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Anomalous transport matrix for cylindrical plasmas

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Abstract. Transport equations are investigated for a cylindrical plasma in the presence of electrostatic fluctuations. In a weakly turbulent regime, the transport matrices that relate the anomalous particle and heat fluxes and the parallel current to the thermodynamical forces are determined by employing drift-kinetic and gyrokinetic orderings for the electrons and ions. The calculation is based on the kinetic equations for the ensemble-averaged and fluctuating distribution functions. The crucial difference with previous works is the inclusion of an extra term in the drift-kinetic equation for the fluctuating electron distribution function. This extra term, which arises from the ensemble-averaged first-order (in a Larmor radius expansion) electron distribution function, leads to the Ware pinch components of the particle and heat fluxes and a correction to the Ohmic current. Furthermore, Shaing's ansatz, which was introduced in the synthetic theory of anomalous and neoclassical transport, is shown to be connected with this extra term in the context of a turbulent plasma, and the physical meaning and the validity of this ansatz are revealed. In drift-wave turbulence, the transport matrices, expressed in an implicit form by considering the frequency of fluctuations as a parameter, are rewritten in an explicit form by determining its frequency through the dispersion relation. The Onsager symmetry is shown to be broken for this explicit form of anomalous transport matrix.

1. Introduction

The main object of transport theory is to derive the transport equations that relate the dissipative fluxes to the thermodynamical forces. These transport equations have been intensively studied for neoclassical transport driven by binary collisions and drift motions in an inhomogeneous magnetic field (Hinton and Hazeltine 1973; Hirshman and Sigmar 1981; Balescu 1988). In neoclassical transport theory, the transport equations linearly relate the radial particle and heat fluxes and the parallel current to the pressure and temperature gradients and the electric field. The transport matrices relating these fluxes and current to the thermodynamical forces have been explicitly calculated in various geometries and collisionality regimes.

It has been observed in most fusion experiments that the cross-field particle and energy fluxes exceed those predicted by neoclassical transport theory (Liewer 1985). This deviation is referred to as anomalous transport. Anomalous transport

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has been attributed to turbulent fluctuations caused by microscopic instabilities in the plasma. There has been much theoretical work on anomalous transport, but most of this has not been connected with neoclassical transport theory.

Shaing (1988a,b, 1990), and later Balescu (1990), Sugama and Horton (1995, 1996, 1997a,b, 1998) and Sugama et al. (1996) proposed a synthesis of anomalous and neoclassical transport theories. In all these papers, the anomalous transport is considered in the weakly turbulent regime for an axisymmetric plasma. The theories of Shaing (1988a,b) and Balescu (1990) are based upon an ansatz (Shaing's ansatz) for the distribution function, which is extended from a similar ansatz used in neoclassical transport theory in the absence of fluctuations. This ansatz in neoclassical transport theory is introduced in the plateau regime in order to preserve the momentum-conservation property of the collision operator. However, the physical meaning of this ansatz is obscure in the context of a turbulent plasma. Sugama and Horton (1995) approached the same problem in a different way. They started with the kinetic equations for the ensemble-averaged and fluctuating parts of the distribution function, and they did not use Shaing's ansatz. These two approaches led to different transport equations and to different claims about the Onsager symmetry of the transport matrices. The situation is further confused with the use of different definitions for the anomalous heat flux by Shaing (1988a,b), Balescu (1990) and Sugama and Horton (1995).

The main discrepancies among the previous works for the transport equations in a weakly turbulent plasma may not be caused by neoclassical effects. The essence of these discrepancies may lie in the synthesis of anomalous transport and *classical* transport. Thus, in the present paper, in order to resolve the conflicting results of the previous works, we investigate the unification of anomalous and classical transport for a cylindrical plasma in a uniform magnetic field. The extension of this work to a toroidal plasma is performed in a companion paper (Taguchi 1999).

Similarly to Sugama and Horton (1995), we start with the kinetic equations for the ensemble-averaged and fluctuating distribution functions, and we also avoid the use of Shaing's ansatz. We differ from Sugama and Horton (1995) in that we keep an extra term in the kinetic equation for the fluctuating electron distribution function. Retaining this term from which the Ware pinch components of radial fluxes and a correction to the Ohmic current are derived is necessary for obtaining the complete transport matrix in the weakly turbulent plasma.

Shaing's ansatz in a turbulent plasma is related to the extra term introduced here. From our formulation, we can clearly understand the physical meaning and the validity of Shaing's ansatz. Shaing (1988a,b) introduced this ansatz in order to synthesize neoclassical and anomalous transport. Thus this sometimes leads to the misunderstanding that Shaing's ansatz is closely connected with the neoclassical effect. However, the neoclassical effect is not essential to Shaing's ansatz or even to Shaing's theory. The results in the neoclassical quasilinear transport theory of Shaing (1988a,b) can be derived even for a cylindrical plasma in a uniform magnetic field.

The present paper is organized as follows. In Sec. 2, we present the kinetic equations for the ensemble-averaged and fluctuating distribution functions in the presence of electrostatic fluctuations. The crucial difference from the previous works in our kinetic equations is also explained. The anomalous particle and heat fluxes are introduced in Sec. 3. Starting from the energy balance equation, we derive the precise definition of the anomalous heat flux, and we discuss the difference between our

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definition and the one used previously. The transport matrix for the electrons is calculated in Sec. 4. The similar transport matrix for the ions is obtained in Appendix A. The imaginary frequency of the fluctuations, which is neglected in the calculation of Sec. 4 and Appendix A, gives an important effect on the ion transport. The ion particle and heat fluxes due to this imaginary frequency and the transport matrix are discussed in Sec. 5. Up to here, we treat the real and imaginary frequencies as parameters in the transport equations. In Sec. 6 we determine these frequencies self-consistently from the dispersion relation in the drift-wave turbulent plasma, and we derive the transport equation in an explicit form. The Onsager symmetry for the anomalous transport matrix is also discussed in this section. In Sec. 7, we consider the physical meaning and the validity of Shaing's ansatz, and we discuss the relation with our formulation. Concluding remarks are given in Sec. 8.

2. Basic equations

A plasma that is weakly turbulent owing to the presence of electrostatic fluctuations is considered in cylindrical geometry. We represent the equilibrium magnetic field in the cylindrical coordinates (r, θ, z) as $\mathbf{B} = B_z\mathbf{e}_z + B_\theta\mathbf{e}_\theta$, with $B_z \gg | B_\theta |$, where $\mathbf{e}_z$ and $\mathbf{e}_\theta$ are the unit vectors in the toroidal and poloidal directions. This cylindrical plasma is assumed to have inhomogeneity in the radial direction r for the equilibrium density, temperature and electrostatic potential.

Let us introduce two small parameters $\delta_\alpha = \rho_\alpha/L$ and $\Delta = k_\parallel/k_\perp$, where ρ_α is the Larmor radius of species α , L is the gradient scale length, and $k_\parallel \sim L^{-1}$ and $k_\perp$ are the parallel and perpendicular wavenumbers of fluctuations. We decompose any physical quantity A_α^t into an ensemble-averaged part A_α and a fluctuation δA_α,

$$A_\alpha^t = A_\alpha + \delta A_\alpha, \tag{2.1}$$

and we express the fluctuation in the form

$$\delta A_\alpha = \sum_{m,n} \delta A_{\alpha mn} \exp\left[i \left(m\theta - \frac{nz}{2\pi R} - \omega t \right) \right]. \tag{2.2}$$

Here the frequency ω, which is, in general, a complex quantity, is determined as a function of the wave vector $\mathbf{k}$ through the dispersion relation $\omega = \omega_\mathbf{k} + i\gamma_\mathbf{k}$. We assume the order of the fluctuating potential $\delta\phi$ to be

$$\frac{e_\alpha \delta\phi}{T_\alpha} \sim \Delta, \tag{2.3}$$

where e_α and T_α are the charge and temperature of species α. Furthermore, the following drift-kinetic and gyrokinetic orderings for the electrons and ions are assumed:

$$\delta_e \ll \Delta, \qquad \delta_i \sim \Delta. \tag{2.4}$$

We now expand the distribution function f_α^t of species α in terms of the small parameters δ_α and Δ as

$$f_\alpha^t = f_{\alpha 0} + f_{\alpha 1} + \delta f_\alpha + ..., \tag{2.5}$$

where $f_{\alpha 0}$ and $f_{\alpha 1}$ are the zeroth- and first-order ensemble-averaged distribution functions in the expansion with respect to δ_α, and $f_{\alpha 0}$ is taken to be Maxwellian; and the fluctuating distribution function δf_α is assumed to be of order Δ. Then the

