Ergod. Th. & Dynam. Sys. (2002), **22**, 375–407
DOI: 10.1017/S0143385702000196
© *2002 Cambridge University Press*
Printed in the United Kingdom

Piece-wise affine maps conjugate to interval exchanges

MILTON COBO
IMECC/Unicamp - CX.P 6065, CEP 13083-970, Campinas-SP, Brazil
(e-mail: miltonc@impa.br)

(*Received* 19 *May* 1999 *and accepted in revised form* 3 *April* 2001)

Abstract. If T is an interval exchange transformation we denote by $C_{\text{aff}}(T)$ (respectively $S_{\text{aff}}(T)$) the set of piece-wise affine maps of the interval which are conjugate (respectively semi-conjugate) to T. In this work we will give a description of the set $C_{\text{aff}}(T)$ for *almost all* T. We present an explicit interval exchange T_0 such that $S_{\text{aff}}(T_0)\backslash C_{\text{aff}}(T_0)$ is non-empty. All the elements of $S_{\text{aff}}(T_0)\backslash C_{\text{aff}}(T_0)$ are uniquely ergodic and have a unique wandering interval.

1. *Introduction and formulation of results*

Let $a = a_0 < a_1 < a_2 < \cdots < a_{m-1} < a_m = b$ be a partition of the interval $X = [a, b)$.

Definition 1. A map $f : X \to X$ is an affine interval exchange transformation if it is a bijective transformation of the form

$$f(x) = w_i x + d_i, \quad x \in [a_{i-1}, a_i),\ i = 1, 2, \ldots, m$$

for some vectors $d = (d_1, \ldots, d_m)$, $w = (w_1, \ldots, w_m) \in \mathbb{R}^m$.

An affine interval exchange is just a bijective piece-wise affine map. We will refer to the vector w as the *slope-vector* of f. In this paper we will assume that f has positive derivatives, i.e. $w_i > 0, i = 1, \ldots, m$. For convenience, we will always assume $X = [0, 1)$.

An *interval exchange transformation* (or interval exchange for short) is an affine interval exchange transformation with slope-vector $w = (1, 1, \ldots, 1)$. It is usual to parameterize the set of all interval exchanges by the set $\Delta^{m-1} \times \mathcal{G}_m$, where

$$\Delta^{m-1} = \left\{ y = (y_1, \ldots, y_m) \in \mathbb{R}^m : \sum_{i=1}^{m} y_i = 1,\ y_i > 0, i = 1, \ldots, m \right\}$$

and $\mathcal{G}_m$ is the set of permutations of $\{1, 2, \ldots, m\}$. Indeed, to each $(\lambda, \pi) \in \Delta^{m-1} \times \mathcal{G}_m$ we will associate the following interval exchange $T_{(\lambda,\pi)}$. Let $0 = a_0 < a_1 < a_2 < \cdots < a_{m-1} < a_m = 1$ be the partition of X given by $a_k = \sum_{i=1}^{k} \lambda_i$, $i = 1, 2, \ldots, m$.

https://doi.org/10.1017/S0143385702000196 Published online by Cambridge University Press

Set $\lambda^\pi = (\lambda_{\pi^{-1}(1)}, \lambda_{\pi^{-1}(2)}, \ldots, \lambda_{\pi^{-1}(m)})$, $a_i^\pi = \sum_{j=1}^{\pi(i)} \lambda_j^\pi$, $1 \leq i \leq m$, $a_0^\pi = 0$ and $a_m^\pi = 1$. Then

$$T_{(\lambda,\pi)}(x) = x - a_{i-1} + a^\pi_{\pi(i)-1}, \quad x \in [a_{i-1}, a_i), \quad i = 1, 2, \ldots, m, \tag{1}$$

is a bijective transformation of X, i.e. an interval exchange transformation. It is not hard to see that, conversely, a pair $(\lambda, \pi) \in \Delta^{m-1} \times \mathcal{G}_m$ can be associated to every interval exchange with a partition of $m + 1$ points. We will consider the standard measure on $\Delta^{m-1} \times \mathcal{G}_m$, i.e. the product of the (normalized) Lebesgue measure in Δ^{m-1} and the counting measure in $\mathcal{G}_m$. In future, the statement 'almost all interval exchange' will always refer to this product measure.

We will say that two affine interval exchanges, f_1 and f_2, are C^k-conjugated, $k = 0, 1, \ldots \infty$ if there is a conjugacy of class C^k between f_1 and f_2.

If $w = (w_1, \ldots, w_m) \in \mathbb{R}_+^m$, the vector $(\log w_1, \ldots, \log w_m)$ will be denoted by $\log w$. In the same way, if $\gamma = (\gamma_1, \ldots, \gamma_m) \in \mathbb{R}^m$, we will denote $(e^{\gamma_1}, \ldots, e^{\gamma_m}) \in \mathbb{R}^m$ by e^γ.

According to [**1**], a necessary condition for an affine interval exchange f with slope-vector w to be conjugate to $T_{(\lambda,\pi)}$ is that $\log w$ belongs to the orthogonal complement of λ, denoted hereafter by $\lambda^\perp$.

THEOREM 1. *For almost all $(\lambda, \pi) \in \Delta^{m-1} \times \mathcal{G}_m$ and $T = T_{(\lambda,\pi)}$, there exists a sequence of subspaces*

$$E_1 \subset E_2 \subset E_3 = \lambda^\perp$$

such that if f is an affine interval exchange conjugate to T and with slope-vector w, then the following properties hold.

(a) *If $\log w \in E_1$, f is C^∞-conjugated to T. The subspace E_1 is one-dimensional.*

(b) *If $\log w \in E_2 \backslash E_1$, f is C^1-conjugated to T.*

(c) *If $\log w \in E_3 \backslash E_2$, then any conjugacy between f and T is not an absolutely continuous function.*

Theorem 1 will be proved in §4. The proof of this result strongly depends on the work of Veech [**10**] and Zorich [**15**].

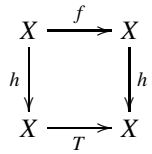
Remark. Liousse and Marzougui [**7**] have proved that, for almost all T, there is a C^2-conjugacy between f and T iff $\log w \in E_1$. Therefore, the conjugacy found in (b) is never of class C^2.

Remember that almost every interval exchange is uniquely ergodic [**8, 10**]. An immediate corollary of this theorem is as follows.

Before we state our next theorem, let us introduce some previous results and the notion of semi-conjugacy.

Definition 2. Let T be an interval exchange and let f be an affine interval exchange transformation. We say that f is *semi-conjugate* to T if there is a map $h : X \to X$ which is a continuous, monotonic and surjective and such that the following diagram commutes

https://doi.org/10.1017/S0143385702000196 Published online by Cambridge University Press



We note that h can be non-injective. In this case, there is an interval $J \subset X$ such that $h(J)$ is just a point $x_0 \in X$ and therefore, $h(f^k(J)) = T^k(x_0)$, $k \in \mathbb{Z}$. If T is a minimal interval exchange, J is a wandering interval for f, that is, J is an interval such that, for all $k \in \mathbb{Z}$, $f^k(J)$ is an interval and the limit set of J is not a periodic orbit.

Let $S_{\text{aff}}(T)$ be the set of all affine interval exchanges semi-conjugated to T. Let $C_{\text{aff}}(T, w)$ (respectively $S_{\text{aff}}(T, w)$) be the set of elements of $C_{\text{aff}}(T)$ (respectively $S_{\text{aff}}(T)$) with slope-vector w. Observe that $C_{\text{aff}}(T, w) \subset S_{\text{aff}}(T, w)$. We will exhibit an explicit interval exchange T_0 such that $C_{\text{aff}}(T_0, w)$ is empty for all w such that $\log w \in E_3 \backslash E_2$. However, for all those w, the set $S_{\text{aff}}(T_0, w) \backslash C_{\text{aff}}(T_0, w)$ is non-empty. In particular, all the elements of this set have a wandering interval.

It is proved in [**2, 3**] that, for almost all interval exchange $T = T_{(\lambda,\pi)}$, $S_{\text{aff}}(T, w) \neq \emptyset$ iff $\log w$ is orthogonal to the vector λ.

Our next theorem is related to the following particular example. Let $\pi_0 \in \mathcal{G}_4$ be the permutation given by $\pi_0(1) = 3$; $\pi_0(2) = 2$; $\pi_0(3) = 4$ and $\pi_0(4) = 1$, and let ρ be the Perron–Frobenius eigenvalue of the matrix

$$A = \begin{pmatrix} 2 & 2 & 1 & 1 \\ 1 & 2 & 0 & 0 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix},$$

and $\lambda_0 = (\lambda_1, \lambda_2, \lambda_3, \lambda_4) \in \Delta^3$ be the corresponding (normalized) positive eigenvector for ρ. Let $T_0 = T_{(\lambda_0,\pi_0)}$ be the associated interval exchange. As we will see, results from [**10**] imply that T_0 is uniquely ergodic; hence any $F \in C_{\text{aff}}(T_0)$ is uniquely ergodic. The characteristic polynomial of A is the reciprocal polynomial

$$x^4 - 7x^3 + 13x^2 - 7x + 1$$

with four real roots $\rho^{-1} < \eta^{-1} < 1 < \eta < \rho$.

Let ${}^tA$ be the transpose matrix of A. Let H^{ss} be the eigenspace of ${}^tA$ corresponding to the minimum eigenvalue ρ^{-1}. Let H^s be the two-dimensional eigenspace of ${}^tA$ corresponding to the eigenvalues ρ^{-1}, η^{-1}, and let H be the eigenspace of ${}^tA$ generated by the eigenvectors corresponding $\rho^{-1}, \eta^{-1}, \eta$. It is easy to see that H is the orthogonal complement of λ_0. Let $H^u = H \backslash H^s$. Observe that H^s divides H into two connected components H_1^u and H_2^u.

Our next theorem is the following one.

THEOREM 2. *For all vector $w \in \mathbb{R}_+^4$ such that $\log w \in H$, $S_{\text{aff}}(T_0, w)$ is a unitary set. If f is the unique element of $S_{\text{aff}}(T, w)$, then we make the following statements.*

(a) *If $\log w \in H^s$ then f belongs to $C_{\text{aff}}(T_0)$ and*
  (a1) *if $\log w \in H^{ss}$ then f is C^∞-conjugated to T_0;*
  (a2) *if $\log w \in H^s \backslash H^{ss}$, then f is C^1-conjugated to T_0.*

https://doi.org/10.1017/S0143385702000196 Published online by Cambridge University Press

(b) *If* $\log w \in H^u$ *then* $C_{\text{aff}}(T_0, w) = \emptyset$, *i.e.* f *is strictly semi-conjugated to* T_0. *Moreover,* f *has a unique wandering interval* J *and a (unique) invariant measure which is supported in a Cantor set of null Lebesgue measure.*
There are points x_1 *and* x_2 *in* X, *such that if* $h : X \to X$ *is a semi-conjugacy between* f *and* T_0, *then* h(*orbit of* J) = *orbit of* x_i *if* $\log w \in H_i^u$, $i = 1, 2$.

