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BANKRUPTCY PROBABILITIES UNDER NON-POISSON INSPECTION

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Abstract

This paper concerns an insurance firm's surplus process observed at renewal inspection times, with a focus on assessing the probability of the surplus level dropping below zero. For various types of inter-inspection time distributions, an explicit expression for the corresponding transform is given. In addition, Cramér–Lundberg-type asymptotics are established. Also, an importance sampling-based Monte Carlo algorithm is proposed, and is shown to be logarithmically efficient.

Keywords: Cramér–Lundberg model; periodic inspection; large deviations; ruin probability; importance sampling

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1. Introduction

Ruin theory constitutes a prominent subdomain of actuarial science. A key model, usually referred to as the *Cramér–Lundberg model*, describes the surplus process of an insurance company that receives income (at a constant rate) via premiums, and that pays out claims to its clients according to a compound Poisson process. The key quantity studied is the *ruin probability*, being the probability that the surplus level drops below 0 ('ruin'), either before a given point in time or over an infinite horizon. Ruin theory is a mature and active branch within applied probability: for a broad variety of types of surplus processes, covering the above-mentioned Cramér–Lundberg model, one has succeeded to evaluate (or approximate) the ruin probability. These findings are of great practical interest, as they facilitate the selection of an initial surplus level that renders the ruin probability below a given threshold.

Mathematically, ruin is usually defined in terms of a certain continuous-time stochastic process reaching a given set (e.g. the surplus level dropping below 0). In many practical situations however, the stochastic process under considerations is not, or can not, be continuously observed, but is rather inspected at discrete epochs. This motivates the analysis of setups in which the firm goes bankrupt when the surplus level drops below zero *at an inspection time*.

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The most basic setup corresponds to Poisson inspections, i.e. there are exponentially distributed 'inter-inspection times' at which it is verified whether the surplus level has become negative. This case, with the surplus level evolving according to a Cramér–Lundberg model or more general processes, has been dealt with in great detail before [6, 12, 14]. Let $p(u,t)$ denote the probability of ruin at an inspection epoch before time t, which in the following we refer to as the *bankruptcy probability*. Given that the initial surplus level is $u > 0$, a main finding concerns an explicit expression for the (double) transform

$$\pi(\alpha, \beta) := \int_0^\infty \int_0^\infty \beta \mathrm{e}^{-\alpha u - \beta t} p(u, t)\, \mathrm{d}u\, \mathrm{d}t,$$

which can be interpreted as bankruptcy before an horizon that is distributed as an exponentially distributed quantity with rate β (in the following denoted by T_β), sampled independently of the surplus process. The time-dependent bankruptcy probability $p(u,t)$ can be numerically approximated by applying well-established numerical inversion techniques, such as the fast and accurate procedures presented in [1, 22]; here [22] also covers the inversion of *double* transforms, with techniques that have been extensively validated in [7]. In addition, the asymptotics of the all-time bankruptcy probability $p(u) := p(u, \infty)$ (as the initial surplus u grows large) have been identified; see e.g. the results in [14] for the cases of both light-tailed and heavy-tailed claim sizes.

Note that, from a practical standpoint, a setup with Poisson inspections is not particularly realistic. The ambition of the present paper is therefore to extend the results for exponentially distributed inter-inspection times to a considerably richer class. Acknowledging that the family of *phase-type distributions* offers a flexible and conceptually appealing way to approximate any distribution on the positive half-line [8, Theorem III.4.2], our focus lies on inter-inspection times stemming from convenient and practically relevant members of this family. The net cumulative claim process $Y(\cdot)$, equalling the initial surplus u decreased by the surplus process $X_u(\cdot)$, is assumed to be described by a spectrally positive Lévy process such that $Y(0) = 0$.

Our objective is to evaluate, for such inter-inspection times, two quantities: (i) the transform $\pi(\alpha, \beta)$ of the bankruptcy probability, uniquely characterizing the time-dependent bankruptcy probability for any initial surplus u; and (ii) the Cramér–Lundberg asymptotics of the all-time bankruptcy probability $p(u)$, characterizing its behavior in the regime that the initial surplus u grows large. The types of phase-type distributions considered are mixtures of exponential and Erlang distributions, through which any distribution can (in principle) be approximated arbitrarily accurately.

Although already covered by e.g. [12, 14, 35], we start our analysis by the case of Poisson inspections, to showcase the approach followed, thus increasing the transparency of the derivations for the other, more involved, cases. Most notably, when analyzing the transform $\pi(\alpha, \beta)$ we systematically follow a uniform approach: we first derive an expression in terms of a set of unknown constants and then develop a procedure to determine these unknown constants. When studying the Cramér–Lundberg asymptotics of $p(u)$, we perform a change of measure, after which determining of the constants involved boils down to analyzing the distribution of the overshoot over level u.

We also present an importance sampling-based Monte Carlo approach to estimate $p(u)$, which can handle, in principle, any inter-inspection time distribution and which we show to be logarithmically efficient. The experiments show that replacing the inter-inspection times by a two-moments-fit based, phase-type counterpart yields almost exact results.

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We proceed by providing a brief, nonexhaustive overview of related literature. The evaluation of ruin probabilities in the conventional, permanently inspected, Cramér–Lundberg context is discussed in great detail in various textbooks, such as [9, 31]. The use of our bankruptcy concept was advocated in [3], acknowledging the fact that the insurance firm may continue doing business for a while, even when it is technically ruined. In [5] the bankruptcy probability has been evaluated for the Cramér–Lundberg setting with Poisson inspections and exponentially distributed claim sizes, while the later papers [12, 14, 35] also cover generally distributed claim sizes. In both [5, 12], the inspection rate was allowed to depend on the current surplus level. It is noted that in [12], and in [6] as well, a related queueing (or inventory) model was analyzed as well, with the special feature that the server works even when there are no customers (or orders), building up storage that is removed at the Poisson inspection epochs. The analysis presented in [14] features a clean decomposition, which provides an explicit relation between the model under Poisson inspection and its counterpart under permanent inspection, which has been further generalized in [13]. The case of Erlang inter-inspection times and a compound Poisson net cumulative claim process (i.e. a subclass of our broader class of spectrally positive Lévy processes) is treated in [2], also covering results for the Gerber–Shiu metrics (of which the bankruptcy probability is a special case), while [34] provides approximations for these objects in the case of deterministic inter-inspection times.

Our paper has been organized as follows. Section 2 formally describes our model, introduces the objects of study, and recalls a number of useful existing results (such as the Wiener–Hopf decomposition for spectrally positive Lévy processes). For exponentially and hyperexponentially distributed inter-inspection times, the transform $\pi(\alpha, \beta)$ is computed in Section 3, whereas Section 4 focuses on Erlang and hyper-Erlang inter-inspection times. In Section 5 it is pointed out what complications arise when one would try to compute $\pi(\alpha, \beta)$ for more general inter-inspection time distributions. Section 6 identifies the Cramér–Lundberg-type asymptotics of $p(u)$ for the types of inter-inspection times considered. Then, in Section 7 a provably optimal Monte Carlo method for estimating $p(u)$ is developed and numerical experiments are performed. Section 8 provides a discussion and concluding remarks.

2. Model, objective, and preliminaries

We denote by $Y(\cdot)$ a *spectrally positive Lévy process* with Laplace exponent

$$\varphi(\alpha) := \log \mathrm{E}\, \mathrm{e}^{-\alpha Y(1)}$$

and $\psi(\cdot)$ its right inverse. To rule out trivial cases, it is assumed that $Y(\cdot)$ is not a subordinator (i.e. a monotonically increasing or decreasing process). Throughout this paper, we denote by $\bar{Y}(\cdot)$ and $\underline{Y}(\cdot)$ the running maximum and running minimum process, respectively, associated with $Y(\cdot)$. In the context of ruin theory, the process $Y(\cdot)$ represents the *net cumulative claim process*, which is equivalent to the initial surplus $u > 0$ decreased by the surplus process $X_u(\cdot)$.

Unlike in the conventional ruin-theoretic setting, in this paper we do not observe the process $Y(\cdot)$ permanently, but rather only according to a renewal inspection process. The inter-inspection times are denoted by the sequence $(\Omega_n)_{n\in\mathbb{N}}$ of independent and identically distributed (i.i.d.) random variables, with the Ω_n distributed as the generic nonnegative random variable Ω, and the corresponding inspection times by

$$\bar{\Omega}_n := \sum_{m=1}^{n} \Omega_m.$$

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The sequence $(\Omega_n)_{n\in\mathbb{N}}$ is assumed to be sampled independently from the net cumulative claim process $Y(\,\cdot\,)$.
The goal of this paper is to analyze the (finite and infinite-time) bankruptcy probability at inspection epochs, assuming that the inter-inspection times are distributed as the random variable Ω. The central object of study is

$$p(u, t \mid \Omega) := \mathrm{P}(\text{there exists } n \in \mathbb{N}{:}\bar{\Omega}_n \leqslant t,\, Y(\bar{\Omega}_n) > u),$$

i.e. the probability of bankruptcy before time horizon t only considering the value of $Y(\,\cdot\,)$ at inspection epochs; for clarity, we have included Ω in the notation, to stress the dependence on the inter-inspection times. In this paper we consider the case of an exponentially distributed time horizon, from which we can obtain its deterministic horizon counterpart by Laplace inversion. We write T_β to denote an exponentially distributed random variable with parameter β (i.e. with mean $1/\beta$). By sending β to 0 we obtain the infinite-time bankruptcy probability [31, Section 1.2]. The random variable T_β, sampled independently from the net cumulative claim process $\bar{Y}(\,\cdot\,)$ and the inter-inspection times $(\Omega_n)_{n\in\mathbb{N}}$, is often referred to as the *killing time*. In the following we use the notation

$$\mathfrak{p}_\beta(u \mid \Omega) := \mathrm{P}\left(\text{there exists } n \in \mathbb{N}{:}\bar{\Omega}_n \leqslant T_\beta,\, Y(\bar{\Omega}_n) > u\right).$$

We intensively use the *Wiener–Hopf decomposition* for spectrally positive Lévy processes [28, Section 6.5.2]. This states that (i) $\bar{Y}(T_\beta)$ and $\bar{Y}(T_\beta) - Y(T_\beta)$ are independent; (ii) that for any $\alpha \geqslant 0$ and $\beta > 0$,

$$\xi(\alpha, \beta) := \mathrm{E}\,\mathrm{e}^{-\alpha\bar{Y}(T_\beta)} = \frac{\alpha - \psi(\beta)}{\varphi(\alpha) - \beta}\frac{\beta}{\psi(\beta)}; \tag{1}$$

and (iii) that $\bar{Y}(T_\beta) - Y(T_\beta)$ is distributed as $-\underline{Y}(T_\beta)$ and that it is exponentially distributed with parameter $\psi(\beta)$. If the net cumulative claim process has a negative drift (i.e. $\mathrm{E}\,Y(1) < 0$, in the ruin literature referred to as the *net profit condition*), then we can send β to 0, so as to obtain

$$\xi(\alpha) := \mathrm{E}\,\mathrm{e}^{-\alpha\bar{Y}(\infty)} = \frac{\alpha\,\varphi'(0)}{\varphi(\alpha)}.$$

In this paper we have two target objects. In the first place we wish to analyze $\mathfrak{p}_\beta(u \mid \Omega)$ through its transform with respect to the exceedance level u. Concretely, we point out, for various types of inter-inspection times Ω, how one can compute

$$\pi(\alpha, \beta \mid \Omega) := \int_0^\infty \mathrm{e}^{-\alpha u}\,\mathfrak{p}_\beta(u \mid \Omega)\,\mathrm{d}u.$$

In the second place, we identify the Cramér–Lundberg asymptotics (in the regime that u is large, that is) of the all-time bankruptcy probability $\mathfrak{p}(u \mid \Omega) := \mathfrak{p}_0(u \mid \Omega)$, for which we also devise an efficient importance sampling-based simulation approach.

Remark 1. The results in this paper concern exponentially distributed killing times. It is noted, though, that these can be converted into results on Erlang killing times (having, for a fixed mean, a lower variance). This procedure has been pointed out in [15]; see also [31, Section 3.4].

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3. Exponential and hyperexponential inter-inspection times

In this section we wish to study the bankruptcy probability $\mathfrak{p}_\beta(u \mid \Omega)$ for exponentially and hyperexponentially distributed inter-inspection times Ω, by considering the corresponding transform $\pi(\alpha, \beta \mid \Omega)$. In the first subsection we consider the case of exponential inter-inspection times, i.e. we associate with Ω a random variable that is distributed as T_ω for some $\omega > 0$. As mentioned, this case has been dealt with already in [14], albeit relying on another argumentation, but we include it to maximally transparently demonstrate the approach followed in this paper, thus making the derivations for the other inter-inspection time distributions more accessible. In the second subsection we show how to use these results to deal with the case of hyperexponential inter-inspection times, in which we associate with Ω a random variable that is with probability $p_i > 0$ distributed as T_{ω_i} for some $\omega_i > 0$, where $\sum_{i=1}^{d} p_i = 1$ for $d \in \mathbb{N}$.

In both cases, and also in the cases dealt with in the next section, our derivation consists of two steps. In the first step, we derive an equation that characterizes the unknown quantity $\pi(\alpha, \beta \mid \Omega)$, by conditioning on the evolution of the process until the first inspection time (if this occurs before killing). This equation, which is of a linear form, can be readily solved, but still contains one or multiple unknowns. In the second step, it is pointed out how these unknowns can be determined.

3.1. Exponentially distributed inter-inspection times

In this subsection we consider the case of inter-inspection times that are exponentially distributed with parameter $\omega > 0$. Note that the time until the first event, which can be either an inspection or killing, is $T_{\beta+\omega}$ (i.e. exponentially distributed with parameter $\beta + \omega$). We define the two nonnegative random quantities

$$Z^+(\beta, \omega) \equiv Z^+ := \bar{Y}(T_{\beta+\omega}), \qquad Z^-(\beta, \omega) \equiv Z^- := -\underline{Y}(T_{\beta+\omega}).$$

The random variable Z^- is exponentially distributed with parameter $\theta := \psi(\beta + \omega)$, and by virtue of the Wiener–Hopf decomposition Z^- and Z^+ are independent. Below we use the compact notation $Z := Z^+ - Z^-$. Note that, due to (1),

$$\mathrm{E}\, \mathrm{e}^{-\alpha Z^+} = \xi(\alpha, \beta + \omega) = \frac{\alpha - \psi(\beta + \omega)}{\varphi(\alpha) - \beta - \omega} \frac{\beta + \omega}{\psi(\beta + \omega)}. \tag{2}$$

3.1.1. *Equation for transform.* We shall evaluate the quantity $\pi(\alpha, \beta \mid T_\omega)$ by conditioning on the first event. If $Y(\,\cdot\,)$ has to exceed level u at an inspection time, then this first event time should be an inspection (i.e. not killing), which happens with probability $\omega/(\beta + \omega)$. Then there are two options: at this inspection time we are either already above u and, hence, ruin occurs or we are at some level v below u and we still need to bridge the remaining $u - v$. By following this line of reasoning, we obtain an expression in terms of our target quantity $\pi(\alpha, \beta \mid T_\omega)$, which we can solve. To this end, we first we write

$$\mathfrak{p}_\beta(u \mid T_\omega) = \frac{\omega}{\beta + \omega} \Big(\mathrm{P}(Z > u) + \int_{-\infty}^{u} \mathrm{P}(Z \in \mathrm{d}v)\, \mathfrak{p}_\beta(u - v \mid T_\omega) \Big).$$

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In order to evaluate the associated transform, we multiply the entire equation by $\mathrm{e}^{-\alpha u}$ and integrate over positive u:

$$\pi(\alpha, \beta \mid T_\omega) = \frac{\omega}{\beta+\omega}\Bigg(\underbrace{\int_0^\infty \mathrm{e}^{-\alpha u}\, \mathrm{P}(Z > u)\, \mathrm{d}u}_{\text{(i)}} + \underbrace{\int_{u=0}^\infty \mathrm{e}^{-\alpha u} \int_{v=-\infty}^{u} \mathrm{P}(Z \in \mathrm{d}v)\, \mathfrak{p}_\beta(u-v \mid T_\omega)\, \mathrm{d}u}_{\text{(ii)}} \Bigg). \tag{3}$$

We proceed by separately analyzing the integrals (i) and (ii) appearing in (3).

(i) As a first step, we condition on the value of Z^- and change the order of integration. With $\theta := \psi(\beta+\omega)$, recalling that Z^- is exponentially distributed with parameter θ, we readily find that

$$\int_0^\infty \mathrm{e}^{-\alpha u}\, \mathrm{P}(Z > u)\, \mathrm{d}u = \theta \int_{u=0}^\infty \mathrm{e}^{-\alpha u} \int_{v=0}^\infty \mathrm{e}^{-\theta v} \mathrm{P}(Z^+ > u+v)\, \mathrm{d}v\, \mathrm{d}u.$$

Performing the change of variable $x := u+v$, we have that this equals

$$\theta \int_{x=0}^\infty \mathrm{e}^{-\theta x} \mathrm{P}(Z^+ > x) \Big(\int_{u=0}^{x} \mathrm{e}^{u(\theta-\alpha)}\, \mathrm{d}u \Big)\, \mathrm{d}x.$$

Upon evaluating the inner integral, this reduces to

$$\frac{\theta}{\theta-\alpha} \Big(\int_0^\infty \mathrm{e}^{-\alpha x} \mathrm{P}(Z^+ > x)\, \mathrm{d}x - \int_0^\infty \mathrm{e}^{-\theta x} \mathrm{P}(Z^+ > x)\, \mathrm{d}x \Big).$$

Recalling that $\xi(\alpha, \beta+\omega) = \mathrm{E}\, \mathrm{e}^{-\alpha Z^+}$, observe that

$$\begin{aligned} \int_0^\infty \mathrm{e}^{-\alpha x} \mathrm{P}(Z^+ > x)\, \mathrm{d}x &= \int_{x=0}^\infty \mathrm{e}^{-\alpha x} \int_{y=x}^\infty \mathrm{P}(Z^+ \in \mathrm{d}y)\, \mathrm{d}x \\ &= \int_{y=0}^\infty \Big(\int_{x=0}^{y} \mathrm{e}^{-\alpha x}\, \mathrm{d}x \Big) \mathrm{P}(Z^+ \in \mathrm{d}y) \\ &= \frac{1}{\alpha} \Big(\int_0^\infty \mathrm{P}(Z^+ \in \mathrm{d}y) - \int_0^\infty \mathrm{e}^{-\alpha y} \mathrm{P}(Z^+ \in \mathrm{d}y) \Big) \\ &= \frac{1}{\alpha} \big(1 - \xi(\alpha, \beta+\omega) \big) \end{aligned}$$

A similar reasoning applies when replacing α by θ. Combining the above, we thus conclude that

$$\begin{aligned} \int_0^\infty \mathrm{e}^{-\alpha u}\, \mathrm{P}(Z > u)\, \mathrm{d}u &= \frac{\theta}{\theta-\alpha} \Big(\frac{1}{\alpha} \big(1 - \xi(\alpha, \beta+\omega) \big) - \frac{1}{\theta} \big(1 - \xi(\theta, \beta+\omega) \big) \Big) \\ &= \frac{\alpha \xi(\theta, \beta+\omega) - \theta \xi(\alpha, \beta+\omega)}{\alpha(\theta-\alpha)} + \frac{1}{\alpha} =: F(\alpha, \beta, \omega). \end{aligned}$$

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(ii) We continue by evaluating the second integral, starting by splitting it into two parts:

$$\int_{u=0}^{\infty} \mathrm{e}^{-\alpha u} \int_{v=-\infty}^{u} \mathrm{P}(Z \in \mathrm{d}v)\, \mathfrak{p}_{\beta}(u-v \mid T_{\omega})\ \mathrm{d}u = \pi_1(\alpha, \beta \mid T_{\omega}) + \pi_2(\alpha, \beta \mid T_{\omega}), \quad (4)$$

where the first term $\pi_1(\alpha, \beta \mid T_{\omega})$ corresponds to the triangular integration area$\{0 \leqslant v \leqslant u\}$ and the second term $\pi_2(\alpha, \beta \mid T_{\omega})$ to the rectangular integration area $\{v \leqslant 0 \leqslant u\}$.

