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Optimisation of hydrophone placement: a dynamical systems approach

D. A. WOOD[1] and D. J. ALLWRIGHT[2]

[1] *Mathematics Institute, University of Warwick, Coventry CV4 7AL, UK*
[2] *Mathematical Institute, University of Oxford, 24-29 St. Giles, Oxford OX1 3LB, UK*

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Tools from equivariant bifurcation theory are applied to the problem of the optimisation of horizontal planar arrays to minimise array noise gain in a 2D isotropic noise field. We specifically use the performance measure derived in the paper by Hayward [3], which is invariant under suitable actions of the symmetry group $\mathbf{S}_n \times \mathbf{O}(2)$, although we would expect most suitable measures to inherit such a symmetry due to the physical properties of the array. An analysis of bifurcations in the presence of this symmetry provides a list of array configurations which one would expect to (locally) optimise performance. This analysis provides a systematic way to search through $2n$-dimensional phase-space, via low dimensional searches, for solutions. We provide an example of the technique for an array with seven hydrophones and a maximum aperture size of $3m$.

1 Introduction

Hayward [3] considered the optimisation of hydrophone placement for acoustic arrays, using simulated annealing (see, for example, Press *et al.* [4]). Both horizontal and vertical arrays were considered, with a variety of different array performance measures. The details of the performance measure for a horizontal array are given by Hayward [3], but in outline suppose the acoustic pressure signals from the hydrophones are delayed and summed to give a response, and the delays are chosen so that that response is maximised for acoustic plane waves incident from a certain 'steer' direction. Suppose also that the normalisation is such that unit amplitude incident wave gives unit amplitude response. Then a major measure of array performance is the response to *noise*, which should be as low as possible. Hayward takes the performance measure to be the noise level in the response, for unit isotropic two-dimensional noise in the water, averaged over steer directions and over a certain frequency interval. The aim is to minimise this performance measure, and it has certain symmetries, namely rotations, reflections, and permutations of the hydrophone positions. As Hayward points out, the optimal hydrophone configurations consist of regular geometric arrangements: lattices and concentric regular polygons (see Figure 3). In this paper, we develop an analytical approach to these results using Equivariant Bifurcation Theory. We show how this leads to a systematic method for classifying the potential minimal configurations. As well as being interesting and informative in itself, this classification also defines those families of configurations that should be searched for

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numerically. This enables the numerical searching to be more efficient because the problem is reduced to a number of low-dimensional searches instead of one high-dimensional search. Consequently, larger problems can be handled, and furthermore solutions can be found more quickly (for deterministic methods) or with higher probability (for stochastic methods such as simulated annealing).

The method of equivariant bifurcation theory is generally applicable to the minimisation of functions with symmetries. These occur widely of course, but in particular the method would apply to some more general versions of Hayward's problem, as we shall note in § 2.

2 Horizontal arrays and formulation of the problem

For a horizontal array in the (x, y)-plane, Hayward denotes the positions of the i^{th} hydrophone by $\mathbf{x}_i = (x_i, y_i)$, so the positions of all n hydrophones are given by the $2n$-dimensional vector $\mathbf{X} = (\mathbf{x}_1, \ldots, \mathbf{x}_n)$. Since consideration is restricted to an *unshaded* array, where the hydrophone signals are simply summed without any weighting, $\mathbf{X}$ contains all the parameters over which the optimisation is to be carried out, and the performance measure is a function which Hayward denotes by $v(\mathbf{X})$. As mentioned earlier, v is the array noise gain averaged over all steer directions in the plane, and over a frequency interval $[f_1, f_2]$. It is given by

$$v(\mathbf{X}) = \frac{1}{k_2 - k_1} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{W(k_2|\mathbf{x}_i - \mathbf{x}_j|) - W(k_1|\mathbf{x}_i - \mathbf{x}_j|)}{|\mathbf{x}_i - \mathbf{x}_j|} \tag{2.1}$$

where $k_i = 2\pi f_i / c$, W is defined by

$$W(u) = \int_0^u J_0(v)^2 \, dv$$

where J_0 is the Bessel function of order 0 and the summand in (2.1) is replaced by its limiting value 1 when $i = j$. The optimisation problem is to minimise $v(\mathbf{X})$, and we shall initially consider this with *no* aperture constraint, so that the $\mathbf{x}_i$ are not restricted to lie in a given region.

It is immediately apparent that v is invariant under permutation of hydrophone positions, that is, if ρ is a permutation of the integers $1, 2, \ldots, n$ then

$$v(\mathbf{x}_{\rho(1)}, \ldots, \mathbf{x}_{\rho(n)}) = v(\mathbf{x}_1, \ldots, \mathbf{x}_n).$$

Clearly v is also invariant under translation, rotation and reflection of the hydrophone positions:

$$v(\mathbf{x}_1 + \mathbf{a}, \ldots, \mathbf{x}_n + \mathbf{a}) = v(\mathbf{x}_1, \ldots, \mathbf{x}_n)$$

$$v(L\mathbf{x}_1, \ldots, L\mathbf{x}_n) = v(\mathbf{x}_1, \ldots, \mathbf{x}_n)$$

where $\mathbf{a}$ is an arbitrary vector in the plane and L is the 2×2 matrix of a reflection or a rotation. The systematic way to study the consequences of such symmetry properties is to let Γ be the group of actions under which v is invariant. So, we let $\mathbf{V}$ be the set of

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$\mathbf{X} = (\mathbf{x}_1, \ldots, \mathbf{x}_n)$, and we let Γ be the group of transformations γ of $\mathbf{V}$ with the property

$$v(\gamma \mathbf{X}) = v(\mathbf{X}) \text{ for all } \mathbf{X} \text{ in } \mathbf{V}.$$

In our case, Γ is the direct product $\mathbf{S}_n \times \mathbf{E}(2)$ where $\mathbf{S}_n$ is the symmetric group (of permutations of $1, \ldots, n$) and $\mathbf{E}(2)$ is the group of all Euclidean transformations (i.e. isometries) of the plane. $\mathbf{E}(2)$ can be considered as the group of 3×3 matrices of the form $\begin{pmatrix} L & \mathbf{a} \\ 0 & 1 \end{pmatrix}$ where L lies in $\mathbf{O}(2)$, the group of 2×2 orthogonal matrices, and $\mathbf{a}$ is a vector in the plane, and this 3×3 matrix acts on the plane by taking $\mathbf{x}$ to $L\mathbf{x} + \mathbf{a}$.

We pause to note the symmetry properties that would arise for a more general, more realistic beamformer. First, if shading weights w_i on the hydrophone signals are used, then the terms summed in (2.1) are multiplied by $w_i w_j$. The permutation group $\mathbf{S}_n$ can then still act invariantly: it merely has to apply the *same* permutation to both the shading weights w_i and the hydrophone positions $\mathbf{x}_i$. Secondly, if the hydrophones do not all lie in a plane, then we have to think about the three-dimensional geometric symmetries but, assuming the noise field is homogeneous, there will still be translation invariance. If the noise field is isotropic in the horizontal but not in the vertical, then the geometric symmetry group is a certain subgroup $\widetilde{\mathbf{E}}(3)$ of $\mathbf{E}(3)$, consisting of 4×4 matrices of the form $\begin{pmatrix} L & 0 & \\ 0 & 1 & \mathbf{a} \\ 0 & 0 & 1 \end{pmatrix}$ where $L \in \mathbf{O}(2)$ and $\mathbf{a} \in \mathbf{R}^3$. However, if the noise was isotropic in three dimensions then the full group $\mathbf{E}(3)$ would act invariantly. Thus, in each case the symmetry group Γ of the function v can readily be identified from the obvious symmetry properties of the physical system being considered.

