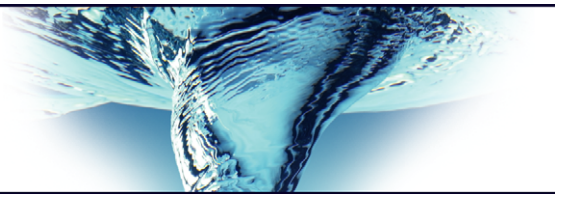
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# Steady water waves with vorticity: spatial Hamiltonian structure

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Spatial dynamical systems are obtained for two-dimensional steady gravity waves with vorticity on water of finite depth. These systems have Hamiltonian structure and Hamiltonian is essentially the flow–force invariant.

**Key words:** Hamiltonian theory, surface gravity waves, waves/free-surface flows

## 1. Introduction

We consider the two-dimensional nonlinear problem of steady waves in a horizontal open channel of uniform rectangular cross-section. The motion of water (or, more generally, an inviscid, incompressible, heavy fluid) occupying the channel is supposed to be rotational with a prescribed vorticity distribution which is a Lipschitz function on $\mathbb{R}$. This type of motion commonly occurs in nature as is indicated by observations (see, for example, Thomas (1990) and Swan, Cummins & James (2001) and references cited therein). We impose no restriction on the type of waves (they may be solitary, periodic with an arbitrary number of crests per period, whatsoever), and the assumption about vorticity yields that our considerations cover flows with counter-currents which were studied in the papers by Kozlov & Kuznetsov (2011*b*, 2013) and several others.

Our aim is to derive Hamiltonian systems equivalent to the original problem of steady waves written in the common form, that is, in terms of a stream function (see, for example, Keady & Norbury 1978). For this purpose we use an approach based on the so-called flow force. Its role in studies of irrotational water waves and especially for developing variational principles for steady waves was permanently emphasized by Benjamin (1971, 1984, 1995), beginning with the pioneering work Benjamin & Lighthill (1954) in which the well-known conjecture was formulated. (It is proved for near-critical values of the Bernoulli constant in Kozlov & Kuznetsov 2010, 2011*a*.)

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In this connection, it is worth quoting Benjamin himself (see Benjamin 1984, p. 4): 'The (...) pivotal property (...) is that (the flow force) is the canonical representative of translation invariance associated with the variational principle that describes steady waves'. Further discussion of the flow–force invariant for irrotational steady waves can be found in § 6.4 of the cited article.

Baesens & MacKay (1992) (see a brief description of their results in § 1.1 below) used this invariant in the context outlined above. Moreover, they supposed that the approach based on it may be applied 'to many other uniformly travelling-wave problems, for example, with density stratification and/or vorticity'. They also posed a problem 'whether all wave systems which have a Hamiltonian formulation in time also have a Hamiltonian formulation in space for their uniformly travelling waves' and conjectured that 'the Hamiltonian is likely to be Benjamin's flow–force invariant in every case but the question is to determine the correct phase space and symplectic form'. In this paper, this question is answered at least with regards to steady waves with vorticity.

### 1.1. *Background*

In Kozlov & Kuznetsov (2013), we characterized briefly the plethora of results obtained for the problem under consideration and a similar one dealing with waves on water of infinite depth; further details can be found in the survey article by Strauss (2010). Therefore, we consider only the work concerning the Hamiltonian approach to water waves mainly focusing on steady waves.

It was Zakharov (1968) who introduced Hamiltonian formulation for time-dependent, irrotational waves on deep water as early as 1968. Considering the case of finite depth, Bridges (1994) noted (see p. 222 of his paper) that if the simple transformation to the frame of reference moving at the speed of wave propagation is applied, then a system arises which looks like a Hamiltonian, but is not a Hamiltonian system. Instead, he developed an explicit account of the Hamiltonian formalism in two and three dimensions based upon physical considerations and using a version of the flow force functional. In the above-mentioned paper by Baesens & MacKay (1992), new variables are introduced instead of physical ones, thus reducing the original problem to a new one in the lower half-plane. Along with Hamiltonian formulations proposed in this paper, the authors demonstrated numerically that there exists a fold in Stokes' family of periodic waves. Groves & Toland (1997) mainly consider waves on water of uniform finite depth and only briefly outline the case of infinite depth. Similar to the approach of Baesens & MacKay (1992), new variables are introduced in their paper so that the original problem is reduced to a new one in the strip of unit width and Hamiltonian formulations for zero and non-zero surface tension are treated simultaneously, unlike in the approach of Baesens & MacKay (1992), where these problems are considered separately. The reason for this is explained on p. 337 of their paper and is as follows. 'The limit (as the coefficient of surface tension tends to zero) is singular: although the variational principle from which we started is good, the Legendre transform cannot be carried out. Nevertheless, a Hamiltonian formulation can still be found'.