As before, to evaluate the first double integral in (4), we interchange the order of the integrals. We obtain

$$\pi_1(\alpha, \beta \mid T_{\omega}) = \pi(\alpha, \beta \mid T_{\omega}) \int_{0}^{\infty} \mathrm{e}^{-\alpha v} \mathrm{P}(Z \in \mathrm{d}v),$$

which, by conditioning on the value of Z^-, equals

$$\theta\ \pi(\alpha, \beta \mid T_{\omega}) \int_{v=0}^{\infty} \mathrm{e}^{-\alpha v} \Big(\int_{x=0}^{\infty} \mathrm{e}^{-\theta x} \mathrm{P}\big(Z^+ - v \in \mathrm{d}x\big) \Big)\, \mathrm{d}v.$$

By evaluating the inner integral with a change of variable $y := v + x$, this expression can be further rewritten as

$$\begin{aligned} &\theta\ \pi(\alpha, \beta \mid T_{\omega}) \int_{v=0}^{\infty} \int_{y=v}^{\infty} \mathrm{e}^{v(\theta-\alpha)} \mathrm{e}^{-\theta y} \mathrm{P}(Z^+ \in \mathrm{d}y)\, \mathrm{d}v \\ &\quad = \theta\ \pi(\alpha, \beta \mid T_{\omega}) \int_{y=0}^{\infty} \Big(\int_{v=0}^{y} \mathrm{e}^{v(\theta-\alpha)}\, \mathrm{d}v \Big) \mathrm{e}^{-\theta y} \mathrm{P}(Z^+ \in \mathrm{d}y), \end{aligned}$$

where we again changed the order of the integrals. Upon evaluating the obtained inner integral, this reduces to

$$\theta\ \pi(\alpha, \beta \mid T_{\omega}) \int_{0}^{\infty} \mathrm{e}^{-\theta y} \Big(\frac{\mathrm{e}^{\theta y - \alpha y}}{\theta - \alpha} - \frac{1}{\theta - \alpha} \Big)\, \mathrm{P}(Z^+ \in \mathrm{d}y).$$

Some elementary manipulations lead us to the conclusion that

$$\pi_1(\alpha, \beta \mid T_{\omega}) = \theta\ \pi(\alpha, \beta \mid T_{\omega}) \frac{\xi(\alpha, \beta + \omega)}{\theta - \alpha} - H(\alpha, \beta, \omega),$$

where

$$H(\alpha, \beta, \omega) := \theta\ \pi(\alpha, \beta \mid T_{\omega}) \frac{\xi(\theta, \beta + \omega)}{\theta - \alpha}.$$

In order to evaluate (4), we are left with considering the second integral, i.e. $\pi_2(\alpha, \beta \mid T_{\omega})$. Performing the change of variable $x := u - v$, and swapping the order of the integrals, we have

$$\begin{aligned} \pi_2(\alpha, \beta \mid T_{\omega}) &= \int_{v=-\infty}^{0} \int_{x=-v}^{\infty} \mathrm{e}^{-\alpha v} \mathrm{e}^{-\alpha x}\, \mathfrak{p}_{\beta}(x \mid T_{\omega})\ \mathrm{d}x\, \mathrm{P}(Z \in \mathrm{d}v) \\ &= \int_{x=0}^{\infty} \int_{v=-x}^{0} \mathrm{e}^{-\alpha v} \mathrm{P}(Z \in \mathrm{d}v)\, \mathrm{e}^{-\alpha x}\, \mathfrak{p}_{\beta}(x \mid T_{\omega})\, \mathrm{d}x. \end{aligned}$$

Now note that the inner integral v is only evaluated in nonpositive values, so $-\infty < v \leqslant 0$. Since both $Z^-, Z^+ \geqslant 0$, we have $Z^+ - Z^- \leqslant 0$ if and only if $Z^- \geqslant Z^+$. The probability of

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$Z^+ - Z^-$ being less than v can hence be evaluated as follows: for $v \leqslant 0$, realizing that $\mathrm{P}(Z^- \geqslant Z^+) = \xi(\theta, \beta + \omega)$,

$$\mathrm{P}(Z < v) = \mathrm{P}(Z < v \mid Z^- \geqslant Z^+)\,\mathrm{P}(Z^- \geqslant Z^+) = \mathrm{e}^{\theta v}\,\xi(\theta, \beta + \omega),$$

using the memoryless property of the exponential distribution. We thus find that

$$\pi_2(\alpha, \beta \mid T_\omega) = \theta\,\xi(\theta, \beta + \omega) \int_{x=0}^{\infty} \Big(\int_{v=-x}^{0} \mathrm{e}^{v(\theta-\alpha)}\,\mathrm{d}v \Big)\, \mathrm{e}^{-\alpha x}\, \mathfrak{p}_\beta(x \mid T_\omega)\,\mathrm{d}x.$$

Evaluating the (standard) inner integral and performing similar manipulations as before, we conclude that

$$\pi_2(\alpha, \beta \mid T_\omega) = H(\alpha, \beta, \omega) - G(\alpha, \beta, \omega),$$

where

$$G(\alpha, \beta, \omega) := \theta\,\xi(\theta, \beta + \omega) \frac{\pi(\theta, \beta \mid T_\omega)}{\theta - \alpha}.$$

Upon combining the above, we obtain

$$\pi_1(\alpha, \beta \mid T_\omega) + \pi_2(\alpha, \beta \mid T_\omega) = \theta\,\pi(\alpha, \beta \mid T_\omega) \frac{\xi(\alpha, \beta + \omega)}{\theta - \alpha} - G(\alpha, \beta, \omega).$$

We now use the results obtained above in order to derive an expression for the transform $\pi(\alpha, \beta \mid T_\omega)$. Upon adding up the expressions that we found in (i) and (ii), we obtain

$$\pi(\alpha, \beta \mid T_\omega) = \frac{\omega}{\beta + \omega} \left(F(\alpha, \beta, \omega) + \frac{\theta\,\xi(\alpha, \beta + \omega)}{\theta - \alpha} \pi(\alpha, \beta \mid T_\omega) - G(\alpha, \beta, \omega) \right). \tag{5}$$

Substituting the known expression for $\xi(\alpha, \beta + \omega)$, as was given in (2), into the relation (5), we obtain the following result characterizing $\pi(\alpha, \beta \mid T_\omega)$.

Proposition 1. *For any $\alpha \geqslant 0$ and $\beta > 0$,*

$$\pi(\alpha, \beta \mid T_\omega) \Big(1 - \frac{\omega}{\beta + \omega - \varphi(\alpha)} \Big) = \frac{\omega}{\beta + \omega} \Big(F(\alpha, \beta, \omega) - G(\alpha, \beta, \omega) \Big). \tag{6}$$

3.1.2. *Determination of the unknown constant.* Whereas $F(\alpha, \beta, \omega)$ is an explicit expression in terms of the model primitives, $G(\alpha, \beta, \omega)$ contains the unknown quantity $\pi(\theta, \beta \mid T_\omega)$. This unknown can be identified by observing that any root (in the right-half of the complex α plane, that is) of the left-hand side of (6) is necessarily also a root of the right-hand side, using the fact that $\pi(\alpha, \beta \mid T_\omega)$ is finite for all $\alpha, \beta > 0$. Solving

$$1 = \frac{\omega}{\beta + \omega - \varphi(\alpha)},$$

leads us to $\varphi(\alpha) = \beta$ or $\alpha = \psi(\beta)$ since $\psi(\,\cdot\,)$ is the right inverse of the Laplace exponent $\varphi(\,\cdot\,)$. It requires an elementary calculation to verify that we thus end up with

$$\pi(\theta, \beta \mid T_\omega) = \frac{\omega\,\psi(\beta)\,\xi(\theta, \beta + \omega) - \beta\,\theta + \beta\,\psi(\beta)}{\theta\,\omega\,\psi(\beta)\,\xi(\theta, \beta + \omega)}.$$

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Plugging this expression into $\pi(\alpha, \beta \mid T_\omega)$, that we found in Proposition 1, after considerable calculus one arrives at the following result.

Theorem 1. *For any $\alpha \geqslant 0$ and $\beta > 0$,*

$$\pi(\alpha, \beta \mid T_\omega) = \frac{1}{\alpha} \frac{\omega\, \varphi(\alpha)(\theta - \alpha)\, \psi(\beta) + \alpha\, \beta\big(\varphi(\alpha) - \beta - \omega\big)(\theta - \psi(\beta))}{(\varphi(\alpha) - \beta)(\beta + \omega)(\theta - \alpha)\psi(\beta)}.$$

Define $\varrho(\alpha, \beta \mid \Omega) := \mathrm{E}\, \mathrm{e}^{-\alpha \bar{Y}_\Omega(T_\beta)}$, where

$$\bar{Y}_\Omega(t) := \sup\{Y(\bar{\Omega}_n) : \bar{\Omega}_n \leqslant t\}.$$

We note that $\pi(\alpha, \beta \mid T_\omega)$ and $\varrho(\alpha, \beta \mid T_\omega)$ can be expressed in one another, using the well-known 'translation formula' $\varrho(\alpha, \beta \mid T_\omega) = 1 - \alpha\, \pi(\alpha, \beta \mid T_\omega)$. We eventually find the following result, which is a true generalization of the time-dependent version of the Pollaczek–Khinchine formula, i.e.

$$\varrho(\alpha, \beta \mid T_\omega) = \frac{\alpha - \psi(\beta)}{\varphi(\alpha) - \beta} \frac{\beta}{\psi(\beta)} \frac{\varphi(\alpha) - \beta - \omega}{\alpha - \psi(\beta + \omega)} \frac{\psi(\beta + \omega)}{\beta + \omega};$$

cf. [14, Proposition 1]. Indeed, as $\omega \to \infty$, which corresponds to the inspection process taking place at an increasingly high frequency, thus approaching permanent inspection of the process $Y(t)$, we find that the transform corresponds to that appearing in the time-dependent Pollaczek–Khinchine formula

$$\lim_{\omega \to \infty} \varrho(\alpha, \beta \mid T_\omega) = \frac{\alpha - \psi(\beta)}{\varphi(\alpha) - \beta} \frac{\beta}{\psi(\beta)};$$

see e.g. [18, Thm.4.1].

3.2. Hyperexponentially distributed inter-inspection times

We now consider the bankruptcy model with hyperexponential-d inter-inspection times, by mimicking the line of reasoning that has been used in the previous subsection for the case of exponentially distributed inter-inspection times.

With probability $p_i > 0$ the inter-inspection time is exponentially distributed with rate $\omega_i > 0$ for $i \in \{1, \ldots, d\}$ and $\sum_{i=1}^{d} p_i = 1$. This means that, with $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_d)$ and $p = (p_1, \ldots, p_d)$, a generic inter-inspection time $T_{\boldsymbol{\omega},p}$ is characterized by the density,

$$\mathrm{P}(T_{\boldsymbol{\omega},p} \in \mathrm{d}t) = \sum_{i=1}^{d} p_i\, \omega_i \mathrm{e}^{-\omega_i t} \mathrm{d}t.$$

Recalling the definition of the two independent, nonnegative random variables $Z^+(\beta, \omega_i) := \bar{Y}(T_{\beta+\omega_i})$ and $Z^-(\beta, \omega_i) := -\underline{Y}(T_{\beta+\omega_i})$, we find that

$$\xi_i(\alpha, \beta) := \mathrm{E}\, \mathrm{e}^{-\alpha Z^+(\beta, \omega_i)} = \xi(\alpha, \beta + \omega_i) = \frac{\alpha - \psi(\beta + \omega_i)}{\varphi(\alpha) - \beta - \omega_i} \frac{\beta + \omega_i}{\psi(\beta + \omega_i)},$$

and that $Z^-(\beta, \omega_i)$ is exponentially distributed with parameter $\theta_i := \psi(\beta + \omega_i)$.

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3.2.1. *Equation for transform.* The main idea is that at time 0 we sample which of the d exponentially distributed inter-inspection times will be used. Given this is the ith inter-inspection time (with parameter ω_i, that is), we can reuse the argument developed for the exponential case, but with $Z^+(\beta, \omega)$ replaced by $Z^+(\beta, \omega_i)$ and with $Z^-(\beta, \omega)$ replaced by $Z^-(\beta, \omega_i)$. Hence, as the counterpart of the relation for the transform $\pi(\alpha, \beta \mid T_\omega)$ that we derived in the case of exponentially distributed inter-inspection times, we now obtain

$$\pi(\alpha, \beta \mid T_{\boldsymbol{\omega},p}) = \sum_{i=1}^{d} p_i \frac{\omega_i}{\beta + \omega_i} \left(\frac{\alpha\, \xi_i(\theta_i, \beta) - \theta_i\, \xi_i(\alpha, \beta)}{\alpha(\theta_i - \alpha)} + \frac{1}{\alpha} + \theta_i \pi(\alpha, \beta \mid T_{\boldsymbol{\omega},p}) \frac{\xi_i(\alpha, \beta) - \xi_i(\theta_i, \beta)}{\theta_i - \alpha} + \theta_i\, \xi_i(\theta_i, \beta) \frac{\pi(\alpha, \beta \mid T_{\boldsymbol{\omega},p}) - \pi(\theta_i, \beta \mid T_{\boldsymbol{\omega},p})}{\theta_i - \alpha} \right);$$

cf. the decomposition featuring in (5). Following the same procedure as in the case of exponentially distributed inter-inspection times, this can be rewritten as

$$\pi(\alpha, \beta \mid T_{\boldsymbol{\omega},p}) = \sum_{i=1}^{d} p_i \frac{\omega_i}{\beta + \omega_i} \Big(F(\alpha, \beta, \omega_i) + \frac{\theta_i\, \xi_i(\alpha, \beta)}{\theta_i - \alpha} \pi(\alpha, \beta \mid T_{\boldsymbol{\omega},p}) - G_i(\alpha, \beta, \boldsymbol{\omega}) \Big).$$

Here the function $F(\alpha, \beta, \omega)$ is as defined in the subsection on exponentially distributed inter-inspection times, so that, for $i = 1, \ldots, d$,

$$F(\alpha, \beta, \omega_i) = \frac{\alpha \xi(\theta_i, \beta + \omega_i) - \theta_i \xi(\alpha, \beta + \omega_i)}{\alpha(\theta_i - \alpha)} + \frac{1}{\alpha};$$

in addition, for $i = 1, \ldots, d$,

$$G_i(\alpha, \beta, \boldsymbol{\omega}) := \bar{G}(\alpha, \beta, \omega_i)\, z_i,$$

where

$$\bar{G}(\alpha, \beta, \omega_i) := \frac{\theta_i \xi_i(\theta_i, \beta)}{\theta_i - \alpha}, \quad z_i \equiv z_i(\beta, \boldsymbol{\omega}) := \pi(\theta_i, \beta \mid T_{\boldsymbol{\omega},p}).$$

We proceed by isolating our target object $\pi(\alpha, \beta \mid T_{\boldsymbol{\omega},p})$. By inserting the expression for $\xi_i(\alpha, \beta)$, we thus end up with the following characterization.

Proposition 2. *For any $\alpha \geqslant 0$ and $\beta > 0$,*

$$\pi(\alpha, \beta \mid T_{\boldsymbol{\omega},p}) \Big(1 - \sum_{i=1}^{d} p_i \frac{\omega_i}{\beta + \omega_i - \varphi(\alpha)} \Big) = \sum_{i=1}^{d} p_i \frac{\omega_i}{\beta + \omega_i} \Big(F(\alpha, \beta, \omega_i) - G_i(\alpha, \beta, \boldsymbol{\omega}) \Big). \quad (7)$$

3.2.2. *Determination of the unknown constants.* Observe that, on the right-hand side of (6) in Proposition 2, the object $F(\alpha, \beta, \omega_i)$ is explicitly expressed in terms of the model primitives, and so is $\bar{G}(\alpha, \beta, \omega_i)$. The constants $z_1, \ldots, z_d$ however are yet to be determined. In the remainder of this subsection we point out how these constants can be identified.

Lemma 1. *For any β and $\boldsymbol{\omega}$, the equation*

$$\sum_{i=1}^{d} p_i \frac{\omega_i}{\beta + \omega_i - \varphi(\alpha)} = 1 \quad (8)$$

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has d roots in the right half-plane, say, $\alpha_1 \equiv \alpha_1(\beta, \boldsymbol{\omega}), \ldots, \alpha_d \equiv \alpha_d(\beta, \boldsymbol{\omega})$.

Proof. The main idea behind the proof (as well as other proofs later in the paper) is the following. Consider a d-dimensional Markov additive process (MAP), say $(W(\cdot), J(\cdot)) \equiv (W(t), J(t))_{t \geqslant 0}$; we refer to, e.g. [8, Section XI.2] for a definition. With q_{ij}, for $i, j = 1, \ldots, d$, denoting the transition rates corresponding to the modulating continuous-time Markov chain $J(\cdot)$ on $\{1, \ldots, d\}$, and with $\varphi_1(\cdot), \ldots, \varphi_d(\cdot)$ Laplace exponents of d Lévy processes, define the (i, j) th entry of the $(d \times d)$-matrix $M(\alpha, \boldsymbol{\eta})$ by

$$m_{ij}(\alpha, \boldsymbol{\eta}) := (\varphi_i(\alpha) - \eta_i)\, 1\{i = j\} + q_{ij}$$

for a given componentwise positive vector $\boldsymbol{\eta}$. From [23, Theorem 1 and Remark 2.1] we know that, if the d Lévy processes featuring in $(W(\cdot), J(\cdot))$ are no subordinators, then the equation $\det M(\alpha, \boldsymbol{\eta}) = 0$ has, for any componentwise positive vector $\boldsymbol{\eta}$, exactly d roots in the right-half of the complex plane. To prove the lemma, the objective is now to construct a MAP $(W(\cdot), J(\cdot))$ such that the corresponding equation $\det M(\alpha, \boldsymbol{\eta}) = 0$ is, for some choice of the vector $\boldsymbol{\eta}$, equivalent to (8).

