \mathscr{K}} \leqslant C (1-r)^{(1-2p)/p} \| f(\mathrm{e}^{\mathrm{i}\theta}) \|_{L^q \mathscr{K}},$$

using a standard estimate [**3**, p. 65] at the last step. Hence if (6.2) holds we can estimate the Bergman norm by

$$\begin{aligned} \| f \|^2_{G_v} &\leqslant \iint_D (1-r) \| f'(r\mathrm{e}^{\mathrm{i}\phi}) \|^2_{L^\infty(\mathrm{d}\phi, \mathscr{K})} \| v(r,\theta) \|_{c^1} \, \mathrm{d}r \, \mathrm{d}\theta \\ &\leqslant \iint_D (1-r)^{-3+2/p} \| v(r,\theta) \|_{c^1} \, \mathrm{d}r \, \mathrm{d}\theta \times \| f(\mathrm{e}^{\mathrm{i}\theta}) \|^2_{L^q \mathscr{K}} \leqslant C \| f(\mathrm{e}^{\mathrm{i}\theta}) \|^2_{L^q \mathscr{K}}. \end{aligned}$$

THEOREM 6.2. *Suppose that $v(r,\theta)$ is twice continuously differentiable on the unit disc with $v(0) = 0$. Suppose further that*

$$\iint_D \left(\left\| \frac{\partial v}{\partial r} \right\|_{c^1} + \left\| \frac{\partial^2 v}{\partial r \, \partial \theta} \right\|_{c^1} \right) \mathrm{d}r \, \mathrm{d}\theta \leqslant K. \tag{6.5}$$

Then

(i) *there is an operator σ belonging to c^2 with $\sigma \geqslant 0$ for which $v(r,\theta) \leqslant \sigma^2$,*

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(ii) *the Hardy space* $H_0^2\mathscr{K}$ *is boundedly contained in* G_v *and*

$$\|f\|_{G_v}^2 \leqslant C\|f(e^{i\theta})\sigma\|_{L^2c^2}^2 + C\|\sigma f(e^{i\theta})\|_{L^2c^2}^2 \quad (\textit{for } f\in H_0^2\mathscr{K}), \tag{6.6}$$

(iii) *the inclusion map* $A_0\overline{\otimes}\mathscr{K}\to G_v$ *is* $2-C^*$*-summing*.

Proof. We obtain σ by integrating the positive parts of the derivatives of $v(r,\theta)$. *Let* σ be the positive operator whose square satisfies

$$\sigma^2 = \iint_D \left(\frac{\partial^2 v}{\partial r\,\partial\theta}\right)_+ dr\,d\theta + \int_0^1 \left(\frac{\partial v(r,\theta_0)}{\partial r}\right)_+ dr.$$

By Condition (6.5) there is a value of θ_0 for which trace $(\sigma^2)\leqslant K$. Also

$$\sigma^2 \geqslant \int_0^u \int_{\theta_0}^{\phi} \frac{\partial^2 v(r,\theta)}{\partial r\,\partial\theta}\,dr\,d\theta + \int_0^u \frac{\partial v(r,\theta_0)}{\partial r}\,dr = v(u,\phi). \tag{6.7}$$

We compare the G_v norm with the Hardy norm using the classical Littlewood–Paley identity given in [**4**, p. 236], and (6.14) below. Since $r(1-r)\leqslant r\log(1/r)$ we obtain from (6.7) that

$$\begin{aligned}\iint_D (1-r)\langle|f'(re^{i\theta})|_s^2, v(r,\theta)\rangle r\,dr\,d\theta &\leqslant \iint_D \langle|f'(re^{i\theta})|_s^2,\sigma^2\rangle r\log\frac{1}{r}\,dr\,d\theta\\ &= 4^{-1}\int_{\mathbb{T}}\langle|f(e^{i\theta})|_s^2,\sigma^2\rangle\,d\theta \leqslant C\|\sigma^2\|_{c^1}\|f\|_{H^2\mathscr{K}}^2 \leqslant C\|\sigma\|_{c^2}^2\|f\|_{H^2\mathscr{K}}^2.\end{aligned} \tag{6.8}$$

These estimates include (6.6). The conclusions (ii) and (iii) are immediate consequences of (6.8).

