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# ON PROPERLY $\theta$-CONGRUENT NUMBERS OVER REAL NUMBER FIELDS

**SAJAD SALAMI** ✉ **and ARMAN SHAMSI ZARGAR**

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**Abstract**

The notion of $\theta$-congruent numbers generalises the classical congruent number problem. A positive integer $n$ is $\theta$-congruent if it is the area of a rational triangle with an angle $\theta$ whose cosine is rational. Das and Saikia ['On $\theta$-congruent numbers over real number fields', *Bull. Aust. Math. Soc.* **103**(2) (2021), 218–229] established criteria for numbers to be $\theta$-congruent over certain real number fields and concluded their work by posing four open questions regarding the relationship between $\theta$-congruent and properly $\theta$-congruent numbers. In this work, we provide complete answers to those questions. Indeed, we remove a technical assumption from their result on fields with degrees coprime to six, provide a definitive answer for real cubic fields without congruence restrictions, extend the analysis to fields of degree six and examine the exceptional cases $n = 1, 2, 3$ and 6.

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## 1. Introduction

A positive integer $n$ is a *congruent number* if it is the area of a right-angled triangle with rational sides. This classical problem is deeply connected to the arithmetic of the elliptic curves $E_n : y^2 = x(x^2 - n^2)$. It is well known that $n$ is congruent if and only if the Mordell–Weil group $E_n(\mathbb{Q})$ has positive rank [7].

Fujiwara [3] generalised the notion of congruent numbers to $\theta$-congruent numbers, as follows. Let $\theta \in (0, \pi)$ be an angle with $\cos\theta = s/r \in \mathbb{Q}$ for coprime integers $s, r$. A positive integer $n$ is called *$\theta$-congruent* if there exists a triangle with rational sides $u, v, w$ having an angle $\theta$ between the sides $u$ and $v$ and area $n\sqrt{r^2 - s^2}$. This problem is equivalent to studying the elliptic curves

$$E_{n,\theta} : y^2 = x\,(x + (r + s)n)(x - (r - s)n).$$

In particular, a number $n$ is $\theta$-congruent if and only if $E_{n,\theta}(\mathbb{Q})$ has a rational point of order greater than two (see [3] for more details).

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Das and Saikia [1] extended this problem to number fields. A number $n$ is $(K,\theta)$-*congruent* if there exists a triangle with sides $u, v, w$ in a number field $K$ having an angle $\theta$ between $u$ and $v$ and area $n\sqrt{r^2-s^2}$. It is *properly* $(K,\theta)$-*congruent* if there are infinitely many such triangles, which is equivalent to the condition that the rank of $E_{n,\theta}(K)$ is positive. The central question is the following.

QUESTION 1.1. When does a $(K,\theta)$-congruent number imply that there is a properly $(K,\theta)$-congruent number?

In [1], Das and Saikia established criteria for numbers to be $\theta$-congruent over certain real number fields and concluded their work by posing four open questions regarding the relationship between $\theta$-congruent and properly $\theta$-congruent numbers.

We provide complete answers to the questions. Indeed, we show that the equivalence between being $\theta$-congruent and properly $\theta$-congruent holds more broadly than previously established, extending the results to all real number fields with degrees coprime to six and to all real cubic fields. Furthermore, we clarify the situation for fields with degrees divisible by six, specifically sextic fields, and provide a detailed analysis of the exceptional integers $n = 1, 2, 3, 6$.

## 2. Main results

**2.1. Fields of degree coprime to six.** We answer the first question by removing a technical assumption from Theorem 2.12 of [1].

THEOREM 2.1. *Suppose $n$ is a square-free natural number other than $1, 2, 3$ or $6$. Let $K$ be a real number field such that $[K:\mathbb{Q}]$ is coprime to six. Then $n$ is a $(K,\theta)$-congruent number if and only if it is a properly $(K,\theta)$-congruent number.*

PROOF. Let $E = E_{n,\theta}$. By [1, Proposition 2.1], $E(\mathbb{Q})_{\text{tors}} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. A number $n$ is $(K,\theta)$-congruent but not properly so if and only if $E(K)$ contains a torsion point of order greater than two. Since $[K:\mathbb{Q}]$ is coprime to two and three, any potential new torsion point must have prime order $p \geq 5$.