Consider the MAP defined by a transition rate matrix Q that is given by

$$Q = p\boldsymbol{\omega}^\top - \operatorname{diag}(\boldsymbol{\omega}),$$

and Laplace exponents $\varphi_1(\alpha) = \cdots = \varphi_d(\alpha) = \varphi(\alpha)$. The entries of the componentwise positive vector $\boldsymbol{\eta}$ are defined to be $\eta_i = \beta$, for $i = 1, \ldots, d$. We now observe that

$$M(\alpha, \boldsymbol{\eta}) = Q + \operatorname{diag}(\boldsymbol{\varphi}(\alpha) - \boldsymbol{\eta}) = p\boldsymbol{\omega}^\top + \operatorname{diag}(\varphi(\alpha)\mathbf{1} - \boldsymbol{\omega} - \beta\mathbf{1}),$$

with $\mathbf{1}$ a d-dimensional all-ones vector. As a consequence, $M(\alpha, \eta)$ is the sum of a diagonal matrix and a rank-one matrix.

The last step is to evaluate $\det M(\alpha, \boldsymbol{\eta})$, to show that the equation $\det M(\alpha, \boldsymbol{\eta}) = 0$ is equivalent to (8). To this end, define $\Delta := \operatorname{diag}(\varphi(\alpha)\mathbf{1} - \boldsymbol{\omega} - \beta\mathbf{1})$, and let δ_i be the (i,i)th entry of this diagonal matrix. Observe that $M(\alpha, \boldsymbol{\eta}) = \Delta + p\boldsymbol{\omega}^\top$. As a consequence, we can write

$$\det M(\alpha, \boldsymbol{\eta}) = \det \Delta \cdot \det (I_d + p\boldsymbol{\omega}^\top \Delta^{-1}).$$

For A of dimension $m \times n$, and B of dimension $n \times m$, we have $\det (I_m + AB) = \det (I_n + BA)$. We thus conclude that

$$\det M(\alpha, \boldsymbol{\eta}) = \det \Delta \cdot \det (I_1 + \boldsymbol{\omega}^\top \Delta^{-1} p) = \left(\prod_{i=1}^{d} \delta_i \right) \left(\sum_{i=1}^{d} p_i \frac{\omega_i}{\varphi(\alpha) - \beta - \omega_i} + 1 \right),$$

so that indeed $\det M(\alpha, \boldsymbol{\eta}) = 0$ is equivalent to (8).

To identify the unknown quantities z_i, we use the property that any root of the left-hand side of (7) in Proposition 2 (in the right-half of the complex α plane) is necessarily a root of the right-hand side as well, noting that $\pi(\alpha, \beta \mid T_{\boldsymbol{\omega},p})$ is finite for all $\alpha \geqslant 0$ and $\beta > 0$. Let $\alpha_1, \ldots, \alpha_d$ be the roots of the left-hand side in the right-half of the complex α plane; there are precisely d of these, by Lemma 1. Consequently, we are to solve a system of linear equations: we have, for any $j = 1, \ldots, d$,

$$\sum_{i=1}^{d} p_i \frac{\omega_i}{\beta + \omega_i} \Big(F(\alpha_j, \beta, \omega_i) - \bar{G}(\alpha_j, \beta, \omega_i)\, z_i \Big) = 0. \tag{9}$$

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It can be argued that the roots α_j are distinct; see Remark 2. Hence, the unknown vector z can be identified by solving the j linear equations (9). We thus arrive at the following result.

Theorem 2. *For any $\alpha \geqslant 0$ and $\beta > 0$,*

$$\pi(\alpha, \beta \mid T_{\boldsymbol{\omega},p}) = \frac{\displaystyle\sum_{i=1}^{d} p_i \frac{\omega_i}{\beta + \omega_i} \Big(F(\alpha, \beta, \omega_i) - \bar{G}(\alpha, \beta, \omega_i)\, z_i \Big)}{\displaystyle 1 - \sum_{i=1}^{d} p_i \frac{\omega_i}{\beta + \omega_i - \varphi(\alpha)}},$$

where the vector z follows from the linear equations (9).

Remark 2. The d roots are actually real, as follows from a reasoning similar to that in [16, Section 3.2]. First realize that without loss of generality we can assume the rates ω_i are distinct (because if they were not, some of the p_i could be 'lumped' to make the ω_i distinct). Putting (without loss of generality) these ω_i in increasing order, we obtain that there is precisely one real root between x_{i-1} and x_i, with $x_0 := 0$ and $x_i := \psi(\beta + \omega_i)$ for $i = 1, \ldots, d$. Observe that the smallest root is $\psi(\beta)$. This implies that the roots are distinct.

4. Erlang and hyper-Erlang inter-inspection times

In this section we study the bankruptcy probability $\mathfrak{p}_\beta(u \mid \Omega)$ for Erlang and hyper-Erlang distributed inter-inspection times Ω, again by considering the corresponding transform $\pi(\alpha, \beta \mid \Omega)$. Interestingly, this will eventually also enable us to approximate deterministically distributed inter-inspection times: observe that the sum of k i.i.d. exponentially distributed random variables with parameter λk has mean $1/\lambda$ and variance $1/(k\lambda^2)$, with the latter decreasing to 0 as $k \to \infty$; cf. [34].

In the first subsection we consider the Erlang distributed inter-inspection time case, where Ω is associated with a random variable $E_{k,\omega}$ that is the sum of $k \in \mathbb{N}$ independent random variables, each of them being distributed as T_ω, for some $\omega > 0$. In the second subsection we show how to use these results to handle the hyper-Erlang inter-inspection time case, in which we associate with Ω a random variable that is with probability p_i distributed as $E_{k_i,\omega}$ for some k_i, $\omega > 0$, where $\sum_{i=1}^{d} p_i = 1$. In both cases our derivation consists of the same two steps as in the previous section.

4.1. Erlang distributed inter-inspection times

We consider the bankruptcy model with Erlang-k inspection times, with shape parameter $k \in \mathbb{N}$ and scale parameter ω. This means that a generic inter-inspection time $E_{k,\omega}$ is characterized via the density

$$\mathrm{P}\big(E_{k,\omega} \in \mathrm{d}t\big) = \omega \frac{(\omega t)^{k-1}}{(k-1)!} \mathrm{e}^{-\omega t}\, \mathrm{d}t.$$

In this subsection again a crucial role is played by the two nonnegative random quantities

$$Z^+(\beta, \omega) \equiv Z^+ := \bar{Y}(T_{\beta+\omega}), \qquad Z^-(\beta, \omega) \equiv Z^- := -\underline{Y}(T_{\beta+\omega}).$$

In what follows, let $Z_k^+ \equiv Z_k^+(\beta, \omega)$ denote the sum of k i.i.d. copies of Z^+, and analogously $Z_k^- \equiv Z_k^-(\beta, \omega)$ the sum of k i.i.d. copies of Z^-.

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Recalling $\mathrm{E}\,\mathrm{e}^{-\alpha Z^+} = \xi(\alpha, \beta+\omega)$, which can be evaluated by applying (1), we write

$$\xi_k(\alpha, \beta+\omega) := \mathrm{E}\,\mathrm{e}^{-\alpha Z_k^+} = (\xi(\alpha, \beta+\omega))^k.$$

In addition, we recall that Z^- is exponentially distributed with parameter $\theta := \psi(\beta+\omega)$. It is also noted that Z^- and Z^+ are, by virtue of the Wiener–Hopf decomposition, independent. This entails that Z_k^- is Erlang distributed with shape parameter k and scale parameter θ, and that Z_k^- and Z_k^+ are independent. Below we use the compact notation $Z_k := Z_k^+ - Z_k^-$.

4.1.1. *Equation for transform.* The objective is to evaluate the transform

$$\pi(\alpha, \beta \mid E_{k,\omega}) := \int_0^\infty \mathrm{e}^{-\alpha u}\, \mathfrak{p}_\beta(u \mid E_{k,\omega})\ \mathrm{d}u.$$

To this end, first note that

$$\mathfrak{p}_\beta(u \mid E_{k,\omega}) = \left(\frac{\omega}{\beta+\omega}\right)^k \left(\mathrm{P}(Z_k > u) + \int_{-\infty}^u \mathrm{P}(Z_k \in \mathrm{d}v)\, \mathfrak{p}_\beta(u-v \mid E_{k,\omega})\right).$$

In order to evaluate the associated transform, we multiply the entire equation by $\mathrm{e}^{-\alpha u}$ and integrate over u, so as to obtain

$$\begin{aligned}&\pi(\alpha, \beta \mid E_{k,\omega})\\ &= \left(\frac{\omega}{\beta+\omega}\right)^k \Bigg(\underbrace{\int_0^\infty \mathrm{e}^{-\alpha u}\, \mathrm{P}(Z_k > u)\, \mathrm{d}u}_{\text{(i)}} + \underbrace{\int_{u=0}^\infty \mathrm{e}^{-\alpha u} \int_{v=-\infty}^u \mathrm{P}(Z_k \in \mathrm{d}v)\, \mathfrak{p}_\beta(u-v \mid E_{k,\omega})\, \mathrm{d}u}_{\text{(ii)}} \Bigg).\end{aligned} \tag{10}$$

As before, we deal with the terms (i) and (ii) separately. Lemma 2 below, proven in Appendix A, provides an expression for (i). Define, with $\xi^{(m)}(\alpha, \beta)$ denoting the mth derivative of $\xi(\alpha, \beta)$ with respect to α,

$$\begin{aligned}&I(\alpha, \beta, \omega)\\ &:= \left(\frac{\theta}{\theta-\alpha}\right)^k \frac{1-\xi_k(\alpha, \beta+\omega)}{\alpha} - \frac{1}{\theta}\sum_{n=0}^{k-1} \left(\frac{\theta}{\theta-\alpha}\right)^{k-n} \left(1 - \sum_{m=0}^n \xi_k^{(m)}(\theta, \beta+\omega)\frac{(-\theta)^m}{m!}\right).\end{aligned} \tag{11}$$

Lemma 2. *For any $\alpha \geqslant 0$,*

$$\int_0^\infty \mathrm{e}^{-\alpha u}\, \mathrm{P}(Z_k > u)\, \mathrm{d}u = I(\alpha, \beta, \omega).$$

The right-hand side of (11) can be further simplified by swapping the order of summation and evaluating the finite geometric series, but we leave this computation out.
We now consider integral (ii). To this end, define

$$\delta_{n,k} := \frac{(-\theta)^n}{n!}\xi_k^{(n)}(\theta, \beta+\omega).$$

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Also,

$$J(\alpha,\beta,\omega) := \sum_{n=0}^{k-1} \delta_{k-1-n,k} \left(\frac{\theta}{\theta-\alpha}\right)^{n+1} \sum_{m=0}^{n} \frac{(\alpha-\theta)^m}{m!} \pi^{(m)}(\theta,\beta \mid E_{k,\omega}) = \sum_{i=0}^{k-1} \bar{J}_i(\alpha,\beta,\omega)\, z_i,$$

where (by swapping the order of the two sums) $J_i(\alpha,\beta,\omega)$ and $z_i \equiv z_i(\beta,\omega)$ are

$$\bar{J}_i(\alpha,\beta,\omega) := \sum_{n=i}^{k-1} \delta_{k-1-n,k} \left(\frac{\theta}{\theta-\alpha}\right)^{n+1} \frac{(\alpha-\theta)^i}{i!}, \qquad z_i := \pi^{(i)}(\theta,\beta \mid E_{k,\omega}). \tag{12}$$

Observe that $\bar{J}_i(\alpha,\beta,\omega)$ is explicitly known (in terms of the model primitives, that is), whereas the k coefficients $z_0,\ldots,z_{k-1}$ are yet to be determined. The following lemma is proven in Appendix A.

Lemma 3. *For any $\alpha \geqslant 0$,*

$$\begin{aligned}\int_{u=0}^{\infty} \mathrm{e}^{-\alpha u} \int_{v=-\infty}^{u} \mathrm{P}(Z_k \in \mathrm{d}v)\, \mathfrak{p}_\beta(u-v \mid E_{k,\omega})\, \mathrm{d}u \\ = -J(\alpha,\beta,\omega) + \left(\frac{\theta}{\theta-\alpha}\right)^k \xi_k(\alpha,\beta+\omega)\, \pi(\alpha,\beta \mid E_{k,\omega}).\end{aligned}$$

We now use the results obtained above to derive an expression for the transform $\pi(\alpha,\beta \mid E_{k,\omega})$. Adding up the expressions that we have found in Lemmas 2 and 3,

$$\begin{aligned}&\pi(\alpha,\beta \mid E_{k,\omega}) \\ &\quad = \left(\frac{\omega}{\beta+\omega}\right)^k \Bigg(\quad I(\alpha,\beta,\omega) - J(\alpha,\beta,\omega) + \left(\frac{\theta}{\theta-\alpha}\right)^k \xi_k(\alpha,\beta+\omega)\, \pi(\alpha,\beta \mid E_{k,\omega}) \Bigg).\end{aligned} \tag{13}$$

Noting that

$$\left(\frac{\theta}{\theta-\alpha}\right)^k \xi_k(\alpha,\beta+\omega) = \left(\frac{\beta+\omega}{\beta+\omega-\varphi(\alpha)}\right)^k,$$

we arrive at the following result.

Proposition 3. *For any $\alpha \geqslant 0$ and $\beta > 0$,*

$$\pi(\alpha,\beta \mid E_{k,\omega}) \left(1 - \left(\frac{\omega}{\beta+\omega-\varphi(\alpha)}\right)^k\right) = \left(\frac{\omega}{\beta+\omega}\right)^k \big(I(\alpha,\beta,\omega) - J(\alpha,\beta,\omega)\big). \tag{14}$$

4.1.2. *Determination of the unknown constants.* We observe that in Proposition 3 the right-hand side of (14) contains the object $J(\alpha,\beta,\omega)$, which contains the unknown quantities $z_0,\ldots,z_{k-1}$. The aim of the remainder of this subsection is to identify these unknown quantities, leading to an expression for our object of study, i.e. $\pi(\alpha,\beta \mid E_{k,\omega})$. The following lemma is proven in Appendix A.

Lemma 4. *For any β, ω and k, the equation*

$$\left(\frac{\omega}{\beta+\omega-\varphi(\alpha)}\right)^k = 1 \tag{15}$$

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has k roots in the right half of the complex plane, say, $\alpha_1 \equiv \alpha_1(\beta, \omega), \ldots, \alpha_k \equiv \alpha_k(\beta, \omega)$.

The unknowns z_i can be found relying on Lemma 4, once more using the fact that any root of the left-hand side (in the right-half of the complex α-plane) of (14) in Proposition 3 is also a root of the right-hand side. It entails that we have to solve the linear system

$$I(\alpha_j, \beta, \omega) = \sum_{i=0}^{k-1} \bar{J}_i(\alpha_j, \beta, \omega)\, z_i \tag{16}$$

for $j = 1, \ldots, k$. Below we write $J(\alpha, \beta, \omega) := \sum_{i=0}^{k-1} \bar{J}_i(\alpha, \beta, \omega)\, z_i$.

Theorem 3. *For any* $\alpha \geqslant 0$ *and* $\beta > 0$,

$$\pi(\alpha, \beta \mid E_{k,\omega}) = \frac{(\omega/(\beta+\omega))^k \big(I(\alpha, \beta, \omega) - J(\alpha, \beta, \omega)\big)}{1 - (\omega/(\beta + \omega - \varphi(\alpha)))^k},$$

where the vector z follows from the linear equations (16).

Remark 3. The entries of z are real, as can be seen as follows.

(i) First observe that the roots α_j solve the equation, with i denoting the imaginary unit,

$$\varphi(\alpha) = \omega - \omega\, \mathrm{e}^{2\pi \mathrm{i} j/k} + \beta, \tag{17}$$

with $j = 0, \ldots, k-1$. This means that any $\varphi(\alpha_j)$ is either real or the complex conjugate of $\varphi(\alpha_{j'})$ for $j \neq j'$.

(ii) Noting that, for $v, w \in \mathbb{R}$,

$$\varphi(v + w\mathrm{i}) = \log \mathrm{E}\mathrm{e}^{-(v+w\mathrm{i})Y(1)} = \log \mathrm{E}\left[\mathrm{e}^{-vY(1)}(\cos(-wY(1)) + \mathrm{i}\sin(-wY(1))\right],$$

it requires a straightforward verification to conclude that if $\alpha_j = v + w\mathrm{i}$ solves 17 then so does its complex conjugate $\alpha_{j'} = v - w\mathrm{i}$. Hence, the roots of 15 are either real or belong to a pair of complex conjugates.

(iii) Consider row j and j' in the linear system (16), with α_j and $\alpha_{j'}$ being complex conjugates. From their definitions it can be verified that $I(\alpha_j, \beta, \omega)$ is the complex conjugate of $I(\alpha_{j'}, \beta, \omega)$ and that $\bar{J}_i(\alpha_j, \beta, \omega)$ is the complex conjugate of $\bar{J}_i(\alpha_{j'}, \beta, \omega)$. Now replace rows j and j' by one row that follows by adding the two rows up, and a second row that follows by subtracting one row from the other, and dividing by i. All coefficients in the new linear system have become real, and hence, the solution z is real.

Remark 4. In the case that $Y(\cdot)$ is of compound Poisson type, an insightful decomposition applies, earlier identified in [2, Equation (3.9)]. In the first place, Z_k^+ is distributed as the sum of k i.i.d. random variables, each of them characterized by the transform $\xi(\alpha, \beta + \omega)$. Realizing that this $\xi(\alpha, \beta + \omega)$ can be interpreted in terms of a geometric sum of ladder heights, as can be found in, for instance, [31, Section 1.4], $(\xi(\alpha, \beta + \omega))^k$ represents a negative binomial number of such ladder heights. In the second place, Z_k^- is distributed as the sum of k independent exponential random variables with parameter $\psi(\beta + \omega)$, and, consequently, has an Erlang distribution (and is independent of Z_k^+).

