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On the stable Hopf invariant

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We present a simplified approach to the stable Hopf invariant. We provide short elementary proofs of the Cartan formula, the composition formula, and the transfer formula. In addition, when π is a discrete group, we show how to extend these results to the stable category of π-spaces. We also consider the extent to which the stable Hopf invariant is unique.

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1. Introduction

Heinz Hopf introduced what is now called the Hopf invariant in his 1931 work on maps between spheres as a way to distinguish homotopy classes of maps that are invisible to homology [11]. In the 1950s, this notion was generalized by James [12, pp. 192–193], who defined invariants associated to desuspending maps of based spaces $\Sigma A \to \Sigma B$. For each positive integer n, the nth James–Hopf invariant is an operation on homotopy classes

$$\gamma_n \colon [\Sigma A, \Sigma B] \longrightarrow [\Sigma A, \Sigma B^{[n]}]$$

which is natural in both A and B, where $B^{[n]}$ denotes the n-fold smash product of B with itself.

Boardman and Steer [2] instead consider the suspended invariants

$$\lambda_n = \Sigma^{n-1}\gamma_n \colon [\Sigma A, \Sigma B] \longrightarrow [\Sigma^n A, \Sigma^n B^{[n]}]$$

and show that they are axiomatically characterized by certain properties encoded in what they call a *Hopf ladder*. A Hopf ladder is a collection of such natural

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transformations satisfying $\lambda_1 = \mathrm{id}$, $\lambda_n(\Sigma f) = 0$ for $n \geq 2$, and the Cartan formula

$$\lambda_n(f+g) = \sum_{i+j=n} \lambda_i(f) \cdot \lambda_j(g)\,.$$

In the 1970s, the *Segal–Snaith Hopf invariants* h_n were introduced using the approximation theorem in combination with the Snaith splitting [23], [17], [6], [7], [21]. Consider the infinite loop space $Q(B) = \Omega^\infty\Sigma^\infty B$ for a based space B. The Snaith splitting provides a decomposition of the suspension spectrum

$$\Sigma^\infty Q(B) \simeq \bigvee_{n\geq 0} \Sigma^\infty D_n(B)\,,$$

where $D_n(B) := B^{[n]} \wedge_{\Sigma_n} (E\Sigma_{n+})$ is the *n-adic construction*, i.e., the reduced homotopy orbits of the symmetric group Σ_n acting on the n-fold smash product $B^{[n]}$. We will use of a version of the Snaith splitting provided by Goodwillie's calculus of functors [9] which has the advantage of being natural.'The argument of [9, ex. 1.20], which is only stated for $\Omega\Sigma X$, also applies to $Q(X)$.'

If we compose the Snaith splitting with projection onto the nth summand and take the adjoint, we obtain a map

$$Q(B) \to Q(D_n(B))\,.$$

The latter induces the operation

$$h_n : \{A, B\} \to \{A, D_n(B)\}$$

by applying homotopy classes $[A, -]$. In the display, $\{A, B\} = [A, Q(B)]$ is the abelian of homotopy classes of stable maps $A \to B$. As far as I am aware, a Hopf-ladder type axiomatic characterization of the invariants h_n is currently unknown.

In this paper, we restrict our attention to the case $n = 2$. In this case, there is a different construction of an operation

$$H : \{A, B\} \to \{A, D_2(B)\}$$

which avoids the Snaith splitting (I first learned of such a construction from Andrew Ranicki). Our main goal is to develop and analyse the fundamental properties of H.

Let

$$E : [A, B] \to \{A, B\}$$

be the *stabilization map*, i.e., the passage from unstable homotopy classes to stable homotopy classes. If $f, g \in \{A, B\}$ then the *cup product* [2]

$$f \cup g \in \{A, B \wedge B\}$$

is the stable composition

$$A \xrightarrow{\Delta_A} \wedge A \xrightarrow{f\wedge g} B \wedge B\,.$$

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The *symmetrized cup product*

$$f \cup_2 g \in \{A, D_2(B)\}$$

is the composition

$$A \xrightarrow{f \cup g} B \wedge B \to D_2(B)$$

where the map $B \wedge B \to D_2(B)$ is the homotopy quotient by the cyclic group $\mathbb{Z}_2 = \Sigma_2$.

We are now in position to state the main result.

THEOREM A. *There is an operation*

$$H : \{A, B\} \to \{A, D_2(B)\}$$

called the stable Hopf invariant *that is natural in both A and B and which satisfies:*

(i) *(Normalization).* $H(E(f)) = 0$*;*
(ii) *(Cartan formula).* $H(f + g) = H(f) + H(g) + f \cup_2 g$*;*
(iii) *(Transfer formula).* $\operatorname{tr} H(f) = f \cup f - \Delta_B \circ f$*;*
(iv) *(Composition formula).* $H(g \circ f) = H(g) \circ f + D_2(g) \circ H(f)$.

REMARK 1.1. The transfer homomorphism $\operatorname{tr} : \{A, D_2(B)\} \to \{A, B \wedge B\}$ arises from the regular $\mathbb{Z}_2$-covering space pair

$$((B \times B) \times E\mathbb{Z}_2, (B \vee B) \times E\mathbb{Z}_2) \to ((B \times B) \times_{\mathbb{Z}_2} E\mathbb{Z}_2, (B \vee B) \times_{\mathbb{Z}_2} E\mathbb{Z}_2)$$

[1, const. 4.1.1]. For the Segal–Snaith Hopf invariant, the Cartan formula appears in [5] (for a geometric description, see [13, thm. 2.2]; see also [14] for additional properties of the Segal–Snaith Hopf invariant). For the *geometric Hopf invariant* of Crabb and Ranicki, a version of the transfer and composition formulas appear in [8, prop. 5.33].

Our approach most closely resembles the one taken by Crabb and Ranicki with some key differences: They fix a $\mathbb{Z}_2$-representation V and define the geometric Hopf invariant as a natural transformation

$$h_V : [S^V \wedge A, S^V \wedge B]_{\mathbb{Z}_2} \to [\Sigma S(\alpha \otimes V)_+ \wedge S^V \wedge A, S^{(\alpha+1)\otimes V} \wedge B]_{\mathbb{Z}_2},$$

where the domain is the set of homotopy classes of $\mathbb{Z}_2$-equivariant maps $S^V \wedge A \to S^V \wedge B$. In the display, S^V denotes the one-point compactification of V, α is the sign representation, and $S(\alpha \otimes V)$ is the unit sphere.

By constrast, the source and target of the operation H do not involve a choice of representation and are expressed in terms of *unequivariant* stable homotopy classes. Even though H is phrased in unequivariant language, its actual construction requires passing through the $\mathbb{Z}_2$-equivariant stable category. The operations H and h_V are related by stabilization by representations and the application of a transfer map.

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1.1. Relation to the Segal–Snaith operation

Let $h := h_2$ be the second Segal–Snaith Hopf invariant.