DEFINITION. A positive Radon measure μ on D is said to be a *Carleson measure* if there is a constant C_* such that $\mu(R_I)\leqslant C_*|I|$ for each subinterval I of $[0,2\pi]$, where R_I is the sector $R_I=\{re^{i\theta}: r\geqslant 1-|I|, \theta\in I\}$ based upon I [**4**, p. 31].

THEOREM 6.3. *Suppose that* v *is twice continuously differentiable and that* $\|\Delta((1-r)v(r,\theta))\|_{c^1} r\,dr\,d\theta$ *defines a Carleson measure in the disc with Carleson constant* C_*. *Suppose also that* $\|v(1,\theta)\|_{L^\infty c^1}\leqslant K$. *Then*

(i) *the vector-valued Hardy space* $H_0^2\mathscr{K}$ *is boundedly contained in* G_v *with*

$$\iint_D (1-r)\langle|f'(re^{i\theta})|_s^2, v(r,\theta)\rangle r\,dr\,d\theta \leqslant C(K+C_*)\|f(e^{i\theta})\|_{L^2\mathscr{K}}^2 \quad (\textit{for } f\in H_0^2\mathscr{K}); \tag{6.9}$$

(ii) *the inclusion map* $A_0\overline{\otimes}\mathscr{K}\to G_v$ *is* $2-C^*$*-summing*.

Proof. (i) We shall prove (6.9) by simplifying the left-hand side. Firstly we observe that, since $f(re^{i\theta})$ and its adjoint are harmonic, we have by [**4**, p. 236] that

$$4|f'(re^{i\theta})|_s^2 = \Delta(|f(re^{i\theta})|_s^2).$$

Now $(1-r)$ vanishes on the boundary so, on applying Green's Theorem, we are left with

$$\begin{aligned}\iint_D (1-r)\langle\Delta(|f(re^{i\theta})|_s^2), v(r,\theta)\rangle r\,dr\,d\theta &= (6.10)_1 + (6.10)_2\\ &= \iint_D \langle|f(re^{i\theta})|_s^2, \Delta((1-r)v(r,\theta))\rangle r\,dr\,d\theta + \int_0^{2\pi}\langle|f(e^{i\theta})|_s^2, v(1,\theta)\rangle\,d\theta.\end{aligned} \tag{6.10}$$

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By the triangle inequality $\|P_r f(e^{i\theta})\|_{\mathscr{K}} \leqslant P_r^\theta(\|f(e^{i\theta})\|_{\mathscr{K}})$. Hence by Carleson's Theorem [**4**, p. 32] the first integral $(6.10)_1$ is at most

$$\iint_D (P_r^\theta \|f(e^{i\theta})\|_{\mathscr{K}})^2 \|\Delta((1-r)\, v(r,\theta))\|_{c^1} r\, dr\, d\theta \leqslant CC_* \int_0^{2\pi} (\sup_r P_r^\theta \|f(e^{i\theta})\|_{\mathscr{K}})^2 d\theta. \tag{6.11}$$

It follows from the Hardy–Littlewood Maximal Theorem [**4**, p. 22, 24] that this is at most

$$CC_* \int_0^{2\pi} \|f(e^{i\theta})\|_{\mathscr{K}}^2\, d\theta. \tag{6.12}$$

For the other integral $(6.10)_2$ we have that

$$\int_0^{2\pi} \langle |f(e^{i\theta})|_s^2, v(1,\theta)\rangle d\theta \leqslant \|v(1,\theta)\|_{L^\infty c^1} \int_0^{2\pi} \|f(e^{i\theta})\|_{\mathscr{K}}^2\, d\theta. \tag{6.13}$$

(ii) The form of the right-hand side of (2.2) suggests that we apply the previous argument to the operator-valued functions

$$f(re^{i\theta}) = \begin{bmatrix} f_1(re^{i\theta}) & 0 & \cdots & 0 \\ f_1(re^{-i\theta})^* & 0 & \cdots & 0 \\ f_2(re^{i\theta}) & 0 & \cdots & 0 \\ f_2(re^{-i\theta})^* & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ f_n(re^{i\theta}) & 0 & \cdots & 0 \\ f_n(re^{-i\theta})^* & 0 & \cdots & 0 \end{bmatrix}, \qquad v(r,\theta) = \begin{bmatrix} v(r, \mp\theta) & 0 \\ 0 & 0 \end{bmatrix}.$$

Since $A_0 \overline{\otimes}\, \mathscr{K}$ is boundedly contained in $H_0^2 \mathscr{K}$, statement (ii) follows from (6.9). See [**6**, p. 247] for further details.