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4.2. Hyper-Erlang distributed inter-inspection times

We consider the bankruptcy model with hyper-Erlang-d inter-inspection times, with a common scale parameter. With probability p_i the inter-inspection time is Erlang-k_i distributed (for some $k_i \in \mathbb{N}$), with scale parameter $\omega > 0$, for $i \in \{1, \ldots, d\}$ and $\sum_{i=1}^{d} p_i = 1$. This means that a generic inter-inspection time $E_{k,p,\omega}$ has the density, with $k = (k_1, \ldots, k_d)$ and $p = (p_1, \ldots, p_d)$,

$$\mathrm{P}(E_{k,p,\omega} \in \mathrm{d}t) = \sum_{i=1}^{d} p_i \, \omega \frac{(\omega t)^{k_i - 1}}{(k_i - 1)!} \mathrm{e}^{-\omega t} \, \mathrm{d}t.$$

Without loss of generality, we assume that the k_i are put in increasing order.

Remark 5. In this subsection we work with a common scale parameter $\omega > 0$. At the expense of more complex computations and the introduction of additional notation, we could have dealt with a setting in which with probability p_i the inter-inspection time is Erlang-k_i distributed with scale parameter $\omega_i > 0$ (i.e. with the parameters ω_i not necessarily identical). We have chosen to work with a common scale parameter, as the proof of [8, Theorem III.4.2] reveals that any distribution on the positive half-line can be approximated arbitrarily closely by mixtures of Erlang distributions with the same scale parameter.

4.2.1. *Equation for transform.* Based on the decomposition that we developed in (13), corresponding to the transform $\pi(\alpha, \beta \mid E_{k,\omega})$ in the case of Erlang distributed inter-inspection times, we now obtain

$$\begin{aligned}
&\pi(\alpha, \beta \mid E_{k,p,\omega}) \\
&\quad = \sum_{i=1}^{d} p_i \Big(\frac{\omega}{\beta + \omega} \Big)^{k_i} \Big(I_i(\alpha, \beta, \omega) - J_i(\alpha, \beta, \omega) + \Big(\frac{\theta}{\theta - \alpha} \Big)^{k_i} \xi_{k_i}(\alpha, \beta + \omega) \, \pi(\alpha, \beta \mid E_{k,p,\omega}) \Big),
\end{aligned} \tag{18}$$

with

$$\begin{aligned}
I_i(\alpha, \beta, \omega) &:= \Big(\frac{\theta}{\theta - \alpha} \Big)^{k_i} \frac{1 - \xi_{k_i}(\alpha, \beta + \omega)}{\alpha} \\
&\quad - \frac{1}{\theta} \sum_{n=0}^{k_i - 1} \Big(\frac{\theta}{\theta - \alpha} \Big)^{k_i - n} \Big(1 - \sum_{m=0}^{n} \xi_{k_i}^{(m)}(\theta, \beta + \omega) \frac{(-\theta)^m}{m!} \Big), \\
J_i(\alpha, \beta, \omega) &:= \sum_{n=0}^{k_i - 1} \delta_{k_i - 1 - n, k_i} \Big(\frac{\theta}{\theta - \alpha} \Big)^{n+1} \sum_{m=0}^{n} \frac{(\alpha - \theta)^m}{m!} \pi^{(m)}(\theta, \beta \mid E_{k,p,\omega}) \\
&= \sum_{j=0}^{k_i - 1} \bar{J}_{i,j}(\alpha, \beta, \omega) \, z_j,
\end{aligned}$$

where, again by swapping the order of the two sums,

$$\bar{J}_{i,j}(\alpha, \beta, \omega) := \sum_{n=j}^{k_i - 1} \delta_{k_i - 1 - n, k_i} \Big(\frac{\theta}{\theta - \alpha} \Big)^{n+1} \frac{(\alpha - \theta)^j}{j!}$$

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and $z_j \equiv z_j(\beta, \omega) := \pi^{(j)}(\theta, \beta \mid E_{k,p,\omega})$. Observe that $\bar{J}_{i,j}(\alpha, \beta, \omega)$ is explicitly known, whereas the coefficients $z_0, \ldots, z_{k_d-1}$ are yet to be determined. As before, the next step is to isolate the object $\pi(\alpha, \beta \mid E_{k,p,\omega})$ in (18). We thus end up with the following characterization of our target quantity.

Proposition 4. *For any $\alpha \geqslant 0$ and $\beta > 0$,*

$$\pi(\alpha, \beta \mid E_{k,p,\omega})\left(1 - \sum_{i=1}^{d} p_i \Big(\frac{\omega}{\beta + \omega - \varphi(\alpha)}\Big)^{k_i}\right) = \sum_{i=1}^{d} p_i \Big(\frac{\omega}{\beta + \omega}\Big)^{k_i} \big(I_i(\alpha, \beta, \omega) - J_i(\alpha, \beta, \omega)\big). \tag{19}$$

4.2.2. *Determination of the unknown constants.* On the right-hand side of (19) in Proposition 4 we have that the object $I_i(\alpha, \beta, \omega)$ is explicitly expressed in terms of the model primitives, and so is $\bar{J}_i(\alpha, \beta, \omega)$. The coefficients $z_0, \ldots, z_{k_d-1}$ however are still unknown. The aim of the remainder of this subsection is to identify these unknown quantities. The following lemma, proven in Appendix A, plays a crucial role.

Lemma 5. *For any β, ω, p and k, the equation*

$$\sum_{i=1}^{d} p_i \Big(\frac{\omega}{\beta + \omega - \varphi(\alpha)}\Big)^{k_i} = 1 \tag{20}$$

has k_d roots in the right half of the complex plane, say, $\alpha_1 \equiv \alpha_1(\beta, \boldsymbol{\omega}), \ldots, \alpha_{k_d} \equiv \alpha_{k_d}(\beta, \boldsymbol{\omega})$.

The unknowns $z_1, \ldots, z_{k_d}$ can be found using Lemma 5 in combination with the property that any root (in the right half of the complex α plane) of the left-hand side of (19) in Proposition 4, is necessarily a root of the right-hand side as well. This reasoning leads to the linear system

$$\sum_{i=1}^{d} p_i \Big(\frac{\omega}{\beta + \omega}\Big)^{k_i} I_i(\alpha_\ell, \beta, \omega) = \sum_{i=1}^{d} p_i \Big(\frac{\omega}{\beta + \omega}\Big)^{k_i} \sum_{j=0}^{k_i - 1} \bar{J}_{i,j}(\alpha_\ell, \beta, \omega)\, z_j \tag{21}$$

for $\ell = 1, \ldots, k_d$. As in Remark 3, the entries of z are real. Defining

$$J_i(\alpha, \beta, \omega) := \sum_{j=0}^{k_i - 1} \bar{J}_{i,j}(\alpha, \beta, \omega)\, z_j,$$

we have found the following result.

Theorem 4. *For any $\alpha \geqslant 0$ and $\beta > 0$,*

$$\pi(\alpha, \beta \mid E_{k,p,\omega}) = \frac{\displaystyle\sum_{i=1}^{d} p_i (\omega/(\beta + \omega))^{k_i} (I_i(\alpha, \beta, \omega) - J_i(\alpha, \beta, \omega))}{\displaystyle 1 - \sum_{i=1}^{d} p_i (\omega/(\beta + \omega - \varphi(\alpha)))^{k_i}},$$

where the vector z follows from the linear equations (21).

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5. Reflections on distribution of inter-inspection times

In this section we discuss the role played by the inter-inspection times Ω. While the first subsection points out that phase-type distributions are a natural candidate to approximate Ω with, the second subsection argues that only a subclass of these phase-type distributions (including those featured in Sections 3 and 4) allows an explicit analysis.

5.1. Approximating general distributions, two-moments fit

The inter-inspection time distributions we have dealt with in the previous sections are all contained in the wider class of *phase-type distributions*. Informally, a phase-type random variable has a distribution that is constructed by taking convolutions and/or mixtures of exponential distributions. Phase-type distributions owe their popularity to the fact that they are capable of approximating any distribution (on the positive half-line, that is) arbitrarily closely [8, Theorem III.4.2], while still allowing (semi-)explicit evaluation.

A phase-type random variable is uniquely characterized as the time until absorption of a finite-state continuous-time Markov chain with a single absorbing state. Following the approach presented in [8, Section III.4], it is encoded by an initial distribution $a \in \mathbb{R}^m$ and a substochastic $m \times m$ transition rate matrix S, where 'substochastic' means that in at least one row the sum of the nondiagonal elements exceeds the diagonal element. For instance, the Erlang distribution with k phases is represented by $a_1 = 1$, $S_{ii} = -\omega$ for $i = 1, \ldots, k$ and $S_{i,i+1} = \omega$ for $i = 1, \ldots, k-1$ (while all other entries of the k-dimensional vector a and of the $k \times k$ matrix S are set equal to 0).

It is noted that the proof of [8, Theorem III.4.2] reveals that actually any distribution on the positive half-line can be approximated arbitrarily closely by elements from a specific subclass of phase-type distributions. This smaller class are the mixtures of Erlang distributions, each of them having the same scale parameter, i.e. the class of distributions dealt with in Section 4.2. The proof also indicates that the drawback of working with this specific class of phase-type distributions is that one may need a large dimension m to get an accurate fit. That is, the reason why a two-moment fit has been proposed in which a relatively low value of m suffices. In the one presented in [33], for distributions with a coefficient of variation less than 1, a mixture of Erlang random variables (with the same scale parameter) is used and, for distributions with a coefficient of variation larger than 1, a hyperexponential random variable is used, all of them of relatively low dimension; this fitting method is compactly summarized in, e.g. [27, Section 3.1]. This two-moment fit renders the inter-inspection time distributions covered in Sections 3 and 4 particularly useful.

5.2. Limitations

In the previous sections we succeeded in identifying the transform $\pi(\alpha, \beta \mid \Omega)$ for (hyper-)exponential and (hyper-)Erlang inter-inspection times, both belonging to the family of phase-type distributions. This suggests that the analysis may extend to the full class of phase-type distributions. Careful inspection however reveals that such an extension is challenging.

To explain what complications arise, let us analyze the distribution of $Z = Y(\Omega)$ if the inter-inspection times Ω are of phase type. Let K_j for $j = 1, \ldots, m$ be the number of visits of the associated underlying Markov chain to state j before absorption. Then $Z = Z^+ - Z^-$, where, in self-evident notation,

$$Z^+ := \sum_{j=1}^{m} \sum_{i=1}^{K_j} Z_{ji}^+, \qquad Z^- := \sum_{j=1}^{m} \sum_{i=1}^{K_j} Z_{ji}^-.$$

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While all pair of random variables Z_{ji}^{+} and Z_{ji}^{-} are independent, for obvious reasons,

$$\sum_{i=1}^{K_j} Z_{ji}^{+} \quad \text{and} \quad \sum_{i=1}^{K_j} Z_{ji}^{-}$$

are *not*. As a consequence, for general phase-type Ω, it is not clear how to evaluate objects like

$$\int_0^\infty \mathrm{e}^{-\alpha u}\, \mathrm{P}(Z > u)\, \mathrm{d}u \quad \text{and} \quad \mathrm{P}(Z < v \mid Z < 0),$$

which were the key quantities in the computations of the previous sections.
As is readily verified, the reason why the computations in Sections 3 and 4 worked out is that, for all of the inter-inspection time distributions considered, the random variables K_j are either 0 or 1, so that Z^{+} and Z^{-} are independent. A sufficient condition for this property to hold is that the substochastic transition rate matrix S does not allow cycles (so that every state is visited at most once), a condition that is met for (hyper-)exponential and (hyper-)Erlang inter-inspection times; see also Appendix B.

Remark 6. We conclude this section by remarking that it *is* true that Z^{-} is of phase type, owing to the facts that (i) all random variables Z_{ji}^{-} are exponentially distributed, and that (ii) the random variables K_j, for $j = 1, \ldots, m$, correspond to the number of visits of a Markov chain to its respective states (before absorption, that is).
A second observation is that the expressions found in the previous sections suggest a relation with associated queueing systems; for instance, the expressions found for the model in Section 4.1 strongly resemble those of the workload in the E_k/G/1 queue (i.e. the queue with Erlang-k interarrival times, general service times and a single server); cf. [32, Theorem 25, p.44]. One would therefore expect that analyzing our system with phase-type Ω is of the same complexity as the workload in a Ph/G/1 queue (i.e. the queue with phase-type interarrival times, general service times and a single server). This is, however, *not* true, as in the Ph/G/1 queue there is independence between the sequence of interarrival times and the sequence of service times, while in our bankruptcy context Z^{-} and Z^{+} are not necessarily independent.

6. Cramér–Lundberg-type asymptotics

The objective of this section is to find the exact asymptotics of

$$\mathfrak{p}(u \mid \Omega) := \mathrm{P}(\text{there exists } n \in \mathbb{N}{:}Y(\bar{\Omega}_n) > u),$$

i.e. we aim to identify an explicit function $f(u)$ such that $f(u)\,\mathfrak{p}(u \mid \Omega) \to 1$ as $u \to \infty$. We focus on the light-tailed regime, in which $\mathfrak{p}(u \mid \Omega)$ asymptotically looks like $\gamma \mathrm{e}^{-\theta^\star u}$ for a positive decay rate $\theta^\star$ and 'preconstant' γ. This extends the classical Cramér–Lundberg ruin probability asymptotics to our bankruptcy setting.
In the first subsection we derive a useful identity, relating $\mathfrak{p}(u \mid \Omega)$ to the likelihood ratio of an exponentially twisted version of the process $Y(\,\cdot\,)$ at a stopping time, revealing the value of $\theta^\star$. This relation is used in the second subsection: by adapting the techniques that we developed earlier in this paper, we succeed in identifying γ for the types of inter-inspection time distributions considered in Sections 3 and 4.

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6.1. Change of measure

We start by introducing some useful notation. As before, we consider the net cumulative claim process $Y(t)$ at the inspection epochs $\bar{\Omega}_1, \bar{\Omega}_2, \ldots$. We introduce the increments of $Y(t)$ between two consecutive inspections, i.e. $Z_n := Y(\bar{\Omega}_n) - Y(\bar{\Omega}_{n-1})$. Note that $(Z_n)_n$ is a sequence with i.i.d. random variables, say, distributed as the generic random variable Z. As a consequence, we can rewrite our target probability as

$$\mathfrak{p}(u \mid \Omega) = \mathrm{P}(T(u) < \infty) = \mathrm{P}(\tau(u) < \infty),$$

where

$$T(u) := \inf\left\{ n \in \mathbb{N} \colon \sum_{i=1}^{n} Z_i > u \right\}, \qquad \tau(u) := \inf_{n \in \mathbb{N} : Y(\bar{\Omega}_n) > u} \bar{\Omega}_n,$$

so that $Y(\tau(u)) = \sum_{i=1}^{T(u)} Z_i$. In this section we assume that the *net profit condition* is in place, i.e. $\mathrm{E}\, Z < 0$ or equivalently $\varphi'(0) > 0$, so that bankruptcy becomes increasingly rare as $u \to \infty$. To find a convenient expression for $\mathfrak{p}(u \mid \Omega)$, we work with an alternative measure $\mathbb{Q}$. We assume that we are in the *light-tailed regime*, in that there is a $\theta^\star > 0$ such that $\mathrm{E}yy\, \mathrm{e}^{\theta^\star Z} = 1$. Then the probability measure $\mathbb{Q}$ is defined through the relation

$$\mathbb{Q}(Z \in \mathrm{d}x) = \mathrm{P}(Z \in \mathrm{d}x)\, \mathrm{e}^{\theta^\star x}, \tag{22}$$

which integrates to 1 by the definition of $\theta^\star$. Note that

$$\mathrm{E}y\, \mathrm{e}^{\theta Z} = \int_0^\infty \mathrm{P}(\Omega \in \mathrm{d}t)\, \mathrm{E}y\, \mathrm{e}^{\theta Y(t)} = \int_0^\infty \mathrm{P}(\Omega \in \mathrm{d}t)\, \mathrm{e}^{\varphi(-\theta)\, t} = \mathrm{E}\, \mathrm{e}^{\varphi(-\theta)\, \Omega}.$$

As $\Omega \geqslant 0$, to obtain $\mathrm{E}yy\, \mathrm{e}^{\theta^\star Z} = 1$, one should necessarily have $\varphi(-\theta^\star) = 0$, implying that $\mathbb{Q}$ is *not* affected by the distribution of the inter-inspection times Ω. Note that due to the convexity of $\varphi(\cdot)$ and $\varphi'(0) > 0$, we have $\varphi'(-\theta^\star) < 0$; throughout we assume that $|\varphi'(-\theta^\star)| < \infty$. Under the measure $\mathbb{Q}$, the Laplace exponent of the driving Lévy process becomes $\varphi_{\mathbb{Q}}(\alpha) := \varphi(\alpha - \theta^\star)$; in the following $\psi_{\mathbb{Q}}(\cdot)$ denotes the right inverse of $\varphi_{\mathbb{Q}}(\cdot)$.

Let $\ell(x)$ denote the likelihood ratio $\mathrm{dP}/\mathrm{d}\mathbb{Q}$ evaluated in the argument x. Then we have the fundamental identity [8, Section XIII.3], describing the 'translation' from sampling under the alternative measure $\mathbb{Q}$ to sampling under the actual measure P:

$$\mathrm{P}(T(u) < \infty) = \mathrm{E}y\, 1\{T(u) < \infty\} = \mathrm{E}_{\mathbb{Q}}\big[1\{T(u) < \infty\}\, \ell(Z_1) \cdots \ell(Z_{T(u)})\big],$$

where we also use the independence of the random variables Z_i.

It is readily verified that

$$\frac{\mathrm{d}}{\mathrm{d}\theta} \mathrm{E}y_{\mathbb{Q}} | \mathrm{e}^{\theta Z} = -\varphi'(-\theta - \theta^\star)\, \mathrm{E}y\, \big[\Omega\, \mathrm{e}^{\varphi(-\theta-\theta^\star)\Omega}\big],$$

so that

$$\mathrm{E}_{\mathbb{Q}} Z = \frac{\mathrm{d}}{\mathrm{d}\theta} \mathrm{E}_{\mathbb{Q}}\, \mathrm{e}^{\theta Z}\Big|_{\theta=0} = -\varphi'(-\theta^\star)\, \mathrm{E}\, \big[\Omega\, \mathrm{e}^{\varphi(-\theta^\star)\Omega}\big] > 0.$$

This means that $T(u) < \infty$ almost surely under $\mathbb{Q}$ (i.e. bankruptcy is not rare under $\mathbb{Q}$), entailing that

$$\mathrm{P}(T(u) < \infty) = \mathrm{E}_{\mathbb{Q}}\big[\ell(Z_1) \cdots \ell(Z_{T(u)})\big].$$

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Due to the fact that $\ell(Z_i) = \mathrm{e}^{-\theta^\star Z_i}$, upon combining the above, we thus find that

$$\mathfrak{p}(u \mid \Omega) = \mathrm{E}_{\mathbb{Q}}\big[\ell(Z_1)\cdots\ell(Z_{T(u)})\big] = \mathrm{E}_{\mathbb{Q}} \exp\left(-\theta^\star \sum_{i=1}^{T(u)} Z_i\right) = \mathrm{E}_{\mathbb{Q}}\, \mathrm{e}^{-\theta^\star Y(\tau(u))}.$$

We have arrived at the following result.

