Proc. Camb. Phil. Soc. (1963), 59, 347
Printed in Great Britain

Some formulae for the G-function

By R. K. SAXENA
*Department of Mathematics, University of Rajasthan, Jaipur, India**

Communicated by J. H. Williamson

(*Received* 3 *March* 1962)

1. The object of the present note is to evaluate some integrals involving Meijer's G-function, in which the argument of the G-function contains a factor $t^m\{t^{\frac{1}{2}}+(1+t)^{\frac{1}{2}}\}^{2n}$ where m and n are positive integers and t is the variable of integration. Two different forms of the general result have been obtained, one for $m > n$ and the other for $m < n$. The value of the corresponding integral when $m = n$ is also obtained. For the definition, properties and the behaviour of the G-function, see (2), §§ 5·3, 5·31 and (5), § 18.

2. In the proof, we require the gamma-function formula ((2), p. 4)

$$\Gamma(mz) = (2\pi)^{\frac{1}{2}-\frac{1}{2}m} m^{mz-\frac{1}{2}} \prod_{k=0}^{m-1} \Gamma(z+k/m), \tag{1}$$

where $m = 2, 3, 4, \ldots$ and the Mellin–Barnes integral for the G-function ((1), p. 207)

$$G^{p,q}_{r,s}\left(x \middle| \begin{matrix} a_1, \ldots, a_r \\ b_1, \ldots, b_s \end{matrix}\right) = \frac{1}{2\pi i} \int_L \frac{\prod_{j=1}^{p} \Gamma(b_j - \xi) \prod_{j=1}^{q} \Gamma(1-a_j+\xi)}{\prod_{j=p+1}^{s} \Gamma(1-b_j+\xi) \prod_{j=q+1}^{r} \Gamma(a_j-\xi)} x^{\xi} d\xi, \tag{2}$$

where L is a suitable contour.

The infinite integral ((3), p. 311)

$$\int_0^\infty t^{\lambda-1}(1+t)^{-\frac{1}{2}}[t^{\frac{1}{2}}+(1+t)^{\frac{1}{2}}]^{2\mu}\, dt = \frac{2^{1-2\lambda}\Gamma(2\lambda)\,\Gamma(\frac{1}{2}-\lambda-\mu)}{\Gamma(\frac{1}{2}+\lambda-\mu)}, \tag{3}$$

where $\Re(\lambda) > 0$ and $\Re(\frac{1}{2}-\lambda-\mu) > 0$ will also be required.

3. *First integral.* The formula to be proved is

$$\begin{aligned}\int_0^\infty t^{\lambda-1}(1+t)^{-\frac{1}{2}}[t^{\frac{1}{2}}+(1+t)^{\frac{1}{2}}]^{2\mu} G^{p,q}_{r,s}\left[zt^m\{t^{\frac{1}{2}}+(1+t)^{\frac{1}{2}}\}^{2n} \middle| \begin{matrix} a_1, \ldots, a_r \\ b_1, \ldots, b_s \end{matrix}\right] dt \\ = \frac{(2\pi)^{1-m-n}\, m^{2\lambda-\frac{1}{2}}}{\Gamma(\frac{1}{2})\,(m+n)^{\lambda+\mu}\,(m-n)^{\lambda-\mu}} \\ \times G^{p+m+n,\,q+2m}_{r+2m,\,s+2m}\left[\frac{zm^{2m}}{(m+n)^{m+n}\,(m-n)^{m-n}} \middle| \begin{matrix} a_1, \ldots, a_q, c_1, \ldots, c_{2m}, a_{q+1}, \ldots, a_r \\ b_1, \ldots, b_p, d_1, \ldots, d_{m+n}, b_{p+1}, \ldots, b_s, e_1, \ldots, e_{m-n} \end{matrix}\right],\end{aligned} \tag{4}$$

where m and n are positive and non-negative integers, respectively, such that $m > n$,

$$c_{i+1} = \frac{1-2\lambda+i}{2m}, \quad d_{j+1} = \frac{1-2\lambda-2\mu+2j}{2m+2n}, \quad e_{k+1} = \frac{1-2\lambda+2\mu+2k}{2m-2n},$$

for $i = 0, 1, \ldots, (2m-1)$, $j = 0, 1, \ldots, (m+n-1)$ and $k = 0, 1, \ldots, (m-n-1)$.