Now we turn to Hamiltonian formulations for waves with vorticity. First, Zakharov's result was extended to the case of constant vorticity by Wahlén (2007), and it was also noted that both of these formulations allow us to derive long-wave approximations retaining the symplectic structure (see Craig & Groves (1994) and Wahlén (2008) for the irrotational and constant-vorticity cases, respectively). Second, Groves & Wahlén (2007, 2008) considered Hamiltonian systems for steady waves



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on water of finite depth in the case of an arbitrary distribution of vorticity. The first of these papers deals only with solitary capillary–gravity waves, whereas both Stokes and solitary gravity waves are investigated in the second paper in the absence of surface tension. In both papers, Hamiltonian systems are obtained for problems reformulated through the use of the partial hodograph transformation. The advantage of this approach is that the Legendre transform can be carried out (see, in particular, Groves & Wahlén (2008)). On the other hand, there is a disadvantage as well: it results from the fact that the approach is applicable only to unidirectional flows, whereas flows with counter-currents are within the scope of our approach. (Studies of the latter flows was initiated by Wahlén (2009), and extended in Ehrnström, Escher & Wahlén (2011), Ehrnström, Escher & Villari (2012) and Kozlov & Kuznetsov (2013).)

### 1.2. *Statement of the problem*

Let an open channel of uniform rectangular cross-section be bounded below by a horizontal rigid bottom and let water occupying the channel be bounded above by a free surface not touching the bottom. The surface tension is neglected, the pressure is assumed to be constant on the free surface, and the water motion is supposed to be two-dimensional and rotational. Combining these and the fact that water is incompressible allows us to seek the velocity field in the form $(\psi_y, -\psi_x)$, where $\psi(x, y)$ is referred to as the stream function (see, for example, the book by Lavrentiev & Shabat (1980)). It is also supposed that the vorticity distribution $\omega$ (it is a function of $\psi$ as is explained in §1 of the cited book) is a prescribed Lipschitz function on $\mathbb{R}$.

We use non-dimensional variables chosen so that the constant volume rate of flow per unit span and the constant acceleration due to gravity are scaled to unity in our equations. For this purpose lengths and velocities are scaled to $(Q^2/g)^{1/3}$ and $(Qg)^{1/3}$, respectively; here $Q$ and $g$ are the dimensional quantities for the rate of flow and the gravity acceleration, respectively. We recall that $(Q^2/g)^{1/3}$ is the depth of the critical uniform stream in the irrotational case (see, for example, Benjamin 1995).

In appropriate Cartesian coordinates $(x, y)$, the bottom coincides with the $x$-axis and gravity acts in the negative $y$-direction. We choose the frame of reference so that the velocity field is time-independent as well as the unknown free-surface profile. The latter is assumed to be the graph of $y = \eta(x)$, $x \in \mathbb{R}$, where $\eta$ is a positive $C^1$-function. Therefore, the longitudinal section of the water domain is $D = \{x \in \mathbb{R}, 0 < y < \eta(x)\}$, and $\psi$ is assumed to belong to $C^2(D) \cap C^1(\bar{D})$.

Since the surface tension is neglected, $\psi$ and $\eta$ must satisfy the following free-boundary problem:

$$\psi_{xx} + \psi_{yy} + \omega(\psi) = 0, \quad (x, y) \in D; \tag{1.1}$$

$$\psi(x, 0) = 0, \quad x \in \mathbb{R}; \tag{1.2}$$

$$\psi(x, \eta(x)) = 1, \quad x \in \mathbb{R}; \tag{1.3}$$

$$|\nabla\psi(x, \eta(x))|^2 + 2\eta(x) = 3r, \quad x \in \mathbb{R}. \tag{1.4}$$

In the last condition (known as Bernoulli's equation), $r$ is a constant considered as the problem's parameter and referred to as the total head (see, for example, Keady & Norbury 1978). This statement has long been known and its derivation from the governing equations and the assumptions about the boundary behaviour of water particles can be found, for example, in Constantin & Strauss (2004). Under some restrictions on $\omega$ (in particular, if the vorticity is a monotonic function of the depth



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which occurs when there are no stagnation points), Constantin, Sattinger & Strauss (2006) derived the whole set of relations (1.1)–(1.4) from a variational principle applied to either of two functionals related to the Hamiltonian/Lagrangian duality. Moreover, Constantin & Strauss (2007) and Hur & Lin (2008) used both of these variational formulations in their investigations of stability properties of steady waves with vorticity.