THEOREM B. *The operations* $h, H : \{A, B\} \to \{A, D_2(B)\}$ *coincide.*

Suppose that A is a CW complex of dimension s and Y is N-connected. Then *stable range* occurs when $s \leq 2r+1$; in this case, $E : [A, B] \to \{A, B\}$ is surjective by the Freudenthal suspension theorem. The *metastable range* occurs when $s \leq 3r+1$. It is well known that the Segal–Snaith Hopf invariant is the complete obstruction to destabilization in the metastable range [18].

COROLLARY C. *Assume* $s \leq 3r + 1$. *Let* $f \in \{A, B\}$ *be a homotopy class. Then* $H(f) = 0$ *if and only if* $f = E(f')$ *for some* $f' \in [X, Y]$.

A natural transformation is said to possess the *EHP property*if it satisfies the conclusion of corollary C.

1.2. Extension to π-spaces

For applications to surgery theory, it is useful to generalize the above to the π-equivariant setting, where π is any discrete group. Let A and B be based π-CW complexes such that the action is free away from the basepoint. In this instance, we have an *equivariant stable Hopf invariant*

$$H^\pi : \{A, B\}_\pi \to \{A, D_2(B)\}_\pi$$

where $\{A, B\}_\pi := [A, Q(B)]_\pi$ denotes the abelian group of homotopy classes of stable equivariant maps $A \to B$.

ADDENDUM D. *The equivariant stable Hopf invariant* H^π *possesses properties (i)–(iv). Furthermore, the* π*-equivariant analogue of corollary C holds: In the metastable range,* $f \in \{A, B\}_\pi$ *destabilizes to* $[A, B]_\pi$ *if and only if* $H^\pi(f)$ *is trivial.*

1.3. Concerning uniqueness

Suppose that $\lambda : \{A, B\} \to \{A, D_2(B)\}$ is a natural transformation possessing the following properties:

(a) $\lambda(f) = 0$ if f is represented by an unstable map,
(b) λ possesses the Cartan formula, and
(c) λ possesses the EHP property.

Let $A(\mathbb{Z}_2)$ denote the Burnside ring of the group $\mathbb{Z}_2$ (cf. [3]). There is an isomorphism

$$A(\mathbb{Z}_2) \cong \mathbb{Z}[\rho]/(\rho^2 - 2\rho)\,,$$

where ρ corresponds to $\mathbb{Z}_2$ considered as a $\mathbb{Z}_2$-set. Let $\hat{A}(\mathbb{Z}_2)$ denote the completion of $A(\mathbb{Z}_2)$ with respect to its augmentation ideal. In §8, we show that $\hat{A}(\mathbb{Z}_2)$ acts on the homotopy classes of natural transformations $D_2(-) \to D_2(-)$.

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Let $\hat{A}(\mathbb{Z}_2)^\times$ denote the group of units of $\hat{A}(\mathbb{Z}_2)$, and let K denote the kernel of the split surjective homomorphism $\hat{A}(\mathbb{Z}_2)^\times \to \hat{A}(e)^\times = \{\pm 1\}$ induced by augmentation. Then

$$K \cong \mathbb{Z}_2 \times \mathbb{Z}_2^\wedge,$$

where $\mathbb{Z}_2^\wedge$ is the group of 2-adic integers. The $\mathbb{Z}_2$-factor is generated by $\rho - 1$ and the factor $\mathbb{Z}_2^\wedge$ is topologically generated by $2\rho - 3$.

THEOREM E. *The set of natural transformations $\lambda : \{A, B\} \to \{A, D_2(B)\}$ which satisfy (a)–(c) above is a free and transitive K-set, i.e., it is a K-torsor.*

Outline. §2 introduces $\mathbb{Z}_2$-equivariant stable homotopy from a low-tech point-of-view. §3 recalls the tom Dieck splitting in the case of the group $\mathbb{Z}_2$. In §4, we provide a construction of the stable Hopf invariant. §5 concerns the proof of theorem A. The proof of theorem B appears in §6. §7 contains the proof of addendum D. In §8, we prove theorem E.

2. $\mathbb{Z}_2$-equivariant stable homotopy

Let T be the category of compactly generated weak Hausdorff spaces and let T_* be the corresponding category of based spaces. We let $T_*(\mathbb{Z}_2)$ be the category of based spaces with (left) $\mathbb{Z}_2$-action. If X is an unbased $\mathbb{Z}_2$-space, we let X_+ denote the corresponding based one obtained from X by taking a disjoint basepoint.

For based $\mathbb{Z}_2$-spaces A, B, the function space of unequivariant based maps

$$\mathrm{map}_*(A, B)$$

is equipped with a preferred $\mathbb{Z}_2$-action, where the action is defined by conjugating functions.

Up to isomorphism, there are exactly two irreducible real orthogonal representations of $\mathbb{Z}_2$ of positive dimension. These are the trivial representation 1 and the sign representation α. We choose a *complete universe* $\mathcal{U}$ for $\mathbb{Z}_2$ given by the direct sum of countably many copies 1 and α. Then every finite dimensional subrepresentation of $\mathcal{U}$ is a direct sum of the form

$$s\alpha + t\,,$$

for non-negative integers s and t.

If $V \subset \mathcal{U}$ is a finite dimensional subrepresentation, then we let S^V denote its one-point compactification. The latter is a sphere of dimension $\dim V$ equipped with a based $\mathbb{Z}_2$-action.

Suppose that Y is a based $\mathbb{Z}_2$-CW complex. We give $S^V \wedge Y$ the diagonal action. The based function space

$$\Omega^V \Sigma^V(Y) := \mathrm{map}_*(S^V, S^V \wedge Y)$$

is equipped with a $\mathbb{Z}_2$-action. Then the colimit

$$Q_{\mathbb{Z}_2}(Y) := \mathrm{colim}_{V \in \mathcal{U}} \Omega^V \Sigma^V(Y)$$

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is equipped with a $\mathbb{Z}_2$-action. The latter is the zeroth space of a $\mathbb{Z}_2$-equivariant spectrum

$$\Sigma^\infty_{\mathbb{Z}_2}(Y)$$

whose V-th space is $Q_{\mathbb{Z}_2}(S^V \wedge Y)$.

Definition 2.1. *Let X be a based $\mathbb{Z}_2$-CW complex. Set*

$$\{X, Y\}_{\underline{\mathbb{Z}_2}} := \operatorname*{colim}_{V \subset \mathcal{U}} \pi_0(\mathrm{map}_*(S^V \wedge X, S^V \wedge Y)^{\mathbb{Z}_2}) .$$

Remark 2.2. Alternatively,

$$\{X, Y\}_{\underline{\mathbb{Z}_2}} = \pi_0(\mathrm{map}_*(X, Q_{\mathbb{Z}_2}(Y))^{\mathbb{Z}_2})$$

where $\mathrm{map}_*(X, Q_{\mathbb{Z}_2}(Y))^{\mathbb{Z}_2}$ is the based mapping space of C-equivariant maps $X \to Q_{\mathbb{Z}_2}(Y)$.

