On Differentiation with respect to a Function. By Mr E. C. Francis, Peterhouse.

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Thirty years ago, in a paper on continued fractions*, Stieltjes published a definition of the integral which bears his name. His replacement of the variable of integration x by a more general "base function" $\phi(x)$—a change which throws so much light upon other theories of integration—received at first little attention, but has later sprung into greater prominence; so much so that Professor Hildebrandt, in summarizing these various theories in a paper to the American Mathematical Society†, makes the statement that "it [the Stieltjes Integral] seems destined to play the central rôle in the integrational and summational processes of the future." Yet even now the integral and the allied theory of differentiation with respect to a function have been subjected to little detailed analysis, and the possibilities of extension have been only touched upon. It is the object of this present paper to establish certain results which are of some value in themselves and which prepare the way for an attack upon the integral.

A theory of differentiation of a function $F(x)$ with respect to a function $\phi(x)$ depends essentially upon the study of the limits of an incrementary ratio of the form $v(F)/v(\phi)$ and so must be preceded by and correlated with a Theory of Variation. This we attempt to do. For various reasons—in part only the consideration of subsequent applications to the theory of the Stieltjes Integrals—the functions $F(x)$, $\phi(x)$ are restricted, and we concentrate upon a *strictly increasing* base function $\phi(x)$ and a "*smooth*" function $F(x)$ (i.e. a function whose discontinuities are of the first species only). Such a restriction enables us to define generalized derivatives and to establish some interesting properties, in particular extensions of Rolle's Theorem and the Mean Value Theorem of ordinary differential calculus; and finally to sum them up in a Generalized Variation Theorem which proves under very general conditions that if a derivative $D(x)$ is such that

$$H \leqslant D(x) \leqslant K$$

at points of a set E, then

$$v(\phi) \,.\, H \leqslant v(F) \leqslant v(\phi) \,.\, K,$$

where variations on E are considered.

* T. J. Stieltjes, "Recherches sur les Fractions Continues," *Ann. de la Fac. des Sc. de Toulouse*, vol. VIII (1894).

† Prof. T. H. Hildebrandt, "On Integrals related to, and extensions of, Lebesgue Integrals," *Bull. Amer. Math. Soc.* vol. XXIV (1918).

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The paper concludes with an extension of this theorem to "special" derivatives with regard to a set.

§ 1. VARIATION.

A useful definition of the variation of a function $F(x)$ on a set of points E—we shall denote this by $v(F \mid E)$ or by $v(F)$ or $v(E)$ when there is no ambiguity—must reduce to the measure mE in the case $F(x) \equiv x$. If $F(x)$ is continuous this at once suggests

$$v(F \mid ab) = F(b) - F(a),$$

and upon this basis Denjoy has dealt with the variation of a continuous function on certain simple sets*. If, however, $F(x)$ is discontinuous the lack of the absolutely additive property is sufficient to condemn the definition.

Considerations which are apparent enough make far the simplest case that of *an increasing function* $F(x)$. Here we distinguish between open and closed intervals (ab) and $\langle ab \rangle$; and defining

$$v(F \mid \langle ab \rangle) = F(b+0) - F(a-0),$$
$$v(F \mid (ab)) = F(b-0) - F(a+0),$$

we can extend our definition via open and closed sets to give for any set E upper and lower variations $v^*(E)$ and $v_*(E)$. If these are equal we define $v(E)$ equal to each and say that the set E is *measurable* (F) (or *measurable* when there is no ambiguity). The whole theory is exactly analogous to de la Vallée Poussin's Theory of Measure†.

Extension to functions of bounded variation is obvious‡, but both the form of definition adopted and Denjoy's work on continuous functions suggest a further extension. Provided $F(x \pm 0)$ exist at all points x we can define $v(F \mid I)$ as above, I being any interval; and we are led to single out such functions. We shall call them *smooth* functions. (They include as special cases continuous functions and functions of bounded variation.) A useful general definition of $v(F \mid E)$ is not possible even in this case, but

* A. Denjoy, "Mémorie sur la Totalization des Nombre Dérivés non-sommables," *Annales de l'Ecole Normale Supérieure* (1916).