* Present address: Maharajas College, Jaipur, India.

https://doi.org/10.1017/S0305004100036975 Published online by Cambridge University Press

This formula is valid under the following sets of conditions.

(i) $$r+s<2(p+q),\quad |\arg z|<(p+q-\tfrac{1}{2}r-\tfrac{1}{2}s)\pi,$$
$$\Re(\lambda+mb_j)>0\quad (j=1,\ldots,p),\qquad \Re[\lambda+\mu-\tfrac{1}{2}+(m+n)(a_k-1)]<0\quad (k=1,\ldots,q);$$

(ii) $$r\leqslant s,\quad r+s\leqslant 2(p+q),\quad |\arg z|\leqslant(p+q-\tfrac{1}{2}r-\tfrac{1}{2}s)\pi,$$
$$\Re(\lambda+mb_j)>0\quad (j=1,2,\ldots,p),$$
$$\Re[\lambda+\mu-\tfrac{1}{2}+(m+n)(a_k-1)]<0\quad (k=1,2,\ldots,q),$$
$$\Re\left[(m+n)\left(\sum_{h=1}^{r}a_h-\sum_{h=1}^{s}b_h-\tfrac{1}{2}\right)+(r-s)(\lambda+\mu-\tfrac{1}{2}-\tfrac{1}{2}m-\tfrac{1}{2}n)\right]>-1;$$

(iii) $$r\geqslant s,\quad r+s\leqslant 2(p+q),\quad |\arg z|\leqslant(p+q-\tfrac{1}{2}r-\tfrac{1}{2}s)\pi,$$
$$\Re(\lambda+mb_j)>0\quad (j=1,2,\ldots,p),$$
$$\Re[\lambda+\mu-\tfrac{1}{2}+(m+n)(a_k-1)]<0\quad (k=1,2,\ldots,q),$$
$$\Re\left[(m+n)\left(\sum_{h=1}^{r}a_h-\sum_{h=1}^{s}b_h-\tfrac{1}{2}\right)+(r-s)(\lambda-\tfrac{1}{2}m-\tfrac{1}{2}n)\right]>-1;$$

(iv) $$r<s,\quad q=0,\quad p>\tfrac{1}{2}r+\tfrac{1}{2}s,\quad |\arg z|<(p-\tfrac{1}{2}r-\tfrac{1}{2}s)\pi,$$
$$\Re(\lambda+mb_j)>0\quad (j=1,2,\ldots,p).$$

To prove (4), substitute the value of the G-function from (2) in the integrand of (4) and get

$$\int_0^\infty t^{\lambda-1}(1+t)^{-\frac{1}{2}}[t^{\frac{1}{2}}+(1+t)^{\frac{1}{2}}]^{2\mu}\times\left\{\frac{1}{2\pi i}\int_L\frac{\prod_{j=1}^{p}\Gamma(b_j-\xi)\prod_{j=1}^{q}\Gamma(1-a_j+\xi)}{\prod_{j=p+1}^{s}\Gamma(1-b_j+\xi)\prod_{j=q+1}^{r}\Gamma(a_j-\xi)}[zt^m\{t^{\frac{1}{2}}+(1+t)^{\frac{1}{2}}\}^{2n}]^\xi d\xi\right\}dt. \tag{5}$$

Here interchange the order of integrations and evaluate the inner integral with the help of (3). Then it becomes