Later, when we use the symmetry properties of v to derive a more general setting in which to locate possible optimal solutions, it turns out that the translational invariance introduces technical difficulties into the analysis. Often in physical applications, this problem is circumvented by imposing suitable periodic boundary conditions. Such a boundary condition with period l replaces the translation group $\mathbf{R}$ by the circle group $\mathbf{R}/l$, and so introduces a new rotational symmetry. (Configurations whose period is a sub-multiple l/k will be invariant under the discrete cyclic subgroup of order k of this circle group.)

Here we take a different approach since there will be no suitable period on which to base such a reduction. Instead we impose the constraint that the centre of mass of the hydrophones must remain fixed, which can be realised by insisting that $\sum \mathbf{x}_i = \mathbf{0}$. In this way, we have removed the translation invariance of v and reduced the symmetries to rotations about the origin, reflections in lines through the origin and permutation of indices. This gives us a function invariant under the group $\Gamma = \mathbf{O}(2) \times \mathbf{S}_n$ acting on $\mathbf{R}_0^{2n} = \{\mathbf{X} \in \mathbf{R}^{2n} : \mathbf{x}_1 + \ldots + \mathbf{x}_n = 0\}$, where $\mathbf{O}(2)$ acts on every $\mathbf{x}_i$ as a rotation or reflection. Also, Γ acts linearly on this space.

It will prove advantageous to identify co-ordinates (x_i, y_i) in $\mathbf{R}^2$ with co-ordinates z_i in the complex plane $\mathbf{C}$, leaving us with a $\mathbf{O}(2) \times \mathbf{S}_n$ invariant function on $\mathbf{C}_0^n = \{\mathbf{z} \in \mathbf{C}^n : z_1 + \ldots + z_n = 0\}$. $\mathbf{S}_n$ acts as before by permutation of indices, rotations about 0 through

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θ radians act as multiplication by $e^{i\theta}$ on each z_i and reflections in the real axis act by complex conjugation $\mathbf{z} \mapsto \bar{\mathbf{z}}$.

To look for local and global minima of v we seek points $\mathbf{X}_e$ in $\mathbf{C}_0^n$ for which $\nabla v(\mathbf{X}_e) = \mathbf{0}$. For a general function f, one method for finding points with $\nabla f = \mathbf{0}$ is to consider the dynamical system $d\mathbf{X}/dt = -\nabla f(\mathbf{X})$. Then local minima of f are stable equilibrium points of this system.

It is a well known fact that if a scalar function $f : \mathbf{V} \to \mathbf{R}$ is invariant under the action of a group Γ acting on linearly $\mathbf{V}$ then the function $\nabla f : \mathbf{V} \to \mathbf{V}$ will be *equivariant* under the same action of Γ, that is, for all $\gamma \in \Gamma$, $v \in \mathbf{V}$,

$$\nabla f(\gamma v)\gamma = \nabla f(v).$$

This follows by using the Chain Rule to differentiate both sides of the condition for invariance that

$$f(\gamma \mathbf{X}) = f(\mathbf{X})$$

and then using the orthogonality of the actions. The simplest example is the function $f(x) = x^2$. This function is invariant under the $\mathbf{Z}_2$ action taking x to $-x$. In this case, $\nabla f = 2x$, which is equivariant under this action, i.e. $f(-x) = -2x = -f(x)$.

We are therefore now trying to solve the set of n equations $\nabla f(\mathbf{X}) = \mathbf{0}$ (actually there are only $n - 1$ equations since they are dependent due to our earlier conditions, but in practice we consider all n equations).

Our approach is now to consider solutions to our problem which may occur through a symmetry breaking bifurcation as some parameter is varied. In order to do this we note that the trivial solution $\mathbf{X} = (0, \ldots, 0)$ always satisfies $\nabla g(\mathbf{X}) = \mathbf{0}$ regardless of the values of k_1 or k_2. In addition this solution is left invariant under the action of the whole group $\mathbf{S}_n \times \mathbf{O}(2)$.

When a parameter is varied, the solution $\mathbf{X} = \mathbf{0}$, in which all hydrophones are clustered at one point, may undergo bifurcations leading to new, non-trivial, equilibrium points. Even if the variation is not one that could occur in practice, this is a valuable method of discovering and classifying such points.

Of course, we are interested in solutions to the problem far from the bifurcation point, and so, while we would expect a large number of these solution branches created by the bifurcation to persist, they may do so with different stabilities from that which they bifurcated off the trivial solution. This may be due to secondary bifurcations which may cause new solution branches to be created. It is also not uncommon for branches to bifurcate *super-critically*, in which case the branch only exists for the parameter decreasing instead of increasing from the point of bifurcation, unless the branch changes direction, as can be observed in problems with $\mathbf{D}_3$ symmetry (see Golubitsky *et al.* [2]). The feature common to all these phenomena however, is that they all lead to the possibility that representatives of all the solution branches due to these phenomena will exist in the parameter ranges far from the bifurcation point, and it is these solutions in which we are interested.

Schematic bifurcation diagrams of all these above examples of solution branches, which represent all the 'simplest' behaviour we would expect to witness in one-parameter bifurcation problems, are shown in Figure 1.

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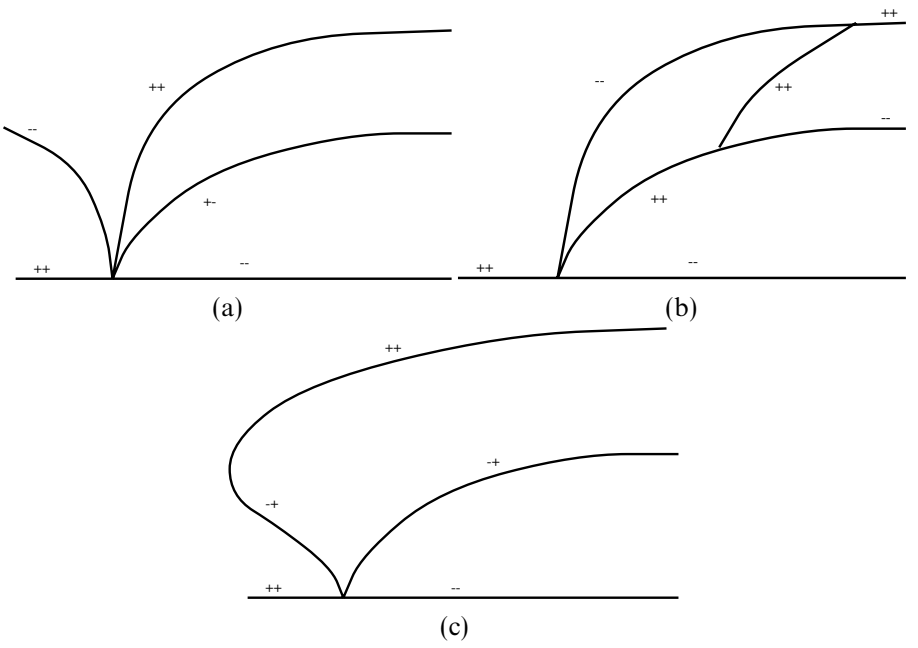


FIGURE 1. Response curves as a function of the bifurcation parameter: (a) typical branches from a primary bifurcation, (b) a secondary bifurcation joining two primary branches, and (c) a supercritical bifurcation turning and changing stability. In each case the '+' and '−' denote signs of eigenvalues.

We now justify the use of a bifurcation theory approach to look for solutions to our specific problem.