Quadratic integrals and Carleson measures

LEMMA 6 (Fefferman–Stein). *If w is a bounded c^2-valued analytic function on the unit disc, then $(1-r)\|w'(re^{i\theta})\|_{c^2}^2 r\, dr\, d\theta$ defines a Carleson measure on D.*

Proof. The proof follows that of the scalar-valued case. The Littlewood–Paley identity

$$\int_{\mathbb{T}} \|w(e^{i\theta}) - w(\zeta)\|_{c^2}^2 P_\zeta(\theta) \frac{d\theta}{2\pi} = \frac{2}{\pi} \iint_D \|w'(re^{i\theta})\|_{c^2}^2 \log \left| \frac{1-\bar{\zeta} re^{i\theta}}{re^{i\theta} - \zeta} \right| r\, dr\, d\theta \tag{6.14}$$

with $\zeta \in D$ holds for c^2-valued analytic functions. One can check the case of $\zeta = 0$ by using Parseval's identity in the Hilbert space $L^2(d\theta, c^2)$, and then deduce the general case by using the conformal transformation $z \mapsto (z+\zeta)/(1+\bar{\zeta} z)$. If w is bounded (or, more generally, if w is in $\mathrm{BMO}(c^2)$) we have that

$$\sup_{\zeta \in D} \int_{\mathbb{T}} \|w(e^{i\theta}) - w(\zeta)\|_{c^2}^2 P_\zeta(\theta) \frac{d\theta}{2\pi} < \infty. \tag{6.15}$$

By the calculations presented in [**4**, p. 238–240] it follows from (6.14) that the measure $(1-r)\|w'(re^{i\theta})\|_{c^2}^2 r\, dr\, d\theta$ is a Carleson measure.

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DEFINITION. *A c^1-valued function $v(re^{i\theta})$ on the unit disc is said to be factorable* if there is a pair of c^2-valued functions $w_L(re^{i\theta})$ and $w_R(re^{i\theta})$ which are analytic on the unit disc and satisfy $v(re^{i\theta}) = w_R(re^{i\theta})\, w_R(re^{i\theta})^* = w_L(re^{i\theta})^* w_L(re^{i\theta})$ for $re^{i\theta} \in D$.

THEOREM 6.4. *Let v be a factorable density with bounded factors w_L and w_R. Then:*

(i) *the Hardy space $H_0^2 \mathscr{K}$ is boundedly contained in G_v and for any finite sequence of $f_j \in H_0^2 \mathscr{K}$ the following holds*

$$(\sum_j \|f_j\|_{G_v}^2)^{1/2} \leqslant C(\|w_R(e^{i\theta})\|_{L^\infty c^2} + \|w_L(e^{i\theta})\|_{L^\infty c^2}) \times \|(\sum_j |f_j(e^{i\theta})|_s^2)^{1/2}\|_{L^2 \mathscr{K}}; \quad (6.16)$$

(ii) *the inclusion map $A_0 \bar{\otimes} \mathscr{K} \to G_v$ is $2-C^*$-summing.*

Proof. The desired inequality for sequences will be deduced from the case of a single function. We write the integrand in (6.1) as

$$2\langle |f'(re^{i\theta})|_s^2, v(re^{i\theta})\rangle = \|f'(re^{i\theta})\, w_R(re^{i\theta})\|_{c^2}^2 + \|w_L(re^{i\theta}) f'(re^{i\theta})\|_{c^2}^2. \quad (6.17)$$

Both summands on the right-hand side of (6.17) are dealt with in a similar way. Since $\frac{d}{dz}$ is a derivation, the first one contributes

$$\|f'(re^{i\theta})\, w_R(re^{i\theta})\|_{c^2}^2 \leqslant 2\|f(re^{i\theta})\, w_R'(re^{i\theta})\|_{c^2}^2 + 2\left\|\frac{d}{dz}(f(re^{i\theta})\, w_R(re^{i\theta}))\right\|_{c^2}^2.$$

Integrating this expression we have

$$\begin{aligned}\iint_D \|f'(re^{i\theta})\, w_R(re^{i\theta})\|_{c^2}^2 (1-r)\, r\, dr\, d\theta &\leqslant (6.18)_1 + (6.18)_2 \\ &= 2\iint_D \|f(re^{i\theta})\|_{\mathscr{K}}^2 \|w_R'(re^{i\theta})\|_{c^2}^2 (1-r)\, r\, dr\, d\theta \\ &\quad + 2\iint_D \left\|\frac{d}{dz}(f(re^{i\theta})\, w_R(re^{i\theta}))\right\|_{c^2}^2 (1-r) r\, dr\, d\theta. \end{aligned} \quad (6.18)$$