In addition, all the elements of $S_{\text{aff}}(T_0)$ *are at least* C^0*-conjugated.* $f_1 \in S_{\text{aff}}(T_0, w_1)$ *and* $f_2 \in S_{\text{aff}}(T_0, w_2)$ *are* C^1*-conjugated iff* $\log w_1 - \log w_2$ *belongs to* H^s.

Theorem 2 will be proved in §5. This result is an improvement to the example given in [**2**] which is the first example of a uniquely ergodic affine interval exchange with wandering interval. Levitt [**6**] had previously constructed examples which were not uniquely ergodic.

We will see that from the arguments of the proof of Theorem 1, it is clear that there exists a proper subspace E_3' of E_3, such that the decomposition $E_1 \subset E_2 \subset E_3' \subset E_4 = \lambda^\perp$ satisfies the same properties of Theorem 1 if we replace E_3 by E_3'. We believe that if $\log w \in E_4 \backslash E_3'$, the set $C_{\text{aff}}(T, w)$ is empty for almost all T, that is, the situation of the previous example is typical (see Conjecture 2).

2. *The convex set* $\Delta_\Phi(f)$

Let $f_1 : X \to X$ be an affine interval exchange transformation with an associated partition $0 = a_0 < a_1 < \cdots < a_m = 1$ and slope-vector $v = (v_1, \ldots, v_m)$. Let $w = (w_1, w_2, \ldots, w_m)$ be a vector of $\mathbb{R}^m$ and let $\Phi : X \to \mathbb{R}$ be a function of the form

$$\Phi(x) = w_i, \quad x \in [a_{i-1}, a_i), \quad i = 1, \ldots, m. \tag{2}$$

Let $\xi : X \to \mathbb{R}^+$ be a positive function satisfying the equation

$$\xi(f_1(x)) = \Phi(x) \cdot \xi(x), \quad x \in X. \tag{3}$$

Set $h(x) = \int_0^x \xi(s)\, ds$ and define $f_2 : X \to X$ by $f_2 = h \circ f_1 \circ h^{-1}$. Then f_2 is an affine interval exchange transformation with an associated slope-vector $v_2 = (w_1 v_1, \ldots, w_m v_m)$. Indeed, a straightforward computation shows that if $y \in h([a_{i-1}, a_i))$ and $y = h(x)$ then $(f_2)'(y) = \Phi(x) \cdot (f_1)'(x) = w_i v_i$.

More generally, let f_1 be an affine interval exchange with slope-vector $v = (v_1, \ldots, v_m)$ and let $\Phi : X \to \mathbb{R}$ be a measurable function. We will denote by $\Delta_\Phi(f_1)$ the set of Borel probability measures μ on X which are positive in open sets and such that

$$\int_X g \circ f_1^{-1}\, d\mu = \int_X g \Phi (f_1)'\, d\mu \tag{4}$$

for all bounded functions $g : X \to \mathbb{R}$. Clearly $\Delta_\Phi(T)$ is a convex set. Observe that if f_1 is minimal, then every probability measure that satisfies (4) belongs to $\Delta_\Phi(f_1)$. In fact, if $\mu(J) = 0$ for some interval J, then $\mu(f_1^k(J)) = 0$ for all $k \in \mathbb{Z}$, which is impossible since the saturated of J, $\bigcup_{k<0} f_1^k(J)$ is the whole X and then we obtain the contradiction $0 = \mu\big(\bigcup_{k<0} f_1^k(J)\big) = 1$.

The reason for the definition of $\Delta_\Phi(T)$ is as follows. Set $g(x) = \int_0^x d\mu$. Then as μ is positive in open sets, g is increasing. It is also right-continuous, injective and the lateral limits exist at the discontinuity points of g. Let $h : X \to X$ be the map defined by

$$h(y) = x \quad \text{iff} \quad \lim_{z \to x^-} g(z) \le y \le g(x). \tag{5}$$

https://doi.org/10.1017/S0143385702000196 Published online by Cambridge University Press

It is easy to see that h is a continuous non-decreasing map, which is injective iff g is continuous (i.e. if μ has no atoms). Observe that if μ has an atom at $p \in X$, say $\mu(\{p\}) = \delta > 0$ then, as $\mu \in \Delta_\Phi(T)$, we have that

$$\mu(\{f_1^k(p)\}) = \left(\prod_{j=0}^{k-1} \Phi(f_1^j(p))(f_1)'(f_1^j(p)) \right) \cdot \mu(\{p\}) > 0, \quad k \in \mathbb{Z}$$

and, therefore, the whole orbit of p is made of atoms of μ.

We now define a map $f_2 : X \to X$ in the following way: if $y \in X$ is such that $h(y)$ is a point of continuity of g, we define

$$f_2(y) = g \circ f_1 \circ h(y)$$

(note that $f_1(h(y))$ is a point of continuity for g). If $x = h(y)$ is a point of discontinuity for g (a jump discontinuity) then there is a maximal interval I_x such that $h(z) = x$, for all $z \in I_x$. Observe that if x is a discontinuity of g so is $f_1(x)$. We define $f_2 : I_x \to I_{f_1(x)}$ in a linear way (hence with slope $\Phi(x)(f_1)'(x)$).

We can easily check that f_2 is a bijective map and that it would have a wandering interval $I_p = h^{-1}(p)$ when p is an atom for μ.

We can check that if $h(y) \in X$ is a point of continuity of Φ then the derivative of f_2 at y satisfies $f_2'(y) = \Phi(h(y))(f_1)'(h(y))$. In particular, when Φ is of the form (2), f_2 is an affine interval exchange transformation with slope-vector $(w_1 v_1, \ldots, w_m v_m)$.

These observations prove the following proposition.

PROPOSITION 2.1. *Let f_1 be an affine interval exchange with slope-vector $v = (v_1, \ldots, v_m)$ and let ξ be a (measurable) positive solution to equation (3) for $w = (w_1, \ldots, w_m)$. Then the map $f_2 = h \circ f_1 \circ h^{-1}$ where $h(x) = \int_0^x \xi \, dm$ is an affine interval exchange with slope vector $(w_1 v_1, \ldots, w_m v_m)$. More generally, let $\Phi : X \to \mathbb{R}$ be a function of the form (2). Then associated to each $\mu \in \Delta_\Phi(f_1)$ there is a map $f_2 \in C_{\text{aff}}(f_1)$ with slope-vector $(w_1 v_1, \ldots, w_m v_m)$. If μ has atoms, f_2 has wandering intervals.*

In particular, when $f_1 = T$ is an interval exchange transformation, equation (4) becomes

$$\int_X g \circ T^{-1} \, d\mu = \int_X g \Phi \, d\mu \tag{6}$$

for all bounded functions $g : X \to \mathbb{R}$.

LEMMA 2.2. *Let $\Phi : X \to \mathbb{R}^+$ be a measurable positive function. Then for almost all $(\lambda, \pi) \in \Delta^{m-1} \times \mathfrak{R}$ and $T = T_{(\lambda,\pi)}$, there exists at most one measure in $\Delta_\Phi(T)$ which is absolutely continuous with respect to the Lebesgue measure.*

Proof. Let us denote by m the Lebesgue measure on X. We may suppose that T is uniquely ergodic. Suppose that $\mu \in \Delta_\Phi(T)$ is absolutely continuous with respect to the Lebesgue measure and let $d\mu = f \, dm$. Put

$$\Phi(k, x) = \prod_{i=0}^{k-1} \Phi(T^i(x)), \quad k \geq 0.$$

https://doi.org/10.1017/S0143385702000196 Published online by Cambridge University Press

Then equation (6) can be written as follows:

$$\int_X g \cdot f \circ T \, dm = \int_X g \cdot \Phi \cdot f \, dm,$$

which implies that f satisfies, m-almost everywhere, the equation $f \circ T = \Phi \cdot f$ or, more generally, f satisfies, m-a.e., the equation $f \circ T^k = \Phi(k, \cdot) \cdot f, k \geq 1$. Now, Birkhoff's Theorem implies that, for almost every $x \in X$,

$$\begin{aligned} \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(T^k(x)) &= \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(x)\Phi(k, x) \\ &= \int_0^1 f \, dm = 1 \end{aligned}$$

and therefore,

$$f(x) = \lim_{n\to\infty} \frac{n}{\sum_{k=0}^{n-1} \Phi(k, x)}. \tag{7}$$

That is, the density f does not depend, m-a.e., on the measure μ and therefore f is unique. □

It is proved in [**3**] that the extreme measures of $\Delta_\Phi(T)$ are exactly the ergodic elements of $\Delta_\Phi(T)$ and that, for almost all T, $\Delta_\Phi(T)$ is not empty if Φ is of bounded variation and satisfies $\int_X \Phi \, dm = 0$.

We introduce now *Rauzy's induction process* and the main result from [**10**].

3. *Rauzy's induction process*

Let $\mathcal{G}_m^o$ be the set of irreducible permutations of $\{1, 2, \ldots, m\}$, i.e. those permutations which do not have any invariant subset of the form $\{1, 2, \ldots, k\}$, $k < m$. In this work we want to consider uniquely ergodic interval exchanges, therefore we restrict our permutations to the set $\mathcal{G}_m^o$. In fact, if $\pi \notin \mathcal{G}_m^o$, the interval exchange $T = T_{(\lambda,\pi)}$, associated to a pair (λ, π), can be decomposed in several independent interval exchanges.