## 2. Reformulation of the problem as spatial dynamical systems

In order to reformulate problem (1.1)–(1.4) as spatial dynamical systems we begin with the problem's weak setting.

### 2.1. *Weak statement of the problem*

Integrating by parts in (1.1) multiplied by a function $v$, one immediately obtains the following integral identity:

$$\int_D [\psi_x v_x + \psi_y v_y - \omega(\psi)v]\,\mathrm{d}x\,\mathrm{d}y = 0. \tag{2.1}$$

It must hold for all $v \in W^{1,2}(D)$ with compact support in $\bar{D}$ and vanishing on $\partial D$. A solution is sought in the space $W^{1,2}_{loc}(D)$ (by $W^{1,2}_{loc}(D)$ we denote the space each element of which belongs to $W^{1,2}(K)$ for some bounded open subset $K \subset D$) and must satisfy the boundary conditions (1.2) and (1.3) as well as Bernoulli's equation.

In order to transform the curved domain $D$ into a strip of constant width we choose a stream solution corresponding to the same value of $r$ as in (1.4). We recall (see Kozlov & Kuznetsov 2011*b*) that every stream solution has the form $(U(y), h)$, $h = \mathrm{const}$, and these solutions exist only for $r \geqslant r_c$, where $r_c$ is the critical value of Bernoulli's constant (see its definition on p. 375 of the cited paper). It is shown in Kozlov & Kuznetsov (2012) that for all $r > r_c$ at least two stream solutions exist and that $r_c$ is the lower bound of admissible values of $r$ for problem (1.1)–(1.4).

Now we map $D$ onto the strip $S = \mathbb{R} \times (0, h)$ by means of changing the vertical coordinate and also introduce new unknown function as follows:

$$z = y\frac{h}{\eta(x)}, \quad \Phi(x, z) = \psi\left(x, \frac{\eta(x)}{h}z\right). \tag{2.2}$$

In identity (2.1), we substitute the variables $(x, z)$ and the function $\phi$ instead of $(x, y)$ and $\Phi$, respectively, and so (2.1) takes the form

$$\int_S \left\{\left[\Phi_x - \frac{z\eta_x}{\eta}\Phi_z\right]\left[v_x - \frac{z\eta_x}{\eta}v_z\right] + \left[\frac{h}{\eta}\right]^2 \Phi_z v_z - \omega(\Phi)v\right\}\frac{\eta\,\mathrm{d}x\,\mathrm{d}z}{h} = 0. \tag{2.3}$$

This must hold for all $v \in W^{1,2}(S)$ with compact support in $\bar{S}$ and vanishing on $\partial S$. The integral identity (2.3) must be augmented by the following boundary conditions

$$\Phi(x, 0) = 0, \quad \Phi(x, h) = 1, \quad x \in \mathbb{R}, \tag{2.4}$$

whereas for the Bernoulli's equation (1.4) we get the following new form:

$$\frac{h^2[1 + \eta_x^2(x)]}{\eta^2(x)}\Phi_z^2(x, h) + 2\eta(x) = 3r, \quad x \in \mathbb{R}. \tag{2.5}$$



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Here the second condition (2.4) is also taken into account. Thus, problem (1.1)–(1.4) for $\psi(x, y)$ and $\eta(x)$ is equivalent to problem (2.3)–(2.5) for $\Phi(x, z)$ and $\eta(x)$, whereas formulae (2.2) allow us to obtain $\psi(x, y)$ knowing the latter functions.

### 2.2. *Derivation of spatial dynamical systems*

In what follows we assume that solutions of problem (1.1)–(1.4) have no stagnation points on the free surface (this will be essential for deriving spatial dynamical systems equivalent to the problem). In terms of the equivalent problem (2.3)–(2.5), this means that $\Phi_z(x, h) \neq 0$ for all $x \in \mathbb{R}$. Indeed, the definition of $\Phi$ and the second condition (2.4) yield that $\Phi_z(x, h)$ and $\nabla\psi(x, \eta(x))$ vanish simultaneously. It should be mentioned that this assumption is not too restrictive. Indeed, it was shown by Wahlén (2009) and Constantin & Varvaruca (2011) that in the case of constant vorticity all possible stagnation points are located under the free surface at least for waves of sufficiently small amplitude.