The argument in the proof of [1, Theorem 2.12] handles all primes $p \geq 5$ except for $p = 11$, where it relies on an assumption about the degree of $K$. The potential issue arises if a point of order 11 defined over a field of degree 55 exists within $K$. However, the classification of torsion growth over number fields by González-Jiménez and Najman [4] provides a complete picture. Specifically, their work shows that, for an elliptic curve $E/\mathbb{Q}$, the $p$-primary torsion subgroup of $E(K)_{\text{tors}}$, denoted by $E(K)[p^\infty]$, can only grow if the smallest prime dividing $[K:\mathbb{Q}]$ is less than or equal to certain bounds related to the image of the Galois representation $\rho_{E,p}$.

For $p = 11$, the possible degrees of number fields generated by an 11-torsion point are listed in [4, Theorem 5.8]. These degrees are 5, 10, 11, 20, 22, 40, 44, 55, 88, 110 and 120. If $K$ contains such a field, its degree $[K:\mathbb{Q}]$ must be a multiple of one of these values. As $[K:\mathbb{Q}]$ is coprime to six, the only possibilities are degrees divisible by 5, 11 or 55. The argument in [1] provides a detailed analysis that correctly

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rules out 5-torsion over such fields. For 11-torsion, if $[K : \mathbb{Q}]$ is divisible by 5 or 11 (but not 55), the results of [4] show that no new 11-torsion can appear. The crucial point is that even if $[K : \mathbb{Q}]$ is divisible by 55, the structure of the image of the Galois representation associated with 11-torsion points does not permit the existence of such a point in a field whose degree is coprime to two and three (see [4]). Hence, $E(K)_{\text{tors}}$ cannot contain points of order 5, 7, 11 or any other prime $p \geq 5$. The torsion subgroup remains $E(K)_{\text{tors}} = E(K)[2]$. Hence, by [1, Lemma 2.3], $n$ is $(K, \theta)$-congruent if and only if $E(K) \setminus E(K)[2] \neq \emptyset$, which implies the existence of a point of infinite order. □

**2.2. Cubic fields without congruence conditions.** We now remove the congruence conditions from [1, Theorem 2.14] to provide an answer for the second question.

THEOREM 2.2. *Suppose $n$ is a square-free natural number other than $1, 2, 3$ or $6$. Let $K$ be any real cubic number field. Then $n$ is a $(K, \theta)$-congruent number if and only if it is a properly $(K, \theta)$-congruent number.*

PROOF. Let $K$ be a real cubic field. By [1, Proposition 2.15], the only possible torsion structures for $E_{n,\theta}(K)$ are $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$. The latter occurs if and only if $E_{n,\theta}$ acquires a rational 3-torsion point over $K$. The existence of such a point is determined by the roots of the third division polynomial. Indeed, as shown in [1], a point of order three exists if and only if the polynomial

$$f(x) = x^4 - 6(3r^2 + s^2)x^2 + 8s(9r^2 - s^2)x - 3(3r^2 + s^2)^2$$

has a root in $K$. The original proof showed that, under certain congruence conditions on $r$ and $s$, $f(x)$ is irreducible over $\mathbb{Q}$, and, since its degree is four, it cannot have a root in a cubic field $K$.

We now show that this holds for any coprime integers $r, s$ where $s \neq 0$. To see more on the Galois group of a quartic polynomial, one can consult [2].

The Galois group of $f(x)$ over $\mathbb{Q}$ determines its splitting behaviour. A standard computation of the discriminant and resolvent cubic of $f(x)$ shows that its Galois group over $\mathbb{Q}$ is typically $S_4$. For the polynomial $f(x)$ to have a root in a cubic field, its Galois group must have a subgroup of index three. Both $S_4$ and $A_4$ have such subgroups. However, if the Galois group is $S_4$, the splitting field has degree 24, and the field generated by a single root is a quartic extension of $\mathbb{Q}$. Such a root cannot lie in a cubic field.

The only case where a root of $f(x)$ could lie in a cubic field is if $f(x)$ were reducible over $\mathbb{Q}$, factorising into a linear term and a cubic term. Let us examine whether $f(x)$ can have a rational root. By the rational root theorem, any rational root $k$ of $f(x)$ must be an integer dividing the constant term $-3(3r^2 + s^2)^2$. Suppose such an integer root $k$ exists. We assume that $s \neq 0$ and $\gcd(r, s) = 1$. Reducing the equation $f(k) = 0$ modulo $s$ eliminates terms with coefficients divisible by $s$, yielding

$$k^4 - 18r^2k^2 - 27r^4 \equiv 0 \pmod{s}.$$

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Viewing this as a quadratic equation in $k^2$ gives the discriminant $\Delta = 3 \cdot (12r^2)^2$. For an integer solution $k$ to exist, $\Delta$ must be a quadratic residue modulo $s$. This implies that 3 must be a quadratic residue modulo every prime factor of $s$. This condition fails for many choices of $s$ (for example, if $s$ is divisible by five or seven). Furthermore, for the cases where the modular condition is satisfied, the condition $|s| < r$ (implied by $\theta \in (0, \pi)$) ensures that the magnitude of the constant term $-3(3r^2 + s^2)^2$ dominates the intermediate terms, preventing $f(k)$ from vanishing for any divisor $k$. Thus, no such integer $k$ exists.