Let $\pi \in \mathcal{G}_m^o$. Define a new permutation $\sigma = \sigma(\pi)$ on the set $\{0, 1, 2, \ldots, m\}$ as follows. Set $\pi(0) := 0$ and $\pi(m + 1) := m + 1$ and let

$$\sigma(j) := \pi^{-1}(\pi(j) + 1) - 1, \quad 0 \leq j \leq m.$$

Let $\Sigma(\pi)$ be the set of periodic orbits of σ. To each $S \in \Sigma(\pi)$ assign the vector $b_S \in \mathbb{R}^m$ whose components are given by (see 2.9 in [**10**]),

$$b_S^j = \chi_S^{j-1} - \chi_S^j, \quad 1 \leq j \leq m \tag{8}$$

where

$$\chi_S^j = \begin{cases} 1, & \text{if } j \in S \\ 0, & \text{otherwise.} \end{cases}$$

https://doi.org/10.1017/S0143385702000196 Published online by Cambridge University Press

Consider now two maps $a, b : \mathcal{G}_m^o \to \mathcal{G}_m^o$ defined by

$$\begin{aligned} a\pi(j) &= \begin{cases} \pi(j), & j \le \pi^{-1}(m) \\ \pi(m), & j = \pi^{-1}(m)+1 \\ \pi(j-1), & \text{otherwise} \end{cases} \\ b\pi(j) &= \begin{cases} \pi(j), & \pi(j) \le \pi(m) \\ \pi(j)+1, & \pi(m) < \pi(j) < m \\ \pi(m)+1, & \pi(j) = m. \end{cases} \end{aligned} \tag{9}$$

These maps are one-to-one and generate a group of maps of $\mathcal{G}_m^o$. Each orbit, $\mathfrak{R} \subset \mathcal{G}_m^o$, of this group is called a *Rauzy class*. To each $(\lambda, \pi) \in \Delta^{m-1} \times \mathcal{G}_m^o$ and $T = T_{(\lambda,\pi)}$, we will associate the $m \times m$ matrix $A(T)$ given by

$$A(T) = \begin{cases} A(\pi, a), & \text{if } \lambda_m < \lambda_{\pi^{-1}(m)} \\ A(\pi, b), & \text{if } \lambda_m > \lambda_{\pi^{-1}(m)} \end{cases} \tag{10}$$

where the matrices $A(\pi, a)$ and $A(\pi, b)$ are given:

$$A(\pi, a) = \left(\begin{array}{c|cccccc} & 0 & 0 & \dots & \dots & \dots & 0 \\ I_{\pi^{-1}(m)} & \vdots & & \dots & \dots & \dots & \vdots \\ & 1 & 0 & \dots & \dots & \dots & 0 \\ \hline & 0 & 1 & 0 & \dots & \dots & 0 \\ & 0 & 0 & 1 & 0 & \dots & 0 \\ & \vdots & \vdots & \ddots & \ddots & \dots & \vdots \\ 0 & 0 & 0 & \dots & \ddots & 1 & 0 \\ & 0 & 0 & \dots & \dots & 0 & 1 \\ & 1 & 0 & \dots & \dots & 0 & 0 \end{array}\right) \tag{11}$$

$$A(\pi, b) = \left(\begin{array}{ccccc|c} & & I_{m-1} & & & 0 \\ & & & & & \vdots \\ & & & & & 0 \\ \hline 0 & \dots & 1 & \dots & 0 & 1 \end{array}\right)$$

(the '1' in the last file of the matrix $A(\pi, b)$ occurred at the $(\pi^{-1}(m))$th position). We can check that $\det A(\pi, c) = \pm 1$, $c = a$ or b.

Rauzy induction is defined as follows. Let $\mathfrak{R}$ be a Rauzy class and let $T = T_{(\lambda,\pi)}$ be an interval exchange with $\pi \in \mathfrak{R}$. Compare the lengths of $\lambda_{\pi^{-1}(m)}$ and λ_m. Suppose they are different. Let $\nu = \min(\lambda_m, \lambda_{\pi^{-1}(m)})$. Consider then the Poincaré induced map T_J of T on the interval $J = [0, 1-\nu)$. According to [**9**], T_J is an interval exchange of m sub-intervals corresponding to a pair (λ', π') given by

$$(\lambda', \pi') = \begin{cases} (A^{-1}(\pi, a)\lambda, a(\pi)), & \lambda_m < \lambda_{\pi^{-1}(m)} \\ (A^{-1}(\pi, b)\lambda, b(\pi)), & \lambda_m > \lambda_{\pi^{-1}(m)}. \end{cases} \tag{12}$$

https://doi.org/10.1017/S0143385702000196 Published online by Cambridge University Press

Re-scaling the vector λ' we get the transformation $\mathcal{T} : \Delta^{m-1} \times \mathfrak{R} \to \Delta^{m-1} \times \mathfrak{R}$ given by

$$(\lambda, \pi) \mapsto \left(\frac{\lambda'}{|\lambda'|_1}, \pi' \right),$$

where $|\cdot|_1$ represents the L^1-norm of $\mathbb{R}^m$. Note that $\mathcal{T}$ is not defined on the set of null measure

$$\{\lambda \in \Delta^{m-1} : \lambda_m = \lambda_{\pi^{-1}(m)}, \pi \in \mathfrak{R}\}.$$

Veech [**10**] proved the following theorem.

THEOREM 3.1. *The map $\mathcal{T}$ is conservative and ergodic (with respect to the Lebesgue measure) and there exists a unique (up to a scalar multiple) absolutely continuous invariant measure. This measure is infinite.*

From this result Veech was able to prove that almost every interval exchange is uniquely ergodic.

When studying the suspension of interval exchanges to flows on compact surfaces without boundary, Veech ([**10**]) introduced the skew-symmetric matrix L^π, $\pi \in \mathfrak{G}_m$,

$$L^\pi_{ij} := \begin{cases} +1, & \text{if } i < j \text{ and } \pi(i) > \pi(j) \\ -1, & \text{if } i > j \text{ and } \pi(i) < \pi(j) \\ 0, & \text{otherwise.} \end{cases} \tag{13}$$

It is not hard to see (see [**10**]) that $(L^\pi \lambda)_i = \sum_{k=1}^{\pi i - 1} \lambda_k^\pi - \sum_{k=1}^{i-1} \lambda_k = a^\pi_{\pi(i)-1} - a_{i-1}$ and then the (λ, π)-interval exchange T is given by

$$Tx = x + (L^\pi \lambda)_i, \quad x \in [a_{i-1}, a_i), i = 1, \ldots, m. \tag{14}$$

The matrix L^π plays an important role in the ergodic properties of interval exchanges (see [**13**]). It has been shown in [**10**] that the set $\{b_S : S \in \Sigma(\pi), 0 \notin S\}$ is linearly independent and forms a basis for the kernel of L^π. It is also proved that L^π satisfies the equation

$${}^{\mathrm{t}}A(T) L^\pi = L^\pi A(T)^{-1}, \quad ({}^{\mathrm{t}}A(T) \text{ is the transpose of } A(T)). \tag{15}$$

3.1. *Zorich's induction process.* Since the operator $\mathcal{T}$ has an infinite invariant measure, it is not possible to apply Oseledet's Theorem to the co-cycle $A(T)$ defined by equation (10). Zorich [**15**] has defined and studied the following map $\mathcal{G}$ on the space of interval exchange transformations for which there is an absolutely continuous invariant 'probability' measure.

Fix $\pi \in \mathcal{G}_m^o$ and let $\mathfrak{R} = \mathfrak{R}(\pi)$ be the Rauzy class of π. Using π we divide Δ^{m-1} into

$$\Delta^+(\pi) = \{\lambda \in \Delta^{m-1} : \lambda_m > \lambda_{\pi^{-1}(m)}\}$$
$$\Delta^-(\pi) = \{\lambda \in \Delta^{m-1} : \lambda_m < \lambda_{\pi^{-1}(m)}\}.$$

By Veech's Theorem (3.1) almost every orbit of $\mathcal{T}$ is dense in $\Delta^{m-1} \times \mathfrak{R}$. Set $(\lambda^{(k)}, \pi^{(k)}) = \mathcal{T}^k(\lambda, \pi)$. For almost every $(\lambda, \pi) \in \Delta^{m-1} \times \mathfrak{R}$, we define

$$n(\lambda, \pi) = \min_{k \geq 1} k \text{ such that } \begin{cases} \lambda^{(k)} \in \Delta^-(\pi^{(k)}), & \text{when } \lambda \in \Delta^-(\pi) \\ \lambda^{(k)} \in \Delta^+(\pi^{(k)}), & \text{when } \lambda \in \Delta^+(\pi). \end{cases}$$

https://doi.org/10.1017/S0143385702000196 Published online by Cambridge University Press

Zorich's map $\mathcal{G} : \Delta^{m-1} \times \mathfrak{R} \to \Delta^{m-1} \times \mathfrak{R}$ is defined by

$$\mathcal{G}(\lambda, \pi) := \mathcal{T}^{n(\lambda,\pi)}(\lambda, \pi). \tag{16}$$

Zorich [**15**] has proved the following theorem.

THEOREM 3.2. *Let $m > 1$ and let $\mathfrak{R}$ be a Rauzy class. The map*

$$\mathcal{G} : \Delta^{m-1} \times \mathfrak{R} \to \Delta^{m-1} \times \mathfrak{R}$$

is conservative, ergodic and admits an absolutely continuous invariant probability measure μ.

For almost every $(\lambda, \pi) \in \Delta^{m-1} \times \mathfrak{R}$ and $T = T_{(\lambda,\pi)}$, consider the matrix-value function

$$B(T) := A(T)A(\mathcal{T}(T)) \dots A(\mathcal{T}^{n(\lambda,\pi)}(T)) \tag{17}$$

where $A(T)$ is defined in (10) and (11). It is proved in [**15**] that $\log \|B^{-1}(T)\|$ is integrable with respect to μ and therefore we can apply Oseledet's Theorem to the co-cycle

$$B(k, T) := B(T)B(\mathcal{G}(T)) \dots B(\mathcal{G}^{k-1}(T)).$$

Let $\theta_1 \geq \theta_2 \geq \cdots \geq \theta_m$ be the Lyapunov exponents of the co-cycle $B(\cdot, \cdot)$ and let $K(\pi)$ be the kernel of the matrix L^π defined in (13). Let $N(\pi)$ be the dimension of $K(\pi)$ and

$$g = \frac{m - N(\pi) + 1}{2}.$$

In [**10**] it is proved that g is an integer number.

THEOREM 3.3. *The middle $m - 2g$ Lyapunov exponents are equal to zero:*

$$\theta_{g+1} = \theta_{g+2} = \dots \theta_{m-g} = 0.$$

The remaining $2g$ Lyapunov exponents are distributed in pairs:

$$\theta_k = -\theta_{m-k+1}, \quad k = 1, 2, \dots, g.$$

The first Lyapunov exponent θ_1 is strictly greater than the second one:

$$\theta_1 > \theta_2.$$

Zorich has also made the following conjecture [**15**].

CONJECTURE 1.

(a) *The top g Lyapunov exponents are strictly positive.*
(b) *The top g Lyapunov exponents are distinct.*

As we mentioned before, this conjecture has been verified for numerous Rauzy classes. In particular, G. Forni has recently announced part (a) of this conjecture [**16**].

We will consider the map $\mathcal{G}$ as acting on the space of all interval exchanges. Let us put $T_n = \mathcal{G}^n(T)$, $n \geq 0$. Observe that T_n is the re-scaling of the Poincaré induced map of T

https://doi.org/10.1017/S0143385702000196 Published online by Cambridge University Press

on a interval J_n of the form $[0, a_n)$, $a_n > 0$. By the definition of $B(n, T)$ we have that if $T_n = T_{(\lambda', \pi')}$ then

$$\lambda' = \frac{B^{-1}(n, T) \cdot \lambda}{|B^{-1}(n, T) \cdot \lambda|_1}, \quad n \geq 0,$$

where $|\cdot|_1$ is the L^1-norm of $\mathbb{R}^m$.