For our specific performance measure $v(\mathbf{X})$, since we are initially considering its local behaviour about $\mathbf{X} = \mathbf{0}$, we can represent v as a power series by expanding the Bessel Functions according to

$$J_0(v) = \sum_{j=0}^{\infty} \frac{(-1)^j (v/2)^{2j}}{(j!)^2}.$$

Using this expansion, we take the gradient of v and convert to complex co-ordinates by defining $g_j(\mathbf{z}) = dv/dx_j + i dv/dy_j$ where $z_j = x_j + iy_j$, yielding, up to fifth order (for the purposes of clarity), the following expression for g_j:

$$g_j(\mathbf{Z}) = -K_1 z_j + K_2 \sum_{k=1}^{n} (z_j - z_k)^2 \overline{(z_j - z_k)} - K_3 \sum_{k=1}^{n} (z_j - z_k)^3 \overline{(z_j - z_k)}^2$$

where $K_1 = (2n/3)(k_2^2 + k_1 k_2 + k_1^2)$, $K_2 = (3/20)(k_2^4 + k_2^3 k_1 + \ldots + k_1^4)$ and $K_3 = (5/336)(k_2^6 + k_2^5 k_1 + \ldots + k_1^6)$.

All of the K_i are non-negative for all values of the k_i, but if we set $\lambda = K_1$ and allow λ to take both negative and positive values (with the proviso that we can only give such a parameter physical meaning when it is positive), and we treat v as a potential function for a dynamical system in $\mathbf{C}_0^n$, then we have a system where the origin $z_i = 0\ \forall i$ is

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linearly stable for $\lambda < 0$ and linearly unstable for $\lambda > 0$. That is, we take the performance measure $\mu(\mathbf{Z}) = v(\mathbf{Z}) + \frac{1}{2}(K_1 - \lambda)(z_1^2 + \ldots + z_n^2)$ which is equal to our original measure when $\lambda = K_1 = (2n/3)(k_2^2 + k_1 k_2 + k_1^2)$.

We have thus reached the stage where we seek solutions to a bifurcation problem which is equivariant under the action of $\mathbf{O}(2) \times \mathbf{S}_n$. Systems with this symmetry have been studied by Stewart [5], where primary Hopf bifurcations were used to study the possible solutions to many body collisionless systems. Here we consider only steady-state solutions.

3 Bifurcations with symmetry

3.1 Primary bifurcations

Assume we have a function $g : \mathbf{V} \times \mathbf{R} \to \mathbf{V}$, depending on a single parameter $\lambda \in \mathbf{R}$ which is equivariant under the action of the group Γ on $\mathbf{V}$. Further assume that $g(0, \lambda) = 0$ for all values of λ, and that this solution is stable for $\lambda < 0$, becoming unstable as λ passes through zero. That is, the Jacobian $dg_{(0,0)}$ is singular and a bifurcation occurs. When symmetry is present, we expect multiple zero eigenvalues at such a bifurcation, giving rise to a centre-manifold whose dimension is greater than one. By assuming we have already performed a suitable reduction such as a centre-manifold or Liapunov-Schmidt reduction which respects the symmetries (see, for example, Golubitsky *et al.* [2]), we can consider the reduced case where $\mathbf{V}$ *is* the centre manifold; thus $dg_{(0,0)} = 0$.

Generically, we expect this $\mathbf{V}$ to be *Γ-irreducible*, that is, the only Γ-invariant subspaces of $\mathbf{V}$ are $\mathbf{V}$ itself and $\{0\}$. We usually assume a slightly stronger condition that $\mathbf{V}$ is *absolutely* irreducible, so that the only linear mapping on $\mathbf{V}$ that commutes with Γ is the identity. (It is a straightforward exercise to show that absolutely irreducible implies irreducible.)

In this way the origin, our trivial solution (stable for $\lambda < 0$), has the full group Γ as its symmetry; applying any element of Γ leaves $\mathbf{0}$ fixed. As λ passes through zero we expect the solution branches which bifurcate from 0 (either sub- or super-critically), to have less symmetry than Γ; a *symmetry-breaking bifurcation*. It turns out that these solutions have lost symmetry in a very precise way, because a class of solutions which are guaranteed to exist in this situation have, in some sense, lost a minimal amount of symmetry. We make this more precise.

Take a point $v \in \mathbf{V}$ (think of it as a solution to $g(v, \lambda) = 0$), then the *Isotropy Subgroup* Σ of this point, $\Sigma \subset \Gamma$, is defined as

$$\Sigma_v = \{\sigma \in \Sigma : \sigma \cdot v = v\}$$

i.e. it is the subgroup consisting of all the elements of Γ which keep v fixed, also called the stabiliser of v.

In a similar vein, for a subgroup $\Sigma \subset \Gamma$ the *Fixed Point Subspace* of Σ is defined as

$$\mathrm{Fix}(\Sigma) = \{v \in \mathbf{V} : \sigma \cdot v = v, \sigma \in \Sigma\}.$$

The Equivariant Branching Lemma [2] then states that at the bifurcation point discussed above $(v, \lambda) = (0, 0)$ we are *guaranteed* solution branches coming off having isotropy

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subgroup Σ if the dimension of $\mathrm{Fix}(\Sigma)$ is one. Solutions with isotropy having higher dimensional fixed point subspaces could also occur, but are not guaranteed in this case.

Further, it is possible to write the most general normal form for a bifurcation problem with a particular symmetry on the reduced space, using a finite number of generators. This allows us to write down the generic stability properties of the problem with conditions for stability in a model-independent way, in the sense that we can say a lot about the expected stability conditions for any set of equations with the prescribed symmetry properties.

Finally, solution branches which are *conjugate*, that is differ by an element of Γ ("look the same up to an action of Γ") will have the same stability. For example, say we have a solution $(x, 0, \ldots, 0)$ in our problem with $\mathbf{O}(2) \times \mathbf{S}_n$ symmetry, then we also have the solution $(0, x, 0, \ldots, 0)$ with the same stability by applying the permutation $(12) \in \mathbf{S}_n$. In this way the analysis of all such solutions can be drastically reduced. By way of an example we reproduce the main symmetry implications for a bifurcation problem with $\mathbf{O}(2)$ symmetry (see, for example, Golubitsky *et al.* [2]).

Example 3.1 The representation of $\mathbf{O}(2)$ acting on $\mathbf{C}$ in which rotations through an angle θ about the origin act by complex multiplication by $e^{i\theta}$, and reflection in the real axis κ acts by complex conjugation, $\kappa.z = \overline{z}$ is absolutely irreducible.

The lattice of Isotropy Subgroups, up to conjugacy, is given by the following subgroups: $\mathbf{O}(2)$ itself, with fixed point subspace $\{0\}$, and the two element subgroup generated by a reflection $\mathbf{Z}_2 = \{id, \kappa\}$ with fixed point subspace the real line $z = \overline{z}$. Note that in this case every non-zero point in $\mathbf{C}$ is fixed by, and has isotropy given by, a reflection in the line which intersects both the point and the origin. This isotropy however will be conjugate to $\mathbf{Z}_2$ and so need not be considered separately.

Therefore the only isotropy subgroup with a one-dimensional fixed point subspace is $\mathbf{Z}_2$, and so at a bifurcation as described above we are guaranteed solution branches with isotropy $\mathbf{Z}_2$.

3.2 Mode interactions

For a bifurcation problem $g = 0$ where there are specifically two parameters, or where a single parameter equation has a two-parameter unfolding at $(0, 0)$, then we might expect a co-dimension 2 singularity and the centre manifold to have the form $\mathbf{V}_1 \oplus \mathbf{V}_2$, where now both $\mathbf{V}_1$ and $\mathbf{V}_2$ are Γ-irreducible. Typically, the action of Γ on each of these spaces will have a different representation. This scenario now gives rise to 'Mode-interactions'. By this we mean that each irreducible representation $\mathbf{V}_i$ corresponds to a 'mode', the interaction coming from each space having a non-trivial solution branch associated with it from the bifurcation assumed to occur at $(v, \lambda, \mu) = (0, 0, 0)$, where λ and μ are our two parameters. Solutions can again be found, and guaranteed to exist under certain generic hypotheses, using the Equivariant Branching Lemma on each irreducible subspace $\mathbf{V}_i$. Now however the action must act simultaneously, but not identically, on each subspace.

