Feedback

On Note 99.14: Alan Beardon writes: In [1] the author refers to [2] and (implicitly) [3]. In these the authors need to find the limit of the iterates $f^n(x)$ of a map $f(x)$ with fixed point α, and they use the familar condition $|f'(\alpha)| < 1$ to guarantee convergence. It is perhaps worth pointing out that, in all cases, it is easy to find the exact form of each iterate, and so obtain the rate of convergence to the limit as well as all acceptable starting points of the process. In fact, in all cases the maps are of the form $f(z) = \dfrac{az + b}{cz + d}$, where $ad - bc \neq 0$, and for all such maps the exact form of $f^n(z)$ is known.

In [2] the author considers the maps $g(x) = \frac{1}{2}(180 - x)$ and $h(z) = \dfrac{z - 3}{2z - 5}$. Clearly, $g^n(x) - 60 = (-\frac{1}{2})^n(x - 60)$. Also,

$$\frac{h^n(z) - \alpha}{h^n(z) - \beta} = \left(\frac{1}{7 + 4\sqrt{3}}\right)^n \left(\frac{z - \alpha}{z - \beta}\right), \qquad \alpha = \frac{3 - \sqrt{3}}{2}, \qquad \beta = \frac{3 + \sqrt{3}}{2};$$

thus if $z \neq \beta$ then $h^n(z) \to \alpha$ as $n \to \infty$. Of course, if we put $n = 1$ and let $z \to \alpha$ we see that $h'(\alpha) = 1/(7 + 4\sqrt{3})$.

In [1], the author considers the function $f(x) = \dfrac{2x}{\sqrt{1 + 4x^2}}$ which is not of the given form. Nevertheless, if we let $S(z) = z^2$ and $F(z) = \dfrac{4z}{1 + 4z}$, we have $Sf(z) = FS(z)$; hence $Sf^n(z) = F^nS(z)$. In this case

$$\frac{F^n(z) - \frac{3}{4}}{F^n(z)} = \left(\frac{1}{4}\right)^n \frac{z - \frac{3}{4}}{z}.$$

so that $F^n(z) \to \frac{3}{4}$ as $n \to \infty$ whenever $z \neq 0$.

References

1. Rob Downes, A recurrence relation for the limit point of nested triangles, *Math. Gaz.* **99** (July 2015) pp. 323-325.
2. S. Y. Trimble, The limiting case of triangles formed by angle bisectors, *Math. Gaz.* **80** (November 1996) pp. 554-556.
3. S. Jones, Two iteration examples, *Math. Gaz.* **74** (March 1990) pp. 58-62.

On 'When is the sum of two triangular numbers triangular?': Paul Stephenson writes: Nick Lord [1] takes Erick Gooding's task and identifies four parameters whose values prescribe the possible solution families for the equation $T_x + T_y = T_z$, where T_n is the nth triangular number, $\frac{1}{2}n(n + 1)$. Here is a nomogram with which we can do this job.

With the two solid lines fixed, we choose a position for the dashed line and adjust the two dotted lines until the number of dots in the parallelogram equals the number in the small triangle (shown by the big dots in Figure 1).

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We then read off the number of dots on the sides of the smaller triangles, x and y, and of the large triangle, z, to give a solution of $T_x + T_y = T_z$.