Recalling the inequality $r(1-r) \leqslant r\log(1/r)$, we can use the Littlewood–Paley identity (6.14) to estimate the last integral by

$$(6.18)_2 \leqslant \pi \int_0^{2\pi} \|f(e^{i\theta})\, w_R(e^{i\theta})\|_{c^2}^2 \frac{d\theta}{2\pi} \leqslant \pi \|w_R(e^{i\theta})\|_{L^\infty c^2}^2 \int_0^{2\pi} \|f(e^{i\theta})\|_{\mathscr{K}}^2 \frac{d\theta}{2\pi}. \quad (6.19)$$

By the Lemma 6, the integral $(6.18)_1$ involves a Carleson measure. We estimate it using the same maximal function argument as with $(6.10)_1$ above to obtain

$$(6.18)_1 \leqslant CC_* \|f(e^{i\theta})\|_{L^2 \mathscr{K}}^2 \leqslant C\|w_R(e^{i\theta})\|_{L^\infty c^2}^2 \|f\|_{H^2 \mathscr{K}}^2. \quad (6.20)$$

To deduce the stated result (6.16) one applies (6.19) and (6.20) to the analytic operator valued function and weight

$$f = \begin{bmatrix} f_1 & 0 & \cdots & 0 \\ f_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ f_n & 0 & \cdots & 0 \end{bmatrix}, \qquad v = \begin{bmatrix} w_R & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} w_R^* & 0 \\ 0 & 0 \end{bmatrix} \quad (6.21)$$

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and the corresponding estimates involving w_L to

$$f = \begin{bmatrix} f_1 & f_2 & \cdots & f_n \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}, \qquad v = \begin{bmatrix} w_L^* & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} w_L & 0 \\ 0 & 0 \end{bmatrix}.$$

The statement (ii) follows from (6.16) since $L^\infty\mathscr{K}$ is boundedly contained in $L^2\mathscr{K}$.

REMARK 6.5. (i) There are functions f in $A \overline{\otimes} \mathscr{K}$ for which

$$(1-r)\|f'(re^{i\theta})\|_{\mathscr{K}}^2 \, r\,dr\,d\theta$$

does not define a Carleson measure on D. One such is the diagonal matrix function

$$f(z) = \operatorname{diag}\left[\frac{z^j}{(\log j)^{1/2}}\right]_{j\geqslant 3} \quad (\text{for } z \in D). \tag{6.22}$$

This is in contrast to the 'dual' statement [**2**, p. 13] for nontangential maximal functions.

(ii) Although the norm $\|\cdot\|_{\mathscr{K}}$ is not differentiable, it is shown in [**7**, Theorem 1.1] that the distributional gradient $\tilde{\nabla}$ satisfies $|\tilde{\nabla}\|f(re^{i\theta})\|_{\mathscr{K}}| \leqslant \|\tilde{\nabla} f(re^{i\theta})\|_{\mathscr{K}}$ almost everywhere with respect to area measure on the disc. Consequently, if $f \in A \overline{\otimes} \mathscr{K}$ has the property that $(1-r)\|f'(re^{i\theta})\|_{\mathscr{K}}^2 \, r\,dr\,d\theta$ defines a Carleson measure, then

$$(1-r)|\tilde{\nabla}\|f(re^{i\theta})\|_{\mathscr{K}}|^2 r\,dr\,d\theta$$

also defines a Carleson measure.

PROPOSITION 6.6. *Let $f_j \in H^\infty$ and let $a_j \in \mathscr{K}$ for $1 \leqslant j \leqslant n$. Then*

$$(1-r)\left\| \sum_{j=1}^n f_j'(re^{i\theta}) \otimes a_j \right\|_{\mathscr{K}}^2 r\,dr\,d\theta \tag{6.23}$$

defines a Carleson measure on D with Carleson constant C_* where

$$C_* \leqslant C \left\| \sum_{j=1}^n \|a_j\|_{\mathscr{K}} |f_j(e^{i\theta})| \right\|_{L^\infty}^2. \tag{6.24}$$