It occurs that (2.3)–(2.5) can be written as different dynamical systems with respect to the spatial variable $x \in \mathbb{R}$. Let us consider $\Phi$ as a differentiable mapping from $\mathbb{R}$ to $W^{2,2}(0, h) \cap W^{1,2}_1(0, h)$, where $W^{1,2}_1(0, h)$ is the subset of $W^{1,2}(0, h)$ whose elements vanish at $z = 0$ and are equal to 1 at $z = h$, and so $\Phi$ satisfies the boundary conditions (2.4). As in the case of irrotational waves considered by Baesens & MacKay (1992) the Legendre transform cannot be carried out for obtaining a variable conjugate to $\Phi$. Therefore, it must be brought into consideration 'by hand', and so we introduce the following function:

$$\Psi(x, z) = \frac{\eta(x)}{h}\left[\Phi_x(x, z) - \frac{z\eta_x(x)}{\eta(x)}\Phi_z(x, z)\right]. \tag{2.6}$$

In what follows, it will become clear that $\Psi$ mapping $\mathbb{R}$ to $W^{1,2}(0, h)$ is conjugate to $\Phi$.

Putting $z = h$ in (2.6), we get that

$$\eta_x(x) = -\frac{\Psi(x, h)}{\Phi_z(x, h)}, \tag{2.7}$$

where the second condition (2.4) is taken into account again. Using formula (2.7) in (2.5), we write Bernoulli's equation in the form

$$\Psi^2(x, h) + \Phi_z^2(x, h) = P(\eta(x)), \quad x \in \mathbb{R}, \tag{2.8}$$

where $P(\eta(x)) = \eta^2(x)\,[3r - 2\eta(x)]/h^2$ which is positive for all $x \in \mathbb{R}$. This follows from the assumption made at the beginning of this section, and it was also supposed in § 1.2 that $\eta(x) > 0$ for all $x \in \mathbb{R}$.

It is clear that the polynomial $P(\eta)$ attains its maximum at $\eta = r$ because

$$P'(\eta) = 6\eta(r - \eta)/h^2. \tag{2.9}$$

Since this maximum is equal to $r^3/h^2$, we see that the cubic equation $P(\eta) = v$ has a root $\eta_\rho(v)$ provided $v \leqslant r^3/h^2$. In particular, if

$$\Psi^2(x, h) + \Phi_z^2(x, h) \leqslant r^3/h^2 \quad \text{for all } x \in \mathbb{R}, \tag{2.10}$$

then (2.8) has a solution

$$\eta_0(x) = \eta_\rho(\Psi^2(x, h) + \Phi_z^2(x, h)). \tag{2.11}$$



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Let us suppose that the last inequality is strict, then $\eta_0$ is not a double root. In other words, $\eta_0(x) \neq r$ for all $x \in \mathbb{R}$, and so $\eta_0$ is either greater than $r$ or less than this value.

2.2.1. *Spatial dynamical system for* $(\Phi, \Psi)$

Let us write (2.6) in the following form:

$$\Phi_x = \frac{h}{\eta}\Psi - \frac{z}{\eta}\left[\frac{\Psi}{\Phi_z}\right]_{z=h}\Phi_z, \quad (x, z) \in \mathbb{R} \times (0, h), \tag{2.12}$$

where (2.7) is also taken into account.

In order to reduce the integral identity (2.3) to a first-order equation for $\Psi$ we proceed as follows. Using the first relation (2.6), we change the first square bracket in the integrand to $(h/\eta)\Psi$ and integrate by parts in each term. This gives

$$\int_S \left[-\Psi_x + \frac{\eta_x}{\eta}(z\Psi)_z - \frac{h}{\eta}\Phi_{zz} - \frac{\eta}{h}\omega(\Phi)\right] v \, \mathrm{d}z = 0, \tag{2.13}$$

because the properties of $v$ guarantee that the integrated terms vanish. Since $v$ is an arbitrary function, we arrive at the following equation:

$$\Psi_x = -\frac{1}{\eta}\left[\frac{\Psi}{\Phi_z}\right]_{z=h}(z\Psi)_z - \frac{h}{\eta}\Phi_{zz} - \frac{\eta}{h}\omega(\Phi), \quad (x, z) \in \mathbb{R} \times (0, h), \tag{2.14}$$

where formula (2.7) is taken into account again.

Thus, (2.12) and (2.14) form the required dynamical system together with constraint (2.8). The latter defines $\eta_0 \neq r$ by formula (2.11) provided inequality (2.10) is strict. Substituting $\eta_0$ into (2.12) and (2.14), we get spatial dynamical system for $(\Phi, \Psi)$. The phase space of this system is $[W^{2,2}(0, h) \cap W_1^{1,2}(0, h)] \times W^{1,2}(0, h)$.

