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Fixed locus of an involution acting on a Godeaux surface

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Abstract

In this paper, we study involutions of Godeaux surfaces whose bicanonical system has no base components. We classify all possible fixed loci and find corresponding examples to the classification.

0. *Introduction*

A Godeaux surface is a minimal surface X of general type with $p_g(X) = 0$ and $K_x^2 = 1$. In this paper, we work over the field of complex numbers.

Denote by $\mathrm{Tors}\,(X)$ the torsion subgroup of $H^2\,(X, \mathbf{Z})$, or $\mathrm{Pic}\,(X)$. Recall that 0, $\mathbf{Z}_2$, $\mathbf{Z}_3$, $\mathbf{Z}_4$, $\mathbf{Z}_5$ are the only possible values for the torsion group of a Godeaux surface. Many of the explicitly constructed examples of Godeaux surfaces have an involution. Examples are the Godeaux surfaces constructed from D_5-invariant quintics ($\mathrm{Tors}\,(X) = \mathbf{Z}_5$) as studied by Reid [**9**], Oort-Peters' example ($\mathrm{Tors}\,(X) = \mathbf{Z}_4$) [**7**], Beauville's example ($\mathrm{Tors}\,(X) = \mathbf{Z}_3$), Werner's example ($\mathrm{Tors}\,(X) = \mathbf{Z}_2$) [**10**], Barlow's example ($\mathrm{Tors}\,(X) = 0$) [**2**] and Craighero, Gattazzo, Dolgachev and Werner's example ($\mathrm{Tors}\,(X) = 0$) [**4**].

Originally Reid [**9**] studied the log canonical models of pairs (S, C) with respect to $K_S + \frac{1}{2}C$ where S is a rational surface and C is a curve in S such that (S, C) admits a double cover. But he did not treat the particular case of Godeaux surfaces with an involution. Recently Dolgachev and Werner have proved that Craighero and Gattazzo's example is simply connected by studying its double plane construction, or equivalently the pair (S, C). It seems likely that the latter example is deformation equivalent to Barlow's example, but this is not known at present. At any rate, it is interesting to study systematically Godeaux surfaces with an involution.

For a Godeaux surface X, we consider the following condition.

Assumption. *The bicanonical system* $|2K_X|$ *has no base components.*

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So far no example of Godeaux surface has been found violating this assumption.

A $-r$-curve is a smooth rational curve with self-intersection $-r$. Our main result is the following:

THEOREM. *Let X be a Godeaux surface satisfying the Assumption. Let σ be an involution acting on X. Then we have the following.*

(1) *The fixed locus X^σ consists of five isolated points and a divisor R with $K_X R = 1$.*
(2) *The fixed divisor R consists of a smooth irreducible curve ℓ of genus $g(\ell)$ with $K_X \ell = 1$ and at most $g(\ell) + 2$ disjoint -2-curves.*
(3) *The genus $g(\ell)$ of the distinguished component ℓ is equal to 0, 1 or 2.*
(4) *If $g(\ell) = 2$, then ℓ is a component of a member of $|2K_X|$.*

By the algebraic index theorem, $g(\ell) = 2$, if and only if $K_X \ell = \ell^2 = 1$ so that ℓ is numerically equivalent to K_X. In particular, if $\mathrm{Tors}\,(X) = 0$ then $g(\ell) \leqslant 1$.

We also obtain in Section 3 some refinements of the Theorem in cases with certain additional conditions. Namely, using the same notation X, σ, R, ℓ as above, we prove the following.

(i) If $g(\ell) = 2$ and every member of $|2K_X|$ is σ-invariant, then $2\ell \in |2K_X|$ (Proposition 3·1).
(ii) Assume that $g(\ell) = 0$ and that $|3K_X - \ell|$ is a pencil without base components. Then $R = \ell$ (Proposition 3·3).

We remark that $|3K_X - \ell|$ is a pencil if and only if the image of ℓ by the tricanonical map is a line in $\mathbf{P}^3 = \mathbf{P}(H^0(3K_X))$.

The known examples of Godeaux surfaces which have an involution with fixed divisor R are:

Oort-Peters' example: R consists of an elliptic curve ℓ with $\ell^2 = -1$ and three -2-curves.

Werner's example: R consists of an elliptic curve ℓ with $\ell^2 = -1$ and a -2-curve.

Godeaux surfaces constructed from D_5-invariant quintics: R is a -3-curve.

Beauville's example, Barlow's example, Craighero, Gattazzo, Dolgachev and Werner's example: R is a -3-curve.

We find a new family of Godeaux surfaces with an involution whose fixed locus consists of an elliptic curve and five isolated points (Example 4·2) and a family of Godeaux surfaces with an involution whose fixed locus consists of a genus 2-curve and five isolated points (Example 4·3).