† Ch. de la Vallée Poussin, *Intégrales de Lebesgue.*

‡ There is an ambiguity here which is usually allowed to pass unnoticed. $F(x)$, if of bounded variation, can be split up into the difference of two increasing functions $F_1(x)$, $F_2(x)$, but *this can be done in many ways.* One definite way is indicated in all developments of the theory of functions of bounded variation and this is best regarded as standard. If we denote the corresponding functions by $F_1(x)$, $F_2(x)$, we can prove the following Consistency Theorem (often tacitly assumed):

If $G_1(x)$, $G_2(x)$ *are any two increasing functions such that*

$$F(x) = G_1(x) - G_2(x),$$

and if a set E *is measurable* (G_1) *and* (G_2)*; then* E *is measurable* (F_1) *and* (F_2) *and*

$$v(F_1 \mid E) - v(F_2 \mid E) = v(G_1 \mid E) - v(G_2 \mid E).$$

Since the subject is somewhat removed from the main object of the paper I omit the proof.

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we can extend Denjoy's method to treat the case when E is a *simple set*—i.e. is formed of a finite or infinite number of intervals u_n or is the complement of such intervals. (Open and closed sets, in particular, are simple.) If $\Sigma \mid v(u_n) \mid$ is convergent we say that $F(x)$ is of *definite variation* on E and define vE as follows:

$$\text{if } E = \Sigma u_n \qquad \text{we define } vE = \Sigma v u_n;$$
$$\text{if* } E = I - \Sigma u_n \text{ we define } vE = vI - \Sigma v u_n.$$

For most purposes we are concerned with the value of $F(x)$ in the neighbourhood of a point rather than at the point itself and we define a *strictly smooth* function as a smooth function for which $F(x) = \frac{1}{2}\{F(x+0) + F(x-0)\}$. Any given smooth function can be replaced by the obvious "corresponding" strictly smooth function without altering its variation on any set.

The properties of the variation of an increasing function (or of a function of bounded variation) will be taken as known. They include in particular the "absolutely additive" property and that expressed by the equation

$$v(\lim E_n) = \lim (vE_n),$$

which holds if the measurable† sets E_n have a limit set. We can clearly define "measurable (F)" functions: their properties, too, are deduced exactly as when $\phi(x) \equiv x$.

In the case of smooth functions which are not necessarily of bounded variation the properties are considerably restricted, but the following results present no difficulty:

A. *A smooth function is bounded.*

B. *If F_1, F_2, F_3 are functions such that $F_1 = F_2 + F_3$, and if any two of them are strictly smooth and of definite variation on a simple set S, so is the third and*

$$v(F_1 \mid S) = v(F_2 \mid S) + v(F_3 \mid S).$$

The following "absolutely additive" property follows at once by transfinite induction:

If (i) *$F(x)$ is smooth,*
(ii) *$\langle I \rangle$ can be split up into a denumerable set of intervals $\langle J \rangle$,*
(iii) *$v(F \mid J)$ has always the same sign,*

then
$$v(F \mid I) = \Sigma v(F \mid J).$$

We have also the product theorem:

If F_1, F_2 are strictly smooth and of definite variation on S then $F_1 F_2$ is smooth and of definite variation on S. ($F_1 F_2$ is strictly smooth if F_1 and F_2 are positive.)

* Throughout this paper I will denote an interval.
† I.e. measurable with respect to the base function in question.

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§ 2. DERIVATIVES.

We can now proceed to a definition of derivatives with respect to a function. To be useful this should be intimately connected with our theory of variation, should have at least the additive property, and should reduce to the ordinary definition in the case when the base function $\phi(x) \equiv x$. A failure to comply with the last requirement condemns a definition given by P. J. Danniell*.