2.2.2. *Spatial dynamical system for* $(\Phi, \Psi, \eta)$

If $\eta$ in (2.12) and (2.14) is considered as an extra unknown, then these equations must be complemented by another one with $\eta_x$ in the left-hand side. Note that (2.7) cannot be considered as such an equation because it is a consequence of the definition of $\Psi$.

Let us differentiate (2.8) which in view of (2.9) gives

$$\eta_x = \frac{h^2}{3\eta(r-\eta)}[\Psi\Psi_x + \Phi_z\Phi_{xz}]_{z=h}. \tag{2.15}$$

Now we substitute $\Psi_x(x, h)$ from (2.14), then get $\Phi_{xz}(x, h)$ by differentiating (2.12) and also substitute it into (2.15), thus getting

$$\eta_x = \frac{h}{3\eta^2(r-\eta)}\left[h^2(\Psi_z\Phi_z + \Psi\Phi_{zz}) + \eta^2\omega(1)\Psi \right.$$
$$\left. - \frac{h\Psi}{2\Phi_z}\left\{2(\Psi^2 + \Phi_z^2) + h\left(\Psi^2 + \Phi_z^2\right)_z\right\}\right]_{z=h}, \quad x \in \mathbb{R}. \tag{2.16}$$

For the sake of brevity, we denote by $K(\Psi, \Phi, \eta)$ the right-hand side of this equation in what follows. Thus, problem (1.1)–(1.4) is equivalent to the spatial dynamical system (2.12), (2.14) and (2.16) for the triple $(\Phi, \Psi, \eta)$ mapping $\mathbb{R}$ to

$$\left[W^{2,2}(0, h) \cap W_1^{1,2}(0, h)\right] \times W^{1,2}(0, h) \times (\mathbb{R}_+ \setminus \{r\}), \tag{2.17}$$

which is the phase space of this system. It is clear that this system has a first integral given by formula (2.8).



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## 3. Description of Hamiltonian structures

Prior to describing structures, which appear to be Hamiltonian for waves with vorticity, we discuss the flow–force invariant for them.

### 3.1. *Flow force*

Let us consider the following expression (cf. Keady & Norbury 1978, p. 135):

$$s = \left[r + \frac{2}{3}\Omega(1)\right]\eta(x) - \frac{1}{3}\left[\eta^2(x) + \int_0^{\eta(x)} \left(\psi_x^2 - \psi_y^2 + 2\Omega(\psi)\right)\,\mathrm{d}y\right], \quad x \in \mathbb{R}, \tag{3.1}$$

where $\Omega(\psi) = \int_0^{\psi} \omega(\tau)\,\mathrm{d}\tau$ and $(\psi, \eta)$ satisfy problem (1.1)–(1.4). Using this problem, it is straightforward to check that $s_x \equiv 0$, and so $s$ is a functional of $(\psi, \eta)$; it is referred to as the non-dimensional flow force for rotational waves.

The role of $(r, s)$ parameter space became clear after the pioneering work of Benjamin & Lighthill (1954). Indeed, it is possible to investigate (at least numerically) the structure of the local bifurcation of waves (both with and without vorticity) in this space. In the case of constant vorticity, it was found by Doole (1998) that this structure is substantially more complicated than in the irrotational case considered in Doole (1997). Moreover, there is a possibility of global brunch connections on the $(r, s)$-plane inside the cusp corresponding to the unidirectional supercritical and subcritical shear flows. This is completely different from the simple parametrization of irrotational waves that takes place at least for near-critical values of Bernoulli's constant (see Kozlov & Kuznetsov 2010, 2011*a*).

In what follows, it is convenient to use $\mathscr{S} = 3s/2$ instead of $s$. Let us write $\mathscr{S}$ in terms of the triple $(\Phi, \Psi, \eta)$, which gives

$$\mathscr{S} = \left[\frac{3}{2}r + \Omega(1)\right]\eta - \frac{\eta^2}{2} - \int_0^h \left[\frac{h}{2\eta}(\Psi^2 - \Phi_z^2) + \frac{\eta}{h}\Omega(\Phi)\right]\mathrm{d}z, \tag{3.2}$$

which will be used for introducing the Hamiltonian.