Proof. For each j we use the canonical factorization [**4**, p. 87] of H^∞ functions to write $f_j = g_j h_j$, where g_j and h_j are analytic functions on the unit disc with $|g_j| = |h_j| = |f_j|^{1/2}$ on $\mathbb{T}$. Then we can decompose each elementary tensor in (6.23) as

$$\begin{aligned} f_j' \otimes a_j = {} & (g_j' \otimes \|a_j\|_{\mathscr{K}}^{1/2} I)(h_j \otimes \|a_j\|_{\mathscr{K}}^{-1/2} a_j) \\ & + (h_j' \otimes \|a_j\|_{\mathscr{K}}^{1/2} I)(g_j \otimes \|a_j\|_{\mathscr{K}}^{-1/2} a_j). \end{aligned} \tag{6.25}$$

Hence by the Cauchy–Schwarz inequality we have a pointwise estimate

$$\begin{aligned} & (1-r)\left\| \sum_{j=1}^n f_j' \otimes a_j \right\|_{\mathscr{K}}^2 \\ & \qquad \leqslant \left((1-r) \sum_{j=1}^n (|g_j'|^2 + |h_j'|^2) \|a_j\|_{\mathscr{K}} \right) \times \left\| \sum_{j=1}^n (|g_j|^2 + |h_j|^2) \|a_j\|_{\mathscr{K}}^{-1} a_j^* a_j \right\|_{\mathscr{K}}. \end{aligned} \tag{6.26}$$

Since $a_j a_j^* \leqslant \|a_j\|_{\mathscr{K}}^2 I$ we can use the Maximum Modulus Theorem to show that the last factor in (6.26) is at most $2\|\sum_{j=1}^n |f_j(e^{i\theta})| \, \|a_j\|_{\mathscr{K}}\|_{L^\infty}$, recalling the fact that the norm on any Banach space defines a convex function [**4**, p. 34].

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By Lemma 6 applied to the l^2-valued function $(\|a_j\|_{\mathscr{K}}^{1/2} g_j(re^{i\theta}))_{j=1}^n$ we have that $(1-r)(\sum_{j=1}^n |g_j'(re^{i\theta})|^2 \|a_j\|_{\mathscr{K}}) r\,dr\,d\theta$ defines a Carleson measure with Carleson constant at most

$$C\|(\|a_j\|_{\mathscr{K}}^{1/2} g_j(e^{i\theta}))_{j=1}^n\|_{L^\infty(l^2)}^2 \leqslant C\left\|\sum_{j=1}^n \|a_j\|_{\mathscr{K}} |f_i(e^{i\theta})|\right\|_{L^\infty}. \tag{6.27}$$

Using a similar estimate on the other terms in (6.26), the stated result follows.

7. *Atomic measures*

We now consider G_ν when ν is an atomic measure. We shall obtain a partial converse to Theorem 6.1.

DEFINITION. A sequence of points $(\lambda)_{\lambda\in\sigma}$ in the unit disc is said to form an *interpolating sequence* if the Carleson interpolation constant η is strictly positive, where

$$\eta = \inf_{\lambda\in\sigma} \prod_{\rho\in\sigma,\rho\neq\lambda} \left|\frac{\lambda-\rho}{1-\bar{\rho}\lambda}\right|. \tag{7.1}$$

PROPOSITION 7.1. *Let $\nu = \sum_{\lambda\in\sigma} \delta_\lambda \otimes \nu_\lambda$ be an atomic c^1-measure with atoms on an interpolating sequence $(\lambda)_{\lambda\in\sigma}$ in the unit disc. Then there is a constant C_1 such that*

$$\iint_D (1-r)\langle |f'(re^{i\theta})|_s^2, \nu(dr\,d\theta)\rangle \leqslant C_1^2 \|f\|_{H^\infty\mathscr{K}}^2 \quad (\textit{for } f\in H_0^\infty \overline{\otimes}\, \mathscr{K}) \tag{7.2}$$

if and only if ν satisfies

$$\iint_D (1-r)^{-1} \|\nu\|_{c^1}(dr\,d\theta) < \infty. \tag{7.3}$$

Proof. In view of Theorem 6.1(ii), we need only to check that the condition (7.2) is necessary. We consider the Blaschke product

$$\varphi(z) = \prod_{\lambda\in\sigma} \left(\frac{|\lambda|}{\lambda}\frac{\lambda-z}{1-\bar{\lambda}z}\right) \quad (\text{for } z\in D)$$

with zeros on σ. We have that

$$\varphi'(\lambda) = \frac{-1}{1-|\lambda|^2}\frac{|\lambda|}{\lambda} \prod_{\mu\in\sigma:\mu\neq\lambda} \frac{|\mu|}{\mu}\frac{\mu-\lambda}{1-\bar{\mu}\lambda} \quad (\text{for } \lambda\in\sigma). \tag{7.4}$$