Let $F \in S_{\text{aff}}(T, w)$ and let $h : X \to X$ be a semi-conjugacy between F and T. We can easily check that the Poincaré induced map F_n of F on the interval $h^{-1}(J_n)$ is an affine interval exchange semi-conjugated to T_n. Moreover, the slope-vector w_n of F_n satisfies the following relation:

$$\log w_n = \log w \cdot B(n, T), \quad n \geq 0,$$

that is, the co-cycle ${}^tB(n, T)$ transports the slope-vectors of the F_ns. Here ${}^tB(n, T)$ is the transpose matrix of $B(n, T)$.

The co-cycle ${}^tB(n, T)$ is called the 'dual' co-cycle with respect to the co-cycle $B^{-1}(n, T)$. It is proved in [**14**] that ${}^tB(n, T)$ has the same Lyapunov exponents as $B^{-1}(n, T)$. It is also proved in [**14**] that for almost all $(\lambda, \pi) \in \Delta^{m-1} \times \mathfrak{R}$ and $T = T_{(\lambda,\pi)}$,

$$\lim_{n\to\infty} \frac{1}{n} \log \|B^{-1}(n, T)\lambda\| = -\theta_1,$$

that is, the direction λ always corresponds to the minimum Lyapunov exponent of the co-cycle $B^{-1}(k, T)$.

Let $\theta_1 \geq \theta_2 \geq \cdots \geq \theta_m$ be the Lyapunov exponents of the co-cycle $B^{-1}(k, T)$, $k \geq 1$. Let us denote by E_i the set of $\gamma \in \mathbb{R}^m$ such that

$$\lim_{n\to\infty} \frac{1}{n} \log \|{}^tB(n, T)\gamma\| \leq \theta_i$$

and by F_i the set of $\gamma \in \mathbb{R}^m$ such that

$$\lim_{n\to\infty} \frac{1}{n} \log \|B^{-1}(n, T)\gamma\| \leq \theta_i$$

$i = 1, 2, \ldots, m$. Set $E_0 = F_0 = \emptyset$. It is proved in [**14**] that

$$E_i = F_{m-i}^{\perp}, \tag{18}$$

i.e. E_i is the orthogonal complement of F_{m-i}, $i = 0, 1, \ldots, m$. It is also proved that F_1 is a one-dimensional subspace spanned by the vector λ.

Let $T = T_{(\lambda,\pi)}$, where $(\lambda, \pi) \in \Delta^{m-1} \times \mathfrak{R}$, and denote by

$$\theta(\gamma, T) = \lim_{n\to\infty} \frac{1}{n} \log \|{}^tB(n, T) \cdot \gamma\|$$

the Lyapunov exponent of $\gamma \in \mathbb{R}^m$. It is not hard to see that

$$H = \{\gamma \in \mathbb{R}^m : \theta(\gamma, T) < \theta_1\}$$

forms the orthogonal complement of λ. Denote by $E^{ss}(T) = \{\gamma \in E : \theta(\gamma, T) = -\theta_1\}$ and let

$$E^s(T) = \{\gamma \in H : \theta(\gamma, T) < 0\}$$
$$E^c(T) = \{\gamma \in H : \theta(\gamma, T) = 0\}$$
$$E^u(T) = \{\gamma \in H : 0 < \theta(\gamma, T) < \theta_1\}.$$

https://doi.org/10.1017/S0143385702000196 Published online by Cambridge University Press

We will call E^s the 'stable space', E^c the 'central space' and E^u the 'unstable space'. E^{ss} will be called the strong stable space.

Hereafter we will suppose that $\mathfrak{R} \subset \mathcal{G}_m^o$ is a Rauzy class where part (a) of Conjecture 1 holds. In particular, $E_g = E^s$ and $E^s \cup E^c = E_{m-g}$. Also $K(\pi) = F^c$ and $F_g \cup K(\pi) = F_{m-g}$.

4. *Proof of Theorem 1*

This section is devoted to the proof of Theorem 1. The sets E_1, E_2 and E_3 of Theorem 1 can be given as follows:

(1) E_1 is equal to E^{ss};
(2) E_2 is equal to E^s;
(3) E_3 is all of $\lambda^\perp$. Thus, $E_3 \backslash E_2 = E^c \cup E^u$.

For $w \in \mathbb{R}_+^m$ set

$$\phi_w(x) = w_i, \quad x \in [a_{i-1}, a_i), i = 1, 2, \ldots, m. \tag{19}$$

To simplify notation if $\gamma = \log w$ we put $\Delta_{\Phi_w}(T) = \Delta_\gamma(T)$. In view of Proposition 2.1, we will study the set $\Delta_\gamma(T)$ for γ in $E^{ss}(T)$, $E^s(T)$, $E^c(T)$ or $E^u(T)$. We will prove in Proposition 4.1 that if $\gamma \in E^{ss}(T)$ then there is an affine interval exchange which is C^∞-conjugated to T and with a slope-vector w. In Proposition 4.3 we will prove that if $\gamma \in E^s(T)$ then there is a unique measure in $\Delta_\gamma(T)$ and this measure is absolutely continuous with respect to the Lebesgue measure. These are parts (a) and (b) of Theorem 1. In Propositions 4.4 and 4.5 we will prove that if $\gamma \in E^c(T)$ or $\gamma \in E^u(T)$, then there is no absolutely continuous (with respect to the Lebesgue measure) measure in $\Delta_\gamma(T)$. This is part (c) of Theorem 1.

4.1. *The stable space E^s.*

From equation (15) we obtain that ${}^tB(k, T)L^\pi B(k, T) = L^\pi$ and therefore

$$\begin{aligned}\lim_{k\to\infty} \frac{1}{k} \log \|{}^tB(k, T)L^\pi \lambda\| &= \lim_{k\to\infty} \frac{1}{k} \log \|L^\pi B^{-1}(k, T)\lambda\| \\ &= \lim_{k\to\infty} \frac{1}{k} \log \|B^{-1}(k, T)\lambda\| = -\theta_1 < 0\end{aligned}$$

where $-\theta_1$ is the minimum Lyapunov exponent of the co-cycle $B^{-1}(k, T)$. Thus, $L^\pi \lambda \in E^{ss}$. Since F_1 is one-dimensional, E^{ss} is one-dimensional and hence generated by the vector $L^\pi \lambda$.

From the relation $Tx = x + (L^\pi \lambda)_i, x \in [a_{i-1}, a_i), i = 1, \ldots, m$, we can see that $\xi(x) = e^x$ satisfies the relation

$$\xi(T(x)) = e^{(L^\pi \lambda)_i} \xi(x), \quad x \in [a_{i-1}, a_i). \tag{20}$$

For all $t \in \mathbb{R}, t \neq 0$, set $A^{-1} = \int_X \xi^t \, dm$ and

$$h(x) = A \int_0^x \xi^t \, dm = \frac{e^{tx} - 1}{e^t - 1}, \quad x \in X.$$

Then we can easily see that $f = h \circ T \circ h^{-1}$ is the following affine interval exchange:

$$f(x) = e^{(tL^\pi \lambda)_i} x + \frac{e^{(tL^\pi \lambda)_i} - 1}{e^t - 1}, \quad x \in h([a_{i-1}, a_i)), i = 1, 2, \ldots, m. \tag{21}$$

https://doi.org/10.1017/S0143385702000196 Published online by Cambridge University Press

Thus, we have the following.

PROPOSITION 4.1. *For almost all* $(\lambda, \pi) \in \Delta^{m-1} \times \mathfrak{R}$ *and* $T = T_{(\lambda,\pi)}$, *if* $\log w = t(L^{\pi}\lambda), t \neq 0$, *the map* f *defined in (21) is an affine interval exchange with slope-vector* w *which is* C^{∞}*-conjugated to* T.

Let T be an interval exchange associated to the partition $0 = a_0 < a_1 < a_2 < \cdots < a_{m-1} < a_m = 1$ of X. For each $\gamma = (\gamma_1, \ldots, \gamma_m) \in \mathbb{R}^m$, we define $\phi_\gamma : X \to \mathbb{R}$ by $\phi_\gamma(x) = e^{\gamma_i}, x \in [a_{i-1}, a_i), i = 1, 2, \ldots, m$. Let $\Phi_\gamma : \mathbb{Z} \times X \to \mathbb{R}$ be defined by $\Phi(0, x) \equiv 1$ and

$$\Phi_\gamma(k, x) = \begin{cases} \prod_{j=0}^{k-1} \phi_\gamma(T^j(x)), & k \geq 1 \\ \prod_{j=1}^{k} \phi_\gamma^{-1}(T^{-j}(x)), & k \leq -1. \end{cases} \tag{22}$$

Observe that $\Phi_{a\gamma} = \Phi_\gamma^a, a \in \mathbb{R}$ and Φ_γ satisfies

$$\Phi_\gamma(k + m, x) = \Phi_\gamma(k, x)\Phi_\gamma(m, T^k(x)), \quad k, m \in \mathbb{Z}, x \in X,$$

i.e. Φ_γ is a multiplicative co-cycle.

From now on, $\mathfrak{R}$ will denote an arbitrary Rauzy class of $\mathcal{G}_m$.

Consider the following proposition from [**1**].

PROPOSITION 4.2. *For almost all* $(\lambda, \pi) \in \Delta^{m-1} \times \mathfrak{R}$ *and* $T = T_{(\lambda,\pi)}$, *and for every* $\gamma \in E^s(T)$ *there exists a constant* $M > 0$ *such that*

$$M^{-1} \leq \Phi_\gamma(n, x) \leq M, \quad n \in \mathbb{Z}, \quad \textit{for all } x \in X.$$

PROPOSITION 4.3. *For almost all* $(\lambda, \pi) \in \Delta^{m-1} \times \mathfrak{R}$ *and* $T = T_{(\lambda,\pi)}$, *and for every* $\gamma \in E^s(T)$ *there is a unique measure in* $\Delta\gamma(T)$. *This measure is equivalent to the Lebesgue measure.*

Proof. We may suppose that $T = T_{(\lambda,\pi)}$ is uniquely ergodic. By Proposition 4.2, the function $\varphi(x) = \sup_{n\in\mathbb{Z}} \Phi_\gamma(n, x)^{-1}$ is well defined and it satisfies $\varphi(T(x)) = \Phi_\gamma(1, x)\varphi(x)$. Therefore, the measure $\mu = \int \varphi \, dm$ satisfies equation (4). Normalizing μ we obtain an absolutely continuous element of $\Delta\gamma(T)$. Now we will prove that every measure $\mu \in \Delta\gamma(T)$ is equivalent to a Lebesgue measure and then, by Lemma 2.2, that there is a unique measure in $\Delta\gamma(T)$.