Theorem 5. *For any $u \geqslant 0$,*

$$\mathfrak{p}(u \mid \Omega) = \mathrm{E}_{\mathbb{Q}}\, \mathrm{e}^{-\theta^\star Y(\tau(u))}.$$

A consequence of this result is the following uniform bound on $\mathfrak{p}(u \mid \Omega)$, which can be seen as a Lundberg-type inequality for our model with i.i.d. inter-inspection times. It follows from the obvious property $Y(\tau(u)) > u$. Remarkably, this upper bound does not depend on the inter-inspection time distribution (and will therefore be a loose bound in case of infrequent inspections).

Corollary 1. *For any $u \geqslant 0$, irrespective of the inter-inspection time Ω,*

$$\mathfrak{p}(u \mid \Omega) \leqslant \mathrm{e}^{-\theta^\star u}.$$

6.2. Exact asymptotics

In the remainder of this section our objective is to use Theorem 5 to identify the exact asymptotics of $\mathfrak{p}(u \mid \Omega)$. More concretely, we aim to identify the constant $\gamma[\Omega]$ featuring in

$$\lim_{u\to\infty} \mathfrak{p}(u \mid \Omega)\, \mathrm{e}^{\theta^\star u} =: \gamma[\Omega] \tag{23}$$

for the inter-inspection times Ω studied in Sections 3 and 4. In view of Theorem 5, this means that, with $R(u)$ denoting the *overshoot* $Y(\tau(u)) - u$, we wish to evaluate the object

$$\gamma[\Omega] = \lim_{u\to\infty} \mathrm{E}_{\mathbb{Q}} \mathrm{e}^{-\theta^\star R(u)}.$$

Following [25], the idea is to evaluate this quantity using, with U_α denoting an exponentially distributed random variable with parameter α (i.e. with mean α^{-1}, having probability density function $\alpha \mathrm{e}^{-\alpha u}$),

$$\gamma[\Omega] = \lim_{\alpha\downarrow 0} \mathrm{E}_{\mathbb{Q}} \mathrm{e}^{-\theta^\star R(U_\alpha)} = \lim_{\alpha\downarrow 0} \alpha\, \pi^\circ(\alpha \mid \Omega), \tag{24}$$

where the transform

$$\pi^\circ(\alpha \mid \Omega) := \int_0^\infty \mathrm{e}^{-\alpha u}\, \mathrm{E}_{\mathbb{Q}} \mathrm{e}^{-\theta^\star R(u)}\, \mathrm{d}u$$

will be evaluated essentially relying on the techniques developed in Sections 3 and 4. In this derivation we heavily use the 'master equation', with Z distributed as $Y(T_\omega)$,

$$\begin{aligned}\pi^\circ(\alpha \mid \Omega) = &\int_0^\infty \mathrm{e}^{-\alpha u} \int_u^\infty \mathbb{Q}(Z \in \mathrm{d}v)\, \mathrm{e}^{-\theta^\star (v-u)}\, \mathrm{d}u \\ &+ \int_0^\infty \mathrm{e}^{-\alpha u} \int_{-\infty}^u \mathbb{Q}(Z \in \mathrm{d}v)\, \mathrm{E}_{\mathbb{Q}} \mathrm{e}^{-\theta^\star R(u-v)}\, \mathrm{d}u,\end{aligned} \tag{25}$$

where the first term corresponds to the scenario of the first Z directly exceeding level u, and the second term to the scenario that the first Z is still below u; cf. (3).

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6.2.1. *Exponentially distributed inter-inspection times.* Our first goal is to evaluate $\pi^{\circ}(\alpha \mid \Omega)$ for exponentially distributed inter-inspection times with parameter ω, essentially following the approach that we developed in Section 3.1. The underlying idea is to separate Z into the independent parts $Z^{+} := \bar{Y}(T_{\omega})$ and $Z^{-} := -\underline{Y}(T_{\omega})$, the latter random variable being exponentially distributed with parameter $\theta := \psi_{\mathbb{Q}}(\omega)$. As a consequence, the first term on the right-hand side of (25) can be rewritten as

$$\begin{aligned}
&\int_0^{\infty} \mathrm{e}^{-\alpha u} \int_u^{\infty} \mathbb{Q}(Z \in \mathrm{d}v)\, \mathrm{e}^{-\theta^{\star}(v-u)}\, \mathrm{d}u \\
&\quad = \int_{u=0}^{\infty} \mathrm{e}^{-\alpha u} \int_{v=u}^{\infty} \int_{w=0}^{\infty} \theta \mathrm{e}^{-\theta w}\, \mathbb{Q}(Z^{+} - w \in \mathrm{d}v)\, \mathrm{e}^{-\theta^{*}(v-u)}\, \mathrm{d}w\, \mathrm{d}u \\
&\quad = \int_{u=0}^{\infty} \mathrm{e}^{-\alpha u} \int_{v=u}^{\infty} \int_{x=v}^{\infty} \theta \mathrm{e}^{-\theta(x-v)}\, \mathbb{Q}(Z^{+} \in \mathrm{d}x)\, \mathrm{e}^{-\theta^{*}(v-u)}\, \mathrm{d}v\, \mathrm{d}u,
\end{aligned}$$

where in the last equality the change of variable $x := v + w$ has been performed. Swapping the inner two integrals and computing the integral over v yields that the expression in the previous display reads

$$\int_{u=0}^{\infty} \mathrm{e}^{-\alpha u} \int_{x=u}^{\infty} \frac{\theta}{\theta^{\star} - \theta} \left(\mathrm{e}^{-\theta(x-u)} - \mathrm{e}^{-\theta^{\star}(x-u)}\right) \mathbb{Q}(Z^{+} \in \mathrm{d}x)\, \mathrm{d}u.$$

After interchanging the two remaining integrals and integrating over u, we readily obtain

$$\frac{\theta}{\theta^{\star} - \theta} \int_0^{\infty} \left(\frac{\mathrm{e}^{-\theta x} - \mathrm{e}^{-\alpha x}}{\alpha - \theta} - \frac{\mathrm{e}^{-\theta^{\star} x} - \mathrm{e}^{-\alpha x}}{\alpha - \theta^{\star}} \right) \mathbb{Q}(Z^{+} \in \mathrm{d}x).$$

We thus end up with the first integral on the right-hand side of (25) equalling

$$\int_0^{\infty} \mathrm{e}^{-\alpha u} \int_u^{\infty} \mathbb{Q}(Z \in \mathrm{d}v)\, \mathrm{e}^{-\theta^{\star}(v-u)}\ \mathrm{d}u = F^{\circ}(\alpha, \omega),$$

where, recalling that $\theta = \psi_{\mathbb{Q}}(\omega)$,

$$F^{\circ}(\alpha, \omega) := \frac{\theta}{\theta^{\star} - \theta} \left(\frac{\xi_{\mathbb{Q}}(\theta, \omega) - \xi_{\mathbb{Q}}(\alpha, \omega)}{\alpha - \theta} - \frac{\xi_{\mathbb{Q}}(\theta^{\star}, \omega) - \xi_{\mathbb{Q}}(\alpha, \omega)}{\alpha - \theta^{\star}} \right) \tag{26}$$

and, using the fact that $\psi_{\mathbb{Q}}(\beta) = \psi(\beta) + \theta^{\star}$ for any $\beta \geqslant 0$,

$$\xi_{\mathbb{Q}}(\alpha, \beta) := \mathrm{E}_{\mathbb{Q}} \mathrm{e}^{-\alpha \bar{Y}(T_{\beta})} = \mathrm{E}_{\mathbb{Q}}\, \mathrm{e}^{-\alpha Z^{+}} = \frac{\alpha - \psi_{\mathbb{Q}}(\beta)}{\varphi_{\mathbb{Q}}(\alpha) - \beta} \frac{\beta}{\psi_{\mathbb{Q}}(\beta)} = \frac{\psi(\beta) + \theta^{\star} - \alpha}{\psi(\beta) + \theta^{\star}} \frac{\beta}{\beta - \varphi(\alpha - \theta^{\star})};$$

cf. (1). Regarding the second integral on the right-hand side of (25), the procedure mimics step by step the one developed in Section 3.1, thus leading to

$$\frac{\theta}{\theta - \alpha} \left(\pi^{\circ}(\alpha \mid T_{\omega})\, \xi_{\mathbb{Q}}(\alpha, \omega) - \pi^{\circ}(\theta \mid T_{\omega})\, \xi_{\mathbb{Q}}(\theta, \omega) \right).$$

Combining the above, we obtain the equation

$$\pi^{\circ}(\alpha \mid T_{\omega}) \left(1 - \frac{\theta}{\theta - \alpha}\, \xi_{\mathbb{Q}}(\alpha, \omega) \right) = F^{\circ}(\alpha, \omega) - G^{\circ}(\alpha, \omega), \quad G^{\circ}(\alpha, \omega) := \theta\, \xi_{\mathbb{Q}}(\theta, \omega) \frac{\pi^{\circ}(\theta \mid T_{\omega})}{\theta - \alpha}.$$

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We are left with determining the unknown constant $\pi^\circ(\theta \mid T_\omega)$. To this end, it is first observed that $\alpha = \theta^\star$ is a root of the left-hand side, and hence, it is a root of the right-hand side as well. This leads to

$$\pi^\circ(\theta \mid T_\omega) = \frac{\theta - \theta^\star}{\theta} \frac{F^\circ(\theta^\star, \omega)}{\xi_{\mathbb{Q}}(\theta, \omega)}.$$

We conclude that

$$\pi^\circ(\alpha \mid T_\omega) = \frac{(\theta - \alpha)F^\circ(\alpha, \omega) - \theta\, \pi^\circ(\theta \mid T_\omega)\, \xi_{\mathbb{Q}}(\theta, \omega)}{\theta - \alpha - \theta\, \xi_{\mathbb{Q}}(\alpha, \omega)}. \tag{27}$$

Now that we have found the transform $\pi^\circ(\alpha \mid T_\omega)$, we are in the position to identify the constant $\gamma[T_\omega]$. Indeed, appealing to relation 24, we find as a direct application of L'Hôpital's rule, that the expression for $\pi^\circ(\alpha \mid T_\omega)$ displayed in (27) leads to

$$\gamma[T_\omega] = \lim_{\alpha \downarrow 0} \alpha\, \pi^\circ(\alpha \mid T_\omega) = \lim_{\alpha \downarrow 0} \frac{\alpha\, \kappa}{\theta - \alpha - \theta\, \xi_{\mathbb{Q}}(\alpha, \omega)} = \frac{\kappa}{-\big(1 + \theta f(0, \omega)\big)},$$

where

$$f(0, \omega) := \left. \frac{\mathrm{d}}{\mathrm{d}\alpha} \xi_{\mathbb{Q}}(\alpha, \omega) \right|_{\alpha = 0} = \frac{1}{\theta} \frac{\theta\, \varphi'_{\mathbb{Q}}(0) - \omega}{\omega},$$
$$\kappa := \theta\big(F^\circ(0, \omega) - \pi^\circ(\theta \mid T_\omega)\, \xi_{\mathbb{Q}}(\theta, \omega)\big) = \theta\, F^\circ(0, \omega) + (\theta^\star - \theta)\, F^\circ(\theta^\star, \omega).$$

Observe that, using $\theta = \psi_{\mathbb{Q}}(\omega) = \psi(\omega) + \theta^\star$,

$$1 + \theta f(0, \omega) = \frac{\theta\, \varphi'_{\mathbb{Q}}(0)}{\omega}.$$

Our next task is to write the constant κ in a more explicit form. By computing $F^\circ(0, \omega)$ and $F^\circ(\theta^\star, \omega)$, it is directly seen that

$$\kappa = \frac{\theta}{\psi(\omega)} \Big(\xi_{\mathbb{Q}}(\theta, \omega) - 1 - \frac{\theta}{\theta^\star} \big(\xi_{\mathbb{Q}}(\theta^\star, \omega) - 1 \big) - \xi_{\mathbb{Q}}(\theta, \omega) + \xi_{\mathbb{Q}}(\theta^\star, \omega) + \psi(\omega) f(\theta^\star, \omega) \Big),$$

where

$$\xi_{\mathbb{Q}}(\theta^\star, \omega) = \frac{\psi(\omega)}{\theta}, \qquad f(\theta^\star, \omega) := \left. \frac{\mathrm{d}}{\mathrm{d}\alpha} \xi_{\mathbb{Q}}(\alpha, \omega) \right|_{\alpha = \theta^\star} = \frac{1}{\theta} \frac{\psi(\omega)\, \varphi'(0) - \omega}{\omega}.$$

Putting the above results together, standard algebra yields that $\kappa = -\psi(\omega)\, \varphi'(0)/\omega$. This leads us to the following conclusion, in line with [14, Proposition 4.1].

Theorem 6. *As $u \to \infty$, we have $\mathfrak{p}(u \mid T_\omega)\, \mathrm{e}^{\theta^\star u} \to \gamma[T_\omega]$, where*

$$\gamma[T_\omega] = -\frac{\varphi'(0)}{\varphi'_{\mathbb{Q}}(0)} \frac{\psi(\omega)}{\theta} = -\frac{\varphi'(0)}{\varphi'_{\mathbb{Q}}(0)} \frac{\psi(\omega)}{\psi(\omega) + \theta^\star}.$$

As the inspection rate ω grows large, we recover the asymptotics under permanent inspection. Indeed, in agreement with the findings of [21], revealing the exact asymptotics for spectrally positive Lévy processes, we obtain $\gamma[T_\omega] \to -\varphi'(0)/\varphi'_{\mathbb{Q}}(0)$ as $\omega \to \infty$.

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6.2.2. *Hyperexponentially distributed inter-inspection times.* Denoting as before $\theta_i := \psi_{\mathbb{Q}}(\omega_i)$, following the same approach as in the case of exponentially distributed inter-inspection times, we obtain the relation

$$\pi^\circ(\alpha \mid T_{\boldsymbol{\omega},p}) \left(1 - \sum_{i=1}^{d} p_i \frac{\theta_i}{\theta_i - \alpha}\, \xi_{\mathbb{Q}}(\alpha, \omega_i)\right) = \sum_{i=1}^{d} p_i \left(F^\circ(\alpha, \omega_i) - G^\circ(\alpha, \omega_i)\right),$$

with $F^\circ(\cdot, \cdot)$ as defined in (26) with θ replaced by θ_i and

$$G^\circ(\alpha, \omega_i) := \theta_i\, \xi_{\mathbb{Q}}(\theta_i, \omega_i) \frac{\pi^\circ(\theta_i \mid T_{\boldsymbol{\omega},p})}{\theta_i - \alpha}.$$

Inserting the expression for $\xi_{\mathbb{Q}}(\alpha, \omega_i)$, we thus obtain

$$\pi^\circ(\alpha \mid T_{\boldsymbol{\omega},p}) = \frac{\displaystyle\sum_{i=1}^{d} p_i (F^\circ(\alpha, \omega_i) - (\theta_i/(\theta_i - \alpha))\, \xi_{\mathbb{Q}}(\theta_i, \omega_i)\, z_i)}{\displaystyle 1 - \sum_{i=1}^{d} p_i (\omega_i/(\omega_i - \varphi(\alpha - \theta^\star)))}, \tag{28}$$

where the d unknowns $z_i := \pi^\circ(\theta_i \mid T_{\boldsymbol{\omega},p})$ are yet to be determined. The denominator on the right-hand side of 28 (and, hence, the numerator as well) has precisely d roots in the right half of the complex plane, say $\alpha_1, \ldots, \alpha_d$. This follows by the same line of reasoning as in the proof of Lemma 1. We consider the same MAP, but now we have $\varphi_1(\alpha) = \ldots = \varphi_d(\alpha) = \varphi_{\mathbb{Q}}(\alpha)$ and $\boldsymbol{\eta} = 0$ (i.e. no killing). From [23, Theorem 2] we know that, since we have positive drift under $\mathbb{Q}$, the equation has exactly d roots in the right half of the complex plane. Applying the same reasoning as in Remark 2, the roots $\alpha_1, \ldots, \alpha_d$ are real-valued, and hence, the solution z of the linear system, with $j = 1, \ldots, d$,

$$\sum_{i=1}^{d} p_i F^\circ(\alpha_j, \omega_i) = \sum_{i=1}^{d} p_i \frac{\theta_i}{\theta_i - \alpha_j}\, \xi_{\mathbb{Q}}(\theta_i, \omega_i)\, z_i$$

is real-valued as well. Defining

$$\Phi(\alpha) := \prod_{i=1}^{d} (\omega_i - \varphi(\alpha - \theta^\star)), \qquad \Phi_i(\alpha) := \frac{\Phi(\alpha)}{\omega_i - \varphi(\alpha - \theta^\star)},$$

and applying the same reasoning as in the exponential case, we have that the expression for $\pi^\circ(\alpha \mid T_{\omega,p})$ displayed in (28) leads to

$$\gamma[T_{\omega,p}] = \lim_{\alpha \downarrow 0} \alpha\, \pi^\circ(\alpha \mid T_{\omega,p}) = \lim_{\alpha \downarrow 0} \frac{\alpha K(\alpha)\, \Phi(\alpha)}{\Phi(\alpha) - \sum_{i=1}^{d} p_i \omega_i \Phi_i(\alpha)},$$

where

$$K(\alpha) := \sum_{i=1}^{d} p_i \Big(F^\circ(\alpha, \omega_i) - \frac{\theta_i}{\theta_i - \alpha} \xi_{\mathbb{Q}}(\theta_i, \omega_i)\, z_i\Big).$$

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We obtain the following result.