Definition 2.3. ($\mathbb{Z}_2$-equivariant Stable Maps) *Suppose that X and Y are based $\mathbb{Z}_2$-CW complexes. An* equivariant stable map $X \to Y$ *is an equivariant map $X \to Q_{\mathbb{Z}_2}(Y)$.*

Equivariant stable maps can be composed and form an ∞-category. If X is finite, then there is a finite dimensional representation $V \subset \mathcal{U}$ such that a stable map is represented by an equivariant map $S^V \wedge X \to S^V \wedge Y$, where two such maps agree if they coincide in the colimit of the mapping space.

3. The tom Dieck splitting for $\mathbb{Z}_2$

Let X be a free unbased $\mathbb{Z}_2$-CW complex of finite type with orbit space $X/\mathbb{Z}_2$. Then one has an *Adams isomorphism* [16, § II.7]

$$\Sigma^\infty(X/\mathbb{Z}_2)_+ \xrightarrow{\simeq} (\Sigma^\infty_{\mathbb{Z}_2} X_+)^{\mathbb{Z}_2} , \tag{1}$$

where the target is the genuine fixed points of $\mathbb{Z}_2$ acting on the genuine $\mathbb{Z}_2$-spectrum $\Sigma_{\mathbb{Z}_2}(X_+)$. Note that $X/\mathbb{Z}_2$ is equipped with the trivial action of $\mathbb{Z}_2$.

The map of spectra (1) represents an element of the equivariant stable homotopy group

$$\{(X/\mathbb{Z}_2)_+, X_+\}_{\underline{\mathbb{Z}_2}}$$

and is induced by the stable transfer map for the regular covering space

$$p : X \to X/\mathbb{Z}_2 .$$

Example 3.1. Here is a sketch of the homotopy class of the Adams isomorphism in a special case: Assume that X is a smooth compact codimension zero $\mathbb{Z}_2$-submanifold of $V := \mathbb{R}^{n\alpha}$ for n sufficiently large. Let $e : X \to V$ be the inclusion. Then the embedding $(e, p) : X \to V \times X/\mathbb{Z}_2$ has normal bundle the trivial

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G-bundle $X \times V \to X$. The Pontryagin–Thom construction of (e, p) is therefore a $\mathbb{Z}_2$-equivariant map

$$p^! : S^V \wedge (X/\mathbb{Z}_2)_+ \to S^V \wedge X_+$$

which defines the desired element of $\{(X/\mathbb{Z}_2)_+, X_+\}_{\underline{\mathbb{Z}_2}}$.

EXAMPLE 3.2. Let $X = \{x_-, x_+\} \subset V$ be an equivariant subset of cardinality two, i.e., $x_- = -x_+$. In this case, the Adams isomorphism is an equivalence

$$S^0 \xrightarrow{\simeq} (\Sigma^\infty_{\mathbb{Z}_2} X_+)^{\mathbb{Z}_2}$$

whose associated homotopy class may be described as follows: Choose an equivariant tubular neighbourhood $U = U_{x_-} \amalg U_{x_+}$ of X. Then the interior of U is identified with $V \amalg V$. Apply the Pontryagin–Thom construction to obtain an equivariant map

$$x_- +_t x_+ : S^V \to S^V \wedge X_+ \,,$$

where in the above the expression $+_t$ indicates the dependence on the choice of tubular neighbourhood. The equivariant homotopy class of $x_- +_t x_+$ is the desired element of $\{S^0, X_+\}_{\mathbb{Z}_2}$. whose associated *un*equivariant homotopy class represents the sum $x_- + x_+ \in \{S^0, X_+\}$, where $x_-, x_+ : S^0 \to X_+$ now denote the evident inclusions.

However, the reader should not make the mistake of thinking of $x_- +_t x_+$ as the result of summing x and y, since the latter do not represent elements of $\{S^0, X_+\}_{\underline{\mathbb{Z}_2}}$.

EXAMPLE 3.3. The previous example indicates how to obtain an explicit description of the Adams isomorphism in the general case. Let $E_n(2)$ be the second space of the little n-cubes operad. A point of $E_n(2)$ is a rectilinear embedding

$$(0,1)^{\times n} \times \{1,2\} \to (0,1)^{\times n} \,.$$

Then $E_n(2)$ is a free $\mathbb{Z}_2$-space having the equivariant homotopy type of S^{n-1} with its antipodal action. The quotient map $E_n(2) \to E_n(2)/\mathbb{Z}_2$ is a regular twofold cover.

Let X be a free $\mathbb{Z}_2$-CW complex which for simplicity we take to be finite. If n is sufficiently large with respect to the dimension of X, then the twofold cover $X \to X/\mathbb{Z}_2$ is classified by a map

$$t : X/\mathbb{Z}_2 \to E_n(2)/\mathbb{Z}_2$$

in the sense that there is a pullback square

$$\begin{array}{ccc} X & \xrightarrow{\tilde{t}} & E_n(2) \\ \downarrow & & \downarrow \\ X/\mathbb{Z}_2 & \xrightarrow[t]{} & E_n(2)/\mathbb{Z}_2 \,. \end{array}$$

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where the top horizontal map $\tilde{t}$ is equivariant.

If $[x] \in X/\mathbb{Z}_2$ is a point, then let $\{x_-, x_+\}$ denote its preimage in X. Then $t(x) \in E_n(2)/\mathbb{Z}_2$ defines a tubular neighbourhood of the centre of each little cube in the configuration. Identify $x_\pm$ with the centre of the first cube in $\tilde{t}(x_\pm)$. Then Pontryagin–Thom construction defines an equivariant map

$$x_- +_t x_+ : S^V \to S^V \wedge \{x_-, x_+\}_+ \xrightarrow{\subset} S^V \wedge X_+ \,,$$

where $V = \mathbb{R}^{n\alpha}$ is $\mathbb{R}^n$ with the antipodal action. We emphasize here that the notation $+_t$ indicates the dependence on the little n-cube configuration $t(x) \in E_n(2)/\mathbb{Z}_2$.

The operation $[x] \mapsto x_- +_t x_+$ defines a map

$$(X/\mathbb{Z}_2)_+ \to (\Omega^V(S^V \wedge X_+))^{\mathbb{Z}/2}$$

which passes to the Adams isomorphism upon stabilization.

3.1. The relative case

More generally, suppose that Y is an arbitrary based $\mathbb{Z}_2$-CW complex. If we replace Y by $Y \times E\mathbb{Z}_2$, then one has a regular covering space pair

$$(Y \times E\mathbb{Z}_2, E\mathbb{Z}_2) \to (Y \times_{\mathbb{Z}_2} E\mathbb{Z}_2, B\mathbb{Z}_2)$$

and therefore a relative transfer map of pairs which induces on quotients an equivalence

$$\Sigma^\infty(Y_{h\mathbb{Z}_2}) \xrightarrow{\simeq} \Sigma^\infty_{\mathbb{Z}_2}(Y \wedge E\mathbb{Z}_{2+})^{\mathbb{Z}_2} \,,$$

where $Y_{h\mathbb{Z}_2} := Y \wedge_{\mathbb{Z}_2} E\mathbb{Z}_{2+}$ is the reduced Borel construction of $\mathbb{Z}_2$ acting on Y.