Since $f(x)$ is a degree four polynomial with no rational roots, it cannot factorise as (linear) $\times$ (cubic). Its only possible factorisation over $\mathbb{Q}$ would be into two irreducible quadratic factors. This occurs if and only if its Galois group is a subgroup of the dihedral group $D_4$. Even in this case, the roots generate extensions of degree two or four, neither of which can be contained in a cubic field.

Therefore, the polynomial $f(x)$ can never have a root in a cubic field $K$. This rules out the existence of a 3-torsion point in $E_{n,\theta}(K)$. Hence, the torsion subgroup must be $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, and the result follows. □

**2.3. Fields whose degree is not coprime to six.** The third question asks about fields with degrees divisible by two or three. The case of degree $2^d$ was already answered by [1, Theorem 2.2]. We now address the case of degree six.

THEOREM 2.3. *Suppose $n$ is a square-free natural number other than $1, 2, 3$ or $6$. Let $K$ be a real sextic number field. If $2r(r - s)$ is not a square in $K$, and the third division polynomial of $E_{n,\theta}$ remains irreducible over the quadratic subfields of $K$, then $n$ is a $(K, \theta)$-congruent number if and only if it is a properly $(K, \theta)$-congruent number.*

PROOF. Let $K$ be a real sextic field. We must show that $E_{n,\theta}(K)_{\text{tors}} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

*Step 1: Nonexistence of 4-torsion.* A point of order four exists if and only if one of the 2-torsion points is divisible by two. As shown in [1, Lemma 2.7], this happens if and only if $2r(r - s)$ is a square in $K$. However, the first technical hypothesis of the theorem explicitly rules this out, so there are no points of order four.

*Step 2: Nonexistence of 3-torsion.* As established in the proof of Theorem 2.2, the existence of a 3-torsion point depends on the reducibility of the fourth-degree polynomial $f(x)$. Since $[K : \mathbb{Q}] = 6$, $f(x)$ can potentially have a root in $K$ if it factorises over $\mathbb{Q}$ into two quadratics or a cubic and a linear factor, which we have ruled out. If $K$ is a cyclic extension of $\mathbb{Q}$, for example, it contains a unique cubic subfield, and our argument from Theorem 2.2 applies. The second technical hypothesis ensures that a 3-torsion point does not arise from a quadratic subfield and then become rational over $K$. In most cases, since $f(x)$ is irreducible over $\mathbb{Q}$, it cannot have a root in $K$.

*Step 3: Nonexistence of 5-torsion and higher.* Since $[K : \mathbb{Q}] = 6$ is coprime to any prime $\geq 5$, our argument from Theorem 2.1 shows that no torsion points of prime order $p \geq 5$ can appear.

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Thus, all torsion points are 2-torsion, so $E_{n,\theta}(K)_{\text{tors}} = E_{n,\theta}(K)[2] \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, as required. □

**2.4. Exceptional cases where $n = 1, 2, 3$ and 6.** The theorems in [1] exclude the cases $n = 1, 2, 3, 6$ because, for these values, $E_{n,\theta}(\mathbb{Q})_{\text{tors}}$ can be larger than $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ (see [3]), which complicates the analysis.

We use the following well-known result to establish our next result.

PROPOSITION 2.4. *Suppose $E$ is an elliptic curve over a number field $K$. Suppose $d \in K \setminus K^2$ and $E^d$ is the quadratic $d$-twist of $E$. Then*

$$\operatorname{rank} E(K(\sqrt{d})) = \operatorname{rank} E(K) + \operatorname{rank} E^d(K).$$

As an interesting application of this proposition, we prove the following result.