1. *Preliminaries*

LEMMA 1·1 (Adjunction formula for a quotient by an involution). *Let σ be an involution acting on a smooth surface X. Let S be the quotient surface X/σ and $\pi: X \to S$ be the double cover. Denote by R the fixed divisor of σ on X. Then*

$$K_X = \pi^*(K_S) + R.$$

Delete the isolated fixed points of σ from X to get an open subset X' and an étale double cover $\pi: X' \to X'/\sigma$. Then $K_{X'} = \pi^*(L)$ for some $L \in \mathrm{Pic}\,(X'/\sigma)$.

LEMMA 1·2 (Topological Lefschetz formula). *Let σ be an involution acting on a smooth surface X. Let $e(X^\sigma)$ be the Euler number of the fixed locus X^σ. Then*

$$e(X^\sigma) = \sum (-1)^i \mathrm{Tr}\,(\sigma^*: H^i(X, \mathbf{Z})).$$

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LEMMA 1·3 (Holomorphic Lefschetz formula [**1**, p. 566]). *Let σ be an involution acting on a smooth surface X. Suppose that the fixed locus X^σ consists of k isolated points and a divisor R. Then*

$$\sum(-1)^i \mathrm{Tr}\,(\sigma^*: H^i(X, \mathcal{O}_X)) = (k - K_X R)/4.$$

In particular, if in addition X is a Godeaux surface then

$$4 = k - K_X R.$$

LEMMA 1·4 ([**3**] or [**6**]). *Let X be a Godeaux surface and D be a divisor with* $\dim |D| \geqslant 1$. *Then $K_X D \geqslant 2$.*

2. *Proof of Theorem*

Throughout this section, X will denote a Godeaux surface satisfying the Assumption and σ an involution acting on X. Also k and R will denote the number of isolated fixed points and the 1-dimensional part of the fixed locus X^σ.

LEMMA 2·1. $K_X R = 1$ *or* 3.

Proof. From Lemma 1·1 we have

$$K_X = \pi^*(K_S) + R.$$

Since $1 = K_X^2 = 2K_S^2 + 2\pi^*(K_S)R + R^2$, R^2 must be odd. So R is not empty and cannot consist of only -2-curves. This implies that $K_X R$ is a positive odd integer. Consider the tricanonical system $|3K_X|$. This linear system has no base components and defines a σ-equivariant birational map from X to $\mathbf{P}^3 = \mathbf{P}(H^0(3K_X))$ [**6**]. The fixed locus of any involution of $\mathbf{P}^3$ is either the union of a plane and a point, or the union of two skew lines. Assume the first case. Write $R = R_1 + R_2$, where R_1 maps to the plane and R_2 to the point by the tricanonical map. Then $3K_X - R_1$ is effective and $\dim |3K_X - R_2| \geqslant 2$. Intersecting with K_X we obtain

$$3 - K_X R_1 \geqslant 0,$$

$$3 - K_X R_2 \geqslant 2,$$

where the second inequality follows from Lemma 1·4. Combining these two inequalities we get

$$K_X R = 1 \quad \text{or} \quad 3.$$

Next assume the second case. Write $R = R_1 + R_2$, where R_1 maps to one of the two skew lines and R_2 to the other by the tricanonical map. Then $\dim |3K_X - R_1| \geqslant 1$ and $\dim |3K_X - R_2| \geqslant 1$. Intersecting with K_X we obtain from Lemma 1·4 that

$$3 - K_X R_1 \geqslant 2, \quad 3 - K_X R_2 \geqslant 2.$$

Thus $K_X R = 1$.

Recall that the bicanonical system $|2K_X|$ has four base points, counted with multiplicities.

LEMMA 2·2. *Either* (1) *or* (2) *of the following cases occurs.*

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(1) *There is an irreducible component of R meeting the general member of $|2K_X|$ in a non-base point. In particular, every member of $|2K_X|$ is σ-invariant.*
(2) *Each irreducible component of R is a part of a member of $|2K_X|$.*

Proof. First note that both (1) and (2) cannot occur simultaneously, as the base points of $|2K_X|$ are exactly the intersection points of any two members of $|2K_X|$. Next we see that a -2-curve is disjoint from the general member of $|2K_X|$ and is a part of a member of $|2K_X|$. We will prove the lemma by showing that if we negate Case (1) then we have Case (2).

Suppose that no component of R meets the general member of $|2K_X|$ in a non-base point. Let C be a component of R with $K_X C > 0$. Then C meets the general member of $|2K_X|$ in only base points and hence the image of the map

$$H^0(2K_X) \to H^0(2K_X|_C)$$

has dimension 1. This implies that $2K_X - C$ is effective.

In Case (1), every member of $|2K_X|$ is σ-invariant as the general member has a fixed point in a non-base point.