Remark 3.4. The classical transfer map $\mathrm{tr} : \Sigma^\infty Y_{h\mathbb{Z}_2} \to \Sigma^\infty Y$ of the above covering space pair of is the composition

$$\Sigma^\infty Y_{h\mathbb{Z}_2} \to (\Sigma^\infty_{\mathbb{Z}_2} Y)^{\mathbb{Z}_2} \xrightarrow{\subset} \Sigma^\infty_{\mathbb{Z}_2}(Y) \simeq \Sigma^\infty(Y) \,.$$

With Y as above, there is always a split cofibre sequence of spectra

$$\Sigma^\infty Y_{h\mathbb{Z}_2} \to (\Sigma^\infty_{\mathbb{Z}_2} Y)^{\mathbb{Z}_2} \xrightarrow{\simeq} \Sigma^\infty(Y^{\mathbb{Z}_2}) \,.$$

The first map in the sequence is a composition:

$$\Sigma^\infty Y_{h\mathbb{Z}_2} \xrightarrow{\simeq} (\Sigma^\infty_{\mathbb{Z}_2}(Y \wedge E\mathbb{Z}_{2+}))^{\mathbb{Z}_2} \to (\Sigma^\infty_{\mathbb{Z}_2} Y)^{\mathbb{Z}_2} \,,$$

where the first map is the Adams isomorphism and the second map is induced by collapsing $E\mathbb{Z}_2$ to a point.

The splitting map $\Sigma^\infty(Y^{\mathbb{Z}_2}) \to (\Sigma^\infty_{\mathbb{Z}_2} Y)^{\mathbb{Z}_2}$ is induced by the inclusion $Y^{\mathbb{Z}_2} \to Y$; more precisely, it is adjoint to the map

$$Y^{\mathbb{Z}_2} \to (Q_{\mathbb{Z}_2} Y)^{\mathbb{Z}_2}$$

which sends a fixed point of $y \in Y^{\mathbb{Z}_2}$ to the point represented by the equivariant map $S^0 = S^0 \wedge \{y\}_+ \to S^0 \wedge Y$.

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3.2. Specialization

Let B be a based CW complex. We will be interested in applying the above observation to the based $\mathbb{Z}_2$-CW complex

$$Y = B \wedge B\,,$$

where the action of $\mathbb{Z}_2$ is given by permuting the factors of the smash product. In this case, the split cofibre sequence is

$$\Sigma^\infty D_2(B) \xrightarrow{\iota B} (\Sigma^\infty_{\mathbb{Z}_2}(B \wedge B))^{\mathbb{Z}_2} \to \Sigma^\infty B\,,$$

since $(B \wedge B)^{\mathbb{Z}_2} = B$. The splitting map for the cofibre sequence is induced by the reduced diagonal $\Delta_B : B \to B \wedge B$.

Consider the commutative square

$$\begin{array}{ccc} (B^{\times 2} \times \mathbb{Z}_2, B^{\vee 2} \times \mathbb{Z}_2) & \longrightarrow & (B^{\times 2} \times_{\mathbb{Z}_2} \mathbb{Z}_2, B^{\vee 2}, \times_{\mathbb{Z}_2}\mathbb{Z}_2) \\ \downarrow & & \downarrow \\ (B^{\times 2} \times E\mathbb{Z}_2, B^{\vee 2} \times E\mathbb{Z}_2) & \longrightarrow & (B^{\times 2} \times_{\mathbb{Z}_2} E\mathbb{Z}_2, B^{\vee 2}, \times_{\mathbb{Z}_2}E\mathbb{Z}_2) \end{array}$$

in which the vertical maps are twofold coverings and the horizontal maps are induced by the equivariant inclusion $\mathbb{Z}_2 \to E\mathbb{Z}_2$ (i.e., inclusion $S^0 \subset S^\infty$).

Using naturality, we infer that there is a homotopy commutative diagram

$$\begin{array}{ccccc} \Sigma^\infty(B \wedge B) & \xrightarrow{\simeq} & (\Sigma^\infty_{\mathbb{Z}_2}(B \wedge B \wedge \mathbb{Z}_{2+}))^{\mathbb{Z}_2} & \longrightarrow & * \\ {\scriptstyle (-)_2}\downarrow & & \downarrow & & \downarrow \\ \Sigma^\infty(D_2(B)) & \xrightarrow[\iota_B]{} & (\Sigma^\infty_{\mathbb{Z}_2}(B \wedge B))^{\mathbb{Z}_2} & \longrightarrow & \Sigma^\infty B \end{array}$$

whose rows are the tom Dieck cofibre sequences. The map $(-)_2$ is induced by the inclusion $\mathbb{Z}_2 \to E\mathbb{Z}_2$:

$$B \wedge B = (B \wedge B) \wedge_{\mathbb{Z}_2} \mathbb{Z}_{2+} \to (B \wedge B) \wedge_{\mathbb{Z}_2} E\mathbb{Z}_{2+} = D_2(B)\,.$$

4. Definition of the stable Hopf invariant.

Given a stable map $f : A \to B$, i.e., a map $A \to Q(B)$, we may associate two equivariant stable maps $\Delta_B \circ f, (f \wedge f) \circ \Delta_A : A \to B \wedge B$.

Without loss in generality, we may assume that A is a finite complex; then $f : S^n \wedge A \to S^n \wedge B$. The map $\Delta_B \circ f$ is given by the composition

$$S^n \wedge A \xrightarrow{f} S^n \wedge B \xrightarrow{\Delta_B} S^n \wedge B \wedge B\,.$$

For the second map, take $f \wedge f$ and shuffle to obtain an equivariant map

$$S^n \wedge S^n \wedge A \wedge A \cong S^n \wedge A \wedge S^n \wedge A \xrightarrow{f \wedge f} S^n \wedge B \wedge S^n \wedge B \cong S^n \wedge S^n \wedge B \wedge B\,,$$

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where $S^n \wedge S^n$ is given the action that switches factors. Next, note that

$$S^n \wedge S^n \cong S^{n\alpha+n},$$

where α is the sign representation, and $n\alpha + n$ is notation for the direct sum of n-copies of α with n-copies of the trivial representation.

Making this identification and taking the composition

$$S^{n\alpha+n} \wedge A \xrightarrow{\Delta_A} S^{n\alpha+n} \wedge A \wedge A \xrightarrow{f\wedge f} S^{n\alpha+n} \wedge B \wedge B$$

yields the desired equivariant stable map $(f \wedge f) \circ \Delta_B$.