Difficulty will clearly be caused if the denominator of our incrementary ratio can become zero; and we are led to restrict the base function $\phi(x)$ to be strictly increasing†.

The function to be differentiated, $F(x)$, we can take to be smooth, and we define as follows four generalized derivatives, denoted (after Young) by $F^+(x)$, $F_+(x)$, $F^-(x)$, $F_-(x)$. When these are all equal we say that a differential coefficient $F_\phi'(x)$ (or $F'(x)$ if there is no ambiguity) exists and is equal to each of them. We consider three cases:

(i) $\begin{cases}\phi(x+0) = \phi(x-0) \\ F(x+0) = F(x-0)\end{cases}$

$$\left.\begin{aligned} F^+(x) &= \overline{\lim_{h \to +0}} \\ F_+(x) &= \underline{\lim_{h \to +0}} \\ F^-(x) &= \overline{\lim_{h \to -0}} \\ F_-(x) &= \underline{\lim_{h \to -0}} \end{aligned}\right\} \left(\frac{F(x+h)-F(x)}{\phi(x+h)-\phi(x)}\right).$$

(ii) $\phi(x+0) \neq \phi(x-0)$

$$F^+(x) = F_+(x) = F^-(x) = F_-(x) = \frac{F(x+0)-F(x-0)}{\phi(x+0)-\phi(x-0)}.$$

(iii)‡ $\begin{cases}\phi(x+0) = \phi(x-0) \\ F(x+0) \neq F(x-0)\end{cases}$

$$F^+(x) = F_+(x) = F^-(x) = F_-(x) = \begin{cases} +\infty \text{ if } F(x+0) > F(x-0), \\ -\infty \text{ if } F(x+0) < F(x-0). \end{cases}$$

* P. J. Danniell, "On differentiation with regard to a function of limited variation," *Trans. Amer. Math. Soc.* vol. XIX (1918).

Danniell defines two derivatives (upper and lower) by

$$\left.\begin{aligned} \overline{D}(F(x)) &= \overline{\lim_{h \to 0}} \\ \underline{D}(F(x)) &= \underline{\lim_{h \to 0}} \end{aligned}\right\} \left(\frac{F(x+h)-F(x-h)}{\phi(x+h)-\phi(x-h)}\right).$$

If $\overline{D} = D$, $F(x)$ has a differential coefficient equal to each.

But on this definition the function which is equal to x when $x \geq 0$, and to $-x$ when $x < 0$, has differential coefficient zero at the origin.

† I.e. $\phi(x_1) > \phi(x_2)$ if $x_1 > x_2$.

‡ For many purposes $F(x)$ is "continuous with ϕ" (i.e. (iii) is excluded).

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§ 3. Elementary Properties.

The following results are easily proved:

C. *A constant has everywhere differential coefficient zero.*

D. *The base function has everywhere differential coefficient unity.*

E. *If D denotes any derivative and c a positive constant, then*

$$D(cF) = c(DF).$$

F. *If $F'(x)$ exists and c is any constant, then $(cF)' = c.F'$.*

G. *If $F_1(x), F_2(x)$ are strictly smooth and $F(x) = F_1(x) + F_2(x)$, then*

$$F_1^+(x) + F_2^+(x) \geqslant F^+(x) \geqslant \begin{Bmatrix} F_1^+(x) + F_{2+}(x) \\ F_{1+}(x) + F_2^+(x) \end{Bmatrix} \geqslant F_+(x) \geqslant F_{1+}(x) + F_{2+}(x).$$

H. *If $F_1(x)$, $F_2(x)$ are strictly smooth and have differential coefficients this is also true of*

$$F(x) \; (= F_1(x) + F_2(x)),$$

and we have

$$F'(x) = F_1'(x) + F_2'(x).$$

Further if $G(x) = F_1(x) \,.\, F_2(x)$, $G(x)$ is smooth and has a differential coefficient ($G(x)$ is strictly smooth if F_1, F_2 are positive), and

$$G'(x) = F_1'(x)\, F_2(x) + F_1(x)\, F_2'(x).$$

I. *If $F_\psi'(x)$, $\psi_\phi'(x)$ both exist and are finite then $F_\phi'(x)$ exists and is equal to their product.*

J. *If strictly smooth functions $F(x)$, $G(x)$ are such that*

$$F_\phi'(x) = G_\psi'(x),$$

then each of these expressions is equal to $(F + G)'_{\phi+\psi}$.