LEMMA 2·3. *Assume Case* (2) *of Lemma* 2·2. *Then* $K_X R = 1$.

Proof. If C is a component of R with $K_X C > 0$, then C is a part of a member of $|2K_X|$ and $K_X C \leqslant 2$. Thus, if $K_X R = 3$, either

(*a*) there are two components C_1, C_2 of R with $K_X C_1 = 1$, $K_X C_2 = 2$, or
(*b*) there are three components C_1, C_2, C_3 of R with $K_X C_i = 1$, for $i = 1, 2, 3$.

Since C_i is a part of a member of $|2K_X|$ and meets the general member of $|2K_X|$ in only base points, either case contradicts the disjointness of the curves $\{C_i\}$ (because the bicanonical pencil has only four base points).

LEMMA 2·4. *Suppose that every member of $|2K_X|$ is σ-invariant. Let p be an intersection point of R and a general member D of $|2K_X|$.*

(1) *If p is a smooth point of D, then the intersection of R and D at p is transversal.*
(2) *If p is a singular point of D, then R is not tangent to any branch of D at p.*
(3) *If p is a base point of $|2K_X|$, then its multiplicity as a base point is* 2 *or* 4.

Proof. By the Bertini theorem, D is smooth away from the base points of $|2K_X|$. In terms of local coordinates x and y, σ and R can be written as follows:

$$\sigma(x, y) = (x, -y),$$

$$R\colon y = 0.$$

If p is a smooth point of D, then the equation of D, being σ-invariant, has ax as its linear part. This proves (1).

If p is a singular point of D, then p is a base point of $|2K_X|$ and general members of $|2K_X|$ have a node at p. Since every member of $|2K_X|$ is σ-invariant, (2) is proved.

If p is a base point of $|2K_X|$ and a singular point of D, then its multiplicity as a base point is 4. If p is a base point of $|2K_X|$ and a smooth point of D, then its multiplicity as a base point cannot be odd because the equation of D is σ-invariant. This proves (3).

LEMMA 2·5. *If $|2K_X|$ contains a σ-invariant smooth member not passing through any isolated fixed point of σ, then* $K_X R = 1$.

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Proof. Let C be the smooth member. By the exact sequence

$$0 \to -K_X \to K_X \to K_X\big|_C \to 0,$$

$H^0(K_X\big|_C) = 0$. Since the degree $K_X\big|_C = 2$, the Hurwitz formula implies that the curve $D = C/\sigma$ has genus $g(D) \leqslant 2$. Delete the isolated fixed points to get a surface X'. Then $K_{X'} = \pi^*(L)$ for some $L \in \mathrm{Pic}\,(X'/\sigma)$. Since the divisor $L\big|_D$ has degree 1 and can not be effective, $g(D) \geqslant 2$. Therefore, $g(D) = 2$ and, by Hurwitz again, $CR = 2K_XR = 2$.

LEMMA 2·6. *Assume Case* (1) *of Lemma* 2·2. *Then* $K_XR = 1$.

Proof. We divide the proof into the following cases according to the multiplicities of the base points of $|2K_X|$:

(a) 4 simple base points;
(b) 1 double base point and 2 simple base points;
(c) 1 triple base point and 1 simple base point;
(d) 2 double base points;
(e-1) 1 quadruple base point; $|2K_X|$ has smooth general members touching to order 4,
(e-2) 1 quadruple base point; $|2K_X|$ has singular general members with a node.

Note first that a general member of $|2K_X|$ is a smooth curve of genus 4 (Case (a), (b), (c), (d), (e-1)) or a curve with exactly 1 node (Case (e-2)).

Case (a). To get a contradiction, assume that $K_XR = 3$. Since R intersects a general member D of $|2K_X|$ transversally and $RD = 6$, the action of σ on D has at least 6 fixed points. Then the Hurwitz formula implies that the number of base points which are also isolated fixed points of σ is 0 or 4.

If the number is 0, then $|2K_X|$ contains a smooth member which is σ-invariant and does not pass through any of the isolated fixed points of σ, so we are done by Lemma 2·5.

Next assume that the number is 4, i.e. all 4 base points are isolated fixed points of σ.

Consider the tricanonical map from X to $\mathbf{P}^3 = \mathbf{P}(H^0(3K_X))$. Since $K_XR = 3$, we see from the proof of Lemma 2·1 that the fixed locus of σ in $\mathbf{P}^3$ is the union of a plane H and a point q, and that all components of R which are not -2-curves must be mapped into the plane H. Fix a general smooth member C of $|2K_X|$. The tricanonical map, restricted to C, gives the canonical map of C. So the image of C in $\mathbf{P}^3$ has degree at most 6. Since $CR = 6$, the 4 base points must be mapped onto the point q. Then the canonical image of C is not smooth, a contradiction.