By taking adjunctions, we have constructed two maps $A \to Q_{\mathbb{Z}_2}(B \wedge B)^{\mathbb{Z}_2}$. Moreover, it is readily checked that the diagram

$$A \underset{\Delta_B \circ f}{\overset{(f\wedge f)\circ\Delta_A}{\rightrightarrows}} Q_{\mathbb{Z}_2}(B \wedge B)^{\mathbb{Z}_2} \xrightarrow{\eta} Q(B^{\mathbb{Z}_2})$$

commutes, i.e., η coequalizes the maps $\Delta_B \circ f, (f \wedge f) \circ \Delta_A$. It follows that the homotopy class of the difference

$$(f \wedge f) \circ \Delta_A - \Delta_B \circ f \in \{A, B \wedge B\}_{\underline{\mathbb{Z}_2}}$$

is in the image of the split injection

$$\iota_B : \{A, D_2(B)\} \to \{A, B \wedge B\}_{\underline{\mathbb{Z}_2}}.$$

Definition 4.1. *The* stable Hopf invariant *is the natural transformation*

$$H : \{A, B\} \to \{A, D_2(B)\}$$

such that $H(f) \in \{A, D_2(B)\}$ *is the unique element such that*

$$\iota_B H(f) = (f \wedge f) \circ \Delta_A - \Delta_B \circ f. \tag{2}$$

Remark 4.2. The natural transformation $\iota_B \circ H$ is induced by a map of spaces

$$Q(B) \to Q_{\mathbb{Z}_2}(B \wedge B)^{\mathbb{Z}_2}$$

which may be described as follows: Let $\gamma : S^n \to S^n \wedge B$ represent a point of $Q(B)$. Then $\Delta_B \circ \gamma : S^n \to S^n \wedge B \wedge B$ represents a point of $Q_{\mathbb{Z}_2}(B \wedge B)^{\mathbb{Z}_2}$. Likewise, so does $\gamma \wedge \gamma : S^n \wedge S^n \to S^n \wedge B \wedge S^n \wedge B = S^n \wedge S^n \wedge B \wedge B$. For a choice of loop multiplication on $Q_{\mathbb{Z}_2}(B \wedge B)^{\mathbb{Z}_2}$, the operation

$$\gamma \mapsto \gamma \wedge \gamma - \Delta_B \circ \gamma$$

induces $\iota_B \circ H$.

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5. Proof of theorem A

5.1. The cup product again

Let A and B be CW complexes. For stable maps $f, g : A \to B$, the *cup product*

$$f \cup g \in \{A, B \wedge B\} \tag{3}$$

is defined by the composition of the equivariant stable maps $f \wedge g$ and Δ_A, i.e.,

$$f \cup g := (f \wedge g) \circ \Delta_A \,.$$

If A is finite, then $f \cup g$ is given as follows: Let $f : \Sigma^j A \to \Sigma^j B$ and $g : \Sigma^j A \to \Sigma^j B$ be representatives. Then $f \cup g$ is given by the composition

$$\Sigma^{2j} A \xrightarrow{\Sigma^{2j}\Delta_A} \Sigma^{2j}(A \wedge A) = \Sigma^j A \wedge \Sigma^j A \xrightarrow{f \wedge g} \Sigma^j B \wedge \Sigma^j B = \Sigma^{2j}(B \wedge B)\,.$$

The cup product induces a pairing

$$\cup : \{A, B\} \times \{A, B\} \to \{A, B \wedge B\}\,.$$

The proof of the following result is straightforward and we leave its details to the reader.

LEMMA 5.1. *The cup product satisfies the following identities:*

(i) $g \cup f = \tau \circ (f \cup g)$, *where* $\tau : B \wedge B \to B \wedge B$ *is the twist map;*
(ii) $(f + f') \cup g = f \cup g + f' \cup g$.

DEFINITION 5.2. *For equivariant stable maps* $f, g : A \to B$, *the* symmetrized cup product

$$f \cup_2 g := (f \cup g)_2 \in \{A, D_2(B)\}$$

is the stable composition

$$A > \xrightarrow{f \cup g} B \wedge B \xrightarrow{(-)_2} D_2(B)\,.$$

Let $\tau : B \wedge B \to B \wedge B$ the map which permutes factors. Consider the map

$$\iota_B : \Sigma^\infty D_2(B) \to (\Sigma^\infty_{\mathbb{Z}_2}(B \wedge B))^{\mathbb{Z}_2}$$

The for $f, g \in \{A, B\}$ we may form

$$\iota_B(f \cup_2 g) \in \{A, B \wedge B\}_{\underline{\mathbb{Z}_2}}$$

Notation 5.3. For $f, g \in \{A, B\}$ we set

$$\iota_B \circ (f \cup_2 g) := f \cup g +_t g \cup f\,,$$

cf. example 3.3.

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REMARK 5.4. The elements $f \cup g, g \cup f \in \{A, B \wedge B\}$ do not make sense as an elements $\{A, B \wedge B\}_{\underline{\mathbb{Z}_2}}$. However, it is an elementary exercise involving the transfer to show that forgetful homomorphism

$$\{A, B \wedge B\}_{\underline{\mathbb{Z}_2}} \to \{A, B \wedge B\}$$

sends the element $f \cup g +_t g \cup f$ to the element

$$f \cup g + g \cup f = (1 + \tau)(f \cup g)\,.$$

This observation partially justifies the notation. Note that symbol $+_t$ exhibits the dependence of the construction on our configuration space model for the Adams isomorphism.

Proof of theorem A. As remarked above, $\iota_B : \{A, D_2(B)\} \to \{A, B \wedge B)\}_{\underline{\mathbb{Z}_2}}$ is a split injection. Recall that $H(f)$ is defined using the identity (2). For the Normalization property, it suffices to note that when $f : A \to B$ is an unstable map, then the square

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle \Delta_A}\downarrow & & \downarrow{\scriptstyle \Delta_B} \\ A \wedge A & \xrightarrow[f \wedge f]{} & B \wedge B \end{array}$$

commutes. It follows that the difference $\iota_B H(f) := (f \wedge f) \circ \Delta_A - \Delta_B \circ f \in \{A, B \wedge B\}_{\underline{\mathbb{Z}_2}}$ is trivial.

The transfer formula is also an easy consequence of the defining identity (2), since the composition

$$\{A, D_2(B)\} \xrightarrow{\iota_B} \{A, B \wedge B)\}_{\underline{\mathbb{Z}_2}} \to \{A, B \wedge B\}$$

gives the transfer, where the second displayed map is the forgetful homomorphism.