### 3.2. *Hamiltonian, the phase space, the symplectic form and the Hamiltonian system*

In the same way as in Baesens & MacKay (1992), we add a fourth variable, say $w$, and choose as the Hamiltonian

$$H(\Phi, \Psi, \eta, w) = \mathscr{S} + \mathscr{F} \quad \text{where } \mathscr{F} = -K(\Phi, \Psi, \eta)\left[w + \frac{1}{\eta}\int_0^h z\Phi_z(x, z)\Psi(x, z)\,\mathrm{d}z\right]. \tag{3.3}$$

As phase space we take $M = [W^{2,2}(0, h) \cap W_1^{1,2}(0, h)] \times W^{1,2}(0, h) \times (\mathbb{R}_+ \setminus \{r\}) \times \mathbb{R}$, the symplectic form which is the standard one (cf. Baesens & MacKay 1992):

$$\varpi = \mathrm{d}\eta \wedge \mathrm{d}w + \int_0^h \mathrm{d}\Phi \wedge \mathrm{d}\Psi\,\mathrm{d}z. \tag{3.4}$$

The corresponding Hamiltonian system is as follows:

$$\Phi_x = -\frac{\delta H}{\delta \Psi}, \quad \Psi_x = \frac{\delta H}{\delta \Phi}, \quad \eta_x = -\frac{\delta H}{\delta w}, \quad w_x = \frac{\delta H}{\delta \eta}. \tag{3.5}$$



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It is straightforward to calculate the right-hand side terms in the first two equations, thus obtaining

$$-\frac{\delta H}{\delta \Psi} = \frac{h}{\eta}\Psi + \frac{z}{\eta}K\Phi_z + \frac{\delta K}{\delta \Psi}\left[w + \frac{1}{\eta}\int_0^h z\Phi_z\Psi \,\mathrm{d}z\right], \tag{3.6}$$

$$\frac{\delta H}{\delta \Phi} = -\frac{h}{\eta}\Phi_{zz} - \frac{\eta}{h}\,\omega(\Phi) + \frac{1}{\eta}K(z\Psi)_z - \frac{\delta K}{\delta \Phi}\left[w + \frac{1}{\eta}\int_0^h z\Phi_z\Psi \,\mathrm{d}z\right]. \tag{3.7}$$

Note that if the following relation

$$w(x) = \frac{-1}{\eta(x)}\int_0^h z\Phi_z(x,z)\Psi(x,z)\,\mathrm{d}z \tag{3.8}$$

holds, then the Hamiltonian coincides with the flow force and the expression in the square brackets vanishes in (3.6) and (3.7).

### 3.3. *Invariant manifold for system (3.5)*

On the phase space $M$, we introduce a submanifold, say $M'$, consisting of all quadruples $(\Phi, \Psi, \eta, w) \in M$ which satisfy relation (3.8). Let us show that $M'$ is an invariant manifold for system (3.5). For this purpose it is sufficient to prove the following assertion: if the first three equations (3.5) hold on $M'$, then the last equation is also fulfilled there.

Prior to proving this, we note that restricting the first three equations (3.5) to $M'$, one obtains several important properties for them. It was mentioned above that the last term vanishes in formulae (3.6) and (3.7). Then putting $z = h$ in the first equation (3.5), and combining the resulting equality with (3.6) and the boundary condition $\Phi(x,h) = 1$, we get that

$$K(\Phi, \Psi, \eta) = -\left[\frac{\Psi}{\Phi_z}\right]_{z=h}. \tag{3.9}$$

Hence, $K(\Phi, \Psi, \eta)$ is independent of $\eta$ on $M'$. Combining the last relation and formulae (3.6) and (3.7), one gets that the first two equations (3.5) restricted to $M'$ coincide with (2.12) and (2.14), respectively. Furthermore, it is clear that the third equation (3.5) and (2.16) are the same.

Now we are in a position to prove that if the first three equations (3.5) hold on $M'$, then the same is true for the last one. In order to check this we differentiate (3.8), thus getting

$$w_x(x) = \frac{1}{\eta(x)}\left[\frac{K}{\eta(x)}\int_0^h z\,\Phi_z(x,z)\Psi(x,z)\,\mathrm{d}z - \int_0^h z(\Phi_{xz}\Psi + \Phi_z\Psi_x)\,\mathrm{d}z\right]. \tag{3.10}$$

Here the definition of $K$ is also taken into account. Substituting $\Phi_{xz}$ and $\Psi_x$ from (2.12) and (2.14), respectively, we obtain after some algebra and integration by parts that the last integral is equal to

$$\int_0^h z\left\{\frac{h}{2\eta}[(\Psi^2)_z - (\Phi_z^2)_z] - \frac{\eta}{h}[\Omega(\Phi)]_z\right\}\mathrm{d}z - \frac{h^2}{\eta}\Psi^2(x,h). \tag{3.11}$$

Note that this form of the second term arises when one applies formula (2.7). Integrating by parts again, we transform this expression into the following form:

$$-\int_0^h \left[\frac{h}{2\eta}\left(\Psi^2 - \Phi_z^2\right) - \frac{\eta}{h}\Omega(\Phi)\right]\mathrm{d}z - \frac{h^2}{2\eta}\left[\Psi^2 + \Phi_z^2\right]_{z=h} - \eta\Omega(1). \tag{3.12}$$