K. *$F'(x)$ and $\lim_{I \to 0} \left(\dfrac{v(F \mid I)}{v(\phi \mid I)} \right)$, where I is any interval containing the point x, exist together and are equal.*

This theorem shows the close connection between the theories of differentiation and variation.

§ 4. Rolle's Theorem and its consequences.

We are now in a position to extend certain theorems well known in ordinary differential calculus. To do so we make use of the following lemma due to Lebesgue*:

If every point of an interval I is the left-hand end-point of an interval J, then I can be completely covered by at most a denumerable set of non-overlapping intervals J.

We require the following extension:

If every point of a closed set Q is the left-hand end-point of an

* Lebesgue, *Leçons sur l'Intégration*, p. 63.

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interval J, then Q can be completely covered by at most a denumerable set of non-overlapping intervals J.

This follows easily from the original lemma. For let a, b be the end-points of Q. Associate with each point x falling in a black interval $(a_n b_n)$ of Q the interval $\langle x b_n \rangle$. $\langle ab \rangle$ can therefore be covered by a finite or denumerable set of intervals—some intervals J and some intervals of type $\langle x b_n \rangle$. Intervals of the latter type contain no points of Q. Omit them and the extended lemma is proved.

L. *Rolle's Theorem:*

If $D(x)$ is any given derivative of $F(x)$ and I an interval, other than a single point, for which $v(F \mid I) \geqslant 0$, then there is some point ξ of I at which $D(\xi) \geqslant 0$.

For simplicity we take $D(x) = F_+(x)$. The theorem follows *a fortiori* for $F^+(x)$, and is proved for $F^-(x)$, $F_-(x)$ by interchanging left and right.

We may take $v(F \mid I) > 0$:
for if $v(F \mid I) = 0$ then either $F(x)$ is constant in I (and the theorem is true) or there is some point x of I at which one of $F(x \pm 0)$ is different from $F(a)$ (where a stands for $a \pm 0$ according as I is open or closed at its left-hand end a). There is therefore an interval contained in I on which $v(F) > 0$.

We may take I closed:
for by our definition of variation I contains a closed interval I' such that $v(F \mid I') > 0$.

Making these assumptions we suppose the theorem untrue. With each point x of I we can associate an interval $J(x) \equiv \langle xx' \rangle$, as small as we please, and such that

$$v(F \mid J(x)) < 0,$$

for

(i) if $F(x)$ is discontinuous at x

$$F(x + 0) < F(x - 0),$$

we can therefore choose $x' > x$ and as near x as we please such that

$$F(x' - 0) < F(x - 0).$$

(ii) if $F(x)$ is continuous at x there is no difficulty. Our definition of $F_+(x)$ shows at once that we can choose x' as required.

We can now apply Lebesgue's Lemma. This shows that I can be covered by at most a denumerable infinity $J_1, J_2, \ldots$ of such intervals $J(x)$, non-overlapping and extending to a point $b' > b$ and as near to b as we please. Only one of these intervals extends beyond b.

By a theorem already stated

$$v(F \mid \langle ab' \rangle) = \Sigma v(F \mid J_n),$$

since the terms are all negative.

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Hence $$v(F \mid \langle ab' \rangle) \leqslant 0.$$

This is true for b' as near b as we please. Hence

$$v(F \mid \langle ab \rangle) \leqslant 0,$$

and we have a contradiction; which proves the theorem.