$$\frac{2^{1-2\lambda}}{2\pi i}\int_L\frac{\prod_{j=1}^{p}\Gamma(b_j-\xi)\prod_{j=1}^{q}\Gamma(1-a_j+\xi)\,\Gamma(\tfrac{1}{2}-\lambda-\mu-(m+n)\xi)\,\Gamma(2\lambda+2m\xi)}{\prod_{j=p+1}^{s}\Gamma(1-b_j+\xi)\prod_{j=q+1}^{r}\Gamma(a_j-\xi)\,\Gamma(\tfrac{1}{2}+\lambda-\mu+(m-n)\xi)}\left(\frac{z}{2^{2m}}\right)^\xi d\xi$$
$$=\frac{(2\pi)^{1-m-n}m^{2\lambda-\frac{1}{2}}}{\Gamma(\frac{1}{2})(m+n)^{\lambda+\mu}(m-n)^{\lambda-\mu}}\frac{1}{2\pi i}\int_L\frac{\prod_{j=1}^{p}\Gamma(b_j-\xi)\prod_{j=1}^{q}\Gamma(1-a_j+\xi)}{\prod_{j=p+1}^{s}\Gamma(1-b_j+\xi)\prod_{j=q+1}^{r}\Gamma(a_j-\xi)}$$
$$\times\frac{\prod_{k=0}^{m+n-1}\Gamma\left(\frac{1-2\lambda-2\mu+2k}{2m+2n}-\xi\right)\prod_{k=0}^{2m-1}\Gamma\left(\frac{2\lambda+k}{2m}+\xi\right)}{\prod_{k=0}^{m-n-1}\Gamma\left(\frac{1+2\lambda-2\mu+2k}{2m-2n}+\xi\right)}\left[\frac{zm^{2m}}{(m+n)^{m+n}(m-n)^{m-n}}\right]^\xi d\xi, \tag{6}$$

by virtue of (1).

The contour L runs from $-i\infty$ to $+i\infty$, such that all the poles of

$$\Gamma(b_j-\xi)\quad (j=1,2,\ldots,p),\quad \Gamma\left(\frac{1-2\lambda-2\mu+2k}{2m+2n}-\xi\right)\quad (k=0,1,\ldots,(m+n-1))$$

https://doi.org/10.1017/S0305004100036975 Published online by Cambridge University Press

are to the right, and all the poles of

$$\Gamma(1-a_k+\xi) \quad (k=1,2,\ldots,q) \quad \text{and} \quad \Gamma\left(\frac{2\lambda+k}{2m}+\xi\right) \quad (k=0,1,\ldots,(2m-1))$$

to the left of the contour.

On interpreting (6) with the help of (2) it gives (4) under the conditions (i).

Regarding the interchange of the order of integrations in (5) it is observed that the 't'-integral is absolutely convergent if $\Re(\lambda+m\xi)>0$, $\Re[\frac{1}{2}-\lambda-\mu-(m+n)\xi]>0$, $\xi>0$; the 'ξ'-integral converges absolutely for $|\arg z|<(p+q-\frac{1}{2}r-\frac{1}{2}s)\pi$ if $p+q>\frac{1}{2}r+\frac{1}{2}s$, $t>0$, and the convergence of the repeated integral follows from that of the integral in (4). Hence the interchange of the order of integrations is justified by virtue of de la Vallée Poussin's theorem ((1), p. 504).

On considering the behaviour of the G-function ((5), § 18) it can easily be seen that the formula (4) also holds under the sets of conditions (ii), (iii) and (iv).

In particular, on taking $p=2, q=r=1, s=2, m=1, n=0, a_1=1+\sigma, b_1=\frac{1}{2}-\nu$, $b_2=\frac{1}{2}+\nu$ in (4), it yields the following known integral ((4), p. 406).

$$\pi^{\frac{1}{2}}\int_0^\infty t^{\lambda-1}(1+t)^{-\frac{1}{2}}[t^{\frac{1}{2}}+(1+t)^{\frac{1}{2}}]^{2\mu}e^{\frac{1}{2}zt}W_{\sigma,\nu}(zt)\,dt = G_{3,4}^{3,3}\left(z\left|\begin{matrix}1+\sigma, \frac{1}{2}-\lambda, 1-\lambda\\ \frac{1}{2}-\nu, \frac{1}{2}+\nu, \frac{1}{2}-\lambda-\mu, \frac{1}{2}-\lambda+\mu\end{matrix}\right.\right), \tag{7}$$

where $\Re(\lambda)>|\Re(\nu)|-\frac{1}{2}$, $\Re(\sigma+\lambda+\mu)<\frac{1}{2}$, $|\arg z|<\frac{3}{2}\pi$.

