Math. Proc. Camb. Phil. Soc. (1996), **119**, 561
Printed in Great Britain

Sublinear discrete-time order-preserving dynamical systems

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(*Received* 18 *November* 1994; *revised* 28 *March* 1995)

Abstract

Suppose that the continuous mapping $T: \mathbf{R}^n_+ \to \mathbf{R}^n_+$ is order-preserving and sublinear. If every positive semi-orbit has compact closure, then every positive semi-orbit converges to a fixed point. This result does not require that the order be strongly preserved.

1. *Introduction*

The system of ordinary differential equations

$$\dot{x} = F(t,x) = F(t+2\pi, x), \quad x \in \mathbf{R}^n_+ \tag{1·1}$$

is said to be *cooperative* if the off-diagonal terms of $D_x F(t,x)$ are non-negative, where $D_x F(t,x)$ is the $n \times n$ matrix derivative of F with respect to x. The periodic cooperative system (1·1) can be viewed as an order-preserving dynamical system in the sense that the Poincaré (period) map T, defined by: $x(0) \overset{T}{\to} x(2\pi)$, preserves the usual partial ordering on $\mathbf{R}^n$ which is defined by $x \leqslant y$ if $x_i \leqslant y_i$ for $i = 1, 2, \ldots, n$ and $x \ll y$ if $x_i < y_i$ for $i = 1, 2, \ldots, n$. There is now an extensive literature on order-preserving (or monotone) dynamical systems, beginning with the path-breaking work of M. W. Hirsch [**1–4**] for monotone semiflows. The results of Hirsch and later improvements by Matano [**5**], Smith and Thieme [**6**, **7**], Poláčik [**9**] and the author [**15**] established that most orbits of a strongly order-preserving semiflow converge to the set of equilibria. Many researchers have tried to find some extra conditions added to strongly order-preserving dynamical systems to prove that every positive semi-orbit converges to an equilibrium (or a fixed point). Convergence for strongly discrete-time order-preserving dynamical systems has been established by Alikakos and Hess [**10**], Alikakos, Hess and Matano [**11**], Takáč [**12**] and Dancer and Hess [**14**] in the case that all fixed points are Liapunov stable and additional technical assumptions are satisfied. The author [**16**, **17**] constructed Liapunov functions for three- and four-dimensional positive feedback systems to give the convergence criteria. Hirsch [**4**] presented the concavity assumption to the autonomous system (1·1) (which is cooperative and irreducible):

$$\text{if} \quad x > y \quad \text{then} \quad DF(x) < DF(y) \tag{1·2}$$

* The research supported by the National Natural Science Foundation of China under grant 19271001.

https://doi.org/10.1017/S0305004100074417 Published online by Cambridge University Press

and proved that if $F(O) = O$ and all trajectories are bounded, then all trajectories tend to the origin, or else there is a unique equilibrium $p \gg O$ and all trajectories in $\mathbf{R}^n_+\backslash\{O\}$ tend to p. Smith [**8**] considered the periodic cooperative system (1·1) (which is irreducible) and replaced (1·2) by a weaker one

$$x \gg y \gg O \quad \text{implies} \quad D_x F(t, y) > D_x F(t, x). \tag{1·3}$$

Under some basic assumptions, he showed that either the origin is globally asymptotically stable in $\mathbf{R}^n_+$ or there exists a unique positive 2π periodic solution of (1·1) which attracts all initial conditions in $\mathbf{R}^n_+\backslash\{O\}$. The author [**18**] provided algebraic criteria of asymptotic behaviour for (1·1) with the concavity assumption (1·2) or (1·3). Takáč [**13**] weakened the concavity hypotheses (1·2) and (1·3). His hypothesis is that the Poincaré map T is strongly order-preserving, that is, $Tx \ll Ty$ whenever $x < y$, and sublinear in the sense that

$$T(\alpha x) \geqslant \alpha T(x) \quad \text{for all} \quad (\alpha, x) \in [0, 1] \times \mathbf{R}^n_+. \tag{1·4}$$

Together with some technical hypotheses, he proved that every positive semi-orbit with compact closure converges to a fixed point. In the case (1·3), the set E of all fixed points for T is either $\{O\}$ or $\{O, p\}$ with $p \gg O$, while in the case (1·4), E is either $\{O\}$, or $\{O, p\}$ with $p \gg O$, or $\{O\} \cup L$, where L is a simply ordered line segment. Therefore, all fixed points are simply ordered by the usual order relation $\ll$.

The above-mentioned results can apply only to the strongly order-preserving dynamical systems. Recently, the author has studied the asymptotic behaviour of order-preserving dynamical systems without the strong assumption on order. In the paper [**19**, **20**], we showed that the unique equilibrium (fixed point) for an order-preserving semiflow (a discrete-time order-preserving dynamical system) is globally asymptotically stable if and only if every positive semi-orbit has compact closure. This result generalizes theorem 9 in [**2**] and solves an interesting problem on the global stability of an equilibrium proposed by Hirsch [**2**]. The author [**21**] considered three- and four-dimensional autonomous cooperative system with every equilibrium stable and verified that every trajectory approaches an equilibrium if every trajectory has compact closure. For analytic discrete-time order-preserving dynamical systems in $\mathbf{R}^n$, the author [**22**] proved that every positive semi-orbit converges to a fixed point if every fixed point is Liapunov stable and every positive semi-orbit has compact closure. We emphasize that if T is order-preserving but is not strong then there are new types of its dynamic behaviour. For example, $T(x, y, z) = (x, y, y+z)$ is an order-preserving mapping but is not strong. It is easy to see that T satisfies (1·4), therefore, is sublinear. However, the set of all fixed points is $E = \{(x, 0, z) : x, z \in \mathbf{R}\}$ which is not simply ordered and is not included in the types of Takáč [**13**].

The main goal of this paper is to study the asymptotic behaviour of discrete-time order-preserving dynamical systems without the strong assumption on order. If such a dynamical system is sublinear and every positive semi-orbit is compact, then every positive semi-orbit is convergent to a fixed point, some of which, in general, are stable and the others of which are unstable. We shall apply this result to the periodic cooperative system (1·1) without the assumption of irreducibility.

https://doi.org/10.1017/S0305004100074417 Published online by Cambridge University Press

2. *The main result*

In this section, we first agree on certain notation and state some known results which will prove to be useful, and then present the main result of this paper.