We also introduce the Jones interpolating functions h_λ for $\lambda\in\sigma$. These are analytic in the unit disc and satisfy $h_\lambda(0) = 0$, $h_\lambda(\mu) = \delta_{\lambda\mu}$, for all $\lambda, \mu\in\sigma$, and

$$\|\sum_{\lambda\in\sigma} |h_\lambda|\,\|_{L^\infty} \leqslant \frac{C_0}{\eta}\log\frac{2}{\eta}, \tag{7.5}$$

where C_0 is an absolute constant and η is the interpolation constant of σ [**5**, Theorem 6]. For each $\lambda\in\sigma$ we select $a_\lambda\in\mathscr{K}$ of unit operator norm so that $\langle |a_\lambda|_s^2, \nu_\lambda\rangle \geqslant \|\nu_\lambda\|_{c^1}/2$. Finally for each finite subset τ of σ we set

$$f_\tau = \sum_{\lambda\in\tau} \varphi h_\lambda \otimes a_\lambda,$$

which is a $\mathscr{K}$-valued analytic function in the unit disc. Its derivative satisfies $f_\tau'(\lambda) = \varphi'(\lambda)\otimes a_\lambda$ for $\lambda\in\tau$ since $\varphi(\lambda) = 0$.

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If (7.2) holds there is $C_1 < \infty$ for which

$$C_1^2 \|f_\tau\|^2_{L^\infty \mathscr{K}} \geqslant \iint_D (1-r)\,\langle |f'_\tau(re^{i\theta})|^2_s, v(dr\,d\theta)\rangle.$$

Using (7.4) and (7.1) we see that this integral is at least

$$\sum_{\lambda\in\tau} (1-|\lambda|)\frac{\eta^2}{4(1-|\lambda|)^2}\langle |a_\lambda|^2_s, v_\lambda\rangle \geqslant \frac{\eta^2}{8}\sum_{\lambda\in\tau}(1-|\lambda|)^{-1}\|v_\lambda\|_{c^1}. \tag{7.6}$$

Since τ is an arbitrary finite subset of σ, we can conclude from (7.6) and (7.5) that

$$\iint_D (1-r)^{-1}\|v(dr\,d\theta)\|_{c^1} \leqslant \frac{8C_1^2}{\eta^2}\left(\frac{C_0}{\eta}\log\frac{2}{\eta}\right)^2.$$

The following result is stated without proof since a more abstract version will be presented in Section 8. The notation is as in the preceding proof, and $\tilde{G}_v$ is defined at the beginning of Section 6.

PROPOSITION 7.2. *Let v be an atomic c^1-measure with atoms on an interpolating sequence σ. Suppose that v satisfies* (7.3) *and let $\mathscr{F}_\lambda: \tilde{G}_v \to \tilde{G}_v$ be the operator*

$$\mathscr{F}_\lambda f = \varphi'(\lambda)^{-1}\varphi h_\lambda \otimes f'(\lambda) \quad (\text{for } f\in\tilde{G}_v) \tag{7.7}$$

with adjoint $\mathscr{F}^*_\lambda$. Then

(i) *each $\mathscr{F}_\lambda: H^\infty_0\mathscr{K} \to H^\infty_0\mathscr{K}$ is bounded, and*

(ii) *there is an increasing sequence of finite subsets τ of σ such that the identity $I: \tilde{G}_v \to \tilde{G}_v$ is the strong operator limit of $\sum_{\lambda\in\tau}\mathscr{F}^*_\lambda\mathscr{F}_\lambda$.*

8. *Operator decomposition of T*

LEMMA 8.1. [**10**, B10]. *Let G_v be as above. Then there is a finitely additive Borel set function $S\mapsto\mathscr{E}(S)$ with values in $B(G_v)$ such that for each Borel subset S of D we have*

(i) $0 \leqslant \mathscr{E}(S) \leqslant I$,

(ii) $\mathscr{E}(D) = I$,

(iii) $$\langle \mathscr{E}(S)f, f\rangle_{G_v} = \iint_S (1-r)\,\langle |f'(re^{i\theta})|^2_s, v(dr\,d\theta)\rangle \quad (\text{for } f\in G_v). \tag{8.1}$$