Observe first that any probability measure $\mu \in \Delta\gamma(T)$ is continuous. In fact, the presence of an atom x_0 implies that the series $\sum_{k\geq 0} \Phi(k, x_0)$ is finite which is a contradiction to Proposition 4.2.

Let $\{J_n\}_{n\geq 1}$ be the sequence of sub-intervals of X associated with Zorich's induction and let μ be a probability measure in $\Delta\gamma(T)$. For $n \geq 1$ consider the Poincaré induced map T_n' of T on J_n. Remember that $T_n := \mathcal{G}^n(T)$ is the same map T_n' but re-scaled to the whole interval X. Let μ_n' be the restriction of the measure μ to J_n and let μ_n be the probability measure on X given by

$$\mu_n(D) = \mu_n'\left(\frac{1}{\mu(J_n)} D\right), \quad \text{for all the Borelian set } D \subset X. \tag{23}$$

https://doi.org/10.1017/S0143385702000196 Published online by Cambridge University Press

Let $\phi_n = \phi_{\gamma_n}$ be the corresponding function associated to the vector

$$\gamma_n = {}^{\mathrm{t}}B(n, T) \cdot \gamma$$

as in equation (22). It is not hard to see that $\mu_n \in \Delta\gamma_n(T_n)$, that is, μ_n satisfies the equation

$$\int_X g \circ T_n^{-1}\, d\mu_n = \int_X g \cdot \Phi_n\, d\mu_n, \quad \text{for all } g \in \mathrm{L}^1(\mu_n). \tag{24}$$

CLAIM. *The only weak* limit point of the sequence μ_n is the Lebesgue measure m, therefore, the whole sequence μ_n converges, in this topology, to m.*

Proof of the claim. Let $\tilde{\mu} = \lim_{j\to\infty} \mu_{n_j}$ be a weak* limit point of the sequence μ_n. We may suppose, taking a sub-sequence if necessary, that the sequence T_{n_j} converges, in the strong topology, to some uniquely ergodic interval exchange $\tilde{T}$. As $\gamma \in E^s(T)$, the sequence $\gamma_n \to 0$ when $n \to \infty$, and then $\Phi_n \to 1$. Then the Dominated Convergence Theorem implies that

$$\int_X g \circ \tilde{T}^{-1}\, d\tilde{\mu} = \int_X g\, d\tilde{\mu},$$

for every measurable bounded function g of X. Therefore $\tilde{\mu}$ is an invariant measure for $\tilde{T}$, thus $\tilde{\mu} = m$. □

Let J_n^k, $1 \le k \le m$ be the sub-intervals of continuity of $T_n' : J_n \to J_n$ and let $r(k)$ be the return time to J_n of the sub-interval J_n^k, i.e. $r(k)$ satisfies

$$T^j(J_n^k) \cap J_n = \emptyset, \quad j = 1, 2, \ldots, r(k) \quad \text{and} \quad T^{r(k)+1}(J_n^k) \subset J_n. \tag{25}$$

Observe that, by definition, the functions $\Phi_\gamma(j, x)$, $j = 1, 2, \ldots, r(k)$ are constant on each J_n^k. Let us set $\Phi = \Phi_\gamma$ and denote by $\Phi_{j,k}$ the constant value of $\Phi_\gamma(j, x)$ at J_n^k, $1 \le k \le m$, $1 \le j \le r(k)$. Since the whole space X can be decomposed as the disjoint union

$$X = \bigcup_{k=1}^{m} \bigcup_{j=0}^{r(k)} T^j(J_n^k)$$

we have

$$1 = \mu(X) = \sum_{k=1}^{m} \sum_{j=0}^{r(k)} \mu_n'(J_n^k)\Phi_{j,k}.$$

From this, using Proposition 4.2 and the fact that $1 = m(X) = \sum_{k=1}^{m} \sum_{j=0}^{r(k)} |J_n^k|$, we obtain that

$$\frac{1}{M} \le \frac{\mu(J_n^k)}{|J_n^k|} \le M, \quad \text{for all } n \text{ and } 1 \le k \le m. \tag{26}$$

In particular, $M^{-1}|J_n| \le \mu(J_n) \le M|J_n|$, for all n. Now since $\mu_n \to m$, the restriction μ_n' of μ to the interval J_n, weak*-converges to the measure $\mu(J_n) \cdot m$ in J_n. Therefore, there is a constant K and $n_0 > 0$ such that, for all $n \ge n_0$ and any subset $D \subset J_n$, we have that

$$K^{-1}m(D) \le \mu_n'(D) \le Km(D). \tag{27}$$

https://doi.org/10.1017/S0143385702000196 Published online by Cambridge University Press

Observe that any subset $C \subset X$ can be written in the form $C = \bigcup_{k=1}^{m} \bigcup_{j=0}^{r(k)} C_{j,k}$ where $C_{j,k} \subset T^j(J_n^k)$. Set $D_{j,k} = T^{-j}(C_{j,k}) \subset J_n^k$. Clearly $m(D_{j,k}) = m(C_{j,k})$. Finally, Proposition 4.2, equation (27) and the relations

$$\mu(C) = \sum_{k=1}^{m} \sum_{j=0}^{r(k)} \mu_n'(D_{j,k})\Phi_{j,k}$$

and

$$m(C) = \sum_{k=1}^{m} \sum_{j=0}^{r(k)} m(C_{j,k}),$$

imply that

$$(MK)^{-1} m(C) \leq \mu(C) \leq (MK) m(C),$$

that is, μ is equivalent to m. □

4.2. *The central space E^c.*

PROPOSITION 4.4. *For almost all $(\lambda, \pi) \in \Delta^{m-1} \times \mathfrak{R}$ and $T = T_{(\lambda,\pi)}$, if $\gamma \in E^c(T)$ then $\Delta_{\Phi_\gamma}(T)$ does not contain any probability measure which is absolutely continuous with respect to the Lebesgue measure.*

Proof. It is proved in [**11**, Theorem 6.3] that if the following equation

$$f(T(x)) = e^{2\pi w_k i} f(x), \quad x \in [a_{k-1}, a_k),\ k = 1, \ldots, m \tag{28}$$

has a measurable solution $\tilde{\varphi} : X \to \mathbb{S}^1$ for the vector $w = (w_1, \ldots, w_m) \in \mathbb{R}^m$ then the inner product $\langle w, b_S \rangle$ is an integer for all $S \in \Sigma(\pi)$ (see equation (8)). Note that if $\varphi : \mathbb{R} \to \mathbb{R}$ is a positive solution to the equation

$$f(T(x)) = e^{w_k} f(x), \quad x \in [a_{k-1}, a_k),\ k = 1, \ldots, m, \tag{29}$$

and if $g : X \to \mathbb{R}$ is the function $g(x) = \log \varphi(x)$, then for all $0 < \alpha < 1$, the function $\tilde{\varphi}(x) = e^{2\pi\alpha g(x) i}$ is a solution to equation (28) for the vector $\alpha w \in \mathbb{R}^m$. This is only possible if w is orthogonal to the set $\{b_S : S \in \Sigma(\pi)\}$. As this set forms a basis for $K(\pi)$, this implies that w is orthogonal to $K(\pi)$.

By relations (18), E_{m-g} is orthogonal to F_g and E_g is orthogonal to F_{m-g}. Recall that we are assuming part (a) of Conjecture 1 which implies that $E^s \cup E^c = E_{m-g}$ and $K(\pi) \cup F_g = F_{m-g}$. Therefore, $E^c \perp F_g$. Hence as $E^c \cap E^s = \{0\}$, no vector of E^c can be orthogonal to $K(\pi)$. Indeed, if $v \in E^c$ is orthogonal to $K(\pi)$, then it belongs to $F_{m-g}^{\perp}$ (because $E^c \perp F_g$), and hence $g = \dim F_{m-g}^{\perp} = \dim E_g + 1 = g + 1$, a contradiction. Thus, if $\log w = \gamma \in E^c$ there are no solutions to equation (29) and therefore there are no measures in $\Delta_\gamma(T)$ which are absolutely continuous measures with respect to the Lebesgue measure (otherwise the Radon–Nikodym density of this measure would be a solution to equation (29)). □

https://doi.org/10.1017/S0143385702000196 Published online by Cambridge University Press

4.3. *The unstable space E^u.* In this section we will prove the following proposition.

PROPOSITION 4.5. *For almost all $(\lambda, \pi) \in \Delta^{m-1} \times \Re$ and $T = T_{(\lambda,\pi)}$, if $\gamma \in E^u(T)$ then $\Delta_{\Phi_\gamma}(T)$ does not contain any probability measure absolutely continuous with respect to the Lebesgue measure m.*

Proof. Suppose, by contradiction, that $\mu \in \Delta_{\Phi_\gamma}(T)$ is absolutely continuous with respect to m. Write $\mu = \int \varphi \, dm$. Hence from Lemma 2.2 we know that φ satisfies the relations

$$\varphi(T^k x) = \Phi_\gamma(k, x)\varphi(x), \quad x \in X, \ k \geq 1.$$

Using Birkhoff's Ergodic Theorem we obtain

$$\begin{aligned}\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi(T^k(x)) &= \varphi(x) \cdot \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \Phi_\gamma(k, x) \\ &= \int_0^1 \varphi \, dm = 1\end{aligned}$$

and, therefore,

$$\varphi(x)^{-1} = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \Phi_\gamma(k, x). \tag{30}$$

This implies, in particular, that for almost every point x the quotients $\Phi_\gamma(k, x)/k$ remain bounded when k goes to infinity. We will now prove that this is impossible when $\theta(\gamma, T) > 0$, and therefore, such a μ cannot exist.

LEMMA 4.6. *For almost all T and $\gamma \in E^u(T)$, the invariant set*

$$G(T) = \left\{ x \in [0, 1) : \limsup_{n\to\infty} \frac{\Phi_\gamma(n, x)}{n} = \infty \right\}$$

has Lebesgue measure one.

Before we can prove this lemma, we need to introduce the following results from [**11**].

Take $\epsilon > 0$ and for each $(\lambda, \pi) \in \Delta^{m-1} \times \Re$, define the properties $P_1(\epsilon, n)$ and $P_2(\epsilon, n)$ (related to $T = T_{(\lambda,\pi)}$) in the following way. Let J_n be the sub-interval of induction associated to $T_n = \mathcal{G}^n(T)$. Let T_n' be the induced map of T on J_n. Then:

- $P_1(\epsilon, n)$ is said to hold for T if there exists an $N \geq \epsilon/|J_n|$ such that $a_k \notin T^j(J_n)$, $0 \leq j < N$ and a_k, $k = 1, 2, \ldots, m$ are the points of the partition associated to T;
- $P_2(\epsilon, n)$ is said to hold if $|J_n^k| \geq \epsilon |J_n|$ where J_n^k, $k = 1, \ldots, m$ are the sub-intervals of continuity of $T_n' : J_n \to J_n$.