Theorem 7. *As $u \to \infty$, we have $\mathfrak{p}(u \mid T_{\boldsymbol{\omega},p})\, \mathrm{e}^{\theta^\star u} \to \gamma[T_{\boldsymbol{\omega},p}]$, where*

$$\gamma[T_{\boldsymbol{\omega},p}] = \frac{\Phi(0) \sum_{i=1}^{d} p_i \big(F^\circ(0,\omega_i) - \xi_{\mathbb{Q}}(\theta_i,\omega_i)\, z_i\big)}{\Phi'(0) - \sum_{i=1}^{d} p_i \Phi_i'(0)\, \omega_i}.$$

6.2.3. *Erlang distributed inter-inspection times.* In the Erlang case the computations become somewhat more involved. It is readily verified that the first term on the right-hand side of (25) can be rewritten as

$$\int_{u=0}^{\infty} \mathrm{e}^{-\alpha u} \int_{x=u}^{\infty} \int_{v=u}^{x} \mathrm{e}^{-\theta(x-v)} \frac{\theta^k (x-v)^{k-1}}{(k-1)!}\, \mathbb{Q}(Z_k^+ \in \mathrm{d}x)\, \mathrm{e}^{-\theta^*(v-u)}\, \mathrm{d}v\, \mathrm{d}u. \tag{29}$$

Then we use the fact that

$$\int_u^x \mathrm{e}^{-(\theta^\star-\theta)v} \frac{\theta^k (x-v)^{k-1}}{(k-1)!} \mathrm{d}v = \left(\frac{\theta}{\theta-\theta^\star}\right)^k \left(\mathrm{e}^{-(\theta^\star-\theta)x} - \sum_{n=0}^{k-1} \mathrm{e}^{-(\theta^\star-\theta)u} \frac{((\theta^\star-\theta)(x-u))^n}{n!}\right),$$

so that (29) reads, after changing the order of integration,

$$\int_{x=0}^{\infty} \int_{u=0}^{x} \left(\mathrm{e}^{-\theta^\star x} \mathrm{e}^{-(\alpha-\theta^\star)u} \mathrm{d}u - \sum_{n=0}^{k-1} \int_{u=0}^{x} \mathrm{e}^{-\alpha x} \mathrm{e}^{-(\theta-\alpha)u} \frac{((\theta^\star-\theta)u)^n}{n!} \mathrm{d}u\right)$$
$$\times \left(\frac{\theta}{\theta-\theta^\star}\right)^k \mathbb{Q}(Z_k^+ \in \mathrm{d}x).$$

After straightforward calculus we conclude that the first term on the right-hand side of 25 equals, with $\xi_{\mathbb{Q},k}(\alpha,\beta)$ defined in the evident manner,

$$I^\circ(\alpha,\omega) := \left(\frac{\theta}{\theta-\theta^\star}\right)^k \frac{\xi_{\mathbb{Q},k}(\theta^\star,\omega) - \xi_{\mathbb{Q},k}(\alpha,\omega)}{\alpha-\theta^\star}$$
$$- \left(\frac{\theta}{\theta-\theta^\star}\right)^k \sum_{n=0}^{k-1} \frac{(\theta-\theta^\star)^n}{(\theta-\alpha)^{n+1}} \left(\xi_{\mathbb{Q},k}(\alpha,\omega) - \sum_{m=0}^{n} \xi_{\mathbb{Q},k}^{(m)}(\theta,\omega) \frac{(\alpha-\theta)^m}{m!}\right).$$

The second term on the right-hand side of (25) can be found analogously to Lemma 3. Defining, recalling the definition of $\bar{J}_i(\alpha,\beta,\omega)$ from (12),

$$\bar{J}_i^\circ(\alpha,\omega) := \sum_{n=i}^{k-1} \delta_{k-1-n,k}^{\mathbb{Q}} \frac{(\alpha-\theta)^i}{i!} \left(\frac{\theta}{\theta-\alpha}\right)^{n+1}, \qquad \delta_{n,k}^{\mathbb{Q}} := \frac{(-\theta)^n}{n!}\, \xi_{\mathbb{Q},k}^{(n)}(\theta,\omega),$$

we end up with, writing $J^\circ(\alpha,\omega) := \sum_{i=0}^{k-1} \bar{J}_i^\circ(\alpha,\omega)\, z_i$ with $z_i := (\pi^\circ)^{(i)}(\theta \mid E_{k,\omega})$,

$$\pi^\circ(\alpha \mid E_{k,\omega}) = I^\circ(\alpha,\omega) - J^\circ(\alpha,\omega) + \left(\frac{\omega}{\omega-\varphi(\alpha-\theta^\star)}\right)^k \pi^\circ(\alpha \mid E_{k,\omega}),$$

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or equivalently

$$\pi^\circ(\alpha \mid E_{k,\omega}) = \frac{I^\circ(\alpha,\omega) - J^\circ(\alpha,\omega)}{1 - (\omega/(\omega - \varphi(\alpha - \theta^\star)))^k},$$

where we still need to identify, for $i = 0, \ldots, k-1$, the constants z_i.
As before, an application of [23, Theorem 2] reveals that the denominator has precisely k roots in the right half of the complex plane, say $\alpha_1, \ldots, \alpha_k$, and, hence, the corresponding numerator as well. It implies that z can be determined by solving, for $j = 1, \ldots, k$,

$$I^\circ(\alpha_j,\omega) = \sum_{i=0}^{k-1} \bar{J}_i^\circ(\alpha_j,\omega)\, z_i.$$

We obtain the following result, by following the same procedure as before; observe that $\varphi'(-\theta^\star) < 0$.

Theorem 8. *As $u \to \infty$, we have $\mathfrak{p}(u \mid E_{k,\omega})\, \mathrm{e}^{\theta^\star u} \to \gamma[E_{k,\omega}]$, where*

$$\gamma[E_{k,\omega}] = \frac{I^\circ(0,\omega) - J^\circ(0,\omega)}{-k\,\varphi'(-\theta^\star)/\omega}.$$

6.2.4. *Hyper-Erlang distributed inter-inspection times.* Following the same approach as in the case of Erlang distributed inter-inspection times, we obtain the relation

$$\pi^\circ(\alpha \mid E_{k,p,\omega}) = \frac{\displaystyle\sum_{i=1}^{d} p_i \big(I_i^\circ(\alpha,\omega) - J_i^\circ(\alpha,\omega)\big)}{\displaystyle 1 - \sum_{i=1}^{d} p_i (\omega/(\omega - \varphi(\alpha - \theta^\star)))^{k_i}},$$

where

$$I_i^\circ(\alpha,\omega) := \left(\frac{\theta}{\theta - \theta^\star}\right)^{k_i} \frac{\xi_{\mathbb{Q},k_i}(\theta^\star,\omega) - \xi_{\mathbb{Q},k_i}(\alpha,\omega)}{\alpha - \theta^\star} - \left(\frac{\theta}{\theta - \theta^\star}\right)^{k_i} \sum_{n=0}^{k_i-1} \frac{(\theta - \theta^\star)^n}{(\theta - \alpha)^{n+1}} \left(\xi_{\mathbb{Q},k_i}(\alpha,\omega) - \sum_{m=0}^{n} \xi_{\mathbb{Q},k_i}^{(m)}(\theta,\omega) \frac{(\alpha - \theta)^m}{m!}\right),$$

and, with $z_j := (\pi^\circ)^{(j)}(\theta \mid E_{k,p,\omega})$,

$$\bar{J}_{i,j}^\circ(\alpha,\omega) := \sum_{n=j}^{k_i-1} \delta_{k_i-1-n,k_i}^{\mathbb{Q}} \frac{(\alpha - \theta)^j}{j!} \left(\frac{\theta}{\theta - \alpha}\right)^{n+1}, \qquad J_i^\circ(\alpha,\omega) := \sum_{j=0}^{k_i-1} \bar{J}_{i,j}^\circ(\alpha,\omega)\, z_j.$$

As before, assuming (without loss of generality) that $k_1 < \cdots < k_d$, we are left with the task to identify the constants z_j for $j = 0, \ldots, k_d - 1$. Again, application of [23, Theorem 2] reveals that the denominator has exactly k_d roots in the right half of the complex plane, say $\alpha_1, \ldots, \alpha_{k_d}$, which are then roots of the numerator as well. Hence, the vector z can be determined by solving the linear system

$$\sum_{i=1}^{d} p_i I_i^\circ(\alpha_\ell,\omega) = \sum_{i=1}^{d} p_i \sum_{j=0}^{k_i-1} \bar{J}_{i,j}^\circ(\alpha_\ell,\omega)\, z_j,$$

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for $\ell = 1, \dots, k_d$. We obtain the following result.

Theorem 9. *As $u \to \infty$, we have $\mathfrak{p}(u \mid E_{k,p,\omega})\, \mathrm{e}^{\theta^\star u} \to \gamma[E_{k,p,\omega}]$, where*

$$\gamma[E_{k,p,\omega}] = \frac{\displaystyle\sum_{i=1}^{d} p_i \big(I_i^\circ(0,\omega) - J_i^\circ(0,\omega)\big)}{\displaystyle -\sum_{i=1}^{d} p_i k_i\, \varphi'(-\theta^\star)/\omega}.$$

7. Numerical evaluation, rare-event simulation

The objective of this section is to demonstrate how to numerically evaluate $\mathfrak{p}(u \mid \Omega)$, the all-time ruin probability at inspection epochs. Two techniques are highlighted: a highly efficient importance sampling technique and the numerical evaluation of the Cramér–Lundberg-type asymptotics. We start by explaining our importance sampling procedure and prove that it is logarithmically efficient. Then we introduce the class of instances considered in our experiments and point out how to numerically evaluate the constants $\theta^\star$ and $\gamma[\Omega]$. We conclude this section by presenting the output of our experiments.

7.1. Importance sampling

As we observed before, the process $(Y(\bar{\Omega}_n))_{n\in\mathbb{N}}$ constitutes a random walk with i.i.d. increments distributed as the random variable Z. The results of [24] reveal that in the light-tailed regime introduced in Section 6, we have that (6) holds, suggesting the approximation

$$\mathfrak{p}(u \mid \Omega) \approx \gamma[\Omega]\, \mathrm{e}^{-\theta^\star u}. \tag{30}$$

This approach, however, has two intrinsic difficulties. In the first place, the exact asymptotics relate to the regime that u grows large, so that the approximation is expected to be accurate for large u. Hence, in the absence of error bounds, it is not *a priori* clear whether for the value of u under consideration, (30) provides an accurate approximation. In the second place, while we have succeeded to compute $\gamma[\Omega]$ for various types of inter-inspection times Ω, no expression is available for general Ω. As detailed in [24, Theorem 3], with

$$T := \inf\left\{ n \in \mathbb{N} \colon \sum_{i=1}^{n} Z_i > 0 \right\}, \qquad a := \mathrm{E}\big[Z\, \mathrm{e}^{\theta^\star Z} 1\{T < \infty\}\big],$$

we do have that

$$\gamma[\Omega] = \frac{1 - \mathfrak{p}(0 \mid \Omega)}{\theta^\star\, a},$$

but it is not clear how to compute, for an arbitrarily distributed inter-inspection time Ω, the quantities $\mathfrak{p}(0 \mid \Omega)$ and a.

The above considerations show that there is a need to develop computational tools to efficiently and accurately evaluate $\mathfrak{p}(u \mid \Omega)$ for any nonnegative random variable Ω and $u > 0$. The numerical procedure we discuss in this subsection is based on simulation: we devise an importance sampling algorithm that applies the change of measure that we worked with in Section 6. It succeeds in efficiently generating precise estimates for $\mathfrak{p}(u \mid \Omega)$, even in the regime that u grows large (and ruin is rare).

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The starting point of the algorithm is Theorem 5. In each (independent) run, the model is simulated under $\mathbb{Q}$ until the hitting time $\tau(u)$, where we recall that u is reached almost surely. Let W_i denote the value of $Y(\tau(u))$ in the ith run. Our Siegmund-type estimator [10, Section VI.2a], based on n runs, is

$$\mathfrak{p}^{(n)}(u) := \frac{1}{n}\sum_{i=1}^{n} \mathrm{e}^{-\theta^\star W_i}, \tag{31}$$

where the sample variance of this estimator is

$$v_n(u) := \frac{1}{n-1}\left(\sum_{i=1}^{n} (\mathrm{e}^{-\theta^\star W_i} - \mathfrak{p}^{(n)}(u))^2\right).$$

Noting that $W_i \geqslant u$, from

$$\mathbb{V}\mathrm{ar}_{\mathbb{Q}}\, \mathrm{e}^{-\theta^\star W_i} = \mathrm{E}_{\mathbb{Q}}\mathrm{e}^{-2\theta^\star W_i} - (\mathrm{E}_{\mathbb{Q}}\mathrm{e}^{-\theta^\star W_i})^2 \leqslant \mathrm{E}_{\mathbb{Q}}\mathrm{e}^{-2\theta^\star W_i} \leqslant \mathrm{e}^{-2\theta^\star u}$$

the following result follows directly; see [10, Section VI.1] for the definition of logarithmic efficiency.

Theorem 10. *The estimator (31) is logarithmically efficient as* $u \to \infty$.

7.2. Change of measure

We proceed by discussing how to sample the process $Y(\,\cdot\,)$ according to the alternative measure $\mathbb{Q}$. As pointed out in Section 6, under $\mathbb{Q}$ the distribution of Z should follow 22. We have

$$\mathrm{E}[\mathrm{e}^{-\alpha Z}] = \int_0^\infty \mathrm{e}^{\varphi(\alpha)\,t}\, \mathrm{P}(\Omega \in \mathrm{d}t) = \mathrm{E}\,\mathrm{e}^{\varphi(\alpha)\,\Omega},$$

so that

$$\mathrm{E}_{\mathbb{Q}}[\mathrm{e}^{-\alpha Z}] = \mathrm{E}\,\mathrm{e}^{\varphi(\alpha-\theta^\star)\,\Omega}.$$

We conclude that under this new measure the net cumulative claim process $Y(\,\cdot\,)$ has Laplace exponent $\varphi(\alpha-\theta^\star)$ instead of $\varphi(\alpha)$, while the distribution of Ω remains unaltered. This leaves us with the question of how to sample a Lévy process with Laplace exponent $\varphi(\alpha-\theta^\star)$? We consider the practically relevant case of $Y(\,\cdot\,)$ corresponding to the sum of two independent processes: a compound Poisson process $C(\,\cdot\,)$ and a Brownian motion $B(\,\cdot\,)$; it is noted that any spectrally positive Lévy process can be approximated arbitrarily closely by such a sum [11]. Let the Brownian motion have variance coefficient σ^2 and let the compound Poisson process have drift r, arrival rate λ, with positive jump sizes characterized by the Laplace–Stieltjes transform $b(\,\cdot\,)$. This means that

$$\varphi(\alpha) = \log \mathrm{E}\exp(-\alpha B(1) - \alpha C(1)) = \frac{\alpha^2\sigma^2}{2} + r\alpha - \lambda(1-b(\alpha)),$$

where we assume that the net profit condition is in place (i.e. $\mathrm{E}\,Y(1) < 0$, or equivalently, $-\lambda\, b'(0) < r$). To identify $Y(\,\cdot\,)$ under $\mathbb{Q}$, observe that

$$\begin{aligned}\varphi(\alpha-\theta^\star) = \varphi_{\mathbb{Q}}(\alpha) &= \frac{(\alpha-\theta^\star)^2\sigma^2}{2} + r(\alpha-\theta^\star) - \lambda(1-b(\alpha-\theta^\star)) + \varphi(-\theta^\star)\\ &= \frac{\alpha^2\sigma^2}{2} + (r-\theta^\star\sigma^2)\alpha - \lambda(b(-\theta^\star) - b(\alpha-\theta^\star))\\ &= \frac{\alpha^2\sigma^2}{2} + (r-\theta^\star\sigma^2)\alpha - \lambda b(-\theta^\star)\left(1 - \frac{b(\alpha-\theta^\star)}{b(-\theta^\star)}\right),\end{aligned}$$

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recalling that $\varphi(-\theta^\star)=0$. This means that under $\mathbb{Q}$ the Brownian part remains unchanged, but that the compound Poisson process now has drift $r-\theta^\star\sigma^2$, the arrival rate becomes $\lambda b(-\theta^\star)$, and the claim sizes are now characterized by the Laplace–Stieltjes transform $b(\,\cdot\,-\theta^\star)/b(-\theta^\star)$.

In the experiments we let the claim sizes be exponentially distributed with parameter μ. It is readily verified that (for $\alpha>-\mu$)

$$\varphi(\alpha)=\alpha\,\frac{\sigma^2\alpha^2+(\mu\sigma^2+2r)\alpha+2(r\mu-\lambda)}{2(\mu+\alpha)},$$

so that

$$\theta^\star=\frac{\mu\sigma^2+2r}{2\sigma^2}-\frac{\sqrt{(\mu\sigma^2+2r)^2-8\sigma^2(r\mu-\lambda)}}{2\sigma^2}.$$

Under $\mathbb{Q}$ the claim sizes are exponentially distributed with parameter $\mu-\theta^\star$.

7.3. Numerical experiments: two-moment approximation for inter-inspection times

In this seriesof numerical experiments we assess $\mathfrak{p}(u\mid\Omega)$ for an instance in which the inter-inspection time Ω does *not* have one of the phase-type distributions covered in this paper, and compare it to its counterpart in which Ω is approximated by $\Omega_{\rm app}$ determined by the two-moment fit that was discussed in Section 5.1. For ease, we use (without losing any generality) the normalization $\mathrm{E}\,\Omega=1$ and vary $\mathbb{V}\,\mathrm{ar}\,\Omega$ (where we cover values both below and above 1). The inter-inspection time Ω we consider has a lognormal distribution, with its two parameters chosen such that the mean is 1 and the variance has the target value.

We proceed by pointing out how to numerically evaluate the constant $\gamma[\Omega_{\rm app}]$ for these $\Omega_{\rm app}$ as follows.

(i) If the coefficient of variation of the inter-inspection times is larger than 1 then the two-moment fit prescribes that we should use a hyperexponential random variable $T_{p,\boldsymbol{\omega}}$ with $d=2$, where the parameter vectors p and $\boldsymbol{\omega}$ are chosen such that the first two moments have the desired values (and in addition the so-called balanced means condition is fulfilled; see e.g. [27, Section 3.1]). It requires some elementary calculus to verify that α_1 and α_2 are, respectively, $\theta^\star$ and the positive root of (with $p:=p_1$)

$$\varphi_{\mathbb{Q}}(\alpha)=(1-p)\omega_1+p\omega_2.$$

As is readily checked from the expression for $\varphi_{\mathbb{Q}}(\alpha)$, finding α_2 requires solving an equation involving a third degree polynomial.