We denote by $\mathbf{R}^n$ the n-dimensional Euclidean space. Let x or v etc. denote a point in $\mathbf{R}^n$ and x_i or v_i denote its ith component. If S is a subset of $\mathbf{R}^n$, let $\operatorname{Int} S$, $\operatorname{Bd} S$ or ∂S, and $\operatorname{Cl} S$ denote its topological interior, boundary, and closure, respectively. If S_1, $S_2 \subset \mathbf{R}^n$, define $S_1 \backslash S_2$ to be the set of points in S_1 that are not in S_2. The non-negative orthant is given by $\mathbf{R}^n_+$, that is,

$$\mathbf{R}^n_+ = \{x \in \mathbf{R}^n : x_i \geqslant 0 \quad \text{for all} \quad 1 \leqslant i \leqslant n\}.$$

At each point $p \in \mathbf{R}^n$, there is a non-negative cone defined by

$$p + \mathbf{R}^n_+ = \{x \in \mathbf{R}^n : x_i \geqslant p_i \quad \text{for all} \quad 1 \leqslant i \leqslant n\}.$$

For $p \in \mathbf{R}^n_+$, $L(O,p)$ and $R(O,p)$ denote the closed line segment with endpoints O and p and the ray in $\mathbf{R}^n_+$ passing through O and p respectively.

Let $x, y \in \mathbf{R}^n$. There is a partial order on $\mathbf{R}^n$ given by $x \leqslant y$ ($x \ll y$) if and only if $x_i \leqslant y_i$ ($x_i < y_i$) for $i = 1, 2, \ldots, n$. If $x \leqslant y$ and $x \neq y$, then we write $x < y$. x is called a positive vector if $x \gg O$. For the set $A \subset \mathbf{R}^n_+$ and $a \in \mathbf{R}^n_+$, we write $a \leqslant A$ whenever $a \leqslant x$ for all $x \in A$; similarly for $A \leqslant a$ and $a < A$ etc.

Given $x, y \in \mathbf{R}^n$, the set $[x,y] = \{z : x \leqslant z \leqslant y\}$ is called a closed order interval, and $[\![x,y]\!] = \{z : x \ll z \ll y\}$ is called an open order interval.

If the dimension n of Euclidean space $\mathbf{R}^n$ is fixed we let $N = \{1, 2, \ldots, n\}$. If $I \subset N$, we denote by $C(I)$ the complement of I in N, $C(I) = N \backslash I$. If $I \subset N$ we let $\mathrm{H}_\mathrm{I} = \{\mathrm{x} \in \mathbf{R}^n : x_i = 0,\ i \in C(I)\}$, $H^+_I = H_I \cap \mathbf{R}^n_+$ and $\operatorname{Int} H^+_I = \{x \in H_I : x_i > 0 \text{ if } i \in I\}$. Thus, the H^+_I, $I \subset N$, make up the boundary of $\mathbf{R}^n_+$ and we sometimes refer to them as 'faces'. Finally,
if $I \subset N$, $I = \{i_1 < i_2 < \cdots < i_m\}$, and $x \in \mathbf{R}^n$ we write $x_I = (x_{i_1}, x_{i_2}, \ldots, x_{i_m}) \in \mathbf{R}^m$.

Definition 1. A continuous mapping $T : \mathbf{R}^n_+ \to \mathbf{R}^n_+$ is called *order-preserving* (respectively, *strictly order-preserving*) if $x, y \in \mathbf{R}^n_+$ and $x \leqslant y$ ($x < y$) implies that $Tx \leqslant Ty$ (respectively, $Tx < Ty$). The mapping T is called *strongly order-preserving* if $Tx \ll Ty$ whenever $x < y$.

Definition 2. An order-preserving mapping $T : \mathbf{R}^n_+ \to \mathbf{R}^n_+$ is said to be *of type K* if for any subset $I \subset N$ and $x < y$ with $x_i < y_i$ for $i \in I$, one has $Tx < Ty$ with $T_i x < T_i y$ for $i \in I$, where $T_i x$ denotes the ith component of Tx.

From definitions 1 and 2, we immediately obtain the following relationship.

Strongly order-preservation $\Rightarrow$ type K order-preservation $\Rightarrow$ strict order-preservation $\Rightarrow$ order-preservation.

However, every converse relationship does not hold. For example, if $T\binom{x}{y} = \binom{1}{1}$ then T is order-preserving but not strict; if $T\binom{x}{y} = \binom{y}{x}$, then T is strict order-preserving but not of type K; if $T\binom{x}{y} = \binom{x^2+y}{y}$, then T is of type K order-preservation but not strong.

Definition 3. A continuous mapping $T : \mathbf{R}^n_+ \to \mathbf{R}^n_+$ is called *sublinear* if

$$T(\alpha x) \geqslant \alpha T(x) \quad \text{for all} \quad (\alpha, x) \in [0,1] \times \mathbf{R}^n_+.$$

An entire orbit for T is a sequence $(z_n)_{n \in Z}$ in $\mathbf{R}^n_+$ such that $z_{n+1} = Tz_n$ for all $n \in Z$,

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where Z denotes the set of all integers. A point p is called *a fixed point* of T if $Tp = p$. The set of all fixed points is denoted by $E(T)$. $p \in E(T)$ is called *Liapunov stable* if for every $\epsilon > 0$ there is $\delta > 0$ such that $x \in \mathbf{R}_+^n$ and $\|x-p\| < \delta$ implies that $\|T^k x - p\| < \epsilon$ for all $k \in Z_+$, where $\|.\|$ denotes the usual norm in $\mathbf{R}^n$ and $Z_+ = \{0, 1, 2, \ldots\}$.

The positive semi-orbit of any $x \in \mathbf{R}_+^n$ is defined by $O_T^+(x) = \{T^k x : k \in Z_+\}$, and the ω-limit set of x is defined by $\omega(x) = \{y \in \mathbf{R}_+^n : T^{n_k} x \to y \ (k \to \infty)$ for some sequence $n_k \to \infty$ in $Z_+\}$. Notice that if $O_T^+(x)$ is relatively compact in $\mathbf{R}_+^n$, then $\omega(x) \neq \varnothing$ and it is totally invariant, that is, $T\omega(x) = \omega(x)$.

PROPOSITION 2·1. *Let $p < q$ be Liapunov stable order-related fixed points of T. Assume that T is strictly order-preserving and that p is isolated fixed point from above or q is isolated fixed point from below. Then there exists an unstable fixed point $u : p < u < q$.*

This proposition is due to Dancer and Hess and is adapted from [**14**, p. 135]. From this proposition, we immediately obtain the following.

PROPOSITION 2·2. *Let $p < q$ be order-related fixed points of T. If T is strictly order-preserving and all fixed points in $[p, q]$ are Liapunov stable, then p is not an isolated fixed point from above and q is not an isolated fixed point from below.*

Our main result is

THEOREM 2·3. *Let $T : \mathbf{R}_+^n \to \mathbf{R}_+^n$ be order-preserving. Assume that T is of type K and sublinear. If $TO \geqslant O$ and every positive semi-orbit has compact closure in $\mathbf{R}_+^n$, then every positive semi-orbit converges, that is, $\omega(x)$ is a singleton for each $x \in \mathbf{R}_+^n$.*

3. *The proof of the main result*

It is convenient to establish some preliminary results before proceeding to the proof of Theorem 2·3. In all of these, we assume that the hypotheses of the theorem hold.