Clearly property $P_1(\epsilon, n)$ implies that each $T^j(J_n)$ is an interval $(0 \leq j < N)$ and that $\left| \bigcup_{j=0}^N T^j(J_n) \right| \geq \epsilon$. Observe that if $P_1(\epsilon, n)$ and $P_2(\epsilon, n)$ hold simultaneously for T, then

$$\left| \bigcup_{j=0}^{N} T^j(J_n^k) \right| \geq \epsilon^2, \quad 1 \leq k \leq m. \tag{31}$$

Consider now the following lemma from [**11**].

https://doi.org/10.1017/S0143385702000196 Published online by Cambridge University Press

LEMMA 4.7. *For almost all* $(\lambda, \pi) \in \Delta^{m-1} \times \mathfrak{R}$, *there exist an* $\epsilon = \epsilon(\lambda, \pi) > 0$ *and an infinite set* $E = E(\lambda, \pi, \epsilon)$ *of integers such that* $T_{(\lambda,\pi)}$ *satisfies* $P_1(\epsilon, n)$ *and* $P_2(\epsilon, n)$ *for all* $n \in E$.

Remark. In the proof of Lemma 4.7 the basic tool is Theorem 3.1 related to Rauzy's operator $\mathcal{T}$; this proof can be easily adapted Zorich's induction operator $\mathcal{G}$ in a straightforward way using the corresponding Theorem 3.2.

Consider now the following lemma from [**14**].

LEMMA 4.8. *For almost all* $(\lambda, \pi) \in \Delta^{m-1} \times \mathfrak{R}$ *and* $T = T_{(\lambda,\pi)}$, *we have* (*using the* L^1*-norm,* $|\cdot|_1$, *of* $\mathbb{R}^m$)

$$\lim_{k\to\infty} \frac{1}{k} \log(|B(k, T) \cdot e_j|_1) = \theta_1(\mathfrak{R}), \quad 1 \le j \le m.$$

Proof of Lemma 4.6. We may suppose that m is the unique ergodic measure of T. As the set $G(T)$ is T-invariant, it has Lebesgue measure zero or one. Fix $\gamma \in E^u(T)$ and suppose, by contradiction, that $m(G(T)) = 0$. Egoroff's Theorem implies that for every $\delta > 0$ there exist $M_\delta > 0$ and a set B_δ with $m(B_\delta) > 1 - \delta$ such that

$$\frac{1}{n}\Phi_\gamma(n, x) \le M_\delta, \quad \text{for all } x \in B_\delta \tag{32}$$

and for every n greater than some integer $n_0 > 0$. Let $\epsilon > 0$ and let $E = E(\lambda, \pi, \epsilon)$ be the set of integers of Lemma 4.7.

Take $0 < \delta < \epsilon^2/2$ and M_δ, B_δ as in (32). Let $J_n, n \ge 1$ be one of the intervals in Zorich's induction process $\mathcal{G}$. Hereafter we will suppose that $n \in E(\lambda, \pi, \epsilon)$. Let $T_n = \mathcal{G}^n(T)$ be the interval exchange associated to the pair (λ_n, π_n). Remember that T_n is the re-scaling of T'_n, the Poincaré induced map of T on J_n. Let J_n^k, $1 \le k \le m$ be the sub-intervals of continuity of $T'_n : J_n \to J_n$. T'_n is associated to the pair (λ'_n, π_n) where

$$\lambda'_n = (|J_n^1|, |J_n^2|, \ldots, |J_n^m|).$$

Set $\gamma_n = {}^tB(n, T) \cdot \gamma$. Since $\gamma \in E^u(T)$ we have (using the L^1-norm, $|\cdot|_1$, of $\mathbb{R}^m$),

$$\theta(\gamma, T) = \lim_{n\to\infty} \frac{1}{n} \log |\gamma_n|_1 = \theta > 0,$$

and, therefore, there exist $1 < \eta < e^\theta$ and an integer $n_1 > 0$ such that

$$|\gamma_n|_1 \ge \eta^n, \quad n \ge n_1.$$

As γ_n is orthogonal to λ'_n and $\lambda'_n = (\lambda_n^1, \ldots, \lambda_n^m)$ satisfies the relations $(P_2(\epsilon, n))$,

$$\lambda_n^k \ge \epsilon|\lambda_n|_1, \quad 1 \le k \le m,$$

there exists a component $\gamma_n^{k_0} > 0$, $k_0 = k_0(n)$, of $\gamma_n = (\gamma_n^1, \ldots, \gamma_n^m)$ such that

$$\gamma_n^{k_0} \ge \epsilon|\gamma_n|_1 \ge \epsilon\eta^n \quad \text{if } n \ge n_1. \tag{33}$$

Let $r(k, n)$, $1 \le k \le m$ be the return time to J_n for the sub-interval J_n^k. It is easy to see that, for each $1 \le k \le m$, $r(k, n)$ is given by the L^1-norm of the vector $B(n, T) \cdot e_k$. Let θ_1

https://doi.org/10.1017/S0143385702000196 Published online by Cambridge University Press

be the largest Lyapunov exponent of the co-cycle $B(n, T)$. Then Lemma 4.8 implies that, for some $\rho > e^{\theta_1}$ and $n_2 \geq 0$, we have

$$r(k, n) < \rho^n, \quad \text{if } n \geq n_2,\ 1 \leq k \leq m. \tag{34}$$

Observe that, by definition, the functions $\Phi_\gamma(j, x)$, $j = 1, 2, \ldots, r(k, n)$ are constant on each J_n^k. It is easy to see that, in particular,

$$\Phi_\gamma(r(k, n), x) = \exp(\gamma_n^k), \quad \text{for all } x \in J_n^k,\ 1 \leq k \leq m. \tag{35}$$

Set $r_n = r(k_0, n)$. In view of (33), (35) and (34) we have,

$$\frac{\Phi_\gamma(r_n, x)}{r_n} \geq \frac{\exp(\epsilon \cdot \eta^n)}{\rho^n}, \tag{36}$$

for all $x \in J_n^{k_0}$ and n sufficiently large. In particular, if n is sufficiently large, say larger than some $n_3 > 0$, the quotient $\Phi_\gamma(r_n, x)/r_n$, $x \in J_n^{k_0}$ is greater than the constant M_δ defined in equation (32).

Observe now that as $T^j(J_n)$ is an interval for $0 \leq j < N$ (by $P_1(\epsilon, n)$), the induction matrix associated to each of these sub-intervals is equal to $B(n, T)$. Moreover, let $K_j = T^j(J_n)$ be one of these sub-intervals and set $K_j^k = T^j(J_n^k)$, $1 \leq k \leq m$. Then K_j^k is a sub-interval of continuity for the induced map of T_{K_j} on K_j and we can check that T_{K_j} is associated to the same pair (λ_n', π_n) as T_n'. As the value of $\Phi(r_n, x)/r_n, x \in K_j$ depends only on the matrix $B(n, T)$, this function has the same value on each of the sub-intervals $K_j^{k_0}, 0 \leq j < N$.

Therefore, for all integers $n \geq n_3$, there is a set $D_n = \bigcup_{j=0}^{N-1} K_j^{k_0}$ and an integer r_n such that

$$\frac{\Phi(r_n, x)}{r_n} > M_\delta, \quad \text{for all } x \in D_n.$$

Observe that from (31) and the fact that $n \in E(\lambda, \pi, \epsilon)$, the Lebesgue measure of D_n is greater than ϵ^2. However, in view of (32), the set D_n must satisfy

$$D_n \cap B_\delta = \emptyset, \quad n \geq n_3.$$

This contradicts the assumptions $0 < \delta < \epsilon^2/2$ and $m(B_\delta) > 1 - \delta$. This contradiction proves the lemma and Proposition 4.5. □

□

Supported by the example given in Theorem 2 we make the following conjecture.

CONJECTURE 2. *For almost all* $(\lambda, \pi) \in \Delta^{m-1} \times \mathfrak{R}$, $T = T_{(\lambda,\pi)}$ *and for all* $\gamma \in E^u(T)$, *the set* $\Delta_\gamma(T)$ *does not contain any continuous measures, that is, if* $\gamma \in E^u(T)$, $C_{\text{aff}}(T, e^\gamma)$ *is empty.*

5. *Proof of Theorem 2*

We will now study the particular example of Theorem 2. First we will prove part (b) of Theorem 2. The proof of part (a) is essentially the same as that of part (a) of Theorem 1, and will be done in Proposition 5.14.

https://doi.org/10.1017/S0143385702000196 Published online by Cambridge University Press

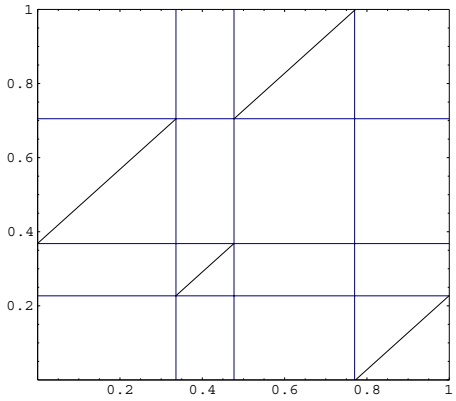


FIGURE 1. Graph of T_0.

Part (b) of Theorem 2 is proved in Theorem 5.1, Corollary 5.12 and Proposition 5.13.