(ii) If the coefficient of variation is smaller than 1 then the two-moment fit entails that we use an $E_{k,p,\omega}$ distribution with $d=2$, $k_1=k$, $k_2=k+1$, and $p_1=p$, where p and k are such that the first two moments have the desired value. In this case $\alpha_1,\ldots,\alpha_{k+1}$ are the roots in the right half of the complex plane of the polynomial equation

$$1-\left(\frac{\omega-\varphi_{\mathbb{Q}}(\alpha)}{\omega}\right)^{k+1}=\frac{p\varphi_{\mathbb{Q}}(\alpha)}{\omega},\tag{32}$$

where, evidently, $\theta^\star$ is one such root. There are two ways of finding the roots $\alpha_1,\ldots,\alpha_{k+1}$, both relying on the availability of a procedure to find the, potentially complex, roots of a polynomial equation, which is available in standard numerical packages. An approach to do so would be to insert the expression for $\varphi_{\mathbb{Q}}(\alpha)$ into (32), and

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to directly solve the resulting polynomial of order $3k+3$, after which one selects the roots with a positive real part. There is an alternative, more efficient, approach, though: (i) first, find the $k+1$ roots for $\varphi_{\mathbb{Q}}(\alpha)$ of (32), say $\varphi_1, \ldots, \varphi_{k+1}$; (ii) then per solution φ_i find the three solutions (in α) of the equation $\varphi_{\mathbb{Q}}(\alpha) = \varphi_i$; (iii) after which one selects, from the resulting $3k+3$ solutions, those with a positive real part (where we know that there are precisely $k+1$ of those).

A second complication lies in the computation of $\delta_{n,k}^{\mathbb{Q}}$ (for $n = 0, \ldots, k-1$), which requires the numerical evaluation of the higher-order derivatives

$$\xi_{\mathbb{Q},k}^{(n)}(\theta, \omega) = \frac{\mathrm{d}^n}{\mathrm{d}\alpha^n} \left(\frac{\alpha - \psi_{\mathbb{Q}}(\beta)}{\varphi_{\mathbb{Q}}(\alpha) - \beta} \frac{\beta}{\psi_{\mathbb{Q}}(\beta)} \right)^k \Bigg|_{\alpha=\theta}.$$

We have streamlined the evaluation of these quantities relying on two ideas. (i) In the first place, it is observed that repeated symbolic derivation of ratios leads to large, unmanageable expressions. To remedy this, first write

$$\xi_{\mathbb{Q},k}(\alpha, \omega) \left((\varphi_{\mathbb{Q}}(\alpha) - \beta)) \, \psi_{\mathbb{Q}}(\beta) \right)^k = \left((\alpha - \psi_{\mathbb{Q}}(\beta)) \, \beta \right)^k,$$

and then differentiate n times using Leibniz' rule. This yields a recursion for the high-order derivatives of $\xi_{\mathbb{Q},k}(\alpha, \omega)$; this we did symbolically using SymPy, the open-source Python library for symbolic computation. (ii) In the second place, inserting $\alpha = \theta$ into this recursion leads to floating point errors due to division by extremely small numbers. To resolve this, we have inserted $\alpha = \theta \pm \varepsilon$ for a small value of $\varepsilon > 0$, and have taken the average of the resulting two numbers.

As mentioned, in the experiments we consider the case of exponentially distributed claims with parameter μ. The parameters chosen are $\lambda = 2$, $r = 1.2$, $\sigma^2 = 0.02$, $\mu = 2$, such that $\theta^\star = 0.32875$.

In Figure 1 we display, for various values of the initial surplus u, $\gamma_u[\Omega] := \mathfrak{p}(u \mid \Omega)\, \mathrm{e}^{\theta^\star u}$ as a function of $\mathbb{V}\mathrm{ar}\, \Omega$, relying on our importance sampling-based Monte Carlo procedure. In Figure 2 we show the corresponding results for its phase-type counterpart $\gamma_u[\Omega_{\mathrm{app}}] := \mathfrak{p}(u \mid \Omega_{\mathrm{app}})\, \mathrm{e}^{\theta^\star u}$, again using importance sampling. Finally, Figure 3 displays the curve $\gamma[\Omega_{\mathrm{app}}]$, obtained by performing the numerical computations described above.

All computer codes developed for this paper can be accessed in a GitHub repository via the following link: https://github.com/saramorcy/ruin-probabilities. In Figures 1 and 2 we have displayed, for any value of the variance of the inter-inspection time (that is, $\mathbb{V}\mathrm{ar}\, \Omega$ and $\mathbb{V}\mathrm{ar}\, \Omega_{\mathrm{app}}$, respectively) in the range $\{0.02, 0.04, \ldots, 2.98, 3.00\}$, the quantity $\gamma_u[\Omega]$ respectively $\gamma_u[\Omega_{\mathrm{app}}]$. Each data point is estimated using our importance sampling-based Monte Carlo procedure, with the number of runs being equal to 10^5; we chose this relatively high number of runs to make sure the corresponding standard errors are extremely low, thus making the curves relatively smooth.

The main conclusions from the figures are as follows. (i) The experiments provide empirical evidence that $\gamma_u[\Omega]$ converges to a constant $\gamma[\Omega]$ as u grows large, also for nonphase-type Ω. (ii) The experiments confirm that $\gamma_u[\Omega_{\mathrm{app}}]$ converges to a constant $\gamma[\Omega_{\mathrm{app}}]$ as u grows large, in line with our theoretical findings. (iii) The curves $\gamma[\Omega]$ and $\gamma[\Omega_{\mathrm{app}}]$ (that is, as functions of $\mathbb{V}\mathrm{ar}\, \Omega$ and $\mathbb{V}\mathrm{ar}\, \Omega_{\mathrm{app}}$, respectively) are hardly distinguishable. This provides backing for the use of the (relatively low-dimensional) two-moment approximation of the inter-inspection times. (iv) Already for relatively low values of the initial surplus u, the value of $\gamma_u[\Omega_{\mathrm{app}}]$ is close to its limiting value $\gamma[\Omega_{\mathrm{app}}]$.

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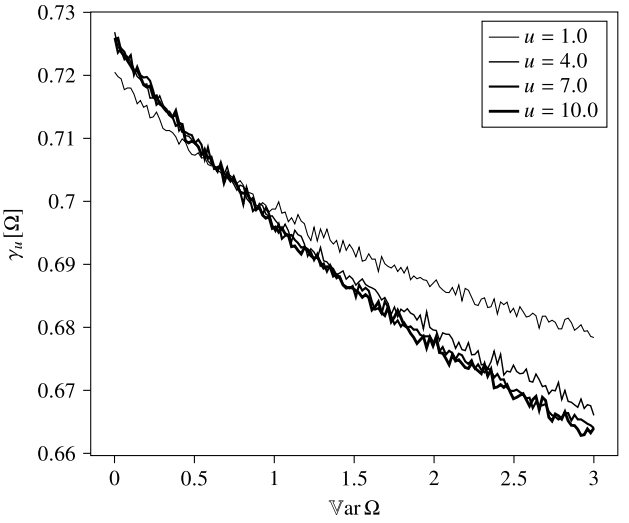


FIGURE 1. Monte Carlo-based estimation of $\gamma_u[\Omega]$.

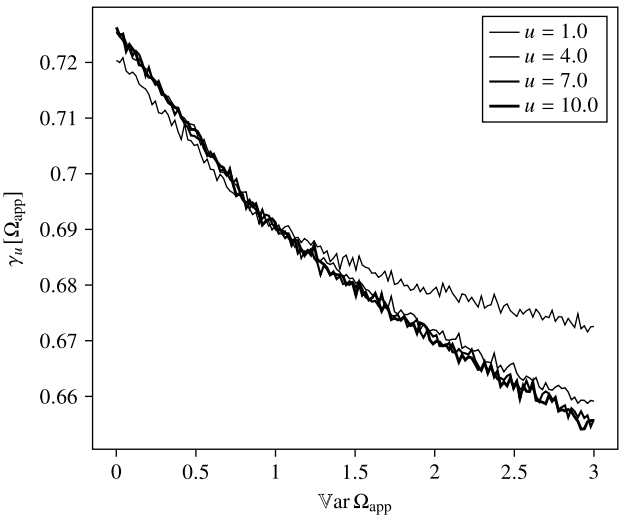


FIGURE 2. Monte Carlo-based estimation of $\gamma_u[\Omega_{\mathrm{app}}]$.

The above observations justify the use of the approximation

$$\mathfrak{p}(u \mid \Omega) \approx \gamma[\Omega_{\mathrm{app}}]\, \mathrm{e}^{-\theta^\star u}.$$

For the specific parameters chosen in this example, observe that the constant $\gamma[\Omega_{\mathrm{app}}]$ only mildly varies in the range $\mathbb{V}\mathrm{ar}\,\Omega \in [0, 3]$, entailing that we could work with the even simpler approximation

$$\mathfrak{p}(u \mid \Omega) \approx \gamma[T_1]\, \mathrm{e}^{-\theta^\star u},$$

where it is recalled that Theorem 6 provides an explicit expression for $\gamma[T_1]$.

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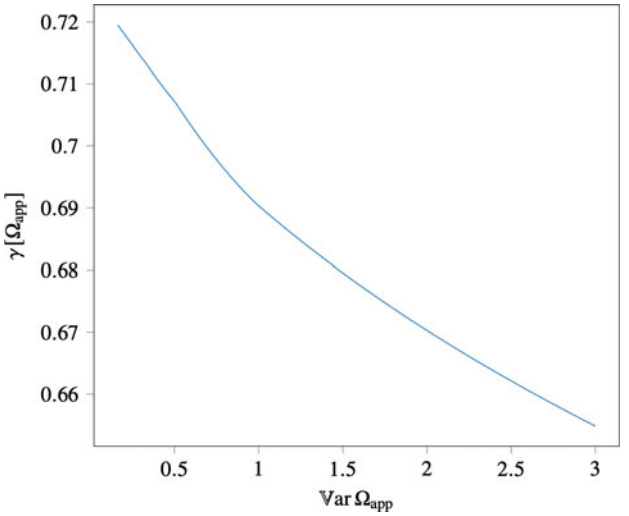


FIGURE 3. Numerical computation for $\gamma[\Omega_{app}]$.

8. Discussion and concluding remarks

This paper has considered the conventional Cramér–Lundberg model, with the distinguishing feature of non-Poissonian inspection. For specific classes of phase-type inter-inspection times, the transform of the bankruptcy probability has been identified, where it is noted that by these classes any distribution on the positive half-line can be approximated arbitrarily closely. We also establish the tail asymptotics of the bankruptcy probability. By using these as a basis for approximations, which in particular apply in the regime of a high initial surplus, numerical Laplace inversion can be avoided. In addition, a provably optimal importance sampling algorithm is proposed, applicable to any inter-inspection time distribution.

Many extensions can be thought of, such as models in which the insurance firm receives interest over its surplus; see e.g. [9, Section VIII.2] and [15] for the case of continuous inspection. Another generalization could concern models with regime switching; in, for example, [9, Section IX.5] and [17, 20] the case of continuous inspection is covered, and [19, 26] specifically consider the case that the background process is not necessarily irreducible. Finally, one could also consider the model of [16], in which the environment is periodically resampled. An evident other topic for future research could be the extension to wider classes of distributions for the inter-inspection times, but the discussion in Section 5.2 indicates that even for general phase-type distributions there are serious challenges.

In this paper the insurance firm goes bankrupt in case the surplus level drops below zero at an inspection time. In the alternative bankruptcy concept of 'Parisian ruin' however, it goes bankrupt as soon as the surplus level process has an excursion below level 0 that is longer than a predefined value [4, 29, 30]; cf. [31, Sections 10.3-10.4]. To the best of our knowledge, no results are known for the mechanism in which there is bankruptcy as soon as there is a negative surplus at a predefined number of inspection times.

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Appendix A. Computations for (hyper-)Erlang inter-inspection times

In order to evaluate the quantity $\pi(\alpha, \beta \mid E_{k,\omega})$, we intensively use the following identities. Some of these are standard, but we include them for completeness.

Lemma 6. *Let X be a nonnegative random variable with LST $\chi(\cdot)$.*

(i) *For any $\kappa \in \mathbb{R}$, $w \geqslant 0$ and $k \in \mathbb{N}$,*

$$\int_0^w \mathrm{e}^{-\kappa v} \frac{\kappa^k v^{k-1}}{(k-1)!}\,\mathrm{d}v = 1 - \sum_{n=0}^{k-1} \mathrm{e}^{-\kappa w} \frac{(\kappa w)^n}{n!}.$$

(ii) *For any $\kappa > 0$ and $k \in \mathbb{N}_0$,*

$$\int_0^\infty \mathrm{e}^{-\kappa x} \frac{(\kappa x)^k}{k!}\,\mathrm{P}(X > x)\,\mathrm{d}x = \frac{1}{\kappa}\left(1 - \sum_{n=0}^{k} \chi^{(n)}(\kappa)\frac{(-\kappa)^n}{n!}\right),$$

where $\chi^{(n)}(\cdot)$ is the nth derivative of $\chi(\cdot)$.

Proof. Part (i) follows from the (known) expression for the cumulative distribution function of an Erlang random variable. Part (ii) follows by noting that, for an Erlang distributed random variable $E_{k,\kappa}$, with shape parameter $k \in \mathbb{N}$ and scale parameter κ, it holds that

$$\begin{aligned}\mathrm{P}(X \geqslant E_{k,\kappa}) &= \int_0^\infty \mathrm{P}(E_{k,\kappa} \leqslant x)\,\mathrm{P}(X \in \mathrm{d}x)\\ &= \int_0^\infty \left(1 - \sum_{n=0}^{k-1} \mathrm{e}^{-\kappa x}\frac{(\kappa x)^n}{n!}\right)\mathrm{P}(X \in \mathrm{d}x)\\ &= 1 - \sum_{n=0}^{k-1} \chi^{(n)}(\kappa)\frac{(-\kappa)^n}{n!},\end{aligned}$$

where the second equality uses the distribution function of the random variable $E_{k,\kappa}$. The claim follows by integration by parts.

Proof of Lemma 2. We start by noting that

$$\int_0^\infty \mathrm{e}^{-\alpha u}\,\mathrm{P}(Z_k > u)\,\mathrm{d}u = \int_0^\infty \int_0^\infty \mathrm{e}^{-\alpha u}\frac{\theta^k v^{k-1}}{(k-1)!}\mathrm{e}^{-\theta v}\mathrm{P}(Z_k^+ > u+v)\,\mathrm{d}v\,\mathrm{d}u,$$

where the equality is due to conditioning on the value of Z_k^-, which by substituting $w := u + v$ and swapping the order of the integrals equals

$$\left(\frac{\theta}{\theta-\alpha}\right)^k \int_0^\infty \left(\int_0^w \mathrm{e}^{-(\theta-\alpha)v}\frac{(\theta-\alpha)^k v^{k-1}}{(k-1)!}\mathrm{d}v\right)\mathrm{e}^{-\alpha w}\,\mathrm{P}(Z_k^+ > w)\;\mathrm{d}w. \tag{33}$$

By Lemma 6(i), a few elementary steps yield that (33) equals

$$\left(\frac{\theta}{\theta-\alpha}\right)^k \int_0^\infty \mathrm{e}^{-\alpha w}\mathrm{P}(Z_k^+ > w)\;\mathrm{d}w - \sum_{n=0}^{k-1}\left(\frac{\theta}{\theta-\alpha}\right)^{k-n}\int_0^\infty \mathrm{e}^{-\theta w}\frac{(\theta w)^n}{n!}\,\mathrm{P}(Z_k^+ > w)\,\mathrm{d}w.$$

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Applying Lemma 6(ii) to both terms, we find that the integral equals $I(\alpha, \beta, \omega)$, as desired.

Proof of Lemma 3. We start by splitting the integral under consideration into two terms: the term $\pi_1(\alpha, \beta \mid E_{k,\omega})$ corresponding to integration over $\{0 \leqslant v \leqslant u\}$ and the term $\pi_2(\alpha, \beta \mid E_{k,\omega})$ corresponding to integration over $\{v \leqslant 0 \leqslant u\}$; cf. the procedure followed in Section 3.1.

First, we evaluate

$$\pi_1(\alpha, \beta \mid E_{k,\omega}) = \int_0^\infty \mathrm{e}^{-\alpha u} \int_0^u \mathrm{P}(Z_k \in \mathrm{d}v)\, \mathfrak{p}_\beta(u - v \mid E_{k,\omega})\, \mathrm{d}u,$$

by conditioning on the value of Z_k^-. We thus obtain

$$\begin{aligned}
&\int_{u=0}^\infty \mathrm{e}^{-\alpha u} \int_{v=0}^u \int_{w=0}^\infty \mathrm{P}(Z_k^+ - w \in \mathrm{d}v)\, \mathrm{e}^{-\theta w} \frac{\theta^k w^{k-1}}{(k-1)!}\, \mathfrak{p}_\beta(u - v \mid E_{k,\omega})\, \mathrm{d}w\, \mathrm{d}u \\
&\quad = \int_{z=0}^\infty \int_{v=0}^\infty \mathrm{e}^{-\alpha(v+z)} \int_{w=0}^\infty \mathrm{P}(Z_k^+ - w \in \mathrm{d}v)\, \mathrm{e}^{-\theta w} \frac{\theta^k w^{k-1}}{(k-1)!}\, \mathfrak{p}_\beta(z \mid E_{k,\omega})\, \mathrm{d}w\, \mathrm{d}z,
\end{aligned}$$

where in the last step the change of variable $z := u - v$ has been performed. We see that this expression can be rewritten as

$$\pi(\alpha, \beta \mid E_{k,\omega}) \int_{v=0}^\infty \mathrm{e}^{-\alpha v} \int_{w=0}^\infty \mathrm{P}(Z_k^+ - w \in \mathrm{d}v)\, \mathrm{e}^{-\theta w} \frac{\theta^k w^{k-1}}{(k-1)!}\, \mathrm{d}w.$$