Case (b). Let q be the double base point. Then q cannot be an isolated fixed point of σ, as every member of $|2K_X|$ is σ-invariant. Thus $q \in R$.

Also note that, by Lemma 2·4, R does not pass through any of the two simple base points. Then the Hurwitz formula implies that the two simple base points are switched by σ. So we are done by Lemma 2·5.

Case (c). From the Hurwitz formula, we see that the two base points cannot be isolated fixed points of σ, so must lie on R. This is a contradiction to Lemma 2·4. So Case (c) can not occur.

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Case (*d*). None of the two base points is an isolated fixed point of σ, as every member of $|2K_X|$ is σ-invariant. So they are either switched by σ or lie on R. Again we are done by Lemma 2·5.

Case (*e*). If a general member of $|2K_X|$ is a smooth curve of genus 4, then the base point cannot be an isolated fixed point of σ, as every member of $|2K_X|$ is σ-invariant. So we are done by Lemma 2·5.

Next assume that a general member of $|2K_X|$ is a curve with exactly 1 node. To get a contradiction, assume that $K_X R = 3$. Then the base point cannot be an isolated fixed point of σ, as a generic member of $|2K_X|$ is not hyperelliptic. This implies that R must pass through the base point. By Lemma 1·3, there are 7 isolated fixed points of σ away from R. Consider again the tricanonical map from X to $\mathbf{P}^3 = \mathbf{P}(H^0(3K_X))$. In this case, the fixed locus of σ in $\mathbf{P}^3$ is the union of a plane H and a point q and all components of R which are not -2-curves must be mapped into the plane H. Furthermore, all 7 isolated fixed points must be mapped to the point q and hence lie on a fundamental cycle F, i.e. a tree of -2-curves. Since $H^2(X, \mathbf{Z})$ has rank 9, the cycle F must be of type A_6, A_8, D_8 or E_8. The last three cases can be ruled out by the following argument. In any of those cases, all components of F are σ-invariant and exactly one component, say N, of F is fixed pointwisely by σ. Let N' be a component of F meeting N. Then N' contains exactly one isolated fixed point of σ. On the blow up of this point the proper transform of N' has self-intersection number -3. This is absurd, as an involution-invariant curve has even self-intersection number, unless it contains an isolated fixed point. Now it remains to show that the first case also leads to a contradiction. Note first that the cycle F is a part of a member of $|2K_X|$. Let $p\colon X' \to X$ be a blow up of the base point of $|2K_X|$ and one of its infinitely near points, and the 7 isolated fixed points of σ. Then $|2K_X|$ induces an elliptic fibration on the quotient X'/σ which is a smooth surface. The fibre of the elliptic fibration coming from the member of $|2K_X|$ containing F must contain a tree of -2-curves of type A_{13}. Call this fibre D. If D contains no -1-curve, then D gives a too big fibre (with too many components) to a relatively minimal elliptic model of X'/σ which has Euler number 12. If D contains a -1-curve which, together with the 13 curves, can be blown down, then a relatively minimal elliptic model has Euler number < 12, still a contradiction.

We are now ready to prove the Theorem.

THEOREM.

(1) *The fixed locus* X^σ *consists of five isolated points and a divisor* R *with* $K_X R = 1$.
(2) *The fixed divisor* R *consists of a smooth irreducible curve* ℓ *with* $K_X \ell = 1$ *and at most* $g(\ell) + 2$ *disjoint* -2*-curves.*
(3) *The genus* $g(\ell)$ *of the distinguished component* ℓ *is equal to* 0, 1 *or* 2.
(4) *If* $g(\ell) = 2$, *then* ℓ *is a component of a member of* $|2K_X|$.

Proof. The first statement follows from Lemma 1·3, 2·3 and 2·6. Since $K_X R = 1$, there is a distinguished component ℓ of R with $K_X \ell = 1$ and other components are -2-curves. The bound for the number of -2-curves in R follows from Lemma 1·2: since $H^2(X, \mathbf{Z})$ has rank 9, the right-hand side of the topological Lefschetz formula

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is bounded by 11. This proves the second statement. By the algebraic index theorem

$$K_X^2\ell^2 \leqslant (K_X\ell)^2 = 1.$$

So $\ell^2 \leqslant 1$ and $g(\ell) \leqslant 2$.

Let us prove the last statement. Assume that $g(\ell) = 2$. Suppose that ℓ is not a part of any member of $|2K_X|$. Then ℓ intersects the general member of $|2K_X|$ in a non-base point (see the proof of Lemma 2·2). So $2K_X\big|_\ell$ gives a g^1_2 on ℓ. Since

$$2K_X|_\ell = K_\ell = (K_X + \ell)|_\ell,$$

we have $K_X|_\ell = \ell|_\ell$. This implies that $h^0(2\ell|_\ell) - h^0(K_\ell) = 2$. Note that

$$h^2(\ell) = h^0(K_X - \ell) \leqslant h^0(K_X) = 0.$$

Since $h^0(\ell) = 1$, from Riemann–Roch we have $h^1(\ell) = 0$. So from the exact sequence

$$0 \to \mathcal{O}_X(\ell) \to \mathcal{O}_X(2\ell) \to \mathcal{O}_l(2\ell\big|_\ell) \to 0$$

we obtain $h^0(2\ell) = 3$.