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gyrophase-averaged part $\bar{f}_{\alpha 1}$ of the first-order distribution function $f_{\alpha 1}$ satisfies the following drift-kinetic equation in the presence of electrostatic fluctuations (Sugama and Horton 1995):

$$C_\alpha(\bar{f}_{\alpha 1}) = -\frac{e_\alpha}{T_\alpha} v_\parallel E_\parallel^{(A)} f_{\alpha 0} - S_\alpha, \tag{2.6}$$

where $v_\parallel = \mathbf{b}\cdot\mathbf{v}$, with $\mathbf{b} = \mathbf{B}/B$, $E_\parallel^{(A)}$ is the externally applied electric field, and C_α is the linearized Fokker–Planck collision operator. The source term due to the fluctuating electrostatic potential is written as

$$S_\alpha = \tfrac{1}{2}\mathrm{Re}\left(e_\alpha v_\parallel \mathbf{b}\cdot\boldsymbol{\nabla}\delta\phi^* \frac{\partial \delta f_\alpha}{\partial E_\alpha}\right)_{\mathrm{ens}}, \tag{2.7}$$

where $E_\alpha = \frac{1}{2} m_\alpha v^2 + e_\alpha \phi$, with the mass m_α of species α and the equilibrium electrostatic potential ϕ; the subscript ens and the superscript $*$ indicate the ensemble average and the complex conjugate respectively; and the partial derivative $\partial/\partial E_\alpha$ is taken keeping the perpendicular velocity $v_\perp = (v^2 - v_\parallel^2)^{1/2}$ fixed.

The linearized drift-kinetic equation for the fluctuating part of the electron distribution function is written up to $O(\Delta\delta_e)$ as

$$\frac{\partial \delta f_e}{\partial t} + (v_\parallel \mathbf{b} + \mathbf{v}_d)\cdot\boldsymbol{\nabla}\delta f_e - C_e(\delta f_e) = -\delta\mathbf{v}_E\cdot\boldsymbol{\nabla} f_{e0} - e(v_\parallel\mathbf{b} + \mathbf{v}_d)\cdot\boldsymbol{\nabla}\delta\phi\frac{\partial f_{e0}}{\partial E_e} - e v_\parallel \mathbf{b}\cdot\boldsymbol{\nabla}\delta\phi\frac{\partial \bar{f}_{e1}}{\partial E_e}, \tag{2.8}$$

where

$$\mathbf{v}_d = \frac{c\mathbf{b}\times\boldsymbol{\nabla}\phi}{B}, \qquad \delta\mathbf{v}_E = \frac{c\mathbf{b}\times\boldsymbol{\nabla}\delta\phi}{B},$$

e is the elementary charge and $\delta\phi$ is the fluctuating electrostatic potential. We divide the distribution function δf_e into adiabatic and non-adiabatic parts as

$$\delta f_e = \frac{e\delta\phi}{T_e} f_{e0} + \delta h_e. \tag{2.9}$$

Then the non-adiabatic part δh_e is determined in the linear approximation from the equation

$$-i(\omega - \omega_E - k_\parallel v_\parallel)\hat{h}_e - C_e(\hat{h}_e) = i(\omega - \omega_E - \omega_{*e}^T)\frac{e\hat{\phi}}{T_e} f_{e0} - \mathrm{i}e k_\parallel v_\parallel \hat{\phi}\frac{\partial \bar{f}_{e1}}{\partial E_e}, \tag{2.10}$$

where $\hat{h}_e$ and $\hat{\phi}$ are the Fourier components of δh_e and $\delta\phi$, i.e. $\hat{h}_e \equiv \delta h_{emn}$ and $\hat{\phi} \equiv \delta\phi_{mn}$. The parallel wavenumber $k_\parallel$, ω_E and $\omega_{*\alpha}^T$ are respectively defined as

$$k_\parallel = \frac{m - nq}{qR},$$

with the safety factor q,

$$\omega_E = \frac{mc}{Br}\frac{d\phi}{dr}$$

and

$$\omega_{*\alpha}^T = \omega_{*\alpha}\left[1 + \eta_\alpha\left(\frac{m_\alpha}{2T_\alpha}v^2 - \frac{3}{2}\right)\right], \tag{2.11}$$

with

$$\omega_{*\alpha} = \frac{mcT_\alpha}{re_\alpha B}\frac{d\ln n_\alpha}{dr}, \qquad \eta_\alpha = \frac{d\ln T_\alpha}{d\ln n_\alpha}.$$

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The second term on the right-hand side of (2.10) is one order higher in Δ compared with the first term, and thus this term is usually neglected in the standard equation for the fluctuating distribution function (Antonsen and Lane 1980). We retain this higher-order term here, since, as is shown in the forthcoming calculation, the Ware pinch and the correction to the Ohmic current due to the fluctuations are derived from it.

Similarly, writing the fluctuating part of the ion distribution function as

$$\delta f_i = -\frac{e_i \delta\phi}{T_i} f_{i0} + \delta h_i \exp\left(\frac{i\mathbf{k}_\perp \cdot \mathbf{v} \times \mathbf{b}}{\boldsymbol{\Omega}_i}\right), \tag{2.12}$$

we can obtain the equation for the Fourier component $\hat{h}_i \equiv \delta h_{imn}$ as (Antonsen and Lane 1980)

$$-i(\omega - \omega_E - k_\parallel v_\parallel)\hat{h}_i - C_i(\hat{h}_i) = -i(\omega - \omega_E - \omega_{*i}^T)\frac{e_i\hat{\phi}}{T_i} J_0(k_\perp \rho_i) f_{i0}, \tag{2.13}$$

where $\rho_i = v_\perp/\Omega_i$, Ω_i is the ion Larmor frequency and J_0 is the Bessel function of order 0.

3. Anomalous fluxes

It is well known that the radial component of the anomalous particle flux $\boldsymbol{\Gamma}_\alpha^{an}$ of species α can be expressed from the momentum balance equation as

$$\Gamma_{\alpha r} \equiv \boldsymbol{\Gamma}_\alpha^{an} \cdot \mathbf{e}_r = -\tfrac{1}{2}\mathrm{Re}\left(\frac{c}{B}\delta n_\alpha \mathbf{e}_\theta \cdot \boldsymbol{\nabla}\delta\phi^*\right)_{\mathrm{ens}}, \tag{3.1}$$

where $\mathbf{e}_r$ is the unit vector in the radial direction.