4. *Second integral.* On proceeding as before, it is found that if m and n are positive integers, such that $m<n$, then under the same sets of conditions of validity as in (4)

$$\int_0^\infty t^{\lambda-1}(1+t)^{-\frac{1}{2}}[t^{\frac{1}{2}}+(1+t)^{\frac{1}{2}}]^{2\mu}G_{r,s}^{p,q}\left[zt^m\{t^{\frac{1}{2}}+(1+t)^{\frac{1}{2}}\}^{2n}\left|\begin{matrix}a_1,\ldots,a_r\\ b_1,\ldots,b_s\end{matrix}\right.\right]dt$$
$$=\frac{(2\pi)^{1-2m}m^{2\lambda-\frac{1}{2}}}{\Gamma(\frac{1}{2})(n+m)^{\lambda+\mu}(n-m)^{\lambda-\mu}}$$
$$\times G_{r+m+n,s+m+n}^{p+m+n,q+2m}\left[\frac{zm^{2m}(n-m)^{n-m}}{(n+m)^{n+m}}\left|\begin{matrix}a_1,\ldots,a_q, c_1,\ldots,c_{2m}, e_1,\ldots,e_{n-m}, a_{q+1},\ldots,a_r\\ b_1,\ldots,b_p, d_1,\ldots,d_{m+n}, b_{p+1},\ldots,b_s\end{matrix}\right.\right], \tag{8}$$

where $c_{i+1}=\dfrac{1-2\lambda+i}{2m}$, $e_{j+1}=\dfrac{1+2\lambda-2\mu+2j}{2n-2m}$, $d_{k+1}=\dfrac{1-2\lambda-2\mu+2k}{2m+2n}$,

for $i=0,1,\ldots,(2m-1)$, $j=0,1,\ldots,(n-m-1)$ and $k=0,1,\ldots,(m+n-1)$.

5. *Third integral.* Similarly when $m=n$, we derive the following result under the same sets of conditions of validity as in (4) with $m=n$.

$$\int_0^\infty t^{\lambda-1}(1+t)^{-\frac{1}{2}}[t^{\frac{1}{2}}+(1+t)^{\frac{1}{2}}]^{2\mu}G_{r,s}^{p,q}\left[zt^n\{t^{\frac{1}{2}}+(1+t)^{\frac{1}{2}}\}^{2n}\left|\begin{matrix}a_1,\ldots,a_r\\ b_1,\ldots,b_s\end{matrix}\right.\right]dt$$
$$=\frac{\pi^{1-2n}n^{\lambda-\mu-\frac{1}{2}}}{2^{\lambda+\mu+2n-\frac{3}{2}}\Gamma(\frac{1}{2}+\lambda-\mu)}$$
$$\times G_{r+2n,s+2n}^{p+2n,q+2n}\left[\frac{z}{2^{2n}}\left|\begin{matrix}a_1,\ldots,a_q, c_1,\ldots,c_{2n}, a_{q+1},\ldots,a_r\\ b_1,\ldots,b_p, d_1,\ldots,d_{2n}, b_{p+1},\ldots,b_s\end{matrix}\right.\right], \tag{9}$$

https://doi.org/10.1017/S0305004100036975 Published online by Cambridge University Press

where $c_{i+1} = \dfrac{1-2\lambda+i}{2n}$, $d_{i+1} = \dfrac{1-2\lambda-2\mu+2i}{4n}$ for $i = 0, 1, \ldots, (2n-1)$

and n is a positive integer.

REFERENCES

(1) BROMWICH, T. J. I'A. *An introduction to the theory of infinite series* (Macmillan; London, 1931).
(2) ERDÉLYI, A. *et al.* *Higher transcendental functions*, vol. I (McGraw-Hill; New York, 1953).
(3) ERDÉLYI, A. *et al.* *Tables of integral transforms*, vol. I (McGraw-Hill; New York, 1954).
(4) ERDÉLYI, A. *et al.* *Tables of integral transforms*, vol. II (McGraw-Hill; New York, 1954).
(5) MEIJER, C. S. On the G-function. VI, VII. *Nederl. Akad. Wetensch. Proc.* **49** (1946), 936–943, 1063–1072.

https://doi.org/10.1017/S0305004100036975 Published online by Cambridge University Press