LEMMA 3·1. *Assume that $p > O$ is a fixed point of T. Then*

(i) *$Tz \geqslant z$ for all $z \in (O, p)$;*

(ii) *$Tz \leqslant z$ for all $z \in R(O, p) \backslash L(O, p)$.*

Proof. For each $i = 1, 2, \ldots, n$, we define the function $g_i(s) = T_i(sp) - sp_i$, $s \in [0, \infty)$. Since p is a fixed point of T, $g_i(1) = 0$ for $i = 1, 2, \ldots, n$. For a given $z \in L(O, p)$, there exists an $s \in [0, 1]$ such that $z = sp$. Therefore,

$$T_i z - z_i = g_i(s) \tag{3·1}$$

for $i = 1, 2, \ldots, n$. It follows from the sublinearity of T that $T_i(sp) \geqslant sT_i(p)$. Thus

$$g_i(s) \geqslant s(T_i p - p_i) = 0 \tag{3·2}$$

for each i. (3·1) and (3·2) imply that $Tz \geqslant z$.

If $z \in R(O, p) \backslash L(O, p)$, then $z = sp$ for some $s > 1$. Thus $0 < 1/s < 1$. From the sublinearity of T, we obtain that $T_i p = T_i(1/s . sp) \geqslant 1/s\, T_i(sp)$. Hence, $T_i(sp) \leqslant sT_i p$. Since p is a fixed point, $T_i p - p_i = 0$ for every i. Therefore, $g_i(s) \leqslant s(T_i p - p_i) = 0$ for $i = 1, 2, \ldots, n$. By (3·1), $Tz \leqslant z$.

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LEMMA 3·2. *If $Tz \geqslant z$ (respectively, $Tz \leqslant z$), then $\omega(z)$ is a singleton $\{u\}$ with $u \geqslant z$ (respectively, $u \leqslant z$).*

This lemma is contained in [**24**, p. 1298].

From Lemma 3·1 and the order-preserving property, we immediately obtain the following stability criterion.

LEMMA 3·3. *If $p \gg O$ is a positive fixed point of T, then p is Liapunov stable.*

Remark 1. From the proofs of Lemmas 3·1 and 3·3, we can conclude that Lemmas 3·1 and 3·3 are valid without the assumption that every positive semi-orbit has compact closure.

LEMMA 3·4. *If T has at least one positive fixed point, then for any $x \gg O$, there exists a positive fixed point u of T such that $u \leqslant \omega(x)$.*

Proof. Suppose that $p \gg O$ is a positive fixed point of T. Then, by Lemma 3·1, $Tz \geqslant z$ for all $z \in L(O, p)$. Since $x \gg O$, there exists a point $z \in L(O, p) \backslash \{O, p\}$ such that $O \ll z \ll x$. By the assumption of Theorem 2·3, T is of type K. Therefore,

$$T^k z \ll T^k x \tag{3·3}$$

for all $k \in Z_+$. It follows from Lemmas 3·1 and 3·2 that $\omega(z) = \{u\}$ for some $u \geqslant z$. This fact and (3·3) imply that $O \ll u \leqslant \omega(x)$.

LEMMA 3·5. *Let $x \in \mathbf{R}^n_+$ and p_0, q_0 be the greatest lower bound and the least upper bound respectively. Then there exist two fixed points p, $q \in E(T)$ such that $\omega(p_0) = \{p\}$, $\omega(q_0) = \{q\}$ and*

$$p \leqslant T^k p_0 \leqslant p_0 \leqslant \omega(x) \leqslant q_0 \leqslant T^k q_0 \leqslant q \tag{3·4}$$

for any $k \in Z_+$.

Proof. Since $\omega(x)$ is compact, it has the greatest lower bound p_0 and the least upper bound q_0. From $\omega(x) \subset \mathbf{R}^n_+$, we have $p_0, q_0 \in \mathbf{R}^n_+$. The definition of p_0 and q_0 implies that $p_0 \leqslant \omega(x) \leqslant q_0$. By the total invariance of $\omega(x)$, $T^k p_0 \leqslant \omega(x) \leqslant T^k q_0$ for any $k \in Z_+$. This shows that $T^k p_0$ is a lower bound of $\omega(x)$ and $T^k q_0$ is an upper bound for $\omega(x)$, whence $T^k p_0 \leqslant p_0$ and $q_0 \leqslant T^k q_0$ for each $k \in Z_+$. Lemma 3·2 implies that there exist $p, q \in E(T)$ such that $\omega(p_0) = \{p\}$, $\omega(q_0) = \{q\}$ and (3·4) holds. This completes the proof. ∎

From (3·4), we know that $\omega(x)$ is a singleton if and only if $\omega(x) = \{p\}$. In order to prove that $\omega(x)$ is a singleton, we can compare p with each element y in $\omega(x)$. For a given $y \in \omega(x)$, we find the index subset $I(y) = \{i : p_i < y_i \text{ for } i \in N\}$ and compute $\#I(y)$, where # denotes the cardinality of a set. $\omega(x)$ is a singleton if and only if the number $M = \max\{\#I(y) : y \in \omega(x)\}$ is zero. Therefore, our central aim is to prove that $M = 0$. This direction idea has been developed by the author in the papers [**21**, **22**]. For this goal, we define the map $M : \omega(x) \to Z_+$ as follows.

$$\delta(y) = \#\{i : y_i \neq p_i \quad \text{for} \quad i \in N\},$$

$$M(x, p) = \max\{\delta(y) : y \in \omega(x)\}.$$

Obviously, $\omega(x)$ is a singleton if and only if $M(x, p) = 0$.

Now, we give the properties on $M(x, p)$.

LEMMA 3·6. *Suppose that T has at least one positive fixed point. Then $M(x, p) \leqslant n - 1$ for all $x \gg O$.*

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Proof. By Lemma 3·4, for any $x \gg O$, there exists a positive fixed point $u \gg O$ such that $u \leqslant \omega(x)$. Together with (3·4) and the order-preserving property for T, we can conclude that $O \ll u \leqslant p$.