Let C be a general member of $|2K_X|$ and consider the exact sequence

$$0 \to \mathcal{O}_X(2\ell - C) \to \mathcal{O}_X(2\ell) \to \mathcal{O}_C(2\ell\big|_C) \to 0.$$

If $2\ell - C$ is effective then $2\ell = C + F$ where F is a sum of -2-curves. Then we get a contradiction, because we obtain $h^0(2\ell) = 2$ from the following sequence

$$0 \to \mathcal{O}_X(C) \to \mathcal{O}_X(2\ell = C + F) \to \mathcal{O}_F((C + F)\big|_F) \to 0.$$

Therefore $h^0(2\ell - C) = 0$ and $h^0(2\ell\big|_C) \geqslant 3$. Since $|3K_X|$ is birational, a general member C of $|2K_X|$ is not hyperelliptic. Since the degree of $2\ell\big|_C$ is 4, we see from Riemann–Roch that $2\ell\big|_C$ is a special divisor. Now the Clifford theorem gives a contradiction. Therefore, we have proved that if $g(\ell) = 2$, then ℓ is a part of a member of $|2K_X|$.

3. *Further discussion*

Throughout this section, X will denote a Godeaux surface satisfying the Assumption and σ an involution acting on X. Also R and ℓ will denote the fixed divisor of σ and its distinguished component with $K_X\ell = 1$, respectively.

Proposition 3·1. *If $g(\ell) = 2$ and every member of $|2K_X|$ is σ-invariant, then $2\ell \in |2K_X|$.*

Proof. By Theorem (4), ℓ is a part of a member D of $|2K_X|$ and intersects a generic member of $|2K_X|$ only in base points. Then, by Lemma 2·4, $|2K_X|$ must have either two base points with multiplicity 2, both of them ℓ passes through transversally, or a node as a base point. In both cases, $|2K_X|$ gives a genus 2 pencil on (the minimal resolution of) the quotient surface. Since ℓ has genus 2, this implies that the member D contains ℓ with multiplicity 2. Then, since ℓ is numerically equivalent to K_X, D contains no -2-curve. Thus $D = 2\ell$.

Recall that the diagonalized action of σ on $\mathbf{P}^3 = \mathbf{P}(H^0(3K_X))$ has diagonal

$$\{1, 1, -1, -1\}, \quad \{1, 1, 1, -1\} \quad \text{or} \quad \{1, -1, -1, -1\}.$$

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Let L be the image of ℓ in $\mathbf{P}(H^0(3K_X))$ under the tricanonical map. Then L is contained in the fixed locus of σ in $\mathbf{P}^3$. We have the following:

(i) if the diagonal is $\{1,1,-1,-1\}$, then L is a line and $|3K_X-\ell|$ is a pencil on which σ acts trivially;

(ii) if the diagonal is $\{1,1,1,-1\}$ or $\{1,-1,-1,-1\}$, then $3K_X = 3\ell + F$, where F is a sum of -2-curves and L is a line or a plane cubic. If L is a line, then $|3K_X-\ell|$ is a pencil on which σ acts non-trivially. If L is a plane cubic, then $\dim|3K_X-\ell| = 0$.

LEMMA 3·2. *Assume that $|3K_X-\ell|$ is a pencil. Then the fixed part of $|3K_X-\ell|$ consists of -2-curves, if non-empty.*

Proof. Write

$$|3K_X-\ell| = |M| + F,$$

where $|M|$ is the moving part and F is the fixed part. Then we have $h^0(M) \geqslant 2$. Thus $K_XM \geqslant 2$ (Lemma 1·4). Since

$$2 = (3K_X-\ell)K_X = MK_X + FK_X,$$

we see that $MK_X = 2$ and F consists of -2-curves, if non-empty.

PROPOSITION 3·3. *Assume that $g(\ell) = 0$ and that $|3K_X-\ell|$ is a pencil without fixed components. Then the fixed divisor R is the -3-curve ℓ.*

Proof. Since $(3K_X-\ell)^2 = 0$, the pencil $|3K_X-\ell|$ has no base points and gives a genus 2 fibration. $(3K_X-\ell)\ell = 6 \neq 0$ implies that all members of $|3K_X-\ell|$ are σ-invariant. (So, in this case the diagonalized action of σ on $\mathbf{P}(H^0(3K_X))$ has diagonal $\{1,1,-1,-1\}$.)