Now substituting $v + w =: z$ and swapping the order of integration, we obtain

$$\begin{aligned}
&\pi(\alpha, \beta \mid E_{k,\omega}) \int_0^\infty \mathrm{e}^{-\alpha z} \left(\int_0^z \mathrm{e}^{-(\theta-\alpha)(z-v)} \frac{\theta^k (z-v)^{k-1}}{(k-1)!} \mathrm{d}v \right) \mathrm{P}(Z_k^+ \in \mathrm{d}z) \\
&\quad = \pi(\alpha, \beta \mid E_{k,\omega}) \int_0^\infty \mathrm{e}^{-\alpha z} \left(\int_0^z \mathrm{e}^{-(\theta-\alpha)v} \frac{\theta^k v^{k-1}}{(k-1)!} \mathrm{d}v \right) \mathrm{P}(Z_k^+ \in \mathrm{d}z) \\
&\quad = \pi(\alpha, \beta \mid E_{k,\omega}) \left(\frac{\theta}{\theta - \alpha} \right)^k \int_0^\infty \mathrm{e}^{-\alpha z} \left(1 - \sum_{n=0}^{k-1} \mathrm{e}^{-(\theta-\alpha)z} \frac{((\theta-\alpha)z)^n}{n!} \right) \mathrm{P}(Z_k^+ \in \mathrm{d}z) \\
&\quad = \pi(\alpha, \beta \mid E_{k,\omega}) \left(\frac{\theta}{\theta - \alpha} \right)^k \int_0^\infty \left(\mathrm{e}^{-\alpha z} - \sum_{n=0}^{k-1} \mathrm{e}^{-\theta z} \frac{((\theta-\alpha)z)^n}{n!} \right) \mathrm{P}(Z_k^+ \in \mathrm{d}z),
\end{aligned}$$

where in the second equality Lemma 6(i) has been relied upon. Note that this expression can be interpreted as

$$\begin{aligned}
&\pi(\alpha, \beta \mid E_{k,\omega}) \left(\frac{\theta}{\theta - \alpha} \right)^k \left(\xi_k(\alpha, \beta + \omega) - \sum_{n=0}^{k-1} \frac{(\alpha - \theta)^n}{n!} \xi_k^{(n)}(\theta, \beta + \omega) \right) \\
&\quad = \pi(\alpha, \beta \mid E_{k,\omega}) \left(\frac{\theta}{\theta - \alpha} \right)^k \left(\xi_k(\alpha, \beta + \omega) - \sum_{n=0}^{k-1} \left(\frac{\theta - \alpha}{\theta} \right)^n \delta_{n,k} \right) \\
&\quad = \pi(\alpha, \beta \mid E_{k,\omega}) \left(\frac{\beta + \omega}{\beta + \omega - \varphi(\alpha)} \right)^k - \pi(\alpha, \beta \mid E_{k,\omega}) \sum_{n=0}^{k-1} \left(\frac{\theta}{\theta - \alpha} \right)^{k-n} \delta_{n,k}. \qquad (34)
\end{aligned}$$

We finally compute

$$\pi_2(\alpha, \beta \mid E_{k,\omega}) = \int_0^\infty \mathrm{e}^{-\alpha u} \int_{-\infty}^0 \mathrm{P}(Z_k \in \mathrm{d}v)\, \mathfrak{p}_\beta(u - v \mid E_{k,\omega})\, \mathrm{d}u.$$

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Here the first task is to compute $\mathrm{P}(Z_k \in \mathrm{d}v)$ for $v < 0$. To this end, define by $M(t)$ a Poisson process with parameter θ (independently of Z_k^+). Observe that the event $\{Z_k^- > Z_k^+\}$ is equivalent to the event $\{M(Z_k^+) < k\}$; see Figures 4 and 5 (covering the scenarios $M(Z_k^+) < k$ and $M(Z_k^+) \geqslant k$, respectively). Hence, for $v < 0$,

$$\begin{aligned}\mathrm{P}(Z_k \in \mathrm{d}v) &= \sum_{\ell=0}^{k-1} \mathrm{P}\big(Z_k \in \mathrm{d}v,\, M(Z_k^+) = \ell\big) \\ &= \sum_{\ell=0}^{k-1} \mathrm{P}\big(-Z_{k-\ell}^- \in \mathrm{d}v\big)\, \mathrm{P}(M(Z_k^+) = \ell) \\ &= \sum_{\ell=0}^{k-1} \mathrm{e}^{\theta v} \frac{\theta^{k-\ell}(-v)^{k-\ell-1}}{(k-\ell-1)!}\, \mathrm{P}(M(Z_k^+) = \ell)\, \mathrm{d}v,\end{aligned}$$

where the second equality follows by the memoryless property and the last equality follows by realizing that $Z_{k-\ell}^-$ has an Erlang distribution with shape parameter $k-\ell$ and scale parameter θ. Also,

$$\begin{aligned}\mathrm{P}(M(Z_k^+) = \ell) &= \int_0^\infty \mathrm{P}(M(z) = \ell)\, \mathrm{P}(Z_k^+ \in \mathrm{d}z) \\ &= \int_0^\infty \mathrm{e}^{-\theta z} \frac{(\theta z)^\ell}{\ell!}\, \mathrm{P}(Z_k^+ \in \mathrm{d}z) \\ &= \frac{(-\theta)^\ell}{\ell!} \xi_k^{(\ell)}(\theta,\, \beta + \omega) = \delta_{\ell,k}.\end{aligned}$$

We thus conclude that the distribution of the negative of $Z_k^+ - Z_k^-$ can be considered as a mixture of Erlang random variables, entailing that we are to evaluate the integral

$$\int_0^\infty \mathrm{e}^{-\alpha u} \int_{-\infty}^0 \sum_{n=0}^{k-1} \mathrm{e}^{\theta v} \frac{\theta^{n+1}(-v)^n}{n!} \delta_{k-1-n,k}\, \mathfrak{p}_\beta(u - v \mid E_{k,\omega})\, \mathrm{d}v\, \mathrm{d}u;$$

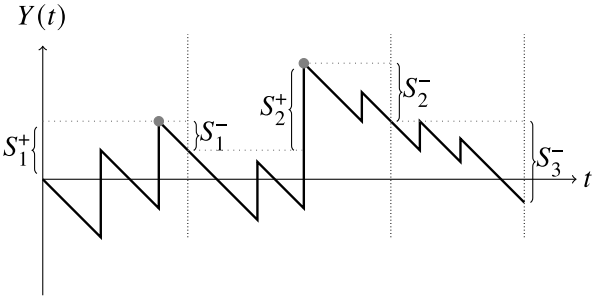
note that we swapped the indices by putting $n := k - \ell - 1$. Then applying the transformation $w := u - v$, and swapping the order of the integrals, after some elementary computations, we arrive at

$$\begin{aligned}&\sum_{n=0}^{k-1} \delta_{k-1-n,k} \left(\frac{\theta}{\theta-\alpha}\right)^{n+1} \int_0^\infty \left(\int_0^w \mathrm{e}^{-(\theta-\alpha)v} \frac{(\theta-\alpha)^{n+1} v^n}{n!} \mathrm{d}v\right) \mathrm{e}^{-\alpha w}\, \mathfrak{p}_\beta(w \mid E_{k,\omega})\, \mathrm{d}w \\ &\quad = \sum_{n=0}^{k-1} \delta_{k-1-n,k} \left(\frac{\theta}{\theta-\alpha}\right)^{n+1} \int_0^\infty \left(1 - \sum_{m=0}^{n} \mathrm{e}^{-(\theta-\alpha)w} \frac{((\theta-\alpha)w)^m}{m!}\right) \mathrm{e}^{-\alpha w}\, \mathfrak{p}_\beta(w \mid E_{k,\omega})\, \mathrm{d}w \\ &\quad = \pi(\alpha,\, \beta \mid E_{k,\omega}) \sum_{n=0}^{k-1} \delta_{k-1-n,k} \left(\frac{\theta}{\theta-\alpha}\right)^{n+1} - J(\alpha,\, \beta,\, \omega). \qquad (35)\end{aligned}$$

Upon adding up the expressions that we have found in (34) and (35), two terms cancel out and we obtain the desired expression.

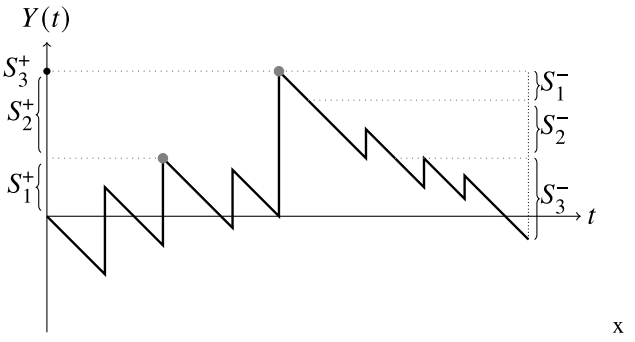
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(a)



The net cumulative claim process in a scenario where $k = 3$, the vertical dotted lines indicate the different phases. In every phase, the decomposition $S_i = S_i^+ - S_i^-$ is displayed. The nodes indicate the location of a maximum over an exponentially distributed interval.

(b)



x

Without loss of generality we may perform a permutation on the $2k$ intervals. First we add the k segments that have gone up by S_i^+, then we place the k segments that have gone down by S_i^-. In this scenario, we observe that $M(Z_k^+) < k$.

FIGURE 4. The decomposition $Z_k = Z_k^+ - Z_k^-$, with $Z_k = \sum_{i=1}^k S_i^+ - \sum_{i=1}^k S_i^-$. Depicted is the event $\{Z_k^+ < Z_k^-\}$, or equivalently $\{M(Z_k^+) < k\}$.

Proof of Lemma 4. We use the precise same line of reasoning as in the proof of Lemma 1. Again $\varphi_1(\alpha) = cldots = \varphi_k(\alpha) = \varphi(\alpha)$ and $\eta_i = \beta$ for $i = 1, \ldots, k$, but now we consider the following MAP, with the entries of the transition rate matrix Q given by the cyclic structure

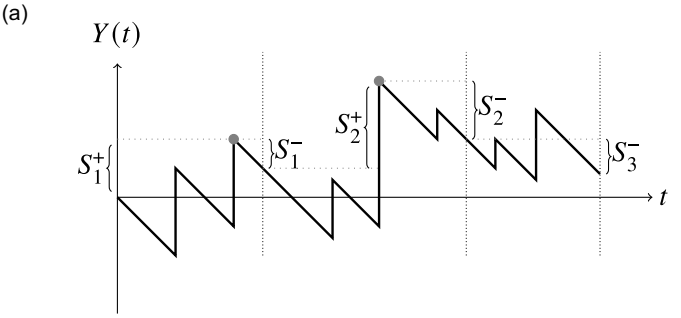
$$q_{i,i} = -\omega, \quad q_{i,i+1} = \omega, \quad q_{k,1} = \omega,$$

for $i = 1, \ldots, k$. Hence,

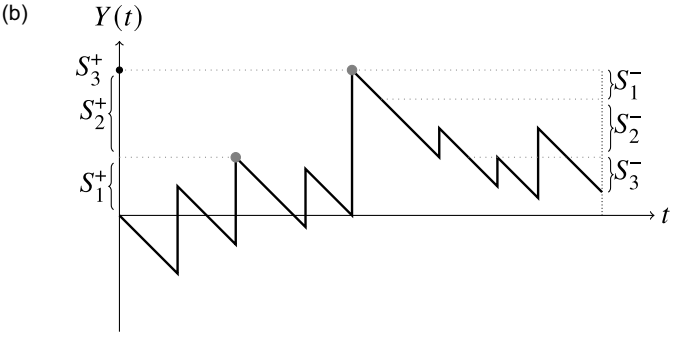
$$M(\alpha, \boldsymbol{\eta}) = Q + \operatorname{diag}(\boldsymbol{\varphi}(\alpha) - \boldsymbol{\eta}),$$

which, unfortunately, does not have the nice property of being the sum of a diagonal matrix and a rank-one matrix (as was the case in the setting of the proof of Lemma 1), since the matrix Q is not of rank one. To show that the equation $\det M(\alpha, \boldsymbol{\eta}) = 0$ is equivalent to 15, we express the determinant in terms of determinants of smaller matrices, by performing a Laplace expansion along the first row. To this end, define $\omega^\star := \beta + \omega - \varphi(\alpha)$ and the matrices $\widehat{Q}$ and $\check{Q}$ given by the entries $\widehat{q}_{ii} = -\omega^\star$, $\widehat{q}_{i,i+1} = \omega$ and $\check{q}_{ii} = \omega$, $\check{q}_{i,i-1} = -\omega^\star$, respectively. As a consequence,

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The cumulative claim process in a scenario where $k = 3$, the vertical dotted lines indicate the different phases. In every phase, the decomposition $S_i = S_i^+ - S_i^-$ is displayed. The nodes indicate the location of a maximum over an exponentially distributed interval.



Without loss of generality we may perform a permutation on the $2k$ intervals. First we add the k segments that have gone up by S_i^+, then we place the k segments that have gone down by S_i^-. In this scenario, we observe that $M(Z_k^+) \geqslant k$.

FIGURE 5. The decomposition $Z_k = Z_k^+ - Z_k^-$, with $Z_k = \sum_{i=1}^k S_i^+ - \sum_{i=1}^k S_i^-$. Depicted is the event $\{Z_k^+ > Z_k^-\}$, or equivalently $\{M(Z_k^+) \geqslant k\}$.

we can write

$$\begin{aligned}\det M(\alpha, \boldsymbol{\eta}) &= -\omega^\star \det \widehat{Q} + (-1)^{k+1}\omega \det \check{Q} \\ &= (-\omega^\star)^k - (-\omega)^k \\ &= (-\omega + \varphi(\alpha) - \beta)^k - (-\omega)^k,\end{aligned}$$

so that $\det M(\alpha, \boldsymbol{\eta}) = 0$ is equivalent to 15. This settles the claim.

Proof of Lemma 5. We use a similar reasoning as in the proofs of Lemmas 1 and 4. Without loss of generality, we had assumed that the shape parameters $k = (k_1, \ldots, k_d)$ are ordered, such that $k_1 < \ldots < k_d$, but in addition we can assume that $k_i = i$. Indeed, the density of a generic inter-inspection time $E_{k,p,\omega}$ can be written as

$$\mathrm{P}(E_{k,p,\omega} \in \mathrm{d}t) = \sum_{i=1}^{k_d} \bar{p}_i \, \omega \frac{(\omega t)^{i-1}}{(i-1)!} \mathrm{e}^{-\omega t} \, \mathrm{d}t,$$

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where $\bar{p}_{k_i} = p_i$ (for $i = 1, \ldots, d$) and 0 otherwise. Hence, we wish to prove that

$$\sum_{i=1}^{k_d} \bar{p}_i \Big(\frac{\omega}{\beta + \omega - \varphi(\alpha)} \Big)^i = 1 \tag{36}$$

has precisely k_d roots in the right half of the complex α plane.
Consider the MAP characterized by $\varphi_1(\alpha) = \cdots = \varphi_{k_d}(\alpha) = \varphi(\alpha)$ and $\eta_i = \beta$ for $i = 1, \ldots, k_d$, while the entries of the first column of the transition rate matrix Q given by, with $\check{p}_i := \sum_{k=1}^{i} \bar{p}_i$ for $i = 1, \ldots, k_d$,

$$q_{1,1} = -(1 - \bar{p}_1)\,\omega, \qquad q_{k_d,1} = \omega, \qquad q_{i,1} = \frac{\bar{p}_i}{1 - \check{p}_{i-1}}\omega \quad \text{for } i = 2, \ldots, k_d - 1,$$

and the other entries given by

$$q_{i,i} = -\omega \quad \text{for } i = 2, \ldots, k_d, \qquad q_{i,i+1} = \frac{1 - \check{p}_i}{1 - \check{p}_{i-1}}\,\omega \quad \text{for } i = 1, \ldots, k_d - 1.$$

As a result,

$$M(\alpha, \boldsymbol{\eta}) = Q + \operatorname{diag}(\boldsymbol{\varphi}(\alpha) - \boldsymbol{\eta}).$$

The rationale behind this choice of Q is the following. Inspection of the proofs of Lemmas 1 and 4 reveals that Q should 'encode' the distribution of Ω. In the present hyper-Erlang case, there are i phases with probability $\bar{p}_i$, which means that with probability $(1 - \check{p}_i)/(1 - \check{p}_{i-1})$ one proceeds to the $(i+1)$th phase after having finished the ith phase (and with the complementary probability $\bar{p}_i/(1 - \check{p}_{i-1})$ one is done).

To show that the equation $\det M(\alpha, \boldsymbol{\eta}) = 0$ is equivalent to (20), we evaluate the determinant by Laplace expansion along the first column. To this end, we define $\omega^\star := \beta + \omega - \varphi(\alpha)$. As a consequence, we can write

$$\begin{aligned}
\det M(\alpha, \boldsymbol{\eta}) &= (\bar{p}_1\omega - \omega^\star)(-\omega^\star)^{k_d-1} + \sum_{j=2}^{k_d} (-1)^{j+1} \frac{\bar{p}_j}{1 - \check{p}_{j-1}} \omega \prod_{i=1}^{j-1} \frac{1 - \check{p}_i}{1 - \check{p}_{i-1}} \omega^{j-1} (-\omega^\star)^{k_d-j} \\
&= (\bar{p}_1\omega - \omega^\star)(-\omega^\star)^{k_d-1} - \sum_{j=2}^{k_d} \bar{p}_j\,(-\omega)^j (-\omega^\star)^{k_d-j} \\
&= (-\omega^\star)^{k_d} - \sum_{j=1}^{k_d} \bar{p}_j\,(-\omega)^j (-\omega^\star)^{k_d-j},
\end{aligned}$$

so that $\det M(\alpha, \boldsymbol{\eta}) = 0$ is equivalent to (36), as desired.

Appendix B. Hyper-Erlang inter-inspection times with distinct scale parameters

In this appendix we briefly point out how the case of hyper-Erlang inter-inspection times with *distinct* scale parameters can be dealt with (recalling that in Section 4.2 this scale parameter ω was held fixed). As can be verified, and bearing in mind the considerations of Section 5, also in this case Z can be decomposed into $Z^+ - Z^-$ with Z^+ and Z^- independent. Note however that in Section 7 we empirically observed that we already obtain a virtually exact result

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when working with hyper-Erlang inter-inspection times with *identical* scale parameters, and therefore, this class discussed in this appendix is of secondary importance.

In this case, the inter-inspection time is Erlang-k_i distributed (for some $k_i \in \mathbb{N}$) with probability p_i, with scale parameter $\omega_i > 0$ (rather than just ω), for $i \in \{1, \dots, d\}$ and $\sum_{i=1}^{d} p_i = 1$. Hence, a generic inter-inspection time $E_{k,p,\boldsymbol{\omega}}$ has the density, with $k = (k_1, \dots, k_d)$, $p = (p_1, \dots, p_d)$, and $\boldsymbol{\omega} = (\omega_1, \dots, \omega_d)$,

$$\mathrm{P}(E_{k,p,\boldsymbol{\omega}} \in \mathrm{d}t) = \sum_{i=1}^{d} p_i\, \omega_i \frac{(\omega_i t)^{k_i - 1}}{(k_i - 1)!} \mathrm{e}^{-\omega_i t}\, \mathrm{d}t.$$

In this case, the 'master equation' becomes

$$\begin{aligned} &\pi(\alpha, \beta \mid E_{k,p,\boldsymbol{\omega}}) \left(1 - \sum_{i=1}^{d} p_i \Big(\frac{\omega_i}{\beta + \omega_i - \varphi(\alpha)}\Big)^{k_i}\right) \\ &\quad = \sum_{i=1}^{d} p_i \Big(\frac{\omega_i}{\beta + \omega_i}\Big)^{k_i} \big(I_i(\alpha, \beta, \omega_i) - J_i(\alpha, \beta, \omega_i)\big), \end{aligned}$$

where the unknown constants can again be found by using [23, Theorem 1 and Remark 2.1].

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Competing interests

There were no competing interests to declare which arose during the preparation or publication process of this paper.

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