M. *The Generalized Mean Value Theorem:*

If $D(x)$ is any given derivative of $F(x)$ and I any interval, then there are points ξ, ξ' contained in I and such that

$$D(\xi') . v(\phi \mid I) \leqslant v(F \mid I) \leqslant D(\xi) . v(\phi \mid I).$$

We may consider $F(x)$, $\phi(x)$ to be *strictly* smooth, for the substitution of the corresponding strictly smooth functions affects neither derivatives nor variation.

Define $\psi(x)$ by

$$\psi(x) = F(x) . v(\phi \mid I) - (\phi(x) - \phi(\alpha)) . v(F \mid I),$$

where α stands for $a \pm 0$ according as I is open or closed at a. $\psi(x)$ is strictly smooth and $v(\psi \mid I) = 0$.

By Theorems **C**, **D**, **F**, **G**

$$D(\psi(x)) = D(F(x)) . v(\phi \mid I) - v(F \mid I).$$

But by the Generalized Rolle's Theorem there are points ξ, ξ' of I such that

$$D(\psi(\xi)) \geqslant 0 \geqslant D(\psi(\xi')).$$

This proves the theorem.

As a corollary we note the following:

N. *If at each point of an interval I, $F(x)$ has a median or extreme derivative zero on a given side, then $F(x)$ is constant in I.*

For suppose the given side is the right and that $v(F \mid I_1) > 0$ for some interval I_1 contained in I. By the last theorem there is a point ξ in I, at which

$$F_+(\xi) \geqslant \frac{v(F \mid I_1)}{v(\phi \mid I_1)} > 0.$$

Hence every median or extreme right derivative is greater than zero, which contradicts our hypothesis. An exactly similar argument for the other possible cases proves the theorem.

The Mean Value Theorem now serves to prove a more general result which embraces it as a special case:

O. *The Generalized Variation Theorem:*

If $D(x)$ is any given derivative of $F(x)$ and E a set measurable (ϕ) on which

$$H \leqslant D(x) \leqslant K^*;$$

then $$H . v(\phi \mid E) \leqslant v(F \mid E) \leqslant K . v(\phi \mid E),$$

* By this statement we mean that if one only of these inequalities is true, then the corresponding inequality of the result holds. We are *not* assuming $D(x)$ bounded.

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provided that either

(i) *E is an interval I.*

(ii) *E is an open set O on which $F(x)$ is of definite variation.*

(iii) *$F(x)$ is of bounded variation* and E is measurable (F).*

Case (i) is contained in the Generalized Mean Value Theorem.

Case (ii). If $O = \Sigma u_n$ we have from case (i)

$$Hv(\phi \mid u_n) \leqslant v(F \mid u_n) \leqslant Kv(\phi \mid u_n).$$

But by our hypothesis both $\Sigma \mid v(\phi \mid u_n) \mid$ and $\Sigma \mid v(F \mid u_n) \mid$ are convergent. Hence

$$Hv(\phi \mid O) \leqslant v(F \mid O) \leqslant Kv(\phi \mid O).$$

Case (iii). This presents considerably more difficulty. The theorem is untrue unless we put some restriction upon $F(x)$. We begin with the case when E is a closed set Q (black intervals u_n). From our assumption that $F(x)$ is of bounded variation we note that ΣW_n is convergent where $W_n = \operatorname{osc}(F \mid u_n)$.

We may assume $H < D(x) < K$:

for if the theorem is true in this case, then, under the given conditions we know that for every $H' < H$, $K' > K$

$$H'v(\phi \mid E) \leqslant v(F \mid E) \leqslant K'v(\phi \mid E).$$

Thus $$Hv(\phi \mid E) \leqslant v(F \mid E) \leqslant Kv(\phi \mid E),$$

and the theorem is proved.

For simplicity we shall consider only the case $F_+(x) > H$. The others are treated by similar methods.