Let N_0 be a -2-curve in R. Then $(3K_X-\ell)N_0 = 0$ implies that N_o is a component of a member D of $|3K_X-\ell|$. It suffices to show that there is no such a member. Since $DK_X = 2$, we can write $D = A+N$, where N is a union of (-2)-curves and $K_XA = 2$. The latter equality implies that A has at most two irreducible components. For any component C of N, we have $C(3K_X-\ell) = -C\ell$, therefore $C\ell = 0$. Thus

$$A\ell = D\ell = (3K_X-\ell)\ell = 6.$$

If A is an irreducible curve of arithmetic genus 2, then $A^2 = 0$ and $D = A$. This is impossible because N is not empty.

Thus A is reducible or multiple. Observe that D, and hence A and N, are σ-invariant. Since a nonsingular rational component of A has at most two fixed points, the equality $A\ell = 6$ shows that A cannot consist of two nonsingular rational components. Also if C is a nonsingular rational component of A, then the remaining component C' must satisfy $K_XC' = 1$ and hence must be of arithmetic genus 1. Since $\ell(C+C') = 6$ the rational component C must have two distinct intersection points with ℓ and no more fixed points of σ. Since $C^2 = -3$ is odd this contradicts the fact that an involution invariant curve must have even self-intersection number unless the involution has an isolated fixed point on it.

Thus each irreducible component of A is of arithmetic genus 1. Notice that since $A\ell = 6$, we must have two irreducible components A_1 and A_2 which of course may coincide. Obviously, each A_i is σ-invariant. Since $(A_1+A_2)\ell = 6$, $A_i\ell \leqslant 4$ and $A_i^2 = -1$

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(odd), we obtain that each A_i contains exactly one nonsingular point which is an isolated fixed point of σ.

Case 1. $A_1 \neq A_2, A_1A_2 \neq 0$. In this case $A_1A_2 = 1$ and A_1 and A_2 intersect in an isolated fixed point of σ. The formulae $0 = DA_i = (A_1 + A_2 + N)A_i = -1 + 1 + NA_i$ imply that $N = 0$. This is a contradiction.

Case 2. $A_1 \neq A_2, A_1A_2 = 0$. As above we obtain that each A_i intersects N at one point. This point must be one of the isolated fixed points of σ and must lie on a component of N. Since N_0 is a pointwisely fixed component of N, we see that there is a σ-invariant component of N which contains exactly one of the isolated fixed points of σ. This is impossible, for all components of σ have self-intersection number -2 (even).

Case 3. $A_1 = A_2$ so that $3K_X - \ell = 2A_1 + N$. Intersecting both sides with K_X we get $A_1K_X = 1$, therefore $A_1^2 = -1$. The formulae

$$0 = DA_1 = (2A_1 + N)A_1 = -2 + NA_1,$$

$$D^2 = -4 + 4A_1N + N^2 = 0$$

show that $NA_1 = 2, N^2 = -4$. From above, we know that A_1 contains exactly one nonsingular point which is an isolated fixed point of σ. If this point lies on N, then there is a σ-invariant component of N which contains exactly one of the isolated fixed points of σ, a contradiction as in Case 2. If this point does not lie on N, then N and A_1 must intersect at two nonsingular points, which must be switched by σ. This is again impossible, for N contains a component which is fixed pointwisely by σ.

We have proved that no member of $|3K_X - \ell|$ can contain a -2-curve which is fixed pointwisely by σ.

4. *Godeaux surfaces with* $\mathrm{Tors}\,(X) = \mathbf{Z}_4$

Let X be a Godeaux surface with $\mathrm{Tors}\,(X) = \mathbf{Z}_4$. The canonical ring was studied by Reid [**8**] via torsion method. Let Y be the cover of X corresponding to $\mathrm{Tors}\,(X)$. For each $j = 1, 2, 3 \in \mathrm{Tors}\,(X)$, let x_j be the unique effective divisor in $H^0(K_X + j)$. Then $\{x_j\}$ intersect each other transversally in one point and $H^0(2K_X + j)$ contains the following elements:

$$x_1x_3,\ x_2^2 \in H^0(2K_X), \tag{1}$$

$$x_2x_3,\ y_1 \in H^0(2K_X + 1), \tag{2}$$

$$x_1^2,\ x_3^2 \in H^0(2K_X + 2), \tag{3}$$

$$x_1x_2,\ y_3 \in H^0(2K_X + 3). \tag{4}$$

Dimension counting shows that there are two relations q_0, q_2:

$$x_1^4,\ x_2^4,\ x_3^4,\ x_1^2x_3^2,\ x_1x_2^2x_3,\ x_1x_2y_1,\ x_2x_3y_3,\ y_1y_3 \in H^0(4K_X)\colon q_0, \tag{5}$$

$$x_1^2x_2^2,\ x_2^2x_3^2,\ x_1^3x_3,\ x_1x_3^3,\ x_1x_2y_3,\ x_2x_3y_1,\ y_1^2,\ y_3^2 \in H^0(4K_X + 2)\colon q_2. \tag{6}$$