Suppose that there is an $x \gg O$ such that $M(x,p) = n$. Then, there exists a point $y \in \omega(x)$ such that $\delta(y) = n$, that is, $y_i \neq p_i$ for all $i \in N$. It follows from (3·4) that $p \leqslant y \leqslant q$, hence $p \ll y \leqslant q$. The argument in the first paragraph shows that $[p,q] \subset \operatorname{Int} \mathbf{R}^n_+$. By Lemma 3·3, all fixed points in $[p,q]$ are Liapunov stable. Therefore, Proposition 2·2 implies that there exists a $w \in E(T)$ such that $p < w \ll y$. It follows from $y \in \omega(x)$ that there exists a positive integer k_0 such that $w \ll T^{k_0}x$. Because T is of type K, $w \ll T^k x$ for $k \geqslant k_0$. Thus $w \leqslant \omega(x)$. Since p_0 is the greatest lower bound of $\omega(x)$ (the notation p_0 is the same as the one in Lemma 3·5), $w \leqslant p_0$. Hence, the order-preserving property implies that $w \leqslant \omega(p_0) = \{p\}$, contradicting $p < w$. This proves $M(x,p) \leqslant n-1$ for any $x \gg 0$.

LEMMA 3·7. *Suppose that T has at least one positive fixed point. Let $M(x,p) = m \leqslant n-1$ for a given $x \gg O$. Then there exist a point $y \in \omega(x)$ and an index subset $I \subset N$ with $\#I = m$ such that $p < T^k y$, $p_i < T^k_i y$ for $k \in Z_+$ and $i \in I$, and $T^k_j y \equiv p_j$ for $k \in Z_+$ and $j \in C(I)$.*

Proof. By Lemma 3·6, there exists a point $y \in \omega(x)$ such that $\delta(y) = m$. Therefore, there is an index subset $I \subset N$ with $\#I = m$ such that $p_i < y_i$ for each $i \in I$ and $p_j = y_j$ for each $j \in C(I)$. Since T is of type K, $p_i < T^k_i y$ for any $k \in Z_+$ and $i \in I$, which implies that $\delta(T^k y) \geqslant m$ for all $k \in Z_+$. However, by total invariance, $T^k y \in \omega(x)$ for $k \in Z_+$. It follows from the definition of $M(x,p)$ that $\delta(T^k y) \leqslant m$ for each $k \in Z_+$. Therefore, $\delta(T^k y) = m$, and $T^k_j y \equiv p_j$ for $k \in Z_+$ and $j \in C(I)$. This shows that such a y is a desired point.

PROPOSITION 3·8. *Suppose that T has at least one positive fixed point. Then $\omega(x)$ is a singleton for any $x \gg O$.*

Proof. By Lemma 3·4, $\omega(x) \subset \operatorname{Int} \mathbf{R}^n_+$ for any $x \gg O$. It follows from Lemma 3·3 that $E(T) \cap \operatorname{Int} \mathbf{R}^n_+$ is composed of Liapunov stable fixed points. By [**14**, lemma 2·2], we know that $\omega(x)$ is a singleton if $\omega(x)$ contains a Liapunov stable fixed point. Therefore, in order to prove this proposition, we only have to prove that $\omega(x) \cap E(T) \neq \varnothing$.

Suppose that there is a point $x \gg O$ such that $\omega(x) \cap E(T) = \varnothing$. Then

$$\omega(z) \cap E(T) = \varnothing$$

for any $z \in \omega(x)$ and $M(x,p) > 0$. By Lemmas 3·6 and 3·7, $M(x,p) = m \leqslant n-1$ and there exist a point $y \in \omega(x)$ and an index subset $I \subset N$ with $\#I = m$ such that $p < T^k y$, $p_i < T^k_i y$ for all $k \in Z_+$ and $i \in I$, and $T^k_j y \equiv p_j$ for all $k \in Z_+$ and $j \in C(I)$. Without loss of generality, we may assume that $I = \{1, 2, \ldots, m\}$. Therefore, $J = C(I) = \{m+1, \ldots, n\}$. We shall keep the notations I and J in mind. Let u^0 and v^0 be the greatest lower bound and the least upper bound of $\omega(y)$ respectively. Then, by Lemma 3·5, there are two fixed points $u, v \in E(T)$ such that $\omega(u^0) = \{u\}$, $\omega(v_0) = \{v\}$, and

$$O \ll p \leqslant u \leqslant T^k u^0 \leqslant \omega(y) \leqslant v^0 \leqslant T^k v^0 \leqslant v \tag{3·5}$$

for all $k \in Z_+$. Since $T^k_j y \equiv p_j$ for all $k \in Z_+$ and $j = m+1, \ldots, n$, we have

$$z_j = u^0_j = v^0_j = u_j = p_j \tag{3·6}$$

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for any $z \in \omega(y)$ and $j = m+1, \ldots, n$.

Now, we claim that

$$M(y, u) \leqslant m-1. \tag{3·7}$$

If this claim were false, then there would be a point $z^* \in \omega(y)$ such that $\delta(z^*) = m$. From (3·5) and (3·6), we have $u_i < z_i^* \leqslant v_i^0$ for $i = 1, 2, \ldots, m$. Since $\omega(z^*) \cap E(T) = \varnothing$, the following order relation holds

$$u < \omega(z^*). \tag{3·8}$$

In the following, we shall prove:

STATEMENT A: *There is a sequence* $\{u^k\} \subset E(T)$ *such that* $u < u^k$, $u_j^k \equiv u_j$ *for each* $k \in Z^+$ *and* $j = m+1, \ldots, n$, *and* $u^k \to u$ *as* $k \to \infty$.

If this statement has been proved, then we can choose a fixed point w of T such that $u < w < z^*$ with $w_i < z_i^*$ for $i = 1, 2, \ldots, m$. It follows from $z^* \in \omega(y)$ and $T_j^k y \equiv p_j (j \in J)$ that there exists a $k_0 > 0$ such that $w < T^{k_0} y$ with $w_i < T_i^{k_0} y$ for $i \in I$. The order-preserving property implies that $w < T^k y$ for all $k \geqslant k_0$, from which it follows that $w \leqslant \omega(y)$. By the definition of u^0, $w \leqslant u^0$. From (3·5), we get that $w \leqslant u$, a contradiction. This proves that (3·7) is true. Thus, in order to prove the claim, it is sufficient to prove Statement A.