To illustrate, in Golubitsky *et al.* [2] they use the example of $\mathbf{O}(2)$ mode-interactions.

Example 3.2 When considering $\mathbf{O}(2)$ acting on $\mathbf{C}$ as in Example 3.1, we would typically expect the centre manifold for mode-interactions to be of the form $\mathbf{V}_1 \oplus \mathbf{V}_2 \cong \mathbf{C} \oplus \mathbf{C}$ where $\mathbf{O}(2)$ acts on each component absolutely irreducibly.

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Let $\mathbf{O}(2)$ acting on $\mathbf{V}_1$ be generated by rotations $\theta z_1 = e^{li\theta} z_1$ and reflection $\kappa z_1 = \overline{z_1}$ and on $\mathbf{V}_2$ by $\theta z_2 = e^{mi\theta} z_1$ and $\kappa z_2 = \overline{z_2}$ where l and m are co-prime. If l and m were not co-prime then we could factor out an action of $\mathbf{O}(2)$ that acts identically on both spaces. Therefore $\mathbf{O}(2)$ acts on $\mathbf{V}_1 \oplus \mathbf{V}_2$ by

$$\theta(z_1, z_2) = (e^{li\theta} z_1, e^{mi\theta} z_2),$$
$$\kappa(z_1, z_2) = (\overline{z_1}, \overline{z_2}).$$

We now seek isotropy subgroups with either a one-dimensional fixed point subspace in a single mode, or two one-dimensional fixed point subspaces in the two modes. This gives us the isotropy subgroups

Isotropy Σ	Fix(Σ)
$\mathbf{O}(2)$	$\{0\}$
$\mathbf{D}_l$	$\mathbf{R}\{(1,0)\}$
$\mathbf{D}_m$	$\mathbf{R}\{(0,1)\}$
$\mathbf{Z}_2$	$\mathbf{R} \oplus \mathbf{R}$
$\mathbf{1}$	$\mathbf{C}^2$

Here $\mathbf{D}_l$, the Dihedral group of order l, acts by the rotation $\theta = 2\pi/l$ and a reflection. This action is the identity on $\mathbf{V}_1$ and as non-trivial rotation on $\mathbf{V}_2$, including the reflection κ forces the $\mathbf{V}_1$ component to be real giving rise to a single mode solution. The only mixed-mode solution is given by isotropy $\mathbf{Z}_2$ which is generated by the reflection κ acting on both spaces.

4 Bifurcations with $\mathbf{O}(2) \times \mathbf{S}_n$ symmetry

Returning to the problem in which we are interested, we consider the types of solutions we would expect to see in bifurcation problems with $\mathbf{O}(2) \times \mathbf{S}_n$ symmetry for both primary and secondary bifurcations. We consider the most general case, and so at this stage we need not know the exact form of the equations to work out the solutions we would expect to see. Indeed, many systems governed by an equivariant equation will exhibit the expected symmetric solutions without a bifurcation occurring *per se*, although without knowing the exact details of every bifurcation, we would not be able to follow specific solution branches. In this way the results can also be used for a wide variety of spatially dependent problems.

The generic situation of problems with direct product symmetries have been considered by Dionne *et al.* [1], but for clarity we present here the complete working for our specific scenario.

4.1 Primary bifurcations

We consider the case where we have one independent parameter, λ; in the hydrophone problem this may equate, for example, to $k_1^2 + k_1 k_2 + k_2^2$ when $\lambda > 0$. The standard (absolutely) irreducible representation for this symmetry where none of its elements acts trivially is the action on $\mathbf{C}_0^n$ given by $\mathbf{S}_n$ acting by permutation of indices, $\theta \in \mathbf{O}(2)$ acting by multiplication by $e^{i\theta}$ and reflection in $\mathbf{O}(2)$ acting by complex conjugation. In this way,

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it can be seen that a Liapunov–Schmidt reduction on the problem posed in § 2 should be trivial.

Up to conjugacy, one class of solution branches we are guaranteed by the Equivariant Branching Lemma are solutions of the form

$$(-kx, \ldots, -kx, (n-k)x, \ldots, (n-k)x)$$

where $x \in \mathbf{R}$. This has the isotropy generated by reflection in the real axis ($\bar{z} = z$) and the permutation group $\mathbf{S}_{(n-k)} \times \mathbf{S}_k$ which permutes independently the first $n-k$ coordinates and the last k. It is because we are considering conjugacy classes that we can group the coordinates into these two distinct blocks. It should be clear that the fixed point subspace has dimension one, and so satisfies the appropriate condition of the equivariant branching lemma. Note the special case where $(n-k) = k$ when we have the additional symmetry in which the two blocks are interchanged followed by rotation through π about the origin.

The other class of solutions do not obviously satisfy the criteria for the equivariant branching lemma to apply, and are denoted $\widetilde{\mathbf{D}}_k^l$. To describe the action of this isotropy we first define a 'block of length l' to be l identical elements which we group together. We then split a vector in $\mathbf{C}_0^n$ into k such blocks with the remaining $n-lk$ elements as a separate block. The other isotropy then acts on the first k blocks by cyclically permuting the blocks and then applying the rotation $\theta = 2\pi/k$, on each block as $\mathbf{S}_l$, and on the remaining $n-lk$ elements as $\mathbf{S}_{(n-l)}$. This has the fixed point subspace

$$(\mathbf{z}, \rho\mathbf{z}, \rho^2\mathbf{z}, \ldots, \rho^{k-1}\mathbf{z}, 0, \ldots, 0)$$

where $\mathbf{z}$ is a block of l elements and $\rho = e^{i\theta}$. By conjugacy (apply a rotation in $\mathbf{O}(2)$) and without loss of generality, $\mathbf{z}$ lies in $\mathbf{R}^l$ (as does $\rho^{k/2}\mathbf{z}$ if k is even) and the solution also has symmetry reflection in the real line (κ) followed by an appropriate order two permutation, along with the other $k-1$ (k even) or $2k-1$ (k odd) lines of reflectional symmetry (of an k-gon). It is this reflectional symmetry that forces the fixed point subspace to have dimension one, and so for every isotropy subgroup $\widetilde{\mathbf{D}}_k^l$ we are guaranteed a branch of solutions to our bifurcation problem with these isotropies as their symmetries.

We now outline the proof that these are the only isotropy subgroups having fixed-point subspace of dimension one (so called 'axial subgroups').

Theorem 4.1 *Up to conjugacy, the only isotropy subgroups of* $\mathbf{O}(2) \times \mathbf{S}_n$ *having a one-dimensional fixed point subspace are given by* $\mathbf{S}_k \times \mathbf{S}_{n-k}$ *and* $\widetilde{\mathbf{D}}_k^l$.

Proof The isotropy subgroups with one-dimensional fixed point subspaces will either be strictly in $\mathbf{S}_n \times \mathbf{Z}_2$ or in $\mathbf{S}_n \times \mathbf{O}(2)$. They cannot be a subgroup of only $\mathbf{S}_n$ since the corresponding fixed-point subspaces would be at least two-dimensional ($\mathbf{C}_0^n$ not irreducible for $\mathbf{S}_n$). This argument applies equally to subgroups of only $\mathbf{O}(2)$, so these can be discounted too. We now consider these two cases.