The composition formula also follows from (2): If $f : A \to B$ and $g : B \to C$, then

$$\begin{aligned} \iota_C H(g \circ f) &= (g \circ f) \wedge (g \circ f) \circ \Delta_A - \Delta_C \circ g \circ f\,, \\ &= (g \wedge g) \circ (f \wedge f) \circ \Delta_A - (\Delta_C \circ g) \circ f\,, \\ &= (g \wedge g) \circ ((f \wedge f) \circ \Delta_B - \Delta_B \circ f) + (g \wedge g) \circ \Delta_B - (\Delta_C \circ g) \circ f\,, \\ &= (g \wedge g) \circ \iota_B H(f) + \iota_C H(g) \circ f\,, \\ &= \iota_C (D_2(g) \circ H(f) + H(g) \circ f)\,. \end{aligned}$$

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For the Cartan formula, we compute

$$\begin{aligned}
\iota_B H(f+g) &= ((f+g)\wedge(f+g))\circ\Delta_A - \Delta_B\circ(f+g)\,,\\
&= (f\wedge f)\circ\Delta_A + ((f\wedge g)\circ\Delta_A +_t (g\wedge f)\circ\Delta_A)\\
&\quad + (g\wedge g)\circ\Delta_A) - (\Delta_B\circ f + \Delta_B\circ g)\,,\\
&= ((f\wedge f)\circ\Delta_A - \Delta_B\circ f)) + ((g\wedge g)\circ\Delta_A - \Delta_B\circ g)\\
&\quad + ((f\wedge g)\circ\Delta_A +_t (g\wedge f)\circ\Delta_A)\,,\\
&= \iota_B H(f) + \iota_B H(g) + (f\cup g +_t g\cup f)\,,\\
&= \iota_B H(f) + \iota_B H(g) + \iota_B(f\cup_2 g)\,,\\
&= \iota_B(H(f) + H(g) + f\cup_2 g)\,.
\end{aligned}$$

□

6. Proof of theorem B

In this paper, we provided an alternative construction of a stable Hopf invariant using the tom Dieck splitting. Here we will verify that our construction coincides with the Segal–Snaith Hopf invariant.

Recall from [17] and [21] that there is a homotopy functor on based spaces $C(-)$ and a natural transformation

$$C(X) \xrightarrow{\eta} Q(X)\,, \tag{4}$$

which is a weak equivalence when X is cofibrant and connected.

The space $C(X)$ is given by a filtration $C_1(X)\subset C_2(X)\subset\cdots$ and

$$\coprod_{n\geq 0}(X^{\times n}\times_{\Sigma_n} E(n))/\sim\,,$$

where $E(n) := \lim_{k\to\infty} E_k(n)$ is the space of n-disjoint little cubes in $\mathbb{R}^\infty$, in which $E_k(n)$ is the space of n little k-cubes in $\mathbb{R}^k$. Note that $E(n)$ is a model for $E\Sigma_n$, i.e., a free contractible Σ_n-space, and the $\{E(n)\}_{n\geq 1}$ collectively form the little ∞-cubes operad. In the display, the equivalence relation is generated by two kinds of operations: The first operation identifies a representative $(x,c)\in X^{\times(n-1)}\times E(n)$ with $(s_j x, c)$, where s_j inserts the basepoint in the j-factor of $X^{\times n}$ and the second operation identifies (x,c) with $(x, d_j(c))$, where $d_j(c): E(n)\to E(n-1)$ is given by deleting the jth cube.

The Pontryagin–Thom construction defines an equivariant map (4). In particular, if $[x,y,t]\in X^{\times 2}\times_{\Sigma_2} E(2)$, then

$$\eta([x,y,t]) = x +_t y = y +_{-t} x \in Q(X)\,.$$

where $-t$ switches the order of the pair of little cubes defined by t.

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Furthermore, there is a natural equivalence

$$\Sigma^\infty C(X) \xrightarrow{\simeq} \bigvee_{n\geq 1} \Sigma^\infty D_n(X)$$

such that the composition

$$\Sigma^\infty((X^{\times n} \times_{\Sigma_n} E_n)_+) \to \Sigma^\infty C(X) \xrightarrow{\simeq} \bigvee_{n\geq 1} \Sigma^\infty D_n(X)$$

is homotopic to the standard map into the n-summand [7].

In addition, if we pass to infinite loop spaces, then the diagram of natural transformations

$$Q(X) \xleftarrow{\simeq} C(X) \xrightarrow{\simeq} \prod_{n\geq 1} Q(D_n(X)) \xrightarrow{\text{project}} Q(D_2(X))$$

induces the second Segal–Snaith Hopf invariant h. Consequently, we may take the above description as a definition of h (see also [22] for Segal's approach).

Proof of theorem B. By the above discussion, the operations h and H are induced by natural transformations of spectrum-valued homotopy functors

$$\mathbf{h}, \mathbf{H} : \bigvee_{n\geq 1} \Sigma^\infty D_n(X) \to \Sigma^\infty D_2(X)\,,$$

where $\mathbf{h}$ is given by projection onto the second summand.

Consequently, for every integer $n \geq 1$, we have a pair if natural transformations of homotopy functors

$$\mathbf{h}_n, \mathbf{H}_n : \Sigma^\infty D_n(X) \to \Sigma^\infty D_2(X)$$

defined respectively by the restrictions of $\mathbf{h}, \mathbf{H}$ to $\Sigma^\infty D_n(X)$. By definition, the natural transformation $\mathbf{h}_n$ is trivial if $n \neq 2$ and $\mathbf{h}_2$ is the identity.

Since the functor $X \mapsto \Sigma^\infty D_n(X)$ is homogeneous of degree n, by standard Goodwillie calculus arguments [9], the natural transformations $\mathbf{H}_n$ are trivial for $n > 2$. It therefore suffices to show that $\mathbf{H}_1$ is homotopically trivial and that $\mathbf{H}_2$ is homotopic to the identity.

The natural transformation $\mathbf{H}_1 : \Sigma^\infty X \to \Sigma^\infty D_2(X)$ is trivial since for the identity map $1_X : X \to X$, we have that $\mathbf{H}_1 = \mathbf{H}_1 \circ 1_X = \mathbf{H} \circ 1_X$ must necessarily vanish, since 1_X is unstable.

It remains to prove that $\mathbf{H}_2 = \mathrm{id}$. This will require some preparation. It will be convenient to redefine $D_2(X)$ as $X^{[2]} \wedge_{\mathbb{Z}_2} E(2)_+$. It is well known that the evident map of based spaces $X_+ \to X$ admits a stable section up to homotopy, i.e., there is a stable map $X \to X_+$ such that the stable composite $X \to X_+ \to X$ is homotopic to the identity. This implies that the induced map $c : D_2(X_+) \to D_2(X)$ also admits a stable section up to homotopy. Then for any infinite loop space Z, the

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induced homomorphism $c^* : [D_2(X), Z] \to [D_2(X_+), Z]$ is injective. Furthermore,

$$D_2(X_+) = (X_+)^{[2]} \wedge_{\mathbb{Z}_2} E(2)_+ \cong (X^{\times 2} \times_{\mathbb{Z}_2} E(2))_+ .$$

Consider the composition

$$D_2(X_+) \xrightarrow{c} D_2(X) \xrightarrow{\hat{\mathbf{h}}_2} Q(D_2(X)) \xrightarrow{\iota_X} Q_{\mathbb{Z}_2}(X \wedge X)^{\mathbb{Z}_2} , \tag{5}$$

where $\hat{\mathbf{h}}_2$ is the inclusion map.

If $(x, y, t) \in X^{\times 2} \times E(2)$ is a point, we write $[x, y, t] \in X^{\times 2} \times_{\mathbb{Z}_2} E(2)$ for the projection of (x, y, t) to its $\mathbb{Z}_2$-orbit. Then by example 3.3, the map (5) is up to homotopy given by

$$[x, y, t] \mapsto x \wedge y +_t y \wedge x , \tag{6}$$

where again we write $x, y : S^0 \to X$ for the based maps defined by the points $x, y \in X$ (cf. examples 3.2 and 3.3).