The anomalous heat flux is defined in several ways by different authors. This situation, confused by the use of various definitions, is discussed in Balescu (1990) and Ross (1989). Using the energy balance equation, we can write the radial component of the anomalous energy flux $\mathbf{Q}_\alpha^{an}$ as

$$Q_{\alpha r} \equiv \mathbf{Q}_\alpha^{an} \cdot \mathbf{e}_r = \tfrac{1}{2}\mathrm{Re}\left[\frac{c}{B}\mathbf{b} \times \boldsymbol{\nabla}\delta\phi^* \cdot (\tfrac{5}{2}\delta p_\alpha \mathbf{I} + \delta\boldsymbol{\Pi}_\alpha) \cdot \mathbf{e}_r\right]_{\mathrm{ens}}, \tag{3.2}$$

where

$$\delta p_\alpha = \tfrac{1}{3}\int m_\alpha v^2 \delta f_\alpha \, d\mathbf{v} \tag{3.3}$$

and

$$\delta\boldsymbol{\Pi}_\alpha = \int m_\alpha(\mathbf{vv} - \tfrac{1}{3}v^2\mathbf{I})\delta f_\alpha \, d\mathbf{v}. \tag{3.4}$$

The total heat flux $\mathbf{q}_\alpha^t$ is related to the total energy flux $\mathbf{Q}_\alpha^t$ as follows:

$$\mathbf{Q}_\alpha^t = \mathbf{q}_\alpha^t + \tfrac{5}{2}T_\alpha^t \boldsymbol{\Gamma}_\alpha^t + \frac{1}{n_\alpha^t}\boldsymbol{\Pi}_\alpha^t \cdot \boldsymbol{\Gamma}_\alpha^t + \tfrac{1}{2}m_\alpha u_\alpha^{t2}\boldsymbol{\Gamma}_\alpha^t, \tag{3.5}$$

where $\boldsymbol{\Gamma}_\alpha^t = n_\alpha^t \mathbf{u}_\alpha^t$ and $\boldsymbol{\Pi}_\alpha^t$ are the total particle flux and viscous tensor. From this exact relation, the radial anomalous heat flux $q_{\alpha r}$ can be expressed as

$$q_{\alpha r} = Q_{\alpha r} - \tfrac{5}{2}T_\alpha \Gamma_{\alpha r} - \tfrac{1}{2}\mathrm{Re}(\tfrac{5}{2}\delta T_\alpha^* \mathbf{e}_r \cdot \delta\boldsymbol{\Gamma}_\alpha)_{\mathrm{ens}} - \tfrac{1}{2}\mathrm{Re}\left(\frac{1}{n_\alpha}\mathbf{e}_r \cdot \delta\boldsymbol{\Pi}_\alpha^* \cdot \delta\boldsymbol{\Gamma}_\alpha\right)_{\mathrm{ens}}, \tag{3.6}$$

where the last term $\frac{1}{2}m_\alpha u_\alpha^{t2}\boldsymbol{\Gamma}_\alpha^t$ in (3.5) is neglected. The fluctuating particle flux

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$\delta\boldsymbol{\Gamma}_\alpha$ is given to the leading order by

$$\delta\boldsymbol{\Gamma}_\alpha = \frac{c}{e_\alpha B}\mathbf{b}\times(\boldsymbol{\nabla}\delta p_\alpha + \boldsymbol{\nabla}\cdot\delta\boldsymbol{\Pi}_\alpha) + \frac{cn_\alpha}{B}\mathbf{b}\times\boldsymbol{\nabla}\delta\phi. \tag{3.7}$$

Substituting (3.7) into (3.6), we obtain the radial anomalous heat flux in the form

$$q_{\alpha r} = -\tfrac{1}{2}\mathrm{Re}\left[\frac{c}{e_\alpha n_\alpha B}\mathbf{e}_r\cdot(\tfrac{5}{2}n_\alpha\delta T_\alpha^*\mathbf{I} + \delta\boldsymbol{\Pi}_\alpha^*)\cdot\mathbf{b}\times\boldsymbol{\nabla}\cdot(\delta p_\alpha\mathbf{I} + \delta\boldsymbol{\Pi}_\alpha)\right]_{\mathrm{ens}}, \tag{3.8}$$

where

$$\delta T_\alpha = \frac{\delta p_\alpha - T_\alpha\delta n_\alpha}{n_\alpha}.$$

In the drift-kinetic ordering ($\delta_e \ll \Delta$) the viscosity tensor for the electrons is expressed in the Chew–Goldberger–Low form:

$$\delta\boldsymbol{\Pi}_e = (\delta p_{\parallel e} - \delta p_{\perp e})(\mathbf{bb} - \tfrac{1}{3}\mathbf{I}),$$

where

$$\delta p_{\parallel e} = \int m_e v_\parallel^2\delta f_e\, d\mathbf{v}, \qquad \delta p_{\perp e} = \tfrac{1}{2}\int m_e v_\perp^2\,\delta f_e\, d\mathbf{v}.$$

Thus the anomalous electron heat flux is simplified as

$$q_{er} = -\tfrac{1}{2}\mathrm{Re}\left[\frac{c}{eB}(\tfrac{3}{2}\delta T_e^* + \delta T_{\perp e}^*)\mathbf{e}_\theta\cdot\boldsymbol{\nabla}\delta p_{\perp e}\right]_{\mathrm{ens}}, \tag{3.9}$$

where

$$\delta T_{\perp e} = \frac{\delta p_{\perp e} - T_e\delta n_e}{n_e}.$$

The expression (3.9) for the anomalous heat flux is presented by Balescu (1990) (in his equation (A19), δp_α should be replaced by $\delta p_{\alpha\perp}$). But, in the main part of his paper, he used a simplified anomalous heat flux defined by neglecting the viscous tensor $\delta\boldsymbol{\Pi}_\alpha$ in the expression (3.8) under the assumption of small pressure anisotropy. This definition is not acceptable in our calculation. For example, as is shown in the next section, the electron heat flux comes from the imaginary part of $\frac{5}{2}n_e\delta T_e\mathbf{I} + \delta\boldsymbol{\Pi}_e$ in (3.8). The contributions to the heat flux from $\frac{5}{2}n_e\delta T_e\mathbf{I}$ and $\delta\boldsymbol{\Pi}_e$ are, in general, of the same order, since $\delta p_{\parallel e} - \delta p_{\perp e} \sim n_e\delta T_e$, and thus we cannot neglect the viscous term $\delta\boldsymbol{\Pi}_e$.

In Shaing (1988b), the following definition (his equation (11)) of the radial anomalous heat flux is used:

$$\begin{aligned} q_{\alpha r}^{(S)} &= -\tfrac{1}{2}\mathrm{Re}\left[\frac{c}{B}\mathbf{e}_\theta\cdot\boldsymbol{\nabla}\delta\phi^*(\tfrac{3}{2}n_\alpha\delta T_\alpha - \delta n_\alpha T_\alpha)\right]_{\mathrm{ens}} \\ &= -\tfrac{1}{2}\mathrm{Re}\left[\frac{c}{B}\mathbf{e}_\theta\cdot\boldsymbol{\nabla}\delta\phi^*\int(\tfrac{1}{2}m_\alpha v^2 - \tfrac{5}{2}T_\alpha)\delta f_\alpha\, d\mathbf{v}\right]_{\mathrm{ens}}. \end{aligned} \tag{3.10}$$

An interesting discussion on this definition is presented in Balescu (1990). Sugama and Horton (1995) chose another definition for the heat flux from their discussion of the entropy production rate due to anomalous transport. They defined the heat flux by choosing a coefficient $\frac{3}{2}$ instead of $\frac{5}{2}$ in (3.10). This '$\frac{3}{2}$–$\frac{5}{2}$' paradox has also been discussed in several papers (Ross 1989; Balescu 1990, 1991).

The precise physical heat flux theoretically derived from the energy balance equation is given by (3.8), and by (3.9) in the drift-kinetic ordering. In this sense the definitions by Shaing and by Sugama and Horton are incorrect. In the present work, we use the anomalous heat fluxes (3.9) for the electrons and (3.8) for the ions.

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4. Anomalous transport coefficients for electrons

Let us introduce a Spitzer function f_s satisfying the equation

$$C_e(f_s) = -\frac{e}{T_e} v_\parallel f_{e0}. \tag{4.1}$$

The pitch-angle variable $v_\parallel/v$ is an eigenfunction of the linearized Fokker–Planck collision operator:

$$C_e\left(\frac{v_\parallel}{v}\varphi(v)\right) = \frac{v_\parallel}{v} C_e^1(\varphi(v)). \tag{4.2}$$

From this property, the Spitzer function is expressed as

$$f_s = \frac{e}{T_e}\frac{v_e}{\nu_{ee}}\frac{v_\parallel}{v} D_e(x) f_{e0}, \tag{4.3}$$

where $x = v/v_e$, with the thermal velocity $v_\alpha = (2T_\alpha/m_\alpha)^{1/2}$,

$$\nu_{ee} = \frac{4\pi n_e e^4 \ln\Lambda}{m_e^2 v_e^3}$$

is the electron–electron collision frequency, and the function D_e is determined from the equation

$$C_e^1(D_e e^{-x^2}) = -\nu_{ee} x e^{-x^2}. \tag{4.4}$$

Multiplying (4.1) by $\bar{f}_{e1}/f_{e0}$ and integrating over velocity space with the use of (2.6) and the self-adjoint property of the collision operator, we obtain the following expression for the parallel current:

$$J_\parallel = -e\int v_\parallel \bar{f}_{e1}\, d\mathbf{v} = -T_e \int \frac{f_s}{f_{e0}} S_e\, d\mathbf{v} + eE_\parallel^{(A)} \int v_\parallel f_s\, d\mathbf{v}. \tag{4.5}$$

In this derivation, we have approximated the electron–ion collision operator by the pitch-angle-scattering term.