(i) $\mathbf{S}_n \times \mathbf{Z}_2$ acting on $\mathbf{C}_0^n$ can be chosen to act (by conjugacy of $\mathbf{O}(2)$) on $\mathbf{R}_0^n \oplus \mathbf{R}_0^n$ by acting on each component simultaneously and as $\mathbf{S}_n$ on the first component, and

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on only the second component as $\mathbf{S}_n \times \mathbf{Z}_2$ ($x \mapsto -x$). On the first component the $\mathbf{Z}_2$ action acts as the identity.

We need isotropy subgroups, Σ, with one-dimensional fixed point subspaces and so must have a one-dimensional fixed-point subspace on the first component and a zero-dimensional on the second. The isotropy subgroups of $\mathbf{S}_n$ are well-known (see for example [6]); those having one-dimensional fixed-point subspaces are of the form $\mathbf{S}_k \times \mathbf{S}_{n-k}$. The $\mathbf{Z}_2$ action acting by itself on the second component will force the fixed point subspace in this component to be the zero vector as required. Thus the required isotropy subgroups are the subgroups generated by $\mathbf{S}_k \times \mathbf{S}_{n-k}$ and the reflection $\kappa \in \mathbf{Z}_2$. Note the special case where $n = 2k$ when the subgroup generated thus is no longer the isotropy subgroup of the corresponding fixed-point subspace. There is an additional symmetry of an order two permutation acting on all the elements followed by the reflection κ.

(ii) $\mathbf{S}_n \times \mathbf{O}(2)$ If $\Sigma = \mathbf{S}_n$ or Σ contains exactly one reflection then we have the above case. If it contains more than one reflection then we have rotations too. Let ρ be the smallest such rotation and so all rotations in Σ will be generated by a multiple of ρ (if we have two rotations of order k_1 and k_2 then the group is generated by a rotation of order $lcm(k_1, k_2)$).

If we temporarily ignore the reflections in Σ, then this subgroup can be thought of as an isotropy subgroup of $\mathbf{S}_n \times \mathbf{SO}(2)$, or $\mathbf{S}_n \times \mathbf{S}^1$, acting on $\mathbf{C}_0^n$ where $\mathbf{S}^1$ is the *circle group* of rotations. Such actions occur naturally when considering Hopf Bifurcations on a reduced space that is so-called Γ-simple (see, for example, Golubitsky *et al.* [2]), where the $\mathbf{S}^1$ action represents phase shifts. This is not the case here, but the conditions for Proposition 7.2 of Golubitsky *et al.* [2] still hold, and so we know that the isotropy subgroups are of the form of *Twisted Subgroups* (see Golubitsky *et al.* [2]). Let $H \subset \mathbf{S}_n \times \mathbf{Z}_2$ then a twisted subgroup is of the form $H_\theta = \{[h, \theta(h)],\ h \in H\}$ where $\theta : H \to \mathbf{S}^1$ is a group homomorphism.

Hence, we can now read off all the isotropy subgroups of $\mathbf{S}_n \times \mathbf{S}^1$. They are well known (see, for example, Stewart [5]), and those with the lowest dimensional fixed-point subspaces are of the form $(\mathbf{z}, \rho\mathbf{z}, \ldots, \rho^{k-1}\mathbf{z}, \mathbf{0})$ where the vector $\mathbf{z} \in \mathbf{C}^l$ is such that $lk \leqslant n$ and all components are equal, the '$\mathbf{0}$' is a vector, and ρ acting as $e^{2\pi i/k}$ is a rotation of order k. As noted earlier, such vectors have in addition the symmetries given by various reflections, if without loss of generality the first component is real, and κ is reflection in the real line, then this vector is also left fixed by this reflection and a suitable permutation along with the reflections $\rho\kappa$, $\rho^2\kappa$, etc. (followed by suitable permutations). In the notation of the twisted subgroups, all such elements h_κ of the isotropy subgroup are mapped to the group identity by θ. The composition of two reflections will give a rotation which is already a multiple of ρ by construction above. These reflections will reduce the dimension of the fixed-point subspaces by at least one, which now forces such fixed-point subspaces to be one-dimensional. □

Realising these isotropies as solutions to our hydrophone placement problem, the isotropy $\mathbf{S}_{(n-k)} \times \mathbf{S}_k$ corresponds to k coincident hydrophones at a point on one side of

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the centre of mass, balanced by $(n-k)$ coincident hydrophones at a point on the other side. This would give the same performance as only two hydrophones (but with shading weights k, $n-k$) and so we are forced to consider the solutions which correspond to all the positions being distinct. This allows us to ignore the majority of the other solution branches guaranteed by the Equivariant Branching Lemma. In fact we are left with (up to conjugacy) only two solutions, with isotropies $\widetilde{\mathbf{D}}^1_{n-1}$ and $\widetilde{\mathbf{D}}^1_n$ with corresponding fixed point subspaces

$$\mathrm{Fix}\big(\widetilde{\mathbf{D}}^1_{n-1}\big) = (s, \rho s, \rho^2 s, \ldots, \rho^{n-2} s, 0),$$

where $\rho = e^{2i\pi/(n-1)}$ and

$$\mathrm{Fix}\big(\widetilde{\mathbf{D}}^1_{n}\big) = (s, \eta s, \eta^2 s, \ldots, \eta^{n-1} s),$$

where $\eta = e^{2i\pi/n}$, $s \in \mathbf{R}$.

One of the two generators of $\widetilde{\mathbf{D}}^1_n$ acts by cyclically permuting all the z_i's followed by multiplication by η, which corresponds to all the hydrophones being equally spaced around a circle; the action of the symmetry is the rotation of this circle so that the jth hydrophone lies on top of where the $j+1$st hydrophone lay and then permuting indices $j \mapsto j+1$ (modulo n). The other generator is the reflection which fixes the orientation of the circle with respect to the axis. $\widetilde{\mathbf{D}_{n-1}}$ acts similarly except that one of the hydrophones remains fixed at the origin and the circle contains $n-1$ equally spaced hydrophones around its circumference.

For the performance measure v, locally augmented by a bifurcation parameter λ, we can work out the branching equations for small λ for each isotropy subgroup above. It turns out that every branch will bifurcate sub-critically, and the stability calculations on the $2n \times 2n$ Jacobian, although messy, will be essentially straightforward since $(dg)_{x_0}$ is forced by the symmetry to have a very specific form. It would now be a plausible exercise to use some path-following numerical methods to follow each branch of solutions and track its stability using a package such as AUTO. However, we apply a more *ad-hoc* method for the purposes of illustration and reduced computational expense in § 6.

4.2 Secondary bifurcations

We first consider the possibility of mode-interactions in our problem. The full phase-space in question has dimension $2n-2$, which is that of the non-trivial irreducible representation of $\mathbf{O}(2) \times \mathbf{S}_n$, and since the centre-manifold for a mode interaction will be of the form $\mathbf{V}_1 \oplus \mathbf{V}_2$, so the representation on each $\mathbf{V}_i$ must involve some of the group acting trivially. However, the only irreducible representations of $\mathbf{S}_n$ are $\mathbf{S}_n$ acting on $\mathbf{R}^n_0$ and the alternating group $\mathbf{A_n}$, also acting on $\mathbf{R}^n_0$. Thus mode-interactions cannot occur in such a system.

The natural question to ask therefore is what solutions would one expect to see occurring from a secondary bifurcation, that is, a new branch of solutions bifurcating from one of the primary branches of solutions detailed above.

There is evidence from considering the generic form of equations with $\mathbf{O}(2) \times \mathbf{S}_n$ that these so called secondary solutions occur in a very natural manner. After an initial primary bifurcation from the trivial solution to solutions with isotropy $\widetilde{\mathbf{D}^l_k}$, there is a secondary bifurcation due to the eigenvalues associated with the stability of the $(n-lk)$ zero entries in the fixed point space, which naturally leads to the appearance of a new branch of solutions.