For this goal, we define a new mapping $F: \mathbf{R}_+^n \to \mathbf{R}_+^n$ given by

$$\left.\begin{aligned} F_i(x) &= T_i(x) \quad \text{if} \quad i = 1, 2, \ldots, m, \\ F_j(x) &= x_j \qquad \text{if} \quad j = m+1, \ldots, n. \end{aligned}\right\} \tag{3·9}$$

Since T is of type K and sublinear, F is also of type K and sublinear. By the order-preserving property of T,

$$u \leqslant T^k z \leqslant T^k z^* \tag{3·10}$$

for all $z \in [u, z^*]$ and $k \in Z_+$. It follows from $T^k z^* \in \omega(y)$ and (3·6) that $T_j^k z^* \equiv u_j$ for every $k \in Z_+$ and $j = m+1, \ldots, n$. Thus, (3·10) implies that $T_j^k z \equiv z_j = u_j$ for any $z \in [u, z^*]$ and $j = m+1, \ldots, n$. This proves that

$$O_F^+(z) = O_T^+(z) \tag{3·11}$$

for any $z \in [u, z^*]$. Hence, u is also a fixed point of F. Applying Remark 1 and Lemma 3·1 to F, we conclude that $F(z) \leqslant z$ for $z \in R(O, u) \backslash L(O, u)$. Together with this fact and the order-preserving property of F, we can easily prove that every positive semi-orbit of F has compact closure in $\mathbf{R}_+^n$. Recall that $I = \{1, 2, \ldots, m\}$, $J = \{m+1, \ldots, n\}$ and that for a given vector $X = (X_1, X_2, \ldots, X_n) \in \mathbf{R}^n$, $X_I = (X_1, \ldots, X_m)$ and $X_J = (X_{m+1}, \ldots, X_n)$. When we restrict our attention to the face $u + H_I^+$, we have $F_I(x) = T_I(x_I, u_J)$, which can be viewed as an m-dimensional mapping $S: u_I + \mathbf{R}_+^m \to u_I + \mathbf{R}_+^m$. Apparently, S is of type K and u_I is a fixed point of S. By the definition of F and S, we have the relation $F^k(x) = (S^k(x_I), u_J)$ for any $x \in u + H_I^+$ and $k \in Z_+$, which implies that every positive semi-orbit of S in $u_I + \mathbf{R}_+^m$ has compact closure. We assert that S has a fixed point in $u_I + \mathbf{R}_+^m$ other than u_I. If not, then u_I is the unique fixed point of S in $u_I + \mathbf{R}_+^m$. Applying the global stability result in [**20**], we know that u_I is globally asymptotically stable in $u_I + \mathbf{R}_+^m$, i.e. $S^k(x_I) \to u_I$ as $k \to \infty$ for any $x_I \in u_I + \mathbf{R}_+^m$. By (3·11), $T^k(z^*) = F^k(z^*) = (S^k(z_I^*), u_J) \to (u_I, u_J) = u$ as $k \to \infty$, that is, $\omega(z^*) = \{u\}$, violating (3·8). This proves our assertion above. Thus, S has a fixed point v_I with

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$v_I > u_I$. Let $v = (v_I, u_J)$. Then $F(v) = (S(v_I), u_J)) = (v_I, u_J) = v$, i.e., v is a fixed point of F with $v > u$. Because F is of type K and sublinear, and $v > u \gg O$, by Remark 1 and Lemma 3·3, u and v are Liapunov stable fixed points of F. Applying Proposition 2·2 to F in $[u, v]$, we obtain that u is not isolated fixed point of F in $[u, v]$ from above. From $u_i < z_i^*$ for $i = 1, 2, \ldots, m$, it follows that there is a sequence $\{u^k\} \subset E(F)$ such that $u < u^k < z^*$, $u_i^k < z_i^*$ for all $k \in Z_+$ and $i = 1, 2, \ldots, m$ and $u^k \to u$ as $k \to \infty$. By (3·11), u^k is also a fixed point of T for each $k \in Z_+$. This shows Statement A. Therefore, (3·7) is verified. This implies that as long as $\omega(x) \cap E(T) = \varnothing$ and $M(x, p) = m$, we can pick a point $y \in \omega(x)$ such that $M(y, u) \leqslant m-1$. Continuing this argument time after time, we obtain a sequence $\{y^k\} \subset \omega(x)$ such that

$$M(y^{k+1}, u^{k+1}) < M(y^k, u^k) \leqslant n-1 \tag{3·12}$$

for each $k \in Z_+$, where $y^0 = x$ and u^k is a fixed point of T given by the manner of (3·4). Obviously, for each $k \in Z_+$, $M(y^k, u^k)$ is a positive integer not more than $n-1$. From (3·12), it is easy to prove that $M(y^k, u^k) < 0$ for all $k \geqslant n+1$. This produces the desired contradiction and completes the proof of this proposition. ∎

For any $x \in \mathbf{R}_+^n$, we define

$$n(x) = \#\{i : x_i > 0 \quad \text{for} \quad i \in N\}.$$

LEMMA 3·9. *If $O_T^+(x) \subset \partial \mathbf{R}_+^n$ for some x, then there exists an index subset $I \subset N$ and a non-negative integer k_0 such that $T^k x \in H_I^+$ for $k \geqslant k_0$.*

Proof. Let $m = \max\{n(T^k x) : k \in Z_+\}$. Then, by the assumption $O_T^+(x) \subset \partial \mathbf{R}_+^n$, $m < n$. From the definition of m, it follows that there exist a $k_0 \in Z_+$ and an index subset $I \subset N$ with $\#I = m$ such that $n(T^{k_0} x) = m$, $T_i^{k_0} x > 0$ for $i \in I$ and $T_j^{k_0} x = 0$ for $j \in C(I)$. Since T is of type K and $T_i^{k_0} x > 0$ for $i \in I$, we have $T_i^k x > 0$ for all $k \geqslant k_0$ and $i \in I$. Therefore, $n(T^k x) \geqslant m$ for $k \geqslant k_0$. However, by the definition of m, $n(T^k x) \leqslant m$. This proves that $n(T^k x) = m$ and $T_j^k x = 0$ for all $k \geqslant k_0$ and $j \in C(I)$. Thus, $T^k x \in H_I^+$ for $k \geqslant k_0$. The proof is complete. ∎

In some cases, we must consider the following m-dimensional mapping $S : \mathbf{R}_+^m \to \mathbf{R}_+^m$ defined by

$$S_i(x_1, \ldots, x_m) = T_i(x_1, \ldots, x_m, 0, \ldots, 0) \quad (i = 1, 2, \ldots, m). \tag{3·13}$$

It is not difficult to check that S is of type K and sublinear.

LEMMA 3·10. *Every positive semi-orbit of S in R_+^m has compact closure.*

Proof. Using (3·9), we know that if $x = (x_1, \ldots, x_m, \ldots, 0, \ldots, 0)$, then $F^k(x) = (S^k(x_I), O)$, where $I = \{1, 2, \ldots, m\}$. To prove the lemma, it suffices to show that all positive semi-orbits of F through points in H_I^+ have compact closures. Clearly, $F(O) \geqslant O$. In the following, we shall prove that $F^k(x) \leqslant T^k x$ for each $x \in H_I^+$ and $k \in Z_+$.

By the definition of F, $F_i(x) = T_i(x)$ for $i \in I$ and $F_j(x) = 0$ for $j \in C(I)$ and $x \in H_I^+$. So $F(x) \leqslant Tx$. Suppose $F^k(x) \leqslant T^k x$. Then, by the order-preserving property of T, $T(F^k(x)) \leqslant T^{k+1} x$.