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Consider the case where $q_0 = q_2 = 0$ does not meet the branch locus of the $\mathbf{Z}_4$-covering:

$$x_1 = x_2 = x_3 = 0, \quad x_2 = y_1 = y_3 = 0, \quad x_1 = x_3 = y_1 = y_3 = 0.$$

Then x_j and y_j generate the canonical ring of Y and $X = X_{q_0,q_2}$ is obtained as a $\mathbf{Z}_4$-quotient of

$$Y_{q_0,q_2} = \mathbf{P}(x_1, x_2, x_3, y_1, y_3)/(q_0, q_2) = \mathbf{P}(1,1,1,2,2)/(q_0, q_2),$$

where $\mathbf{Z}_4$-action sends $(x_1, x_2, x_3, y_1, y_3)$ to $(ix_1, -x_2, -ix_3, iy_1, -iy_3)$ for $i = \sqrt{-1}$. There is an 8-dimensional family of Godeaux surfaces with Tors $(X) = \mathbf{Z}_4$, which is parameterized by

$$q_0\colon\ x_1^4 + x_2^4 + x_3^4 + x_1^2 x_3^2 + a_1 x_1 x_2^2 x_3 + a_2 x_1 x_2 y_1 + a_3 x_2 x_3 y_3 + a_4 y_1 y_3, \tag{7}$$

$$q_2\colon\ x_1^2 x_2^2 + x_2^2 x_3^2 + y_1^2 + y_3^2 + b_1 x_1^3 x_3 + b_2 x_1 x_3^3 + b_3 x_1 x_2 y_3 + b_4 x_2 x_3 y_1. \tag{8}$$

For X_{q_0,q_2}, the bicanonical pencil is obtained by $\lambda x_1 x_3 + \mu x_2^2 = 0$ for $(\lambda, \mu) \in \mathbf{P}^1$. The base points of $|2K_X|$ are two double points. Consider the family of curves that can belong to the bicanonical pencil of a Godeaux surface with Tors $= \mathbf{Z}_4$.

For the case of Godeaux surfaces with Tors $= \mathbf{Z}_5$, the second author [**5**] proved that the family of curves in the bicanonical pencil of this family of surfaces has dimension five via studying the bicanonical pencil and torsions. We conjecture that each torsion element gives an obstruction for the dimension of the family of curves. The next proposition gives more evidence for this.

PROPOSITION 4·1. *Curves belonging to the bicanonical pencil of a Godeaux surface with* Tors $= \mathbf{Z}_4$ *form a* 6*-dimensional family.*

Proof. Let

$$q_0'\colon q_0 + (\lambda x_1 x_3 + \mu x_2^2)(\mathbf{Z}_4\text{-invariant polynomial of degree 2}), \tag{9}$$

$$q_2'\colon q_2 + (\lambda x_1 x_3 + \mu x_2^2)(\mathbf{Z}_4\text{-anti-invariant polynomial of degree 2}). \tag{10}$$

Then $X_{q_0',q_2'}$ and X_{q_0,q_2} have the same bicanonical curves. Therefore the dimension of the family of curves that can belong to the bicanonical pencil of a Godeaux surface with Tors $= \mathbf{Z}_4$ is at most 6.

Consider the special member in the bicanonical pencil, $\mu = 0$. The dimension of the family of curves belong to this special member of the bicanonical pencil is at least five by the following reason. There is a three-dimensional family of curves $C_{q_0,q_2} = X_{q_0,q_2}\big|_{x_1=0}$,

$$x_2^4 + x_3^4 + a_3 x_2 x_3 y_3 + a_4 y_1 y_3 = 0, \tag{11}$$

$$x_2^2 x_3^2 + y_1^2 + y_3^2 + b_4 x_2 x_3 y_1 = 0. \tag{12}$$

Also there is a three-dimensional family of curves $D_{q_0,q_2} = X_{q_0,q_2}\big|_{x_3=0}$,

$$x_2^4 + x_1^4 + a_2 x_1 x_2 y_1 + a_4 y_1 y_3 = 0, \tag{13}$$

$$x_1^2 x_2^2 + y_1^2 + y_3^2 + b_3 x_1 x_2 y_3 = 0. \tag{14}$$

Finally, the normal direction of curves in the bicanonical pencil adds one more dimension.

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We can easily construct two different types of families of Godeaux surfaces with an involution and Tors $= \mathbf{Z}_4$.