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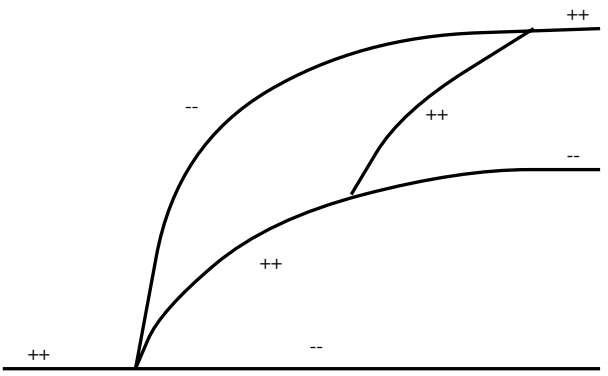


FIGURE 2. Schematic view of two primary branches joined by a secondary bifurcation.

Such solutions we expect to either have an isotropy subgroup of $\mathbf{O}(2)\times\mathbf{S}_n$ with fixed point subspace of dimension larger than one, or to be a branch of solutions connecting two primary branches, as shown schematically in Figure 2. The normal form analysis is beyond the scope of this paper but we investigate the possible consequences of such behaviour.

It will become evident that those 'sub-maximal' isotropies, i.e. isotropy subgroups with fixed point subspaces greater than one that lead to distinct coordinates (as we require) are in one-to-one correspondence with solutions of the branch joining type, and so it is on these latter solutions that we now concentrate.

Assume we have two primary branches of solution corresponding to isotropies $\widetilde{\mathbf{D}}_k^1$ and $\widetilde{\mathbf{D}}_{n-k}^1$, and secondary bifurcations create a new branch of solutions between the two (as outlined above). Then at one end of this secondary branch we have isotropy $\widetilde{\mathbf{D}}_k^1$, at the other end isotropy $\widetilde{\mathbf{D}}_{n-k}^1$, and in between a mixture of the two. In fact the solutions along this joining branch are equivalent to a mode-interaction of the sub-problem of bifurcations in systems with symmetry $\mathbf{O}(2)\times\mathbf{S}_k\times\mathbf{S}_{n-k}$ acting on the space that has been split in the afore-described manner into $\mathbf{R}_0^k\oplus\mathbf{R}_0^{n-k}=\mathbf{V}_1\oplus\mathbf{V}_2$. It should be clear that, since the full problem is equivariant under $\mathbf{O}(2)\times\mathbf{S}_n$, then it is necessarily equivariant under $\mathbf{O}(2)\times\mathbf{S}_k\times\mathbf{S}_{n-k}$.

With this interpretation let $\theta\in\mathbf{O}(2)$ act on $\mathbf{V}_1$ as multiplication by $e^{il\theta}$ and on $\mathbf{V}_2$ as multiplication by $e^{im\theta}$ where l,m are co-prime integers. (There is no reason why the rotation θ should act identically on both spaces). In the case where l,m are not co-prime we can factor out the rotation common to both and the resulting solutions found will not be any different because if p is the greatest common divisor of l and m, $p\neq 1$, then the rotation $\theta=2\pi/p$ acts as the identity on both $\mathbf{V}_1$ and $\mathbf{V}_2$. Different values of l and m give rise to different mode-interactions, but it turns out that only a few of these give rise to solution branches corresponding to distinct coordinates. The reflection $\kappa\in\mathbf{O}(2)$ can be assumed to act identically as $z\mapsto\bar{z}$ in both spaces to make the analysis simpler to envisage. As will be seen later, we can almost always assume that this is the case anyway.

If we do not explicitly require solutions predicted for primary bifurcations, we can seek solutions with a fixed point subspace of dimension two, where the projections into each $\mathbf{V}_i$ give a one-dimensional subspace (where the projection is defined in the obvious manner).

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This gives analogous results to the previous section. Consider the situation described above where we take the irreducible actions of $\mathbf{O}(2) \times \mathbf{S}_k$ on $\mathbf{V}_1$ and $\mathbf{O}(2) \times \mathbf{S}_{n-k}$ on $\mathbf{V}_2$. Assume that we seek solutions where all co-ordinates are non-zero. Then distinct co-ordinates will occur when we choose l and m in the following way. Let $p = gcd(k, n-k)$, and take the mode-interactions where $l = k/p$, $m = (n-k)/p$. The two special cases, namely $k = 1$, and $l = m = 1$, will be dealt with separately after the general case.

Distinct non-zero coordinates with a one-dimensional fixed point subspace in each $\mathbf{V}_i$ implies that in each space we have a twisted subgroup on all $k(n-k)$ entries. That is on each space we have a cyclic permutation followed by a rotation of $2\pi/k$ ($2\pi/(n-k)$). The only way this can happen is with the isotropy $(\widetilde{\mathbf{D}}_k^1, \widetilde{\mathbf{D}}_{(n-k)}^1)$ acting by a rotation of $2\pi/lmp$ applied to both spaces, followed by the permutation $(1 \dots k)([k+1] \dots n)$ which acts as $(1 \dots k)$ on $\mathbf{V}_1$ and as $([k+1] \dots n)$ on $\mathbf{V}_2$. A reflection in (without loss of generality) the real line again fixes orientation and ensures one-dimensional fixed point subspaces in each $\mathbf{V}_i$. In particular note that we may assume both V_1 and V_2 share *at least one common axis of reflection*. (If they do not we can rotate by an action of $\mathbf{O}(2)$ so that we have a conjugate solution that does have a shared axis of reflection).

The case where $k = 1$ must be treated with care since now $\mathbf{V}_1$ in our decomposition becomes the action of $\mathbf{O}(2)$ on $\mathbf{C}$ with isotropy subgroup $\mathbf{Z}_2$ having one-dimensional fixed point subspace and on $\mathbf{V}_2$ we have the action of $\mathbf{O}(2) \times \mathbf{S}_{n-1}$ on $\mathbf{C}_0^{n-1}$. However, now the full space we consider no longer appears to obey our 'fixed centre of mass condition' for a non-zero solution branch in $\mathbf{C}_0^n$. There is no contradiction for the hydrophone problem when we remember that the analysis still occurs in a $2n-2$ dimensional space.

Mode-interactions where there is a single zero entry with the remaining entries distinct can also occur, and in a multitude of ways. The most obvious such occurrence when we take $p = gcd(k-1, n-k)$, $l = (k-1)/p$ and $m = (n-k)/p$. The solution, which can be justified in the same manner as above now has isotropy $(\widetilde{\mathbf{D}}_{k-1}^1, \widetilde{\mathbf{D}}_{(n-k)}^1)$ where the solution branch projected onto $\mathbf{V}_1$ consists of $k-1$ distinct points surrounding a point at zero, and in $\mathbf{V}_2$, $(n-k)$ all coordinates are distinct and non-zero as before.

The case when $l = m = 1$ will only occur when $2k = n$, i.e. n is even and the spaces $\mathbf{V}_1$ and $\mathbf{V}_2$ in our construction are both $2k-2$ dimensional. There is now no reason why the solutions in the full space should have a common axis of rotation and so we have a continuous set of possible solution branches each corresponding to a different relative position of the 'inner' and 'outer' circle of points.