(iv) *Further, if $S\subset V$ are Borel sets of D, there is a contraction $C(S,V)$ with $\mathscr{E}(S)^{1/2} = C(S,V)\,\mathscr{E}(V)^{1/2}$.*

Proof. Denote the right-hand side of (8.1) by $Q_S(f)$. Clearly $0 \leqslant Q_S(f) \leqslant \|f\|^2_{G_v}$. By polarization $Q_S(f)$ defines a bounded positive quadratic form on G_v. Hence by the Riesz Representation Theorem there is a unique self-adjoint operator $\mathscr{E}(S)$ with $Q_S(f) = \langle \mathscr{E}(S)f, f\rangle_{G_v}$. Properties (i), (ii) and (iii) of $\mathscr{E}$ are evident. It is clear that $Q_{S\cup V} = Q_S + Q_V$ if S, V are disjoint, and hence the set function $S\mapsto\mathscr{E}(S)$ is finitely additive.

The statement (iv) follows immediately from Douglas' Lemma and the inequality $Q_S(f) \leqslant Q_V(f)$ between the respective quadratic forms.

THEOREM 8.2. *Let T, v be as in Theorem* 4.1. *Then:*

(i) *there is a bounded linear operator $T: G_v \to X$ extending $T: A_0\otimes\mathscr{K} \to X$;*

(ii) *for each finite partition of the disc $D = \bigcup_j S_j$ by Borel sets S_j, there is a decomposition $T = \sum_j T_j$ as a sum of linear operators $T_j: G_v \to X$ with $\sup_\pm \|\sum_j \pm T_j\|_{G_v, X} \leqslant 2$;*

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(iii) *if v satisfies the hypotheses of any one of Theorems* 6.1–6.4, *there is a constant C_v such that, for each choice of signs, $\sum_j \pm T_j$ defines a bounded operator $A_0 \overline{\otimes} \mathscr{K} \to X$ with $\| \sum_j \pm T_j \|_{A_0 \overline{\otimes} \mathscr{K}, X} \leqslant C_v$.*

Proof. (i) By Theorem 4.1, the operator T, defined initially on the linear subspace of G_v consisting of polynomials with coefficients in $\mathscr{K}$, extends to define a bounded linear operator from G_v to X with $\|T\|_{G_v, X} \leqslant 1$.

(ii) Given a partition of the disc as above we can introduce $T_j = T\mathscr{E}(S_j)$. By additivity of $\mathscr{E}$ we have $I = \sum_j \mathscr{E}(S_j)$ and we thereby obtain the desired decomposition of T. For any choice of signs $\pm$, we can gather together the like signs and use finite additivity of $\mathscr{E}$ to obtain

$$\begin{aligned} \| \sum_j \pm T_j \|_{G_v, X} &\leqslant \| \sum_+ T_j \|_{G_v, X} + \| \sum_- T_j \|_{G_v, X} \\ &\leqslant \|T\|_{G_v, X} \| \sum_+ \mathscr{E}(S_j) \|_{G_v, G_v} + \|T\|_{G_v, X} \| \sum_- \mathscr{E}(S_j) \|_{G_v, G_v} \\ &\leqslant \| \mathscr{E}(\bigcup_+ S_j) \|_{G_v, G_v} + \| \mathscr{E}(\bigcup_- S_j) \|_{G_v, G_v} \leqslant 2. \end{aligned} \tag{8.2}$$

(iii) If v satisfies the hypotheses of any of the specified results, the inclusion map $A_0 \overline{\otimes} \mathscr{K} \to G_v$ is bounded and by composition each $\Sigma_j \pm T_j$ defines a bounded operator $A_0 \overline{\otimes} \mathscr{K} \to G_v \to X$.

9. *Example*

Let $T: A_0 \overline{\otimes} \mathscr{K} \to L^2 l^2$ be the bounded linear operator defined by $Tf(e^{i\theta}) = ([f(e^{i\theta})]_{1n})_{n=1}^{\infty}$ which compresses the matrix of $f(e^{i\theta})$ to its top row. Evidently T is not compact. By examination of (6.8) in the proof of Theorem 6.2 it will be seen that T extends to define a bounded linear operator on G_{σ^2}, where σ is the constant weight $\sigma = e_{11}$.

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