We proceed to show that every point x of Q is the left-hand end-point of an interval $J(x) = \langle xx' \rangle$, as small as we please, such that

$$v(F \mid J(x)) > H \,.\, v(\phi \mid J(x)) \qquad \ldots\ldots(1).$$

There are two cases:

(a) x a point of discontinuity of $\phi(x)$ or of $F(x)$.

Therefore

$$F(x+0) - F(x-0) > H\{\phi(x+0) - \phi(x-0)\}.$$

There is therefore some point $x' > x$ and as near x as we please, such that

$$F(x'-0) - F(x-0) > H\{\phi(x'-0) - \phi(x-0)\}.$$

(b) x a point of continuity of $\phi(x)$ and of $F(x)$.

The result is now an immediate consequence of our definition of $F_+(x)$.

Hence in all cases $J(x)$ can be chosen so that (1) holds: but this result is insufficient as it stands. We must still further restrict our choice of $J(x)$ by means of our assumptions as to $F(x)$.

* This condition is made less stringent by later results in §§ 5, 6

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Choose positive numbers $\epsilon_1, \epsilon_2, \ldots$ of sum less than ϵ. We can find N such that

$$\sum_{N+1}^{\infty} W_n < \epsilon.$$

Now $J(x)$ can be as small as we please.

If x is the left-hand end-point a_n of a black interval u_n where $1 \leqslant n \leqslant N$, choose x' such that

$$| F(x' - 0) - F(x + 0) | < \epsilon_n.$$

In all other cases choose $J(x)$ so small as to contain no portion of the intervals $u_1, u_2, \ldots u_N$.

The extension of Lebesgue's Lemma already proved now enables us to choose a denumerable (or finite) set of non-overlapping intervals $J(x)$ to contain every point of Q. For each of these

$$v(F \mid J(x)) > Hv(\phi \mid J(x)) \qquad \ldots\ldots(1).$$

Further, we see by choice of our intervals and by the fact that the variation of a function of bounded variation is absolutely additive, that if

$$v(F \mid J) = v(F \mid QJ) + \eta \qquad \ldots\ldots(2),$$

then

$$\Sigma \mid \eta \mid \leqslant \epsilon_1 + \epsilon_2 + \ldots + \epsilon_N + \epsilon < 2\epsilon \qquad \ldots\ldots(3).$$

But by the same absolutely additive property

$$v(F \mid Q) = \Sigma v(F \mid QJ).$$

Combined with equations (2) and (3) above we have therefore

$$\begin{aligned} v(F \mid Q) + 2\epsilon &> \Sigma v(F \mid J) \\ &> H \,.\, \Sigma v(\phi \mid J) \\ &\geqslant H \,.\, v(\phi \mid Q). \end{aligned}$$

Since ϵ is arbitrary we deduce the result required at once:

$$v(F \mid Q) \geqslant H \,.\, v(\phi \mid Q).$$

The theorem is now proved for a closed set. The extension to the general set E contemplated in case (iii) is immediate. For we know that for every closed set Q contained in E

$$H \,.\, v(\phi \mid Q) \leqslant v(F \mid Q) \leqslant K \,.\, v(\phi \mid Q).$$

Since $v(\phi \mid E)$, $v(F \mid E)$ both exist it follows that

$$H \,.\, v(\phi \mid E) \leqslant v(F \mid E) \leqslant K \,.\, v(\phi \mid E).$$

P. As a corollary we may notice that *if $F(x)$ is an increasing function, and if, on any set E, a derivative $D(x)$ of $F(x)$ is such that*

$$H \leqslant D(x) \leqslant K,$$

then

$$H \,.\, v_*(\phi \mid E) \leqslant v_*(F \mid E) \leqslant K \,.\, v_*(\phi \mid E).$$

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Q. If, under the same conditions, $F(x)$ *is a function of bounded variation only, we have*

$$v_*(F \mid E) \leqslant Kv_*(\phi \mid E),$$
$$v^*(F \mid E) \geqslant Hv_*(\phi \mid E).$$

§ 5. Special Derivatives.