The full selection of mode-interactions, even to lowest order, becomes rather complicated (especially for general l and m, for all possible k) but we have at least shown that:

Theorem 4.2 *The only isotropy subgroups corresponding to mode-interactions with* $\mathbf{S}_k \times \mathbf{S}_{n-k} \times \mathbf{O}(2)$ *symmetry acting on* $\mathbf{C}_0^n$ *as detailed above having distinct elements are given by the subgroups considered above.*

Considering these 'mode-interactions' as solutions to our hydrophone placement problem then, the isotropy $(\widetilde{\mathbf{D}}_k^1, \widetilde{\mathbf{D}}_{n-k}^1)$ corresponds to two rings of hydrophones, equally spaced within each ring, both rings having a common centre and without loss of generality both rings sharing an axis of reflection. $(\widetilde{\mathbf{D}}_{k-1}^1, \widetilde{\mathbf{D}}_{n-k}^1)$ will look the same but with one less hydrophone in one of the rings and with a single hydrophone at the centre of both rings.

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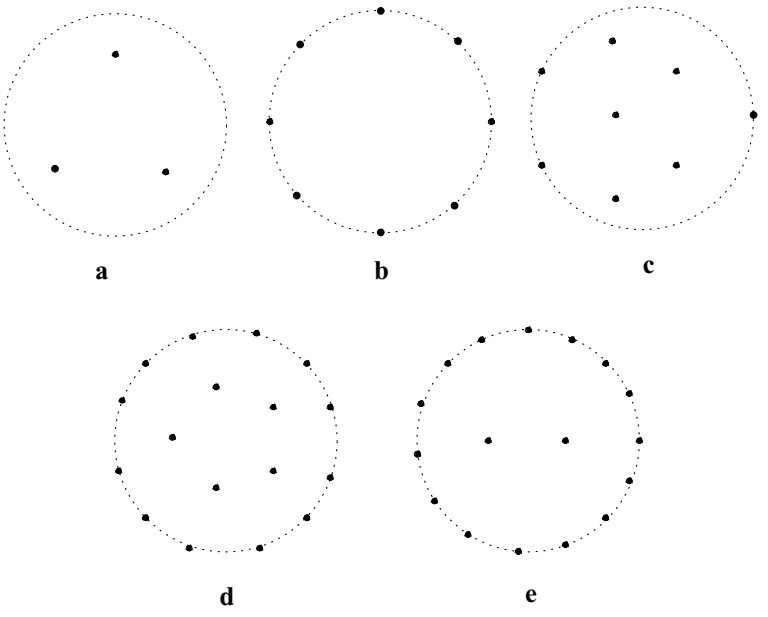


FIGURE 3. Hayward's results from simulated annealing [3].

5 Classifying Hayward's results and numerical results

The results obtained by simulated annealing in Hayward [3] are reproduced in Figure 3. We now classify these according to their isotropy subgroups when considered as solutions of an $\mathbf{O}(2) \times \mathbf{S}_n$ equivariant function (where we drop the '1' superscripts).

The solutions in Figures 3(a) and 3(b) corresponding to systems of 3 and 8 hydrophones, respectively, correspond to primary bifurcations with isotropy $\widetilde{\mathbf{D}}_3$ and $\widetilde{\mathbf{D}}_8$.

The solution in Figure 3(d), for 17 hydrophones, corresponds to a primary bifurcation with isotropy $(\widetilde{\mathbf{D}}_{12}, \widetilde{\mathbf{D}}_5)$, and the solution in Figure 3(e), again for 17 hydrophones but for a differing frequency range, corresponds to a 'mixed-mode' isotropy of $(\widetilde{\mathbf{D}}_{15}, \widetilde{\mathbf{D}}_2)$.

Finally, the solution in Figure 3(c) again corresponds to a secondary bifurcation for 8 hydrophones, where the one isotropy is $\widetilde{\mathbf{D}}_6 \subset \mathbf{O}(2) \times \mathbf{S}_7$ and the other is $\mathbf{Z}_2 \subset \mathbf{O}(2)$.

The preceding results would suggest that the solutions often found are of the form found by the above bifurcations, and so to find possible global minima numerically we will not use path following techniques from the point of bifurcation, $\lambda = 0$ to the point where the performance measure is attained, namely $\lambda = k_1^2 + k_1 k_2 + k_2^2$. Rather, we assume that the solutions persist, and compute the values of v found for each such solution. The assumption is that if we compute the value of a branch which either no longer exists for that value of λ, or is not stable, then we should find another branch which yields a lower value for v.

In this way, we at least find *one* set of co-ordinates which gives a good solution to our optimisation problem, and if it turns out not to be a global minimum, it will surely give us a good set of initial conditions on which to base a numerical search (if the solution is not even a minima it will at least be a saddle point with few unstable directions). In

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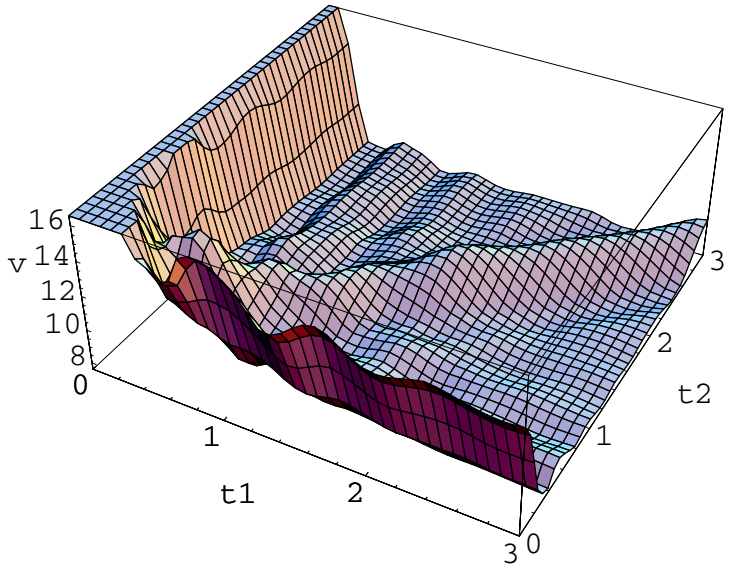


FIGURE 4. $(n,k) = (7,3)$, 3D plot.

this way, we also reduce the problem of optimisation over a $2n$-dimensional space to an optimisation problem over only one- or two-dimensional spaces.

For illustration, we investigate the placement of seven hydrophones, a frequency range of between 900 and $1100Hz$, and a maximum aperture size of $3m$. Using the value of the speed of sound under water as $c = 1500$ this gives us values of $k_1 = 2*\pi*900/1500 = 6\pi/5$ and $k_2 = 2 * \pi * 1100/1500 = 22\pi/15$.

For the numerical and graphical aspects of this problem, routines written in Mathematica were used, which included, for the sake of speed, construction of a look-up table for values of components of the performance measure. The authors therefore accept no responsibility for perfect accuracy to the original problem, but hope that the following will still adequately express the ease of the proposed method of minimisation.

From the above analysis we therefore reduce the problem to looking for minima in the reduced problems of a ring of seven hydrophones (one dimensional problem), two rings of size 1 and 6, 2 and 5, 3 and 4 and then two rings surrounding a single hydrophone at the centre of size 1 and 5, 2 and 4 (each a two-dimensional problem) and finally two rings of three hydrophones surrounding a single hydrophone at their centres, which is a three dimensional problem (radius of each ring and the orientation of the rings with respect to each other). That is, we need to study seven separate minimisation problems on one, two or three dimensions, rather than the original minimisation problem in 14 dimensions.

For example take the case of a ring of four hydrophones and a ring of three (corresponding to the 'mixed mode' solution with isotropy $(\widetilde{\mathbf{D}}_4, \widetilde{\mathbf{D}}_3)$. The various values of the performance measure as the radii of both rings is varied between zero and the maximum aperture allowed (three) is shown as a 3D plot in Figure 4.