Consider the permutation $\pi \in \mathcal{G}_4^o$ given by $\pi(1) = 3$; $\pi(2) = 2$; $\pi(3) = 4$; $\pi(4) = 1$ and let $\lambda_0 = (\lambda_1, \lambda_2, \lambda_3, \lambda_4) \in \mathbb{R}^4$ be the (normalized) positive Perron–Frobenius eigenvector of the following matrix:

$$A = \begin{pmatrix} 2 & 2 & 1 & 1 \\ 1 & 2 & 0 & 0 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}. \tag{37}$$

We can check that for A^2 all its entries are positive and $\lambda_0 \approx (0.33, 0.14, 0.29, 0.22)$. Let $T_0 = T_{(\lambda_0,\pi_0)}$ be the associated interval exchange transformation (see Figure 1). The eigenvalues of A are real and of the form $\rho^{-1} < \eta^{-1} < 1 < \eta < \rho$. Let ${}^tA$ be the transpose matrix of A and let H (respectively H^s) be the vector 3-space (respectively vector 2-space) ${}^tA$-invariant generated by the eigenvectors corresponding to the eigenvalues ρ^{-1}, η^{-1}, η of ${}^tA$ (respectively ρ^{-1}, η^{-1}). It is easy to see that H is the orthogonal complement of λ. Let $\zeta \approx (-0.33, -0.73, 0.54, 0.24)$ be one eigenvector for η. Observe that H^s divides H into two connected components H_1^u and H_2^u. Suppose that $\zeta \in H_1^u$, and let $z_0 \in X$ be the point

$$z_0 = 1 - \lambda_4(1 + \rho^{-1} + \rho^{-2} + \cdots). \tag{38}$$

THEOREM 5.1. *For every vector γ in H^u there is a unique map $F \in S_F(T, e^\gamma)$. This map is strictly semi-conjugated to T_0, i.e. F has a wandering interval. Moreover, there is a unique wandering interval V for F and the orbit of V is associated (via semi-conjugation) with the orbit of $x = 0$ if $\gamma \in H_1^u$ and with the orbit of $x = z_0$ if $\gamma \in H_2^u$.*

Let $F \in S_F(T, e^\gamma)$ and let h be a semi-conjugacy between F and T. Observe that if $h(y) = x$ then $(F^k)'(y) = \Phi_\gamma(k, x), k \in \mathbb{Z}$ (recall definition (22)) and if F has a wandering interval V, then for each point $y \in V$ and $x = h(y)$, the series

$$\sum_{k=-\infty}^{\infty} (F^k)'(y) = \sum_{k=-\infty}^{\infty} \Phi_\gamma(k, x)$$

https://doi.org/10.1017/S0143385702000196 Published online by Cambridge University Press

is finite. This is due to the fact that $\sum_{k=-\infty}^{\infty} |F^k(V)| < 1$ and $|F^k(V)| = (F^k)'(y)|V|$. Conversely, if there exists a point $x \in X$ such that $\sum_{k\in\mathbb{Z}} \Phi_\gamma(k, x) = R < \infty$, then the atomic probability measure

$$\mu_x = R^{-1} \sum_{k=-\infty}^{\infty} \Phi_\gamma(k, x)\delta_{T^k x} \tag{39}$$

is an element of $\Delta_{\Phi_\gamma}(T)$. Therefore, by Proposition 2.1, there is a map $F \in S_F(T, e^\gamma)$ with a wandering interval V whose orbit is associated, via the semi-conjugacy, to the orbit of x. This will be the approach in the proof of Theorem 5.1. Put $x \sim y$ if y is in the T-orbit of x. We will prove the following.

THEOREM 5.2. *Let $z_0 \in X$ be defined by equation (38) and let Φ be the co-cycle associated to the map T_0. Then for all $\gamma \in H^u$ the following are true:*

(a) *if $\gamma \in H_1^u$, then the series $\sum_{k=-\infty}^{\infty} \Phi_\gamma(k, 0)$ is finite; and*

(b) *if $\gamma \in H_2^u$, then the series $\sum_{k=-\infty}^{\infty} \Phi_\gamma(k, z_0)$ is finite.*

Moreover, if the series $\sum_{k=-\infty}^{\infty} \Phi_\gamma(k, x)$ is finite for some point $x \in X$, then $x \sim 0$ if $\gamma \in H_1^u$ and $x \sim z_0$ if $\gamma \in H_2^u$.

Observe that Theorem 5.1 follows immediately from Theorem 5.2. Theorem 5.2 will be proved in §5.2.

We will now consider a slightly more general situation than that of Theorem 5.2. Let $T = T_{(\lambda,\pi)}$, $(\lambda, \pi) \in \Delta^{m-1} \times \mathfrak{R}$ be an interval exchange which is self-induced in some interval J_1, that is, the Poincaré induced map T_1 of T in J_1 is, up to re-scaling, the same map T. We will suppose that J_1 does not contain any discontinuity points of T or T^{-1}.

Let J_1^k, $1 \le k \le m$ be the sub-intervals of continuity of $T_1 : J_1 \to J_1$ and let $r(k)$ be the return time to J_1 of the sub-interval J_1^k, i.e. $r(k)$ satisfies

$$T^j(J_1^k) \cap J_1 = \emptyset, \quad j = 1, 2, \ldots, r(k) \quad \text{and} \quad T^{r(k)+1}(J_1^k) \subset J_1. \tag{40}$$

Let $B = (B_{ik})$ be the induction matrix

$$B_{ik} = \text{card}\{l : T^l(J_1^k) \subset [a_{i-1}, a_i), l = 0, 1, \ldots, r(k)\}. \tag{41}$$

We will suppose that the eigenvalues of B are all real and will denote by H^s the vector space generated by the eigenvectors associated to all the eigenvalues < 1.

Observe that there is a sub-interval $J_2 \subset J_1$ such that T_1 is also self-induced in J_2 with an associated matrix B. In the same way, T is self-induced in J_2 with an associated matrix B^2. In general, we have a nested sequence $J_n, n \ge 1$ of sub-intervals of X, such that T is self-induced in J_n with induction matrix B^n. Let $\{x_0\} = \cap_{n\ge1} J_n$. x_0 will be call the distinguished point of T.

Let $\zeta = (\zeta_1, \ldots, \zeta_m)$ be an eigenvector of B associated to some eigenvalue $\eta > 1$. Consider the co-cycle $\Phi_\zeta(k, x)$, $(k, x) \in \mathbb{Z}\times X$ defined in (22). Observe that, by definition, $\Phi_\zeta(j, x)$ is constant in J_1^k for each $1 \le k \le m$ and $0 \le j \le r(k)$. Let us denote this constant value by $\Phi_{j,k}$.

https://doi.org/10.1017/S0143385702000196 Published online by Cambridge University Press

THEOREM 5.3. *Suppose that the following two conditions hold:*
(i) *if* $\zeta_k < 0$ *then* $\max\{\Phi_{j,k} : 0 \le j \le r(k)\} = 1$;
(ii) *if* $\zeta_k \ge 0$ *then* $\max\{\Phi_{j,k} : 0 \le j \le r(k)\} = \Phi_{r(k),k}$.
Then for all $\gamma \in a\zeta + H^s$, $a > 0$ *the series* $\sum_{k=-\infty}^{\infty} \Phi_\gamma(k, x_0)$ *is finite.*

Let us denote by X^+ (respectively X^-) the union of all sub-intervals $[a_{i-1}, a_i)$ such that $\zeta_i > 0$ (respectively $\zeta_i < 0$). Denote by $X^\pm$ the intersection $X^- \cap T(X^+)$. A simple corollary of Theorem 5.3 follows.

COROLLARY 5.4. *Let* $a > 0$ *and suppose that* $X^\pm \subset J_1$. *Then for all* $\gamma \in a\zeta + H^s$ *the series* $\sum_{k=-\infty}^{\infty} \Phi_\gamma(k, x_0)$ *is finite.*

The next subsection is dedicated to the proof of Theorem 5.3 and Corollary 5.4.

5.1. *Substitutions and self-induced interval exchanges.* Consider the function $\mathcal{I} : X \to \{1, 2, \dots, m\}$ given by

$$\mathcal{I}(x) := i \iff x \in [a_{i-1}, a_i), \quad i = 1, 2, \dots, m.$$

Let $\mathcal{W}$ be the set of words on the alphabet $\mathcal{A} = \{1, 2, \dots, m\}$. To any piece of the orbit $\{x, T(x), \dots, T^k(x)\}$ we can associate the word $a_0 a_1 \dots a_k$ where $a_j = \mathcal{I}(T^j x)$, $j = 0, \dots, k$. The infinite word $a_0 a_1 \dots a_k \dots$ associated to the complete positive (respectively negative) of x is called the positive (respectively negative) *itinerary* of x. Consider the function $\theta : \mathcal{A} \to \mathcal{W}$ defined in the following way. For each $k \in \mathcal{A}$, $\theta(k)$ is the word W_k associated with the piece of orbit

$$\{x, Tx, T^2x, \dots, T^{r(k)}x\}$$

where $x \in J_1^k$ and $r(k)$ is the return time of x to J_1, which is constant in J_1^k. Clearly, W_k does not depend on the point x. The map θ is called a *substitution*. θ can be extended to a map defined in $\mathcal{W}$ (or $\mathcal{A}^\natural$), simply by replacing the words:

$$\theta(a_1 a_2 \dots a_k) = \theta(a_0)\theta(a_1) \dots \theta(a_k).$$

The matrix B^θ, where B^θ_{ik} is the number of times that the symbol i appears in the word $\theta(k)$, $1 \le i, k \le m$ is associated with this substitution. It is not hard to see that B^θ is the same matrix B associated with the induced map in J_1 (see (41)).

Set $\theta^n(k) = \theta^{n-1}(\theta(k))$, $n \ge 1$. Let $r(k, n)$ be the return time of the sub-interval J_n^k to J_n. Observe that if $x \in J_n^k$ the itinerary associated to the piece of orbit $\{x, T(x), \dots, T^{r(k,n)}(x)\}$ is given by the word $\theta^n(k)$. On the other hand the sequences $\theta^n(k)$, $1 \le k \le m$ 'converge' to some infinite sequence $b_0 b_1 \dots b_k \dots$ in the following sense: for all $N_1 > 0$ there is a $N_0 > 0$ such that if $n > N_0$, the first N_1 symbols in the word $\theta^n(k)$ are exactly $b_0 b_1 \dots b_{N_1}$. As J_1 is contained in some sub-interval of continuity of T, the sequence $b_0 b_1 \dots b_k \dots$ is exactly the positive itinerary of the distinguished point $x_0 \in \cap_{n \ge 1} J_n$.