Similarly, using definition of H, we claim that the composition

$$D_2(X_+) \xrightarrow{c} D_2(X) \xrightarrow{\hat{\mathbf{H}}_2} Q(D_2(X)) \xrightarrow{\iota_X} Q_{\mathbb{Z}_2}(X \wedge X)^{\mathbb{Z}_2} \tag{7}$$

(in which $\hat{\mathbf{H}}_2$ is adjoint to $\mathbf{H}_2$) is given by

$$[x, y, t] \mapsto (x +_t y) \wedge (x +_t y) - ((x \wedge x) + y \wedge y)) . \tag{8}$$

To see this, note that $[x, y, t]$ maps via η to the point of $Q(X)$ defined by the composition

$$x +_t y : S^n \xrightarrow{p_{x,y}} S^n \wedge \{x, y\}_+ \xrightarrow{1_{S^n} \wedge i_{x,y}} S^n \wedge X ,$$

in which $p_{x,y}$ is the Pontryagin–Thom construction of the configuration and $i_{x,y} : \{x, y\}_+ \to X$ is the based map induced by the inclusion of subsets. By remark 4.2, it's enough to identify (8) with the expression '$(x +_t y) \wedge (x +_t y) - \Delta_X \circ (x +_t y)$'. In other words, it suffices to prove that $(1_{S^n} \wedge \Delta_X) \circ (x +_t y) = x \wedge x +_t y \wedge y$.

Since

$$\Delta_X \circ i_{x,y} := i_{(x,x),(y,y)} : \{(x, x), (y, y)\}_+ \to X \times X ,$$

we have

$$\begin{aligned} (1_{S^n} \wedge \Delta_X) \circ (x +_t y) &= (1_{S^n} \wedge \Delta_X) \circ (1_{S^n} \wedge i_{x,y}) \circ p_{x,y} , \\ &= (1_{S^n} \wedge i_{(x,x),(y,y)}) \circ p_{x,y} , \\ &= x \wedge x +_t y \wedge y . \end{aligned}$$

Observe that as a homotopy class, the map $[x, y, t] \mapsto x \wedge x +_t y \wedge y$ does not depend on t. We have therefore established the claim that (7) is given by (8).

By functoriality of the little cubes operad action with respect to the diagonal, the map $D_2(X_+) \to Q_{\mathbb{Z}_2}(X \wedge X)^{\mathbb{Z}_2}$ defined by $[(x, y), t] \mapsto (x +_t y) \wedge (x +_t y)$ is

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homotopic to the sum of three maps

$$[(x,y),t] \mapsto (x \wedge x) + (x \wedge y +_t y \wedge x) + (y \wedge y)\,,$$

with each map indicated by parentheses. Therefore, (8) is homotopic to the signed sum of five maps

$$[(x,y),t] \mapsto (x \wedge x) + (x \wedge y +_t y \wedge x) + (y \wedge y) - (x \wedge x) - (y \wedge y)\,. \tag{9}$$

On the level of homotopy classes, we may cancel the diagonal terms appearing in (9), since the group $\{D_2(X_+), X \wedge X\}_{\underline{\mathbb{Z}_2}}$ is abelian. Consequently, the map (9) is homotopic to the map (6). We infer that the (5) and (7) are homotopic.

As ι_X is a homotopy retract it follows that $\hat{\mathbf{H}}_2 \circ c$ is homotopic to $\hat{\mathbf{h}}_2 \circ c$, i.e., on homotopy classes $c^*(\hat{\mathbf{H}}_2) = c^*(\hat{\mathbf{h}}_2)$. Since c^* is injective, we conclude that $\mathbf{H}_2$ is homotopic to $\mathbf{h}_2$. □

7. Proof of addendum D

Let T be the Quillen model category of compactly generated weak Hausdorff spaces [19]. A weak equivalence of T is a weak homotopy equivalence. A fibration is a Serre fibration. A cofibration is a map which possesses the left lifting property with respect to the acyclic fibrations.

Let $T_*(\pi)$ be the category based left π-spaces and π-equivariant maps; this category has a zero object, i.e., the space $*$ consisting of a single point. Declare a map to be a weak equivalence (fibration) if and only if it is a weak equivalence (resp. fibration) upon application of the forgetful functor $T_*(\pi) \to T$. A map is a cofibration if it possesses the left lifting property with respect to the acyclic fibrations. Then with respect to these choices, $T_*(\pi)$ is a model category by application of [10, thm. 11.6.1]. Note that the cofibrant objects of $T_*(\pi)$ are retracts of objects built up from a point by attaching free cells, where a free cell of dimension j is $D^j \times \pi$. In particular, a free π-CW complex (in the base sense) is cofibrant. Note that every object is fibrant.

For cofibrant objects $X, Y \in T_*(\pi)$, let

$$[X,Y]_\pi = \hom_{\mathrm{ho}T_*(\pi)}(X,Y)\,.$$

Note that $Q_{\mathbb{Z}_2}(Y)$ has the structure of an object of $T_*(\pi \times \mathbb{Z}_2)$. Suppose in addition X has the structure of a based $(\pi \times \mathbb{Z}_2)$-CW complex which is π-free after forgetting the $\mathbb{Z}_2$-action. In this case, we set

$$\{X,Y\}_{\pi \times \underline{\mathbb{Z}_2}} := \pi_0(\mathrm{map}_{T_*(\pi)}(X, Q_{\mathbb{Z}_2}(Y))^{\mathbb{Z}_2})\,,$$

where $\mathrm{map}_{T_*(\pi)}(A,B)$ denotes the mapping space of based equivariant maps $A \to B$.

Proof of addendum D. With respect to the above conventions, the equivariant stable Hopf invariant is defined in the same way as when π is trivial: For a cofibrant

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π-space Y, the tom Dieck fibre sequence

$$\Sigma^\infty Y_{h\mathbb{Z}_2} \to (\Sigma^\infty_{\mathbb{Z}_2} Y)^{\mathbb{Z}_2} \to \Sigma^\infty Y^{\mathbb{Z}_2}$$

is a fibre sequence of naive spectra with π-action which splits equivariantly making use of the inclusion $Y^{\mathbb{Z}_2} \to Y$. If we set $Y = B \wedge B$ where $B \in T_*(\pi)$ is a cofibrant object, we see then the construction of §4 yields

$$H^\pi : \{A, B\}_\pi \to \{A, D_2(B)\}_\pi \,.$$

The proof that properties (i)–(iv) hold is exactly the same as the proof of theorem A, where the maps and homotopy classes are to be interpreted equivariantly.