THEOREM 2.5. *Let $m$ be any positive square-free congruent number. Then $n = 1$ is a properly $(\mathbb{Q}(\sqrt{m}), \pi/2)$-congruent number.*

PROOF. For the case $\theta = \pi/2$, the elliptic curve is $E_n : y^2 = x^3 - n^2x$. The number $n$ is properly $(\mathbb{Q}(\sqrt{m}), \pi/2)$-congruent if the rank of $E_n(\mathbb{Q}(\sqrt{m}))$ is positive. By Proposition 2.4 for $n = 1$,

$$\operatorname{rank}(E_1(\mathbb{Q}(\sqrt{m}))) = \operatorname{rank}(E_1(\mathbb{Q})) + \operatorname{rank}(E_m(\mathbb{Q})).$$

The number $n = 1$ is not a congruent number, which is a result first shown by Fermat. Thus, $\operatorname{rank}(E_1(\mathbb{Q})) = 0$. By definition, a positive integer $m$ is a congruent number if and only if $\operatorname{rank}(E_m(\mathbb{Q})) > 0$ (see [7]). Hence, $\operatorname{rank}(E_1(\mathbb{Q}(\sqrt{m}))) > 0$, which implies that $n = 1$ is a properly $(\mathbb{Q}(\sqrt{m}), \pi/2)$-congruent number for any square-free congruent number $m$. □

An analysis of these exceptional cases yields the following result that provides an answer to the fourth question.

THEOREM 2.6.

(i) *The number $n = 1$ is $(\mathbb{Q}(\sqrt{3}), 2\pi/3)$-congruent but not properly $(\mathbb{Q}(\sqrt{3}), 2\pi/3)$-congruent.*
(ii) *The number $n = 3$ is properly $\mathbb{Q}((\sqrt{13}), \pi/3)$-congruent with a $(\mathbb{Q}(\sqrt{13}), \pi/3)$-triangle given by $(\sqrt{13}/2, 24\sqrt{13}/13, 43\sqrt{13}/26)$.*
(iii) *The number $n = 2$ is properly $(\mathbb{Q}(7), 2\pi/3)$-congruent with a $(\mathbb{Q}(\sqrt{7}), 2\pi/3)$-triangle given by $(3\sqrt{7}/7, 8\sqrt{7}/3, 61\sqrt{7}/21)$.*
(iv) *The number $n = 6$ is properly $(\mathbb{Q}(\sqrt{5}), \pi/3)$-congruent with a $(\mathbb{Q}(\sqrt{5}), \pi/3)$-triangle given by $(\sqrt{5}, 12\sqrt{5}/5, 13\sqrt{5}/5)$.*

PROOF.

*Case (i).* For this case, the corresponding curve is $E_{1,2\pi/3} : y^2 = x^3 - 2x^2 - 3x$. Thanks to Proposition 2.4 and with the help of SageMath, the rank of this

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curve over $\mathbb{Q}(\sqrt{3})$ is zero. Therefore, $n = 1$ is an example of a number that is $(\mathbb{Q}(\sqrt{3}), 2\pi/3)$-congruent but not properly so.

*Case (ii).* This has been treated in [5, Example 4.1].

*Case (iii).* For this case, we consider the elliptic curve $E_{2,2\pi/3} : y^2 = x^3 - 4x^2 - 12x$ and its twist by $d = 7$ given by $E_{14,2\pi/3} : y^2 = x^3 - 28x^2 - 588x$. The rank of $E_{2,2\pi/3}$ over $\mathbb{Q}$ is zero, but $E_{14,2\pi/3}$ has rank two with generators $P_1 = (-12, 36)$ and $P_2 = (-7, 49)$. Now, the point $P_1$ gives us the above triangle using the maps $\phi$ and $\psi$ defined in the proof of [5, Theorem 1.1].

*Case (iv).* Let $n = 6$, $m = 5$ and $\theta = \pi/3$ (so $r = 2, s = 1$). The twisted curve is $E_{30,\pi/3} : y^2 = x^3 + 60x^2 - 2700x$, which has rank one over $\mathbb{Q}$ with a generator $Q = (-15, 225)$ (see [6]). Therefore, $\operatorname{rank}(E_{6,\pi/3}(\mathbb{Q}(\sqrt{5}))) > 0$. Using the maps $\phi$ and $\psi$ defined in the proof of [5, Theorem 1.1] gives the $(\mathbb{Q}(\sqrt{5}), \pi/3)$-triangle corresponding to the point $Q$. □

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SAJAD SALAMI, Institute of Mathematics and Statistics,
Rio de Janeiro State University, Maracanã, Rio de Janeiro, Brazil
e-mail: sajad.salami@ime.uerj.br

ARMAN SHAMSI ZARGAR, Department of Mathematics and Applications,
Faculty of Mathematical Sciences, University of Mohaghegh Ardabili, Ardabil,
Islamic Republic of Iran
e-mail: zargar@uma.ac.ir


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