From the definition of F and the order-preserving property of T, we obtain that $F_i^{k+1}(x) = T_i(F_1^k(x), \ldots, F_m^k(x), 0, \ldots, 0) = T_i(F^k(x))$ for $i \in I$ and $0 = F_j^{k+1}(x) \leqslant T_j(F^k(x))$ for $j \in C(I)$, that is, $F^{k+1}(x) \leqslant T(F^k(x)) \leqslant T^{k+1}(x)$. By induction, $F^k(x) \leqslant T^k x$ for all

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$k \in Z_+$. So far we have proved that $O \leqslant F^k(x) \leqslant T^k x$ for all $x \in H_I^+$ and $k \in Z_+$. Since every positive semi-orbit of T has compact closure, every positive semi-orbit of F through $x \in H_I^+$ also has compact closure.

PROPOSITION 3·11. *For each $x \gg O$, $\omega(x)$ is a singleton.*

Proof. Let $n(E) = \max\{n(p) : p \in E(T)\}$. If $n(E) = n$, then the conclusion follows immediately from Proposition 3·8. Suppose $n(E) = m < n$. Then, there exist $p \in E(T)$ and $I \subset N$ with $\#I = m$ such that $p_i > 0$ for $i \in I$ and $p_j = 0$ for $j \in C(I)$.

We shall prove that $E(T) \subset H_I^+$. Otherwise, there exist $q \in E(T)$ and $J \subset N$ such that $J \not\subset I$ and $q \in H_J^+ \backslash H_I^+$. Let $r = 1.u.b\{p, q\}$. Then, $r_i = \max\{p_i, q_i\}$ for $i \in N$. Since $q \bar{\in} H_I^+$, $r \bar{\in} H_I^+$ and $n(r) > m$. Obviously, $p + \mathbf{R}_+^n$ and $q + \mathbf{R}_+^n$ are positively invariant. So $r + \mathbf{R}_+^n$ is also positively invariant since it is the intersection of two positively invariant sets. For any $x \in r + \mathbf{R}_+^n$, we may argue as in the proof of Lemma 3·5 to conclude that there exists a fixed point u of T such that $r \leqslant u \leqslant \omega(x)$. Thus, $n(u) \geqslant n(r) > m$. But, $u \in E(T)$ and by the definition of $n(E) = m$, $n(u) \leqslant m$, a contradiction. This proves that $E(T) \subset H_I^+$. Without loss of generality, we may assume that $I = \{1, 2, \ldots, m\}$ and $p = (p_1, p_2, \ldots, p_m, 0, \ldots, 0) \in E(T)$, where $p_i > 0$ for $i \in I$.

For any $x \gg O$, there exists a point $z \in L(O, p)$ with $z_i > 0$ for $i \in I$ such that $O < z \ll x$. Lemmas 3·1 and 3·2 imply that $\omega(z) = \{u\}$ and $u \geqslant z$. Hence, $\omega(x) \geqslant u$. By (3·4), $\omega(x) \leqslant v$ for some $v \in E(T)$. Therefore, $\omega(x) \subset \operatorname{Int} H_I^+$ for each $x \gg O$.

To study the structure of $\omega(x)$, we consider the m-dimensional mapping on $\mathbf{R}_+^m$, which is defined in (3·13). Clearly, $p_I = (p_1, p_2, \ldots, p_m)$ is a positive fixed point of S. We deduce from Lemma 3·10 and Proposition 3·8 that for a given $x_I \gg O$, S has a fixed point u_I such that $S^k x_I \to u_I$ as $k \to \infty$. Obviously,

$$O_T^+(x) \subset H_I^+ \quad \text{implies} \quad T^k x = (S^k(x_I), O). \tag{3·14}$$

Now, we assert that $\omega(x) \subset E(T)$ for each $x \gg O$. Suppose not, then there is an $x \gg O$ such that $Ty \neq y$ for some $y \in \omega(x)$. Since $\omega(x)$ is totally invariant, i.e. $T\omega(x) = \omega(x)$, there exists a sequence $\{y^{-k}\} \subset \omega(x)$ such that

$$y \xleftarrow{T} y^{-1} \xleftarrow{T} y^{-2} \xleftarrow{T} \cdots y^{-(k-1)} \xleftarrow{T} y^{-k} \cdots .$$

Usually, $\{y^{-k}\}$ is called a negative semi-orbit of T through y which is probably not unique if T is not a diffeomorphism. It follows from $\{y^{-k}\} \subset \omega(x) \subset \operatorname{Int} H_I^+$ and (3·14) that

$$y_I \xleftarrow{S} y_I^{-1} \xleftarrow{S} y_I^{-2} \xleftarrow{S} \cdots y_I^{-(k-1)} \xleftarrow{S} y_I^{-k} \cdots , \tag{3·15}$$

which is a negative semi-orbit of S through y_I. Let $\alpha(y)$ denote the set of all limit points of the sequence $\{y^{-k}\}$. Then $\alpha_I(y_I) = \{z_I : z \in \alpha(y)\}$ is the set of all limit points of the sequence $\{y_I^{-k}\}$. It is easy to prove that $\alpha(y)$ (respectively, $\alpha_I(y_I)$) is compact and totally invariant with respect to T (respectively, S).

The third paragraph has shown that $\omega(x) \subset \operatorname{Int} H_I^+$. Therefore, for a given $z \in \alpha(y) \subset \omega(x)$, $z_I \gg O$ and $O_T^+(z) \subset \operatorname{Int} H_I^+$. By (3·14), $T^k z = (S^k(z_I), O)$. Since S has a positive fixed point in $\mathbf{R}_+^m$, say p_I, Proposition 3·8 implies that $S^k(z_I) \to u_I$ as $k \to \infty$ for some $u_I \gg O$, where u_I is a Liapunov stable fixed point of S by Lemma 3·3. From the invariance and compactness of $\alpha_I(y_I)$, it follows that $u_I \in \alpha_I(y_I)$, we deduce

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from $Ty \neq y$ that $u_I \neq y_I$. Thus, $\mu = \|y_I - u_I\| > 0$. The stability of u_I implies that there exists a $\delta > 0$ such that if $\|x_I - u_I\| < \delta$ and $k \in Z_+$, then

$$\|S^k(x_I) - u_I\| < \frac{\mu}{2}. \tag{3·16}$$

Notice that there exists a positive integer k_0 such that $\|y_I^{-k_0} - u_I\| < \delta$ since u_I is a limit point of the sequence $\{y_I^{-k}\}$. Applying (3·16), we obtain that $\|S^{k_0}(y_I^{-k_0}) - u_I\| < \mu/2$. However, from (3·15), $S^{k_0}(y_I^{-k_0}) = y_I$. This entails $\|y_I - u_I\| < \mu/2$, a contradiction. This shows that $\omega(x) \subset E(T)$.