It remains to show that in the metastable range, a homotopy class $f \in \{A, B\}_\pi$ destabilizes if and only if $H^\pi(f) = 0$. We will provide a crude sketch of the argument and leave the details to the reader. The Snaith splitting is natural, 'More precisely, the proofs that appear in [7] and [9] are natural.' so the weak homotopy equivalence

$$\Sigma^\infty Q(B) \xrightarrow{\simeq} \bigvee_{n\geq 1} \Sigma^\infty D_n(B)$$

is also π-equivariant. It follows that a π-equivariant version h^π of the second Segal–Snaith Hopf invariant h is defined. Moreover, the sequence

$$B \to Q(B) \xrightarrow{h^\pi} Q(D_2(B))$$

is an equivariant fibre sequence in the metastable range by the Blakers–Massey excision theorem, where the map h^π is induced by projection onto the second summand of the Snaith splitting. Then the equivariant stable Hopf invariant h^π is the effect of applying $[A, -]_\pi$ to the map of the same name. We infer that h^π yields the total obstruction to destabilization in the metastable range. Lastly, the proof we gave that $H = h$ is valid in the equivariant setting, and we conclude that $H^\pi = h^\pi$. □

8. Proof of theorem E

Consider a natural transformation

$$\lambda : \{A, B\} \to \{A, D_2(B)\}$$

which satisfies properties (a)–(c).

By naturality and the Snaith splitting, λ is induced by a natural transformation of spectrum-valued functors

$$\bigvee_{n\geq 1} \lambda_n : \bigvee_{n\geq 1} \Sigma^\infty D_n(B) \to \Sigma^\infty D_2(B) \,.$$

Since the functor $B \mapsto \Sigma^\infty D_n(B)$ is homogeneous of degree n, it follows from standard Goodwillie calculus arguments [9] that the natural transformation λ_n is homotopically trivial for $n > 2$. Moreover, by property (i), λ_1 is trivial. Hence, it

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suffices to determine λ_2. Note that for the Segal–Snaith Hopf invariant, h_2 is the identity [15, app. B].

For homotopy functors F, G from based spaces to spectra, let $\mathrm{nat}(F, G)$ be the abelian group of homotopy classes of natural transformations from F to G; when $F = G$, this is a ring with respect to composition. Let $\{S^0, S^0\}_{\mathbb{Z}_2}$ be the ring of homotopy classes of equivariant stable self-maps of $S^0 \wedge E\mathbb{Z}_{2+}$. Again by Goodwillie calculus, the ring homomorphism

$$\{S^0, S^0\}_{\mathbb{Z}_2} \to \mathrm{nat}(D_2(-), D_2(-))$$

induced by $\alpha \mapsto \alpha \wedge_{\mathbb{Z}_2} 1$ is an isomorphism. In particular, $\hat{A}(\mathbb{Z}_2) = \{S^0, S^0\}_{\mathbb{Z}_2}$ acts on $\mathrm{nat}(D_2(-), D_2(-))$.

Let $\theta \in \{S^0, S^0\}_{\mathbb{Z}_2}$ correspond to λ_2; then $\lambda_2 = \theta h_2$. I claim that θ is a unit. To see this, represent θ by an equivariant self-map $S^0 \wedge E\mathbb{Z}_{2+} \to S^0 \wedge E\mathbb{Z}_{2+}$ and let W_θ be its homotopy fibre. Then the diagram

$$\begin{array}{ccccc} B & \longrightarrow & Q(B) & \xrightarrow{h} & Q(D_2(B)) \\ \| & & \| & & \downarrow \theta \\ B & \longrightarrow & Q(B) & \xrightarrow[\lambda]{} & Q(D_2(B)) \end{array}$$

homotopy commutes where the horizontal arrows form the EHP sequences for h and λ. The homotopy fibre of the right vertical map is given by the infinite loop space corresponding to the spectrum $W_\theta \wedge_{h\mathbb{Z}_2} (B \wedge B)$. By property (iii), the latter is $(3r)$-connected whenever B is r-connected. If we let $B = S^{r+1}$ then

$$W_\theta \wedge_{h\mathbb{Z}_2} (S^{r+1} \wedge S^{r+1}) \simeq \Sigma^{r+1} W_\theta \wedge_{h\mathbb{Z}_2} S^{(r+1).\alpha}$$

Suppose that W_θ is s-connected. If r is odd, then $\mathbb{Z}_2$ acts trivially on the reduced homology of $S^{(r+1)\alpha}$. It follows that $\Sigma^{r+1} W_\theta \wedge_{h\mathbb{Z}_2} S^{(r+1)\alpha}$ is $(s+2r+2)$-connected. Hence $s + 2r + 2 \geq 3r$ for r-odd. Since r is arbitrary, it follows that W_θ is weakly contractible. We infer that θ is a unit, proving the claim.

By the Segal conjecture [4], the ring $\{S^0, S^0\}_{\mathbb{Z}_2}$ is canonically isomorphic to the completed Burnside ring $\hat{A}(\mathbb{Z}_2)$. Consider the commutative diagram

$$\begin{array}{ccccc} \hat{A}(\mathbb{Z}_2) & \xrightarrow{\cong} & \{S^0, S^0\}_{\mathbb{Z}_2} & \xrightarrow{\cong} & \mathrm{nat}(D_2(B), D_2(B)) \\ \downarrow & & \downarrow & & \downarrow \pi_B^* \\ \hat{A}(e) & \xrightarrow[\cong]{} & \{S^0, S^0\} & \xrightarrow[\cong]{} & \mathrm{nat}(B \wedge B, D_2(B)) \end{array}$$

where the right vertical arrow is induced by the evident map $\pi_B : B \wedge B \to D_2(B)$. The lower right arrow is an isomorphism by Goodwillie calculus since a natural transformation $B \wedge B \to D_2(B)$ is determined by an equivariant map $\Sigma^\infty \mathbb{Z}_{2+} \to S^0$ and the latter is the same as specifying a non-equivariant map $S^0 \to S^0$. We observe that $B \mapsto \pi_B$ corresponds to the identity element of $\{S^0, S^0\} = \mathbb{Z}$.

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Let $p_i : B_+ \wedge B_+ = (B \times B)_+ \to B$ denote the projections $i = 1, 2$. Let $q_{B_+} : B_+ \wedge B_+ \to D_2(B)$ be the composition

$$B_+ \wedge B_+ \xrightarrow{\pi_{B_+}} D_2(B_+) \to D_2(B)\,.$$

By the Cartan formula for λ and H, we have

$$q_{B_+} = \lambda(p_1 + p_2) = \theta h(p_1 + p_2) = \theta q_{B_+}\,.$$

Since $\{B \wedge B, D_2(B)\} \to \{B_+ \wedge B_+, D_2(B)\}$ is injective, and the image of q_{B_+} is π_B, we infer that $\pi_B = \theta\pi_B$. Consequently, $\theta \in \hat{A}(\mathbb{Z}_2)^\times$ augments to the identity element. We conclude that θ lies in the kernel K of the homomorphism $\hat{A}(\mathbb{Z}_2)^\times \to \hat{A}(e)^\times$.

Conversely, it is clear that any element $\theta \in K$ determines a natural transformation λ satisfying the properties (a)–(c).

Note Added in Proof. If is not hard to show that the EHP property (c) is a consequence of properties (a) and (b). □

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