We consider the anomalous effect up to order $|\, e\delta\phi/T_e\,|^2$. Then, in the calculation of $-iek_\parallel v_\parallel \hat{\phi}\, \partial \bar{f}_{e1}/\partial E_e$ in (2.10), the gyrophase-averaged electron distribution function can be approximated as

$$\bar{f}_{e1} = -f_s E_\parallel^{(A)}. \tag{4.6}$$

With the use of this expression, the imaginary part of the non-adiabatic distribution function $\hat{h}_e$ is obtained in the limit of infinitesimally small collision frequency as

$$\mathrm{Im}\,\hat{h}_e = \pi\delta(\omega_{\mathbf{k}} - \omega_E - k_\parallel v_\parallel)\left(\omega_{\mathbf{k}} - \omega_E - \omega_{*e}^T + k_\parallel v_\parallel \frac{T_e}{f_{e0}}\frac{\partial f_s}{\partial E_e} E_\parallel^{(A)}\right)\frac{e\hat{\phi}}{T_e} f_{e0}, \tag{4.7}$$

where $\gamma_{\mathbf{k}}$ is neglected.

Substituting (4.7) into (3.1), (3.9) and (4.5), we write the transport equation in the form

$$\begin{bmatrix} \Gamma_{er} \\ q_{er}/T_e \\ J_\parallel/T_e \end{bmatrix} = \begin{bmatrix} L_{11}^e & L_{12}^e & L_{13}^e \\ L_{21}^e & L_{22}^e & L_{23}^e \\ L_{31}^e & L_{32}^e & L_{33}^e \end{bmatrix} \begin{bmatrix} X_1^e \\ X_2^e \\ X_3^e \end{bmatrix}, \tag{4.8}$$

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where the thermodynamical forces X_k^e are defined as

$$X_1^e = -\frac{\omega_{\mathbf{k}} - \omega_E}{m}\frac{eBr}{cT_e} - \frac{d\ln p_e}{dr}, \tag{4.9a}$$

$$X_2^e = -\frac{d\ln T_e}{dr}, \tag{4.9b}$$

$$X_3^e = E_{\parallel}^{(A)}. \tag{4.9c}$$

The heat flux q_{er} has quadratic terms with respect to the thermodynamical forces X_k^e. In the expression (4.8), we have adopted the linear approximation for these forces, in which the fluctuating perpendicular pressure $\delta p_{\perp e}$ in (3.9) is replaced by the fluctuating potential term $en_e\delta\phi$. Therefore the electron anomalous heat flux written as

$$\begin{aligned} q_{er} &= -\tfrac{1}{2}\mathrm{Re}\left[\frac{cn_e}{B}\left(\tfrac{3}{2}\delta T_e + \delta T_{\perp e}\right)\mathbf{e}_\theta\cdot\boldsymbol{\nabla}\delta\phi^*\right]_{\mathrm{ens}} \\ &= -\tfrac{1}{2}\mathrm{Re}\left[\frac{c}{B}\mathbf{e}_\theta\cdot\boldsymbol{\nabla}\delta\phi^*\int\left[\tfrac{1}{2}m_e(v^2+v_\perp^2) - \tfrac{5}{2}T_e\right]\delta h_e\, d\mathbf{v}\right]_{\mathrm{ens}} \end{aligned} \tag{4.10}$$

is used in (4.8).

The transport coefficients in (4.8) are given by

$$\begin{bmatrix} L_{11}^e \\ L_{12}^e \\ L_{21}^e \\ L_{22}^e \end{bmatrix} = \frac{n_e\rho_{pe}^2 v_e}{qR}\sum_{m,n}\frac{m^2}{|\,m-nq\,|}\left|\frac{e\delta\phi_{mn}}{T_e}\right|^2\frac{\pi^{1/2}}{8}e^{-\xi_e^2}\begin{bmatrix} 1 \\ \xi_e^2 - \frac{3}{2} \\ \xi_e^2 - \frac{1}{2} \\ \xi_e^4 - 2\xi_e^2 + \frac{11}{4} \end{bmatrix}, \tag{4.11a}$$

$$\begin{aligned} \begin{bmatrix} L_{13}^e \\ L_{23}^e \end{bmatrix} &= -\frac{cT_e}{eBr}\frac{\pi}{2}\sum_{m,n} m\left|\frac{e\delta\phi_{mn}}{T_e}\right|^2 \\ &\quad\times\int\delta(\omega_{\mathbf{k}} - \omega_E - k_\parallel v_\parallel)k_\parallel v_\parallel T_e\frac{\partial f_s}{\partial E_\alpha}\begin{bmatrix} 1 \\ 2x^2 - \frac{5}{2} - \xi_e^2 \end{bmatrix} d\mathbf{v} \\ &= -\frac{n_e c v_e}{B\nu_{ee}r}\sum_{m,n}\frac{m(m-nq)}{|\,m-nq\,|}\left|\frac{e\delta\phi_{mn}}{T_e}\right|^2\frac{\pi^{1/2}}{2} \\ &\quad\times\int_{|\xi_e|}^{\infty} e^{-x^2}\left(\frac{\xi_e^2}{x}D_e' + D_e - \frac{\xi_e^2}{x^2}D_e - 2\xi_e^2 D_e\right)\begin{bmatrix} 1 \\ 2x^2 - \frac{5}{2} - \xi_e^2 \end{bmatrix} dx, \end{aligned} \tag{4.11b}$$

$$\begin{aligned} \begin{bmatrix} L_{31}^e \\ L_{32}^e \end{bmatrix} &= \frac{cT_e}{eBr}\frac{\pi}{2}\sum_{m,n} m\left|\frac{e\delta\phi_{mn}}{T_e}\right|^2 \\ &\quad\times\int\delta(\omega_{\mathbf{k}} - \omega_E - k_\parallel v_\parallel)k_\parallel v_\parallel T_e f_{e0}\frac{\partial}{\partial E_\alpha}\left(\frac{f_s}{f_{e0}}\right)\begin{bmatrix} 1 \\ x^2 - \frac{5}{2} \end{bmatrix} d\mathbf{v} \\ &= \frac{n_e c v_e}{B\nu_{ee}r}\sum_{m,n}\frac{m(m-nq)}{|\,m-nq\,|}\left|\frac{e\delta\phi_{mn}}{T_e}\right|^2\frac{\pi^{1/2}}{2} \\ &\quad\times\int_{|\xi_e|}^{\infty} e^{-x^2}\left(\frac{\xi_e^2}{x}D_e' + D_e - \frac{\xi_e^2}{x^2}D_e\right)\begin{bmatrix} 1 \\ x^2 - \frac{5}{2} \end{bmatrix} dx, \end{aligned} \tag{4.11c}$$