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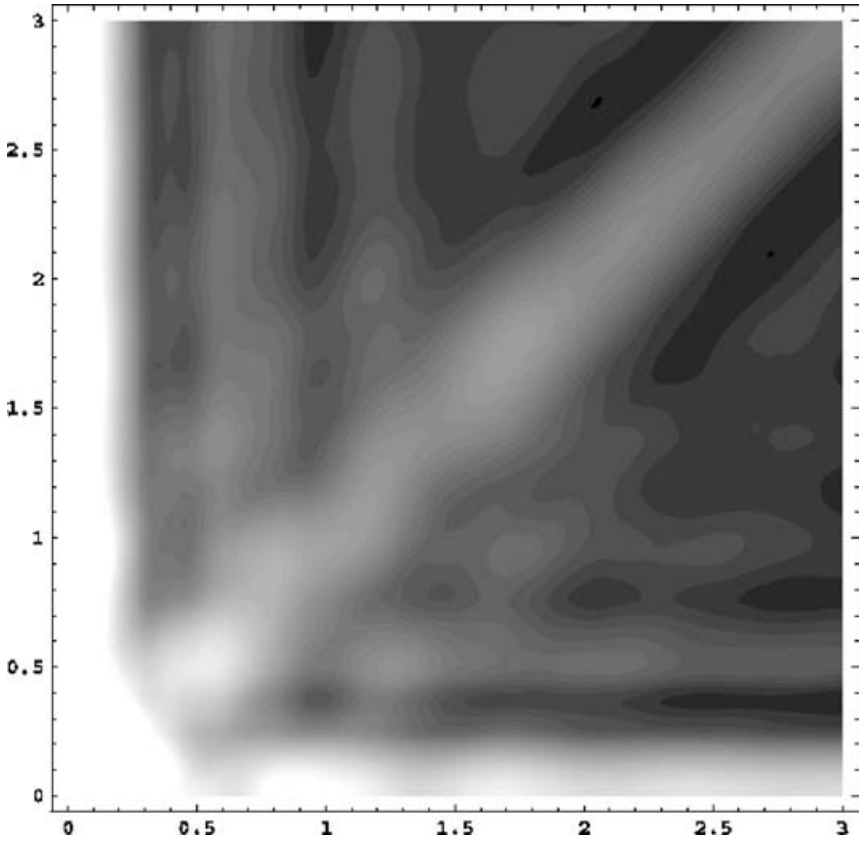


FIGURE 5. $(n, k) = (7, 3)$ contour plot.

To make it easier to see the minima and maxima this plot can be redrawn as a contour plot, see Figure 5. In particular, note that the diagonal corresponds to both rings of hydrophones being the same radius, and that in general we would not expect such a plot to be symmetrical about this line.

In this figure there are clearly two possible choices for a global minimum (the two black spots in the upper right hand quadrant of the plot). Computation of these minima using a local method of steepest descent gives values of 7.8186 for the ring of four hydrophones having radius $2.054m$ and the ring of three hydrophones having radius $2.693m$. The other minima highlighted has a value of 7.8189. Thus for this configuration we know the global minimum, which is plotted in Figure 6.

With the bare minimum of a commercially available package and a generated look-up table these calculations took a matter of minutes, which should translate favourably to a more sophisticated numerical set-up, and computations for the remaining symmetrically derived configurations is also a computationally painless task. Further justification for this method was obtained by the invariance of the minimum value to a rotation of a single ring in a two ring network by an arbitrary angle. In the group theory, this is equivalent to a conjugate solution, and so is as we would expect.

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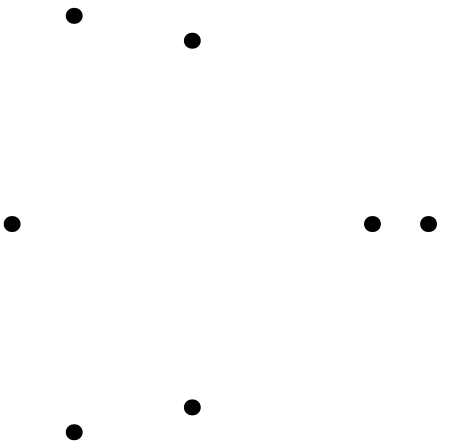


FIGURE 6. Best position for hydrophones when we have two rings, one of 4 hydrophones, one of 3.

Table 1. *Best results for each symmetric configuration of seven hydrophones*

Ring 1	Ring 2	Origin	Radius 1	Radius 2	Min. value
7	0	no	2.445	n/a	7.729
6	1	no	3	2.36	7.738
5	2	no	1.116	3	7.812
4	3	no	2.054	2.693	7.8186
6	0	yes	2.821	n/a	7.7573
5	1	yes	2.8	2.149	7.7758
4	2	yes	2.8	2.149	7.73085
3	3	yes	2.756	2.107	7.698

The best results for all eight networks we must consider are shown in Table 1 where the first two columns are the number of hydrophones in each ring, the third is whether or not there is a single hydrophone situated at the centre of the ring(s), the fourth and fifth columns the radii of the rings and the final column the value of the performance measure at these minima.

Note that the final entry of Table 1 of two rings of three hydrophones centred about a single hydrophone in the middle corresponds to both rings sharing the same orientation. The array corresponding to this configuration is shown in Figure 7.

6 Conclusions

By using knowledge of the solutions expected in systems with strong symmetrical properties (in the sense of equivariance under a group action) we have reduced a minimisation problem in $2n$ dimensions (where n is typically greater than ten) to a series of minimisation

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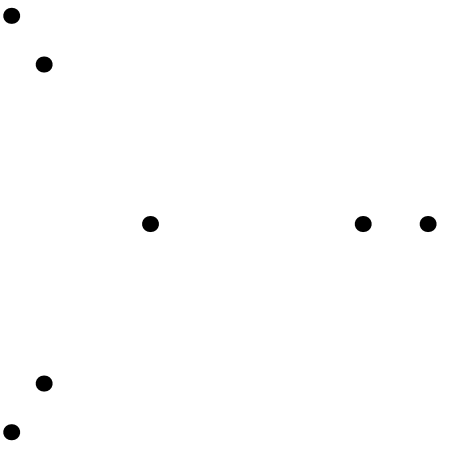


FIGURE 7. Best configuration for 7 hydrophones.

problems in one, two and three dimensions. Such an approach is obviously computationally cheaper, and will locate all the minima found by Hayward [3], and at the very least provide a way to classify such solutions.

That is not to say that this method does not have its limitations. We do not claim that the solutions predicted by a symmetry approach are the only local/global minima which exist, only that they are the most likely, and from the work carried both here and previously, we feel justified in this assumption.

Acknowledgements

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References

[1] DIONNE, B., GOLUBITSKY, M. & STEWART, I. (1996) Coupled cells with internal symmetry. Part II: dDirect products. *Nonlinearity*, **9**, 575–600.

[2] GOLUBITSKY, M., STEWART, I. N. & SCHAEFFER, D. G. (1988) *Singularities and Groups in Bifurcation Theory vol. II*. Applied Mathematical Sciences **69**, Springer-Verlag.

[3] HAYWARD. T. J. (1994) Optimisation of hydrophone placement for acoustic arrays using simulated annealing. *J. Acoust. Soc. Am.* **95**, 201–212.

[4] PRESS, W. H., FLANNERY, B. P., TEUKOLSKY, S. A. & VETTERLING, W. T. (1989) *Numerical Recipes*. Cambridge University Press.

[5] STEWART, I. (1996) Symmetry methods in collisionless many-body problems. *J. Nonliniear Sci.* **6**, 543–563.

[6] WOOD, D. (2001) A cautionary tale of coupling cells with internal symmetries. *Int. J. Bifurcation & Chaos*, **11**, 121–132.

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