It happens often that we are concerned only with the values which our functions $F(x)$, $\phi(x)$ take *at points of a set* E. We therefore introduce the idea of a *special derivative on* E which shall take into account only such values.

At a point x of a set E we define four special derivatives $(F^+(x))_E, (F_+(x))_E, (F^-(x))_E, (F_-(x))_E$ by repeating the definitions of § 2 with the added proviso that in case (i) the points $(x+h)$ must belong to E. Thus when $\phi(x)$ is discontinuous at x all four special derivatives are defined (and are equal), while if x is the limit point of points of E on either side the two corresponding special derivatives exist.

The extension to special differential coefficient is obvious: and the Theorems **C, D, E, F, G, H, I, J** are seen still to be true for special derivatives.

We shall conclude this paper by proving the following extended form of the Variation Theorem:

R. *If* $F(x)$ *is a function of bounded variation and* $D(x)$ *is any given special derivative of* $F(x)$ *on a closed set* Q *which is such that*

$$H \leqslant D(x) \leqslant K^*$$

at all points of Q*; then*

$$H \,.\, v(\phi \mid Q) \leqslant v(F \mid Q) \leqslant K \,.\, v(\phi \mid Q).$$

We can extend the previous proof to deal with this case.

No modification is needed until we come to choose intervals $J(x)$ in case (b); and here the difficulty occurs only when x is a point of continuity of both $F(x)$ and $\phi(x)$ and is the left-hand end-point a_n of a black interval u_n of Q. We know nothing of the right-hand derivatives at such a point x. We therefore choose $\langle xx' \rangle$ so small that x' lies in u_n and

$$|\, v(F \mid \langle xx' \rangle) \,| < \epsilon_n.$$

Equation (1) may now be false for such intervals $\langle xx' \rangle$.

As before we have

$$v(F \mid J) = v(F \mid QJ) + \eta \qquad \ldots\ldots(2),$$
$$\Sigma \mid \eta \mid < 2\epsilon \qquad \ldots\ldots(3),$$
$$v(F \mid Q) = \Sigma v(F \mid QJ) \qquad \ldots\ldots(4),$$

so that
$$v(F \mid Q) + 2\epsilon \geqslant \Sigma v(F \mid J)$$
$$> H \,.\, \Sigma' v(\phi \mid J) - \epsilon,$$

* See note on p. 930.

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where Σ' extends over those J intervals for which (1) is true,

$$\geqslant H \,.\, v\,(\phi \mid Q) - \epsilon,$$

since at most a denumerable set of points a_n of Q, at each of which $\phi\,(x)$ is continuous, are excluded from the intervals of Σ'. Hence as before

$$v\,(F \mid Q) \geqslant H \,.\, v\,(\phi \mid Q).$$

§ 6. A Further Extension. Total Variation on a Closed Set.

The *result* of Theorem **R** is concerned only with values of $F\,(x \pm 0)$ and of $\phi\,(x \pm 0)$ at points x of Q. The *hypotheses* include the assumption that $F\,(x)$ is of bounded variation—an assumption which involves values of $F\,(x)$ at other points x. We are naturally led to suppose that the theorem can be improved, as is indeed the case.

We define the phrase $F\,(x)$ *is of bounded total variation on* Q in the case of a function $F\,(x)$ for which points of Q are discontinuities of the first species only (i.e. which is *smooth on* Q), as meaning that, if I is any interval containing Q, and if I is divided in any way into a finite number of distinct intervals

$$I = I_1 + I_2 + \ldots + I_n,$$

then

(i) $F\,(x)$ *is of definite variation on* Q.

(ii) $\Sigma \mid v\,(QI_r) \mid$ *is bounded for all divisions of the type contemplated.*

The upper bound of the sum in (ii) we define as the *total variation of* $F\,(x)$ *on* Q.

It can now be shown that Theorem **R** is true if the condition "$F\,(x)$ *is of bounded variation*" is replaced by the less stringent one that "$F\,(x)$ *is of bounded total variation on* Q."

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