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$$L^e_{33} = e\int v_\parallel f_s d\mathbf{v} - \frac{\pi}{2}\sum_{m,n}\left|\frac{e\delta\phi_{mn}}{T_e}\right|^2$$
$$\times \int \delta(\omega_\mathbf{k} - \omega_E - k_\parallel v_\parallel) k_\parallel^2 v_\parallel^2 T_e^2 \frac{\partial f_s}{\partial E_\alpha}\frac{\partial}{\partial E_\alpha}\left(\frac{f_s}{f_{e0}}\right) d\mathbf{v}$$
$$= \frac{n_e e^2}{m_e \nu_{ee} T_e}\left[\frac{8}{3}\frac{1}{\pi^{1/2}}\int_0^\infty e^{-x^2} x^3 D_e dx - \frac{v_e}{\nu_{ee}}\sum_{m,n}\frac{|\, m - nq\,|}{qR}\right.$$
$$\times\left|\frac{e\delta\phi_{mn}}{T_e}\right|^2 \frac{\pi^{1/2}}{2}\int_{|\xi_e|}^\infty e^{-x^2}\frac{1}{x}\left(\frac{\xi_e^2}{x}D_e' + D_e - \frac{\xi_e^2}{x^2}D_e - 2\xi_e^2 D_e\right)$$
$$\left.\times\left(\frac{\xi_e^2}{x}D_e' + D_e - \frac{\xi_e^2}{x^2}D_e\right) dx\right], \tag{4.11d}$$

where $D_e' = dD_e/dx$, $\rho_{pe} = v_\alpha/\Omega_{pa}$, with $\Omega_{pa} = e_\alpha B_\theta/m_\alpha c$, and

$$\xi_a = \frac{\omega_\mathbf{k} - \omega_E}{|\, k_\parallel\,|\, v_\alpha}. \tag{4.12}$$

In the small-$|\,\xi_e\,|$ limit, the transport coefficients L^e_{jk} reduce to the following simple forms:

$$L^e_{11} = \frac{n_e \rho_{pe}^2 v_e}{qR}\sum_{m,n}\frac{m^2}{|\, m - nq\,|}\left|\frac{e\delta\phi_{mn}}{T_e}\right|^2\frac{\pi^{1/2}}{8}, \tag{4.13a}$$

$$L^e_{12} = -\frac{3}{2}L^e_{11}, \qquad L^e_{21} = -\frac{1}{2}L^e_{11}, \qquad L^e_{22} = \frac{11}{4}L^e_{11}, \tag{4.13b,c,d}$$

$$L^e_{13} = -L^e_{31} = -\frac{n_e c v_e \tau_{ei}}{Br}\sum_{m,n}\frac{m(m-nq)}{|\, m - nq\,|}\left|\frac{e\delta\phi_{mn}}{T_e}\right|^2\frac{\pi^{1/2}}{4}, \tag{4.13e}$$

$$L^e_{32} = \frac{3}{\sqrt{2}(\sqrt{2}+Z)}L^e_{13}, \qquad L^e_{23} = \frac{-2+5\sqrt{2}Z}{2\sqrt{2}(\sqrt{2}+Z)}L^e_{13}, \tag{4.13f,g}$$

$$L^e_{33} = \frac{n_e e^2 \tau_{ei}}{m_e T_e}\frac{\sqrt{2}+\frac{13}{4}Z}{\sqrt{2}+Z}$$
$$\times\left[1 - v_e\tau_{ei}\sum_{m,n}\frac{|\, m - nq\,|}{qR}\left|\frac{e\delta\phi_{mn}}{T_e}\right|^2\frac{\pi^{1/2}}{4}\frac{\frac{13}{4}Z^2 + 2\sqrt{2}Z + 2}{(\sqrt{2}+Z)(\sqrt{2}+\frac{13}{4}Z)}\right], \tag{4.13h}$$

where $\tau_{ei} = \frac{3}{4}\pi^{1/2}/Z\nu_{ee}$, with the ion charge number Z. Here the function D_e is approximately solved by the moment method: we approximate the function D_e by the two lower moments as $D_e = x[a_1 + a_2(x^2 - \frac{5}{2})]e^{-x^2}$ and calculate the coefficients a_1 and a_2 from the two lower moment equations of (4.4).

5. Anomalous transport coefficients for ions

The transport equation for the particle and heat fluxes of the ions can be calculated in a similar way to the previous section by considering the finite-Larmor-radius effect. We do not repeat this calculation here, and show only the results in Appendix A.

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In this section, we consider the fluctuations satisfying the condition $|\xi_i| \gg 1$, which is applied to drift-wave turbulence. Up to now we have neglected the imaginary frequency $\gamma_{\mathbf{k}}$ for the fluctuations. In the large-$|\xi_i|$ regime, this $\gamma_{\mathbf{k}}$ cannot be neglected, since the radial fluxes due to the $\gamma_{\mathbf{k}}$ dominate over those due to the ion Landau damping shown in Appendix A.

Taking account of the $\gamma_{\mathbf{k}}$ and neglecting the ion Landau damping, we can obtain from (2.13) the non-adiabatic fluctuating distribution function for the ions as

$$\hat{h}_i = \left(1 - \frac{\omega_{*i}^T}{\omega - \omega_E}\right) \frac{e_i \hat{\phi}}{T_i} J_0(k_\perp \rho_i) f_{i0}. \tag{5.1}$$

Substitution of this expression into (3.1) and (3.8) yields the transport equation for the particle flux Γ_{ir} and the heat flux q_{ir} (Appendix B) as

$$\begin{bmatrix} \Gamma_{ir} \\ q_{ir}/T_i \end{bmatrix} = \begin{bmatrix} L_{11}^i & L_{12}^i \\ L_{21}^i & L_{22}^i \end{bmatrix} \begin{bmatrix} -d\ln p_i/dr \\ -d\ln T_i/dr \end{bmatrix}. \tag{5.2}$$

The transport coefficients L_{jk}^i in this equation are obtained as

$$\begin{aligned} \begin{bmatrix} L_{11}^i \\ L_{12}^i \\ L_{21}^i \\ L_{22}^i \end{bmatrix} = {} & \frac{n_i \rho_{pi}^2 v_i^2}{q^2 R^2} \sum_{m,n} m^2 \left| \frac{e_i \delta\phi_{mn}}{T_i} \right|^2 \frac{\gamma_{\mathbf{k}}}{(\omega_{\mathbf{k}} - \omega_E)^2} \frac{\Gamma_0}{8} \\ & \times \left\{ \begin{bmatrix} 1 \\ -1 - b_i M_i \\ 1 - (1+3b_i) M_i \\ \frac{3}{2} - 3b_i + (6b_i^2 + b_i + 1) M_i \end{bmatrix} \right. \\ & \left. + \Gamma_0 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 4b_i - 4 + (\frac{3}{2} + 3b_i - 8b_i^2) M_i + 4b_i^2 M_i^2 \end{bmatrix} \right\}, \end{aligned} \tag{5.3}$$

where

$$\Gamma_0 = e^{-b_i} I_0(b_i),$$

$$b_i = \tfrac{1}{2} k_\perp^2 \bar{\rho}_i{}^2 \qquad \text{with} \quad \bar{\rho}_i = \frac{v_i}{\Omega_i},$$

$$M_i = 1 - \frac{I_1(b_i)}{I_0(b_i)};$$

I_0 and I_1 are the modified Bessel functions of order 0 and 1; and $|\gamma_{\mathbf{k}}| \ll |\omega_{\mathbf{k}}|$ is assumed.

This heat flux q_{ir} is a pure finite-Larmor-radius (FLR) effect, since it vanishes as the ion Larmor radius goes to zero (Appendix B). The heat flux defined by Balescu (1990) is also a pure FLR effect, although the heat flux $q_{ir}^{(S)}$ remains finite in this limit.

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6. Transport matrix in drift-wave turbulence

The transport equations derived in Secs 4 and 5, and those obtained by Shaing (1988b) and Sugama and Horton (1995), include the frequency $\omega_{\mathbf{k}} - \omega_E$ as a parameter. Balescu (1990) claimed that these transport equations are presented in an implicit form since that frequency must be determined self-consistently through the dispersion relation. Following this philosophy, he derived the explicit form of the transport equation for the drift-wave turbulent plasma. In this section, we also derive the explicit form of the transport equation from (4.8) and (5.3) with the use of the dispersion relation of drift waves.