Finally, we prove that $\omega(x)$ is a singleton for each $x \gg O$. Suppose this is not the case, i.e. there exists an $x \gg O$ such that $\omega(x) \subset E(T)$ is not a singleton. Then there are $u, v \in E(T)$ such that $u \neq v$. Without loss of generality, we may assume that $I = \{i : u_i < v_i \text{ for } i \in N\} \neq \varnothing$. By Lemma 3·1, $Tz \geqslant z$ for any $z \in L(O, v)$. Let $\mu = \max\{u_i/v_i : i \in I\}$. Then $0 < \mu < 1$. Choose $z^* = t_0 v$ with $\mu < t_0 < 1$. Then $z^* \in L(O, v)$. The following three properties on the cone $z^* + \mathbf{R}_+^n$ are easily checked:

(i) $z^* + \mathbf{R}_+^n$ is positively invariant;
(ii) $u \overline{\in} z^* + \mathbf{R}_+^n$;
(iii) $v_I \in z_I^* + \operatorname{Int} \mathbf{R}_+^m$.

Since $x \gg O$ and $v \in \omega(x)$, $T^k x \gg O$ for $k \in Z_+$ and there is a subsequence $\{T^{n_k} x\}$ such that $T^{n_k} x \to v$ as $k \to \infty$. From (iii), there exists a $k_0 > 0$ such that $T^{n_k} x \gg z^*$ for $k \geqslant k_0$. (i) implies that $T^k x \in z^* + \mathbf{R}_+^n$ for $k \geqslant n_{k_0}$. Therefore, $\omega(x) \subset z^* + \mathbf{R}_+^n$, contradicting to (ii). This completes the proof.

Proof of Theorem 2·3. For a given $x \in \mathbf{R}_+^n$, if there exists $k_0 \in Z_+$ such that $T^{k_0} x \gg O$, then the conclusion follows immediately from Proposition 3·11. It remains to consider the case $O_T^+(x) \subset \partial \mathbf{R}_+^n$. Let $m = \max\{n(T^k x) : k \in Z_+\}$. Then $m < n$. Applying Lemma 3·9, we know that there exists an index subset $I \subset \mathbf{N}$ and a non-negative integer k_0 such that $T^k x \in H_I^+$ for $k \geqslant k_0$. Without loss of generality, we may assume that $k_0 = 0$ and $I = \{1, 2, \ldots, m\}$. Therefore, $x_I \gg O$ and $T_j^k x \equiv 0$ for $k \in Z_+$ and $j = m+1, \ldots, n$. This shows that $\{T_I^k x\}$ is a positive semi-orbit of the m-dimensional sublinear mapping S in (3·13) which is of type K. By Lemma 3·10, every positive semi-orbit of S has compact closure in $\mathbf{R}_+^m$. Applying Proposition 3·11 to (3·13) with $x_I \gg O$, we know that $T_I^k x \to u_I$ as $k \to \infty$ for some $u_I \in \mathbf{R}_+^m$. This proves $T^k x \to (u_I, O)$ as $k \to \infty$ and completes the proof of the theorem. ∎

Remark 2. If the domain of T is the closed order interval $[O, p]$ and $TO \gg O$, $Tp \leqslant p$, then Theorem 2·3 is also valid.

4. *Applications*

In this section, we shall apply Theorem 2·3 to the system (1·1). To formulate the condition exactly, we make the following.

Definition 4·1. A vector function $f = (f_1(x), \ldots, f_n(x))$ of a vector variable $x = (x_1, \ldots, x_n)$ will be said to be of *type* K in a set $S \subset \mathbf{R}^n$ if for each subscript $i = 1, \ldots, n$ we have $f_i(a) \leqslant f_i(b)$ for any two points $a = (a_1, \ldots, a_n)$, $b = (b_1, \ldots, b_n)$ in S with $a_i = b_i$ and $a_k \leqslant b_k (k = 1, \ldots, n; k \neq i)$.

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Fundamental to any study of (1·1) in this section is the Kamke theorem (see [**23**]). We quote it here.

KAMKE'S THEOREM. *Let the system* (1·1) *satisfy the assumption* (A_1): $F(t,x)$ *is continuous in* $\mathbf{R}_+^{n+1}$ *and of type K for each fixed value of* t, *and the system* (1·1) *has the uniqueness property for initial value problems. Assume that* $x(t)$ *and* $y(t)$ *are the solutions of* (1·1) *on* $[a,b]$. *Then*

(i) $x(b) \leqslant y(b)$ *whenever* $x(a) \leqslant y(a)$;
(ii) $x(b) < y(b)$ *and* $x_i(b) < y_i(b)$ *for* $i \in I$ *whenever* $x(a) < y(a)$ *and* $x_i(a) < y_i(a)$ *for* $i \in I$, *where* $I \subset N$.

The other two assumptions we need are given here.

(A_2) $F(t,x)$ is 2π-periodic in t for each fixed value of x and all right maximally extended solutions are contained in $\mathbf{R}_+^n$ and bounded.

(A_3) Each $F_i (i = 1, \ldots, n)$ is sublinear in $x \in \mathbf{R}_+^n$:

$$F_i(t, \alpha x) \geqslant \alpha F_i(t,x) \quad \text{for all} \quad \alpha \in [0,1], \quad x \in \mathbf{R}_+^n \quad \text{and} \quad t \in \mathbf{R}^+. \tag{4·1}$$

Let $\varphi(t;\, s,\, x)$ be the solution map for (1·1), i.e. $x(t) = \varphi(t;s,x)$ is the right maximally extended solution of (1·1) satisfying $x(s) = x$. We are now ready to define the primary subject of study in this section, the Poincaré map. The Poincaré map for (1·1) is defined for $x \in \mathbf{R}_+^n$ as follows

$$T(x) = \varphi(2\pi; 0, x).$$

(A_1) and (A_2) imply that T is well defined and it is a diffeomorphism on $\mathbf{R}_+^n$, whose fixed points correspond to 2π, periodic solutions of (1·1). It follows from the Kamke theorem that T is of type K if (A_1) and (A_2) hold. We shall prove that if (A_3) holds then

$$\varphi(t; 0, \alpha x) \geqslant \alpha \varphi(t; 0, x) \tag{4·2}$$

for a fixed number $\alpha \in [0,1]$ and a fixed point $x \in \mathbf{R}_+^n$ and all $t \geqslant 0$.