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Now we substitute this into the last equation, thus obtaining for $x \in \mathbb{R}$:

$$w_x = \frac{K}{\eta^2}\int_0^h z\Phi_z\Psi\,\mathrm{d}z + \int_0^h \left[\frac{h}{2\eta^2}\left(\Psi^2 - \Phi_z^2\right) - \frac{\eta}{h}\Omega(\Phi)\right]\mathrm{d}z - \eta + \frac{3}{2}r + \Omega(1). \quad (3.13)$$

It is straightforward to show that this equation and the last equation (3.5) are the same which completes the proof. Note that the presence of an integro-differential operator in the right-hand side of (3.13) distinguishes this equation from (2.12), (2.14) and (2.16).

Summarizing our considerations in §§ 3.2 and 3.3, we formulate the following assertion.

*Let $(H, M, \varpi)$ be the spatial Hamiltonian structure with Hamiltonian $H$ given by (3.3), the phase space $M$ consisting of all quadruples $(\Phi, \Psi, \eta, w)$ that belong to*

$$[W^{2,2}(0,h) \cap W_1^{1,2}(0,h)] \times W^{1,2}(0,h) \times (\mathbb{R}_+ \setminus \{r\}) \times \mathbb{R}, \quad (3.14)$$

*whereas the symplectic form $\varpi$ is given by formula (3.4). If $M' \subset M$ is defined by relation (3.8), then $M'$ is an invariant manifold for the Hamiltonian equations of motion (3.5) and the restriction of system (3.5) to $M'$ gives (2.12), (2.14) and (2.16) describing steady water waves with vorticity.*

At the end of § 2.2.2, it was pointed out that system (2.12), (2.14) and (2.16) has a first integral given by formula (2.8). According to the last assertion, it also gives an integral of system (3.5) restricted to $M'$.

### 3.4. *Hamiltonian structure with two constraints*

Another Hamiltonian structure arises if we take as phase space the manifold $M''$ consisting of all quadruples $(\Phi, \Psi, \eta, w) \in M$ satisfying constraints (2.8) and (3.8) simultaneously. It is clear that $M''$ is an invariant manifold for system (3.5). Indeed, it is shown in § 3.3 that the fourth equation (3.5) holds on $M' \supset M''$. The third equation holds on $M''$ because it is a consequence of (2.8) defining this manifold and the remaining two equations. According to its definition, $M''$ is parametrized by $\Phi$ and $\Psi$. Indeed, (2.8) implies that we have to put $\eta = \eta_0 \neq r$, where $\eta_0$ is defined by formula (2.11), whereas (3.8) gives the dependence of $w$ on $\Phi$ and $\Psi$.

Let us consider the restriction of the symplectic form $\varpi$ to $M''$. Let two elements of the cotangent space at $(\Phi, \Psi, \eta, w) \in M''$ be $m_1 = (\Phi_1, \Psi_1, \eta_1, w_1)$ and $m_2 = (\Phi_2, \Psi_2, \eta_2, w_2)$. Since we have, for $i = 1, 2$,

$$\left.\begin{aligned} \eta_i &= 2\eta'_\rho[\Psi\Psi_i + \Phi_z\Phi_{iz}]_{z=h}, \\ w_i &= \frac{\eta_i}{\eta^2}\int_0^h z\Phi_z(x,z)\Psi(x,z)\,\mathrm{d}z - \frac{1}{\eta}\int_0^h z\,[\Phi_{iz}\Psi + \Phi_z\Psi_i]\,\mathrm{d}z, \end{aligned}\right\} \quad (3.15)$$

we get that $\varpi(m_1, m_2)_{(\Phi,\Psi,\eta,w)}$ takes the following form:

$$\int_0^h (\Psi_2\Phi_1 - \Phi_2\Psi_1)\,\mathrm{d}z + \frac{h^2}{3\eta_0^2(r-\eta_0)}\left\{[\Psi\Psi_2 + \Phi_z\Phi_{2z}]_{z=h}\int_0^h z(\Phi_{1z}\Psi + \Phi_z\Psi_1)\,\mathrm{d}z\right.$$
$$\left. - [\Psi\Psi_1 + \Phi_z\Phi_{1z}]_{z=h}\int_0^h z\,(\Phi_{2z}\,\Psi + \Phi_z\,\Psi_2)\,\mathrm{d}z\right\}. \quad (3.16)$$