In drift-wave turbulence, the fluctuations satisfy the conditions $\mid \xi_e \mid \ll 1$ and $\mid \xi_i \mid \gg 1$. Thus $\omega_{\mathbf{k}}$ and $\gamma_{\mathbf{k}}$ for the drift wave are calculated from the quasineutrality condition $\hat{n}_e = Z\hat{n}_i$ with (2.10) and (5.1) as

$$\omega_{\mathbf{k}} - \omega_E = \omega_{*e}(1 - \eta_i b_i M_i)\frac{\Gamma_0}{1 + \Theta(1 - \Gamma_0)}, \tag{6.1}$$

$$\gamma_{\mathbf{k}} = \frac{\Gamma_0(1 - \eta_i b_i M_i)[(1 - \Gamma_0)(1 + \Theta) + \eta_i b_i M_i \Gamma_0 - \frac{1}{2}(1 + \Theta - \Gamma_0\Theta)\eta_e]}{[1 + \Theta(1 - \Gamma_0)]^3}\frac{\pi^{1/2}\omega_{*e}^2}{\mid k_\parallel \mid v_e}, \tag{6.2}$$

where $\Theta = ZT_e/T_i$. Using (6.1) and (6.2), we find that $\gamma_{\mathbf{k}}/(\omega_{\mathbf{k}} - \omega_E)^2$ appearing in the ion transport coefficients contains the thermodynamical forces nonlinearly:

$$\frac{\gamma_{\mathbf{k}}}{(\omega_{\mathbf{k}} - \omega_E)^2} = \frac{(1 - \Gamma_0)(1 + \Theta) + \eta_i b_i M_i \Gamma_0 - \frac{1}{2}(1 + \Theta - \Gamma_0\Theta)\eta_e}{\Gamma_0(1 - \eta_i b_i M_i)[1 + \Theta(1 - \Gamma_0)]}\frac{\pi^{1/2}}{\mid k_\parallel \mid v_e}. \tag{6.3}$$

The transport coefficient for the electron heat flux (3.9) also contains the electron thermodynamical forces X_k^e nonlinearly. In (4.10), we have neglected these quadratic nonlinear term for the electron heat flux q_{er}. But we cannot do the same linearization for the ion transport coefficients, since these coefficients are homogeneous functions in the thermodynamical forces.

Substituting (6.2) into the thermodynamical force X_1^e, we can rewrite the transport equations in the form

$$\begin{bmatrix} \Gamma_{er} \\ q_{er}/T_e \\ q_{ir}/T_i \\ J_\parallel/T_e \end{bmatrix} = \begin{bmatrix} l_{11} & l_{12} & l_{13} & l_{14} \\ l_{21} & l_{22} & l_{23} & l_{24} \\ l_{31} & l_{32} & l_{33} & l_{34} \\ l_{41} & l_{42} & l_{43} & l_{44} \end{bmatrix} \begin{bmatrix} \bar{X}_1 \\ \bar{X}_2 \\ \bar{X}_3 \\ \bar{X}_4 \end{bmatrix}, \tag{6.4}$$

where we newly define the following thermodynamical forces:

$$\bar{X}_1 = -\frac{d\ln p_e}{dr}, \qquad \bar{X}_2 = -\frac{d\ln T_e}{dr} = X_2^e, \tag{6.5a,b}$$

$$\bar{X}_3 = -\frac{d\ln T_i}{dr}, \qquad \bar{X}_4 = E_\parallel^{(A)} = X_3^e. \tag{6.5c,d}$$

As is well known, the radial particle fluxes satisfy the ambipolar condition $\Gamma_{er} = Z\Gamma_{ir}$ if we permit the quasineutrality condition $\hat{n}_e = Z\hat{n}_i$.

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According to (4.13), the transport coefficients for this transport equation are obtained with the use of (6.3) as

$$\begin{bmatrix} l_{11} \\ l_{12} \\ l_{13} \\ l_{21} \\ l_{22} \\ l_{23} \end{bmatrix} = \frac{n_e \rho_{pe}^2 v_e}{qR} \sum_{m,n} \frac{m^2}{|\, m - nq \,|} \left| \frac{e\delta\phi_{mn}}{T_e} \right|^2 \frac{\pi^{1/2}}{8} \begin{bmatrix} 1 - c_i \\ c_i - \frac{3}{2} \\ c_i b_i M_i \\ -\frac{1}{2}(1 - c_i) \\ \frac{11}{4} - \frac{1}{2} c_i \\ -\frac{1}{2} c_i b_i M_i \end{bmatrix}, \tag{6.6a}$$

$$\begin{bmatrix} l_{14} \\ l_{24} \\ l_{41} \\ l_{42} \\ l_{43} \end{bmatrix} = -\frac{n_e c v_e \tau_{ei}}{Br} \sum_{m,n} \frac{m(m - nq)}{|\, m - nq \,|} \left| \frac{e\delta\phi_{mn}}{T_e} \right|^2 \frac{\pi^{1/2}}{4} \times \begin{bmatrix} 1 \\ (5Z - \sqrt{2})/[2(\sqrt{2} + Z)] \\ -c_i - 1 \\ 3/(2 + \sqrt{2}Z) - c_i \\ -c_i b_i M_i \end{bmatrix}, \tag{6.6b}$$

$$l_{44} = L_{33}^e, \qquad l_{31} = -l_{32} = L_{21}^i, \qquad l_{33} = L_{21}^i + L_{22}^i, \qquad l_{34} = 0,$$

where

$$c_i = \frac{\Gamma_0}{1 + \Theta(1 - \Gamma_0)}.$$

The fluctuations in drift-wave turbulence are considered in the region of small $|\, \xi_e \,|$. As is shown in the next section, the transport equation obtained using Shaing's ansatz should coincide with that by our formulation in this small $|\, \xi_e \,|$ limit. However, the transport equation obtained by using Shaing's ansatz in Balescu (1990) do not reduce to our expression (6.4) in the uniform-magnetic-field limit. In Balescu (1990), there is a flaw in his drift-kinetic equation (14). The right-hand side of this equation includes the unphysical term $i\omega_{mn}^E$, which should be omitted. All the subsequent calculations affected by this term must be modified in Balescu (1990).

Finally in this section, we consider the Onsager symmetry of the anomalous transport matrix. This topic has been studied in many papers. Shaing (1988a,b) and Sugama and Horton (1995) discussed the Onsager symmetry for the transport matrices that contain the frequency $\omega_{\mathbf{k}}$ as a parameter. The transport matrix in (4.8) of the present paper corresponds to the matrices that they discuss except for the definition of the heat flux. On the other hand, Balescu (1990) claimed that the Onsager symmetry must be discussed for a transport matrix in which the frequency $\omega_{\mathbf{k}}$ is self-consistently determined by the dispersion relation, since the frequency is still a function of the thermodynamical forces. The structure of the transport matrix and the breakdown of the Onsager symmetry in a magnetically confined plasma are also discussed from a general point of view by Balescu (1991). In the present paper, we take the standpoint of Balescu for the Onsager symmetry. Therefore we conclude that, as is clear from (6.4) with the transport coefficients (6.6), the Onsager symmetry is broken for the anomalous transport matrix $[l_{jk}]$ in a drift-wave turbulent plasma.

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7. Meaning of Shaing's ansatz

Let us define the transport coefficients $\bar{L}^e_{2k}$ for the heat flux $q^{(S)}_{er}$ by

$$\frac{q^{(S)}_{er}}{T_e} \equiv \sum_{k=1}^{3} \bar{L}^e_{2k} X_k.$$

Comparing (3.10) with (4.10), we find that these transport coefficients are related to the transport coefficients L^e_{2k} for the correct heat flux q_{er} as

$$\bar{L}^e_{2k} = \tfrac{1}{2} L^e_{2k} - (\tfrac{5}{2} - \xi_e^2) T_e L^e_{1k}.$$

In the small-$|\xi_e|$ limit, the transport coefficients L^e_{jk} ($j = 1, 3, k = 1, 2, 3$) and $\bar{L}^e_{2k}$ agree with those obtained by Shaing (1988a,b) in the plateau regime. Shaing claimed his theory to be 'neoclassical', but his results are derived in a cylindrical geometry. In Shaing's theory, his ansatz for the distribution function plays an important role. In conventional neoclassical transport theory in the absence of fluctuations, a similar ansatz is used in the plateau regime in order to preserve the momentum-conservation property of the collision operator. However, the meaning of this ansatz for the anomalous transport is different from that for conventional neoclassical transport theory.