Example 4·2. Consider a $\mathbf{Z}_2 = \langle a \rangle$-action on $\mathbf{P}(1, 1, 1, 2, 2)$ by

$$a\colon (x_1, x_2, x_3, y_1, y_3) \to (x_3, x_2, x_1, y_3, y_1).$$

Then $ab = b^{-1}a$ implies that the group $\langle a, b \rangle$ is D_4 where b is the generator of $\mathbf{Z}_4$. If we choose q_0 and q_2 that are compatible with this D_4-action, we obtain a five-dimensional family of Godeaux surfaces X_{q_0,q_2} with the involution and Tors $(X) = \mathbf{Z}_4$ (general elements are smooth).

$$x_1^4 + x_3^4, x_2^4, x_1^2x_3^2, x_1x_2^2x_3, x_1x_2y_1 + x_2x_3y_3, y_1y_3 \in H^0(4K_X)\colon q_0, \tag{15}$$

$$x_1^2x_2^2 + x_2^2x_3^2, x_1^3x_3 + x_1x_3^3, x_1x_2y_3 + x_2x_3y_1, y_1^2 + y_3^2 \in H^0(4K_X + 2)\colon q_2. \tag{16}$$

Since this involution interchanges x_1 and x_3, it does not fix the base points of $|2K_X|$. Consider the points $(x_1, x_2, ix_1, y_1, -iy_1)$. Then

$$ab\colon (x_1, x_2, ix_1, y_1, -iy_1) \to (ix_1, x_2, x_1, -iy_1, y_1) \to (-x_1, -x_2, -ix_1, y_1, -iy_1).$$

In $\mathbf{P}(1, 1, 1, 2, 2)$, $(-x_1, -x_2, -x_3, y_1, y_3) = (x_1, x_2, x_3, y_1, y_3)$. So the above points are fixed points of ab. Since every possible term of q_2 is zero at these points, fixed divisor C is determined by $q_0 = 0$ in $\mathbf{P}(1, 1, 2) = \mathbf{P}(x_1, x_2, y_1)$. Therefore $K_C = \mathcal{O}_C$ and C is an elliptic curve. By $\mathbf{Z}_4$-action, there are four elliptic curves induced by C in Y_{q_0,q_2}. Since X_{q_0,q_2} is obtained by canonical ring, there is no -2-curve if X_{q_0,q_2} is smooth. So the fixed divisor in X_{q_0,q_2} is an elliptic curve by the Theorem. In this example, $|3K_X - \ell|$ is the pencil obtained by $\lambda(x_1^2\, x_2 + x_2\, x_3^2) + \mu(x_1\, y_3 + x_3\, y_1)$.

Example 4·3. Consider a $\mathbf{Z}_2 = \langle a' \rangle$-action on $\mathbf{P}(1, 1, 1, 2, 2)$ by

$$a'\colon (x_1, x_2, x_3, y_1, y_3) \to (-x_1, x_2, -x_3, y_1, y_3).$$

This gives $\mathbf{Z}_4 \times \mathbf{Z}_2$-action on $\mathbf{P}(x_1, x_2, x_3, y_1, y_3)$. Consider q_0 and q_2 which are compatible with this action:

$$q_0\colon\ x_1^4,\ x_2^4,\ x_3^4,\ x_1^2\, x_3^2,\ x_1\, x_2^2\, x_3,\ y_1\, y_3,$$

$$q_2\colon\ x_1^2\, x_2^2,\ x_2^2\, x_3^2,\ x_1^3\, x_3,\ x_1\, x_3^3,\ y_1^2,\ y_3^2.$$

This four-dimensional family of surfaces is not degeneration equivalent to the family of surfaces in Example 4·2. Since $(-x_1, 0, -x_3, y_1, y_3) = (x_1, 0, x_3, y_1, y_3)$, $x_2 = 0$ gives a fixed divisor C on Y_{q_0,q_2}. C is a complete intersection of $q_0\big|_{x_2=0}$ and $q_2\big|_{x_2=0}$ in $\mathbf{P}(1, 1, 2, 2) = \mathbf{P}(x_1, x_3, y_1, y_3)$. Write $Y = Y_{q_0,q_2}$, then $K_Y = \mathcal{O}_Y(1)$ and $C = \mathcal{O}_Y(1)$. By the genus formula, $2_g(C) - 2 = C(C + K_Y) = 2K_Y^2 = 8$. Then Hurwitz's Theorem gives $g(\ell) = 2$ where $\ell = C/\mathbf{Z}_4$, so the genus of the fixed divisor induced by a' is two in X_{q_0,q_2}. In this example, ℓ passes through the two double base points of $|2K_X|$ and $2\ell \in |2K_X|$. The linear system $|3K_X - \ell|$ is the pencil obtained by $\lambda(x_1^2\, x_2) + \mu(x_2\, x_3^2)$.

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