Here $\eta_0$ is given by formula (2.11) and is taken into account that $\eta'_\rho = 1/P'(\eta_0)$ (see (2.9) for its reciprocal). Thus, $\varpi$ depends only on $(\Phi_1, \Psi_1)$ and $(\Phi_2, \Psi_2)$ on $M''$. It is straightforward to show that (3.16) is a non-degenerate form, and so $M''$ is a



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symplectic manifold. If we take

$$\mathscr{S}(\Phi,\Psi)=\left[\frac{3}{2}r+\Omega(1)\right]\eta_0-\frac{\eta_0^2}{2}-\int_0^h\left[\frac{h}{2\eta_0}(\Psi^2-\Phi_z^2)+\frac{\eta_0}{h}\Omega(\Phi)\right]\mathrm{d}z, \quad (3.17)$$

as Hamiltonian, then $(\mathscr{S}, M'', \varpi|_{M''})$ is a Hamiltonian structure.

Equating the coefficients at $\delta\Phi$ and $\delta\Psi$ in $\varpi(\delta\Phi,\delta\Psi;\Phi_x,\Psi_x)=\mathrm{d}\mathscr{S}(\delta\Phi,\delta\Psi)$, one obtains the Hamiltonian equations of motion on $M''$. In view of formula (3.16) for $\varpi$, these equations take the form

$$\Phi_x-\frac{h^2z\Phi_z}{6\eta_0^2(r-\eta_0)}[(\Psi^2+\Phi_z^2)_x]_{z=h}=-\frac{\delta'\mathscr{S}}{\delta\Psi}, \quad (3.18)$$

$$\Psi_x-\frac{h^2(z\Psi)_z}{6\eta_0^2(r-\eta_0)}[(\Psi^2+\Phi_z^2)_x]_{z=h}=\frac{\delta'\mathscr{S}}{\delta\Phi}. \quad (3.19)$$

Here $\delta'$ means that $\eta_0$ is considered fixed while calculating these variational derivatives which are as follows:

$$-\frac{\delta'\mathscr{S}}{\delta\Psi}=\frac{h}{\eta_0}\Psi, \qquad \frac{\delta'\mathscr{S}}{\delta\Phi}=-\frac{h}{\eta_0}\Phi_{zz}-\frac{\eta_0}{h}\omega(\Phi). \quad (3.20)$$

If we put $z=h$ in (3.18), then it follows from the second condition (2.4) that

$$\frac{h^2}{6\eta_0^2(r-\eta_0)}[(\Psi^2+\Phi_z^2)_x]_{z=h}=\frac{1}{\eta_0}\left[\frac{\Psi}{\Phi_z}\right]_{z=h}. \quad (3.21)$$

Using this equality in (3.18), (3.19) and taking into account formulae (3.20), we see that this system coincides with (2.12) and (2.14) derived in § 2.2.1.

Summarizing the above considerations, we formulate the following assertion.

*The submanifold $M''\subset M$ defined by constraints (2.8) and (3.8) is invariant for system (3.5). The restriction of the spatial Hamiltonian structure $(H, M, \varpi)$ to $M''$ is also a Hamiltonian structure, namely, $(\mathscr{S}, M'', \varpi|_{M''})$. The Hamiltonian equations of motion corresponding to the last structure are (3.18) and (3.19). They coincide with (2.12) and (2.14), where $\eta=\eta_0$ and the latter is defined by formula (2.11). Therefore, (3.18) and (3.19) describe steady water waves with vorticity.*

## 4. Conclusion

We have considered the problem describing steady, rotational, water waves in the case when counter-currents might be present in the corresponding flow of finite depth. This problem is represented as spatial dynamical systems in an appropriately chosen strip of constant width. In order to show that these systems are Hamiltonian the following approach is applied. First, an extra fourth variable is introduced and the corresponding spatial Hamiltonian structure is defined. Then an invariant submanifold of codimension two of the phase space is considered and it is shown that the restriction of the original structure to this submanifold is also Hamiltonian. Two systems of Hamiltonian equations of motion are obtained, one of which consists of four equations on the whole phase space, whereas only two equations for variables parametrizing the mentioned submanifold form the second system. These results could lead to prediction of waves with vorticity other than periodic and solitary waves whose existence is established so far (see, for example, Groves & Wahlén (2008) and Kozlov & Kuznetsov (2013) and references cited therein).



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The results of the present paper extend those by Groves & Wahlén (2008) concerning Hamiltonian structure for rotational waves on unidirectional flows. In addition, our results are in agreement with the conjecture that for steady travelling waves the Hamiltonian is likely to be the flow force whenever a Hamiltonian formulation exists for a time-dependent wave system (see Baesens & MacKay 1992, p. 345).

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