In our formulation, Shaing's ansatz is understood to be equivalent to the following two approximations in the calculation of $ek_\parallel v_\parallel \hat{\phi} \partial \bar{f}_{e1}/\partial E_e$ in (2.10):

(i) approximate the distribution function $\bar{f}_{e1}$ by

$$\bar{f}_{e1} = \frac{2 v_\parallel}{v_e^2} \left[u_{\parallel e} - \frac{2}{5} \left(\frac{5}{2} - \frac{v^2}{v_e^2} \right) \frac{q_{\parallel e}}{p_e} + \ldots \right] f_{e0}, \tag{7.1}$$

(ii) and then neglect the term proportional to $v_\parallel$, i.e.

$$ie k_\parallel v_\parallel \hat{\phi} \frac{\partial \bar{f}_{e1}}{\partial E_e} = ie k_\parallel v_\parallel \hat{\phi} \left[\frac{\partial v_\parallel}{\partial E_e} \left(\frac{\partial \bar{f}_{e1}}{\partial v_\parallel} \right)_v + \frac{\partial v}{\partial E_e} \left(\frac{\partial \bar{f}_{e1}}{\partial v} \right)_{v_\parallel} \right]$$
$$\sim i k_\parallel \left[u_{\parallel e} - \frac{2}{5} \left(\frac{5}{2} - \frac{v^2}{v_e^2} \right) \right] \frac{e\hat{\phi}}{T_e} f_{e0}, \tag{7.2}$$

where we note that $\partial/\partial E_e$ is performed holding $v_\perp$ fixed, and we have used $\partial v_\parallel / \partial E_e = 1/m_e v_\parallel$.

In the first approximation, the function $\bar{f}_{e1}$ is set to be proportional to $v_\parallel$. From (4.3) and (4.6), this approximation is valid in the evaluation of transport coefficients at least for a cylindrical plasma if we neglect the fluctuations $O(|\, e\hat{\phi}/T_e \,|^4)$. The validity of the truncation by the two (a few) lower moments depends on the form of the source term S_e. The second approximation is justified for $|\xi_e| \ll 1$ because of the resonance condition $\delta(\omega - \omega_E - k_\parallel v_\parallel) = \delta(\xi_e - v_\parallel/v_e)/|k_\parallel| v_e$ in S_e. The parallel flows $u_{\parallel e}$ and $q_{\parallel e}$ in (7.1) can be calculated from the following parallel momentum and heat flux balance equations:

$$F^1_{\parallel e} - e n_e E^{(A)}_\parallel + \int m_e v_\parallel S_e \, d\mathbf{v} = 0, \tag{7.3}$$

$$F^2_{\parallel e} + \int m_e v_\parallel \left(\frac{v^2}{v_e^2} - \frac{5}{2} \right) S_e \, d\mathbf{v} = 0, \tag{7.4}$$

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where

$$F^1_{\|e} = \int m_e v_\| C_e \, d\mathbf{v}, \qquad F^2_{\|e} = \int m_e v_\| \left(\frac{v^2}{v_e^2} - \frac{5}{2} \right) C_e \, d\mathbf{v}.$$

Using the approximations (7.1) and (7.2) with $u_{\|e}$ and $q_{\|e}$ computed from (7.3) and (7.4), we can derive all the transport coefficients obtained by Shaing in the presence of electrostatic fluctuations. In the lowest order of $|e\hat{\phi}/T_e|^2$, these transport coefficients reduce to our results in the small-$|\xi_e|$ limit for a cylindrical plasma. Our calculation shows that Shaing's ansatz for anomalous transport is valid in the small-$|\xi_e|$ limit and in the lowest order of $|e\hat{\phi}/T_e|^2$ for a *cylindrical plasma*. The validity of this ansatz for a toroidal plasma is discussed in Taguchi (1999).

8. Conclusions

We have investigated the transport equations for a cylindrical plasma in the weakly turbulent regime. The complete transport matrices that relate the anomalous particle and heat fluxes and the parallel current to the thermodynamical forces are obtained by employing the drift-kinetic and gyrokinetic orderings for the electrons and ions.

We have started with the kinetic equations for the ensemble-averaged and fluctuating distribution functions in the standard way. But, contrary to previous works, we have introduced an extra term $-iek_\|\hat{\phi}\partial\bar{f}_{e1}/\partial E_e$ in the drift-kinetic equation (2.10) for the fluctuating electron distribution function. Although this term is one order higher in Δ compared with the conventional term, and thus has been neglected in previous works, we must retain this extra term to obtain the complete transport matrices. The transport coefficients L^e_{13} and L^e_{23} in (4.8), which are referred to as the Ware pinch components of the radial particle and heat fluxes, are derived from the newly introduced term. In addition, the correction to the Ohmic current that corresponds to the second term in L^e_{33} is also evaluated from this term.

We have obtained the transport equations (4.8) with the transport coefficients (4.11) for the electrons and (A 1) with (A 2a) for the ions. In these equations we have neglected the effect of the imaginary frequency $\gamma_\mathbf{k}$ of fluctuations. In the large-$|\xi_i|$ regime, which is important in application to drift-wave turbulence, the imaginary frequency $\gamma_\mathbf{k}$ cannot be neglected, because the radial fluxes due to this $\gamma_\mathbf{k}$ dominate over those due to ion Landau damping. The ion transport equation for the fluctuations satisfying $|\xi_i| \gg 1$ is written as (5.2) with the transport coefficients (5.3). The ion heat flux q_{ir} in the large-$|\xi_i|$ regime is shown to be a pure FLR effect, since it vanishes as the ion Larmor radius goes to zero.

The transport equations (4.8), (5.2) and (A 1) are presented in an implicit form, since the frequencies $\omega_\mathbf{k} - \omega_E$ and $\gamma_\mathbf{k}$, which must be determined self-consistently by the dispersion relation, are included as parameters. In order to eliminate these frequencies by using the dispersion relation, we must specify the microscopic instability that causes the turbulent fluctuations. In performing this elimination for the drift-wave turbulent plasma, the particle and heat fluxes, together with the parallel current, are obtained in explicit form as (6.4). From this explicit form, we conclude that the Onsager symmetry is broken for the transport matrix $[l_{kl}]$ in (6.4) for a drift-wave turbulent plasma.

Our formulation reveals the physical meaning and the validity of Shaing's ansatz in the context of a turbulent plasma. This ansatz is shown to be equivalent to

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the two approximations (7.1) and (7.2) in the calculation of the newly introduced extra term $-\mathrm{i}ek_{\|}\hat{\phi}\partial\bar{f}_{e1}/\partial E_e$ in (2.10). These two approximations and thus Shaing's ansatz are valid in the small-$|\,\xi_e\,|$ limit and in the lowest-order of $|\,e\hat{\phi}/T_e\,|^2$ for a cylindrical plasma. In this regime, the transport coefficients obtained by Shaing in the presence of electrostatic fluctuations can be derived in our calculation for a cylindrical plasma.

In this paper we have neglected toroidal effects and magnetic fluctuations. The generalizations to a toroidal plasma and the inclusion of electromagnetic fluctuation in our formulation are straightforward. These are done in Taguchi (1999) and forthcoming papers.