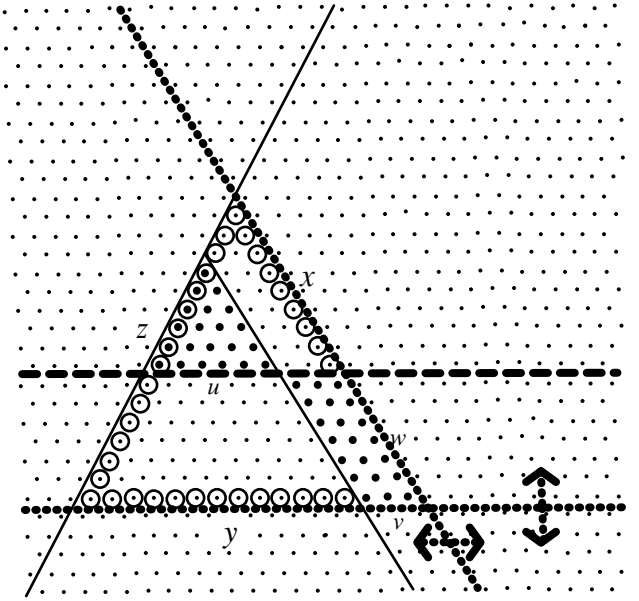


FIGURE 1

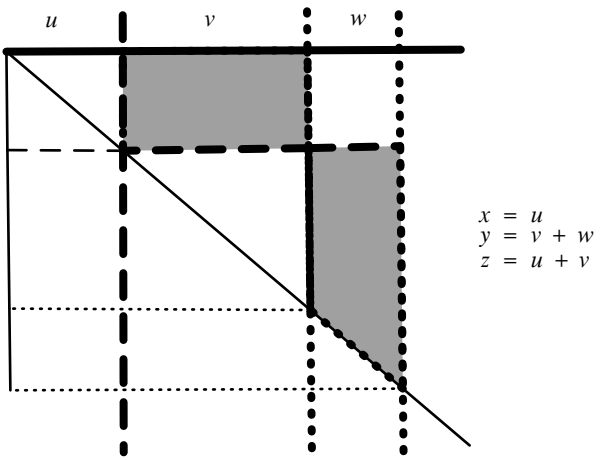


FIGURE 2

Following ancient practice, we shall use the term *gnomon* to mean a part we can add to a shape to make a similar one. Readers will see that the task in figurate terms is to find a gnomon to a triangle (the large trapezium)

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which is itself triangular. We must be able to dissect this trapezium into a smaller trapezium and a parallelogram, and dissect the parallelogram into a triangle to which the smaller trapezium is a gnomon. This is what the nomogram achieves.

We have $x = u + v$, $y = u + w$, $z = u + v + w$. By taking every u value and every factorisation $T_u = vw$, we generate all possible solutions. [Figure 2] shows an analogous construction for the equation $S_x + S_y = S_z$, where S_n is the nth square number, n^2. Solutions are restricted by the Pythagorean condition, namely, the gnomon to a square S_x is itself a square S_y if, and only if, x, y form the shorter sides of a right triangle with integral hypotenuse z. In the triangular case by contrast we have a range of obtuse-angled triangles, as Nick Lord observes.

Reference

1. N. Lord, When is the sum of two triangular numbers triangular?, *Math. Gaz.* **100** (March 2016) pp. 152-154.

lated to the area of the spherical triangle by an equation called Lexell's equation. In fact this δumber has its own geometrical meaning and it is neither unknown nor nameless.

Let ABC be a triangle on the unit sphere centred at O. This number is six times the volume of the tetrahedron $OABC$. It has at least two names. In [2] it is called the *norm* of a triangle, whereas in German literature it is also known as the *Eckensinus* [sine of a (spatial) vertex] – this name is due to Karl von Staudt in the excellent book [3]. The same number is also known as the *solid angle of the spherical triangle*. As literature please consult [4].

Of course the number is not an intrinsic number of spherical geometry but is a property of the embedding Euclidean space. Nevertheless many theorems of plane Euclidean geometry that depend on the area of a triangle have two counterparts. For example the area of a plane triangle is given by

$$A = \tfrac{1}{2}ab \sin\gamma.$$

The norm δ of the spherical triangle is then

$$\delta = \sin a \sin b \sin\gamma,$$

where the spherical area ε is given by the useful but litte known formula

$$\tan\frac{\varepsilon}{2} = \frac{\sin\gamma}{\cot\frac{a}{2}\cot\frac{b}{2} + \cos\gamma}.$$

A second example is the locus of a point C such that the area of the triangle ABC, for two fixed points A and B, remains unchanged. In plane geometry this locus is a line parallel to the line through A and B. In spherical geometry there are two possible loci. If we use the norm of a triangle and if the great circle through A and B is the equator, the locus is a circle of latitude. If we use the spherical area, this locus is the circle of Lexell given

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by C and the antipodal points A^c and B^c.

References

1. John Conway and Alex Ryba, Remembering Spherical Geometry, *Math. Gaz*. 100 (March 2016) pp. 1-8.
2. Murray Klamkin, *Mathematics Magazine 46* (1973), pp. 208-211.
3. Rudolf Sigl, *Ebene und Sphärische Trigonometrie*, Herbert Wichmann Verlag, Karlsruhe (1977).
4. Folke Eriksson, On the Measure of Solid Angles *Mathematics Magazine* **63** (3), (1990) pp. 184-187.

On Note 100.09: Lienhard Wimmer writes: This note [1] contains two excellent elementary proofs, both related to the circle of Apollonius. But one thing *can be made better*. In the second proof the author uses some elementary algebra to find two circles that intersect a given Apollonian circle orthogonally.

If we dig a little bit deeper into the geometric context we find that the angle bisector is not only connected to the circle of Apollonius but also to the *harmonic division* of points. Using this it is easy to prove the following:

> Let two circles intersect orthogonally and let a line intersect both circles in four points. Then these points divide each other harmonically.

For the proof and for further details consult [2, 3].

References

1. Gregoire Nicollier, Extremal Distance Ratios, *Math. Gaz*. **100** (March 2016) pp. 129-130.
2. Stan Oglivy, *Excursions in Geometry*, Oxford University Press, New York (1969).
3. Howard Eves, *College Geometry*, Jones and Bartlett Publishers Boston (1995).

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