If $\alpha = 0$, (4·2) is trivial by (A_2). Suppose $\alpha \in (0,1]$ and set $y(t) = 1/\alpha \varphi(t; 0, \alpha x)$. Then $\dot{y}(t) = 1/\alpha F(t, \alpha y(t)) \geqslant F(t, y(t))$. Therefore (4·2) follows immediately from Kamke's theorem (see theorem 10 in [**23**, p. 29]). Letting $t = 2\pi$ in (4·2), we obtain

$$T(\alpha x) \geqslant \alpha T(x). \tag{4·3}$$

for all $(\alpha, x) \in [0,1] \times \mathbf{R}_+^n$, that is, T is sublinear.

Definition 4·2. A solution $\varphi(t; 0, x)$ of (1·1) is called to be *compact* if it remains in a compact subset of $\mathbf{R}_+^n$ for all $t \geqslant 0$. A compact solution $\varphi(t; 0, x)$ is said to be *asymptotically periodic* of period 2π if there is a periodic solution $\psi(t)$, $\psi(t+2\pi) = \psi(t)$, such that

$$\varphi(t; 0, x) - \psi(t) \to O \quad \text{as} \quad t \to \infty.$$

Applying Theorem 2·3 to the system (1·1), we obtain:

THEOREM 4·1. *Suppose that the system* (1·1) *satisfies the assumptions* (A_1) *through* (A_3). *Then every solution of* (1·1) *is asymptotically periodic of period* 2π.

In case (1·1) is autonomous, that is, F is independent of t:

$$\dot{x} = F(x), \tag{4·4}$$

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Theorem 4·1 implies:

COROLLARY 4·2. *Assume that $F(x)$ satisfies the hypotheses of Theorem* 4·1. *Then every solution of* (4·4) *tends to an equilibrium as* $t \to \infty$.

Theorem 4·1 and Corollary 4·2 are related to the results of [**4**, theorem 6·1], [**8**, theorem 3·1] and [**13**, theorem 3·2′] where they require the additional assumption that F is C^1 with respect to x and $D_x F(t,x)$ is irreducible for each (t,x).

Finally, we shall compare our Theorem 2·3 (or Theorem 4·1) with theorem 1·1 in [**13**]. First, we note that Takáč's domain X of T is more general than ours. However, he assumed that the mapping T is strongly order-preserving in the sense that $Tx \ll Ty$ whenever $x < y$. So, to apply theorem 1·1 of Takáč[**13**] to the system (1·1), one usually has to assume that (1·1) is irreducible. Thus the Poincaré map T generated by (1·1) is C^1 and

$$DT(x) \gg O \tag{4·5}$$

for each $x \in \mathbf{R}^n_+$. A sufficient condition for (4·1) to hold is that

$$D_x F(t,x) \geqslant D_x F(t,y) \tag{4·6}$$

whenever $x \leqslant y$ and $t \in \mathbf{R}^+$. Because if (4·6) is true, then

$$\begin{aligned} F(t,\alpha x) &= F(t,O) + \int_0^1 D_x F(t, s\alpha x)\, \alpha x\, ds \\ &\geqslant F(t,O) + \alpha \int_0^1 D_x F(t,sx)\, x\, ds \\ &= (1-\alpha) F(t,O) + \alpha F(t,x) \\ &\geqslant \alpha F(t,x) \quad (\alpha \in [0,1]). \end{aligned}$$

(4·5) and the proof of [**8**, lemma 3·4] imply that

$$DT(x) \geqslant DT(y) \tag{4·7}$$

whenever $x \leqslant y$. Therefore, if (A_1), (A_2) and (4·6) hold, then theorem 1·1 of Takáč[**13**] implies that every solution of (1·1) is asymptotically periodic of period 2π and the set $E(T)$ of fixed points for the Poincaré map T takes one of the following forms:

(i) $E = \{O\}$;
(ii) $E = \{O, p\}$ with $p \gg O$;
(iii) $E = I.p = \{\alpha p : \alpha \in I\}$, where $I = [\alpha_0, 1)$ with $\alpha_0 \geqslant 0$ and $p \gg O$; and
(iv) $E = I.p \cup \{O\}$.

Moreover, in the case (iii) and (iv), every fixed point in $I.p$ is Liapunov stable and $[\alpha_0 p, p]$ attracts every $x \in \mathbf{R}^n_+ \backslash \{O\}$. We shall prove that T is linear in the closed order interval $[\alpha_0 p, p]$ if (A_1), (A_2) and (4·6) hold.

The irreducibility of F implies (4·5), that is, every $DT(x)$ is a positive matrix. The Perron–Frobenius theorem (see [**8**, p. 1040]) asserts that the spectral radius $\rho(A)$ of a positive matrix A is a simple root of its characteristic equation and $\rho(A)$ increases when any entry of A increases. Since every u in $I.p$ is a Liapunov stable fixed point, $\rho(DT(u)) \leqslant 1$ for each $u \in I.p$. However, every u in $I.p$ is not asymptotically stable. This shows that $\rho(DT(u)) = 1$ for each $u \in I.p$. The Perron–Frobenius theorem and (4·7) imply that $DT(\alpha_0 p) = DT(p)$, and furthermore, $DT(x) \equiv DT(p)$ for all $x \in [\alpha_0 p, p]$,

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that is, the system (1·1) is linear in $[\alpha_0 p, p]$, which is the most interested domain because it attracts every $x \in \mathbf{R}^n_+ \backslash \{O\}$. If the cooperative and irreducible system (1·1) satisfies (A_2) and (4·6), then it has the following three characteristics:

(1) $E(T)$ is compact and totally ordered;

(2) if $E(T)$ has more than two points, then for any $u, v \in E(T)$ with $O < u \ll v$ (1·1) is linear in $[u, v]$;

(3) there is at most one unstable fixed point in $E(T)$.

The following example shows the difference between our Theorem 2·3 and theorem 1·1 of Takáč[13].

Example. Consider the three-dimensional system of differential equations on $\mathbf{R}^3_+$

$$\dot{x}_1 = -x_1 + x_2, \quad \dot{x}_2 = x_1 - x_2, \quad \dot{x}_3 = x_3(1 - x_3). \tag{4·8}$$

It is easy to check that (4·8) is cooperative but not irreducible and satisfies (A_2) and (4·6). The equilibrium set of (4·8) is $E = \{(a, a, b) : a \geqslant 0, b = 0 \text{ or } 1\}$, which is the union of two rays, and hence is unbounded. $u = (2, 2, 0)$, $v = (1, 1, 1) \in E$ are not related by $<$, and if $p, q \in E$ with $O < p \ll q$, then (4·8) is nonlinear in $[p, q]$. All equilibria in $\{(a, a, 0) : a \geqslant 0\}$ are unstable. Therefore, the system (4·8) violates the characteristics (1) through (3). However, our Theorem 2·3 can be applied to (4·8) to conclude that every solution of (4·8) tends to an equilibrium as $t \to \infty$.

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