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Appendix A

The ion particle and heat fluxes due to ion Landau damping are calculated in a similar way to the electron fluxes. By taking account of the finite-Larmor-radius effect, we can obtain the ion transport equation for the fluxes Γ_{ir} and q_{ir} as

$$\begin{bmatrix} \Gamma_{ir} \\ q_{ir}/T_i \end{bmatrix} = \begin{bmatrix} L_{11}^{i(L)} & L_{12}^{i(L)} \\ L_{21}^{i(L)} & L_{22}^{i(L)} \end{bmatrix} \begin{bmatrix} X_1^i \\ X_2^i \end{bmatrix}, \tag{A 1}$$

where the thermodynamical forces are defined by

$$X_1^i = \frac{\omega_{\mathbf{k}} - \omega_E}{m}\frac{e_i B r}{cT_i} - \frac{d\ln p_i}{dr}, \tag{A 2a}$$

$$X_1^i = -\frac{d\ln T_i}{dr}. \tag{A 2b}$$

The transport coefficients $L_{jk}^{i(L)}$ are given by

$$\begin{bmatrix} L_{11}^{i(L)} \\ L_{12}^{i(L)} \\ L_{21}^{i(L)} \\ L_{22}^{i(L)} \end{bmatrix} = \frac{n_i\rho_{pi}^2 v_i}{qR}\sum_{m,n}\frac{m^2}{|\,m-nq\,|}\left|\frac{e_i\delta\phi_{mn}}{T_i}\right|^2\frac{\pi^{1/2}}{8}\Gamma_0 e^{-\xi_i^2} \times \begin{bmatrix} 1 \\ -\frac{3}{2}+\xi_i^2-b_iM_i \\ \xi_i^2+\frac{1}{2}-(3b_i+1)M_i \\ \xi_i^4-\xi_i^2+\frac{5}{4}-3b_i+M_i[-(1+4b_i)\xi_i^2+6b_i^2+3b_i+\frac{3}{2}] \end{bmatrix}, \tag{A 3}$$

where the quadratic nonlinear terms in these thermal forces are neglected. From (A 3), we see that the transport coefficients $L_{jk}^{i(L)}$ are quite small in the large-$|\,\xi_i\,|$ limit.

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Appendix B

Let us rewrite the heat flux (3.8) for the ions in the form

$$q_{ir} = -\tfrac{1}{2}\mathrm{Re}\left[\frac{icm}{e_i n_i r B}\left(\tfrac{3}{2}n_i\delta T_i^* - T_i\delta n_i^*\right)\mathbf{e}_1\cdot(\delta p_i\mathbf{I}+\delta\mathbf{\Pi}_i)\cdot\mathbf{e}_1\right]_{\mathrm{ens}}, \tag{B 1}$$

where $\mathbf{e}_1 = \mathbf{k}_\perp\times\mathbf{b}/k_\perp$ and we have used the relation

$$\mathbf{e}_1\cdot\left(\tfrac{5}{2}n_i\delta T_i\mathbf{I}+\delta\mathbf{\Pi}_i\right)\cdot\mathbf{e}_1 = \mathbf{e}_1\cdot(\delta p_i\mathbf{I}+\delta\mathbf{\Pi}_i)\cdot\mathbf{e}_1 + \tfrac{3}{2}n_i\delta T_i - T_i\delta n_i. \tag{B 2}$$

The fluctuating quantities in (B 1) are written using (5.1) as

$$\mathbf{e}_1\cdot(\hat{p}_i\mathbf{I}+\hat{\mathbf{\Pi}}_i)\cdot\mathbf{e}_1 = -e_i n_i\hat{\phi} - \frac{e_i\hat{\phi}}{T_i}\int\tfrac{1}{2}m_i v_\perp^2[J_2(k_\perp\rho_i)-J_0(k_\perp\rho_i)]J_0(k_\perp\rho_i)\left(1-\frac{\omega_{Ti}^*}{\omega-\omega_E}\right)f_{i0}\,d\mathbf{v}, \tag{B 3}$$

$$\tfrac{3}{2}n_i\hat{T}_i - T_i\hat{n}_i = e_i n_i\hat{\phi} + \frac{e_i\hat{\phi}}{T_i}\int\left(\tfrac{1}{2}m_i v^2-\tfrac{5}{2}T_i\right)J_0(k_\perp\rho_i)^2\left(1-\frac{\omega_{Ti}^*}{\omega-\omega_E}\right)f_{i0}\,d\mathbf{v}, \tag{B 4}$$

where

$$\hat{p}_i = \delta p_{imn}, \qquad \hat{\mathbf{\Pi}}_i = \delta\mathbf{\Pi}_{imn}, \qquad \hat{n}_i = \delta n_{imn}, \qquad \hat{T}_i = \delta T_{imn}.$$

Substituting (B 3) and (B 4) into (B 1), we can obtain the heat flux q_{ir} in (5.2).

When the ion Larmor radius ρ_i goes to zero, the expressions (B 3) and (B 4) reduce to the following forms:

$$\mathbf{e}_1\cdot(\hat{p}_i\mathbf{I}+\hat{\mathbf{\Pi}}_i)\cdot\mathbf{e}_1 = \frac{\text{real quantity}}{\omega-\omega_E}, \tag{B 5}$$

$$\tfrac{3}{2}n_i\hat{T}_i - T_i\hat{n}_i = \frac{\text{real quantity}}{\omega-\omega_E}. \tag{B 6}$$

Therefore we find from (B 1) that the heat flux q_{ir} due to the fluctuations satisfying the condition $|\xi_i|\gg 1$ is a pure finite-Larmor-radius effect, since it vanishes in the limit of $\rho_i = 0$.

References

Antonsen, Jr. T. M. and Lane, B. 1980 Kinetic equations for low frequency instabilities in inhomogeneous plasmas. *Phys. Fluids* **23**, 1205–1214.

Balescu, R. 1988 *Transport Processes in Plasmas*, Vol. 2: *Neoclassical Transport*. North-Holland, Amsterdam.

Balescu, R. 1990 Anomalous fluxes in the plateau regime for a weakly turbulent, magnetically confined plasma. *Phys. Fluids* **B2**, 2100–2112.

Balescu, R. 1991 Is Onsager symmetry relevant in the transport equations for magnetically confined plasmas? *Phys. Fluids* **B3**, 564–579.

Hinton, F. L. and Hazeltine, R. D. 1976 Theory of plasma transport. *Rev. Mod. Phys.* **48**, 239–308.

Hirshman, S. P. and Sigmar, D. J. 1981 Neoclassical transport of impurities in tokamak plasmas. *Nucl. Fusion* **21**, 1079–1201.

Liewer, P. C. 1985 Measurements of microturbulence in tokamaks and comparisons with theories of turbulence and anomalous transport. *Nucl. Fusion* **25**, 543–621.

Ross, D. W. 1989 On standard forms for transport equations and fluxes. *Comments Plasma Phys. Contr. Fusion* **12**, 155–163.

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Shaing, K. C. 1988a The effects of electric field fluctuations on bootstrap current and resistivity in toroidal plasmas. *Phys. Fluids* **31**, 8–11.

Shaing, K. C. 1988b Neoclassical quasilinear transport theory of fluctuations in toroidal plasmas. *Phys. Fluids* **31**, 2249–2265.

Shaing, K. C. 1990 Neoclassical quasilinear transport theory of fluctuations in toroidal plasmas: further considerations. *Phys. Fluids* **B2**, 764–772.

Sugama, H. and Horton, W. 1995 Neoclassical and anomalous transport in axisymmetric toroidal plasmas with electrostatic turbulence. *Phys. Plasmas* **2**, 2989–3006.

Sugama, H. and Horton, W. 1997a Transport processes and entropy production in toroidally rotating plasmas with electrostatic turbulence. *Phys. Plasmas* **4**, 405–418.

Sugama, H. and Horton, W. 1997b Nonlinear electromagnetic gyrokinetic equation for plasmas with large mean flows. *Phys. Plasmas* **5**, 2560–2573.

Sugama, H. and Horton, W. 1998 Symmetries of and entropy production by transport in toroidal confinement systems. *J. Plasma Phys.* **59**, 695–706.

Sugama, H., Okamoto, M., Horton, W. and Wakatani, M. 1996 Transport processes and entropy production in toroidal plasmas with gyrokinetic electromagnetic turbulence. *Phys. Plasmas* **3**, 2379–2394.

Taguchi, M. 1999 Neoclassical quasilinear transport theory in tokamak plasmas. *J. Plasma Phys.* **62**, 287–303.

https://doi.org/10.1017/S0022377899007916 Published online by Cambridge University Press