Let $\eta > 1$ be an eigenvalue of B and let $\zeta = (\zeta_1, \zeta_2, \dots, \zeta_m)$ be an associated eigenvector for ${}^tB^\theta$, i.e. $\zeta B = \eta\zeta$. We will suppose that none of the entries of ζ is zero.

https://doi.org/10.1017/S0143385702000196 Published online by Cambridge University Press

We will say that a word $W = a_1 a_2 \dots a_k$ is admissible (with respect to the map T) if there is a point $x \in X$ such that $a_i = \mathcal{I}(T^j(x))$, $0 \le j \le k$, i.e. if W is part of the itinerary of some point x. In particular, if W is admissible we may define $\Phi_\zeta(W) = \Phi_\zeta(k, x)$ (see (22)). Let $\phi = \phi_\zeta$ be the function $\log \Phi_\zeta$ defined in the set of all admissible words by the formula

$$\phi(a_1 a_2 \dots a_k) = \zeta_{a_1} + \zeta_{a_2} + \cdots + \zeta_{a_k}. \tag{42}$$

Set $\phi(\emptyset) = 0$. For $W = a_1 \dots a_k$ we define the vector $r(W) = (r_1, \dots, r_m) \in \mathbb{R}^m$ where

$$r_j = \text{card}\{1 \le s \le k : a_s = j\}.$$

Observe that the ith column of B^n is the vector $r(\theta^n(\alpha))$. Then we have (see [**2**]),

$$\begin{aligned} \phi(\theta^n(\alpha)) &= \zeta \cdot r(\theta^n(\alpha)) = \zeta \cdot B^n(r(\alpha)) \\ &= \eta^n \cdot \zeta \cdot r(\alpha) = \eta^n \cdot \phi(\alpha) \\ &= \eta^n \cdot \zeta_\alpha. \end{aligned} \tag{43}$$

In the same way, if $W = a_1 a_2 \dots a_k$ is a word in $\mathcal{W}$, then

$$\begin{aligned} \phi(\theta^n(W)) &= \zeta \cdot (r(\theta^n(a_1)) + \cdots + r(\theta^n(a_k))) \\ &= \eta^n \cdot (\zeta_{a_1} + \cdots + \zeta_{a_k}) = \eta^n \cdot \phi(W). \end{aligned} \tag{44}$$

We will denote by $|W|$ the length of the word $W \in \mathcal{W}$, i.e. the number of symbols in W. $r(k, n)$ will denote the length of the word $\theta^n(k)$, $k \in \mathcal{A}$. Let us denote by $r(k)$ the length of $\theta(k)$, $k = 1, \dots, m$. Hereafter, the symbols α, β will denote arbitrary elements of the alphabet $\mathcal{A} = \{1, 2, \dots, m\}$.

Recall that we have defined $\Phi_\gamma(k, x) = \prod_{j=1}^{k} {\Phi_\gamma}^{-1}(T^{-j}(x))$ if $k < 0$. Let $X_i = [a_{i-1}, a_i)$ and $Y_i = T(X_i)$, $i = 1, \dots, m$. For each $\gamma \in \mathbb{R}^m$ define $\tilde{\Phi}_\gamma : X \to \mathbb{R}$ by $\tilde{\Phi}_\gamma(x) = e^{\gamma_i}$ if $x \in Y_i$. Set $\gamma^\pi = (\gamma_{\pi^{-1}(1)}, \dots, \gamma_{\pi^{-1}(m)})$. Observe that $T^{-j+1}(x) \in Y_{\pi(k)} \iff T^{-j}(x) \in X_k$ and, therefore,

$$\begin{aligned} \Phi_\gamma^{-1}(T^{-j}(x)) &= e^{-\gamma_k} = e^{\gamma^\pi_{\pi(k)}} \\ &= \tilde{\Phi}_{-\gamma^\pi}(T^{-j+1}(x)). \end{aligned}$$

Then if $k < 0$ we have

$$\Phi_\gamma(k-1, x) = \tilde{\Phi}_\gamma(k, x) = \sum_{j=0}^{|k|} \tilde{\Phi}_{-\gamma^\pi}(T^{-j}(x)). \tag{45}$$

Observe that T^{-1} is self-induced in J_n, $n \ge 1$. Let $T_{-n} : J_n \to J_n$ be the induced map of T^{-1} in J_n. We can see that if $\tilde{J}_n^k$, $1 \le k \le m$ are the sub-intervals of continuity of T_{-n}, then $\tilde{J}_n^{\pi(k)} = T_n(J_n^k)$. In the same way, if $\theta(k) = a_1 a_2 \dots a_{r(k,n)}$ is the itinerary of J_n^k, then the itinerary (by T^{-1}) of $\tilde{J}_n^{\pi(k)}$ is given by $\tilde{\theta}(\pi(k)) = \pi(a_{r(k,n)}) \dots \pi(a_2)\pi(a_1)$ (with respect to the corresponding partition of T^{-1}).

Let $\tilde{B}$ be the matrix associated to the Poincaré induced map of T^{-1} in J_1. ($\tilde{B}$ is the same matrix $B^{\tilde{\theta}}$ associated to the substitution $\tilde{\theta}$). Let ζ^π be the vector $\zeta^\pi = (\zeta_{\pi^{-1}(1)}, \dots, \zeta_{\pi^{-1}(m)})$ and let Let $M = M^\pi$ be the matrix defined by $M_{ij}^\pi = \delta_{i\pi(i)}$.

https://doi.org/10.1017/S0143385702000196 Published online by Cambridge University Press

Then we can check that $M^{-1} = {}^tM = M^{\pi^{-1}}$ and ${}^t\tilde{B} = M^{-1}\,{}^tBM$, and also $M\zeta^\pi = \zeta$. Hence,

$$\begin{aligned} {}^t\tilde{B}\zeta^\pi &= M^{-1}\,{}^tBM\zeta^\pi = M^{-1}\,{}^tB\zeta \\ &= \eta \cdot M^{-1}\zeta \\ &= \eta \cdot \zeta^\pi \end{aligned}$$

and ζ^π is and eigenvector for ${}^t\tilde{B}$ associated to $\eta > 1$.

It is not hard to see that the $\tilde{\theta}^n(\alpha)$ 'converge' to the negative itinerary of x_0 when $n \to \infty$ for all $\alpha \in \{1, 2, \ldots, m\}$.

PROPOSITION 5.5. *Suppose that the following two conditions hold for every $\alpha \in \mathcal{A}$:*

(a) *if $\theta(\alpha) = a_1 a_2 \ldots a_{r(\alpha)}$ and $\zeta_\alpha < 0$, then*

$$\max_{1 \le j \le r(\alpha)} \zeta_{a_1} + \cdots + \zeta_{a_j} < 0;$$

(b) *if $\theta(\alpha) = a_1 a_2 \ldots a_{r(\alpha)}$ and $\zeta_\alpha > 0$, then*

$$\max_{1 \le j \le r(\alpha)} \zeta_{a_1} + \cdots + \zeta_{a_j} = \sum_{i=1}^{r(\alpha)} \zeta_{a_i}.$$

Then there is a constant $M > 0$ such that for all $n \ge 1$ the following are true: if $\zeta_\alpha < 0$,

$$\sum_{l=0}^{r(\alpha,n)} \Phi_\zeta(l, x_0) < M, \quad x \in J_n^\alpha$$

and if $\zeta_\alpha > 0$,

$$\sum_{l=1}^{r(\alpha,n)+1} \Phi_\zeta(-l, x) = \sum_{l=0}^{r(\alpha,n)} \tilde{\Phi}_\zeta(l, x) < M, \quad x \in \tilde{J}_n^{\pi(\alpha)}.$$

Proof. Let $n \ge 1$ and $\alpha \in \{1, 2, \ldots, m\}$. Set $\theta^n(\alpha) = a_1 a_2 \ldots a_{r(\alpha,n)}$ and define the function $S_{\alpha,n}$ on the set of integers $0 \le l \le r(\alpha, n)$ by the formula

$$S_{\alpha,n}(l) = \phi(a_1 a_2 \ldots a_l) = \zeta_{a_1} + \cdots + \zeta_{a_l}. \tag{46}$$

Set $S_{\alpha,n}(0) = 0, n \ge 1$ by definition and observe that (44) implies that

$$S_{\alpha,n}(r(\alpha)) = \eta^n \cdot \zeta_\alpha, \quad n \ge 1.$$

Define also the function $\tilde{S}_{\pi(\alpha),n}$ on the set of integers $0 \le l \le r(\alpha, n)$ in the following way. If $\tilde{\theta}^n(\alpha) = b_1 b_2 \ldots b_{r(\alpha,n)}$, then

$$\tilde{S}_{\pi(\alpha),n}(l) = -\phi_\zeta(b_1 b_2 \ldots b_l) = -(\zeta_{b_1}^\pi + \cdots + \zeta_{b_l}^\pi). \tag{47}$$

Set $\tilde{S}_{\pi(\alpha),n}(0) = 0, n \ge 1$ by definition and note that, by (44),

$$\tilde{S}_{\pi(\alpha),n}(r(\alpha)) = -\eta^n \cdot \zeta_{\pi(\alpha)}^\pi = -\eta^n \cdot \zeta_\alpha, \quad n \ge 1,$$

because $-\zeta^\pi$ is an eigenvector of ${}^t\tilde{B}$ associated to $\eta > 1$. Observe that $\Phi_\zeta(m, x)$ is equal to $\exp(S_{\alpha,n}(m))$ if $x \in J_n^\alpha$ and $0 \le m \le r(\alpha, n)$ and equal to $\exp(\tilde{S}_{\pi(\alpha),n}(m))$ if $x \in \tilde{J}_n^{\pi(\alpha)}$ and $r(\alpha, n) \le m < 0$ (see (45)).

The conditions (a) and (b) can be written:

https://doi.org/10.1017/S0143385702000196 Published online by Cambridge University Press

(a) if $\zeta_\alpha < 0$, $\max\{S_{\alpha,1}(j) : 0 \le j \le r(\alpha)\} = S_{\alpha,1}(0) = 0$; and
(b) if $\zeta_\alpha > 0$, $\max\{S_{\alpha,1}(j) : 0 \le j \le r(\alpha)\} = S_{\alpha,1}(r(\alpha)) = \zeta_\alpha$.

Note that these two conditions imply that:

(c) if $\zeta_\alpha < 0$, $\max\{\tilde{S}_{\pi(\alpha),1}(j) : 0 \le j \le r(\alpha)\} = \tilde{S}_{\pi(\alpha),1}(r(\alpha)) = -\zeta_\alpha$; and
(d) if $\zeta_\alpha > 0$, $\max\{\tilde{S}_{\pi(\alpha),1}(j) : 0 \le j \le r(\alpha)\} = \tilde{S}_{\alpha,1}(0) = 0$.

Indeed, if $\alpha \in \mathcal{A}$ is such that $\tilde{S}_{\pi(\alpha),1}$ does not satisfy one of the conditions (c) or (d), then for some $1 < p < r(\alpha)$, $\tilde{S}_{\pi(\alpha),1}(p) = \max\{\tilde{S}_{\pi(\alpha),1}(j) : 0 \le j \le r(\alpha)\}$. Let $\theta(\alpha) = a_1 a_2 \dots a_{r(\alpha)}$ and $\tilde{\theta}(\pi(\alpha)) = \pi(a_{r(\alpha)}) \dots \pi(a_1) = b_1 b_2 \dots b_{r(\alpha)}$. Observe that

$$\begin{aligned} S_{\alpha,1}(j) &= \sum_{i=1}^{j} a_i = \sum_{i=1}^{r(\alpha)} \zeta_{a_i} - \sum_{i=j+1}^{r(\alpha)} \zeta_{a_i} \\ &= \sum_{i=1}^{r(\alpha)} \zeta_{a_i} - \sum_{i=1}^{r(\alpha)-j+1} \zeta_{b_i}^{\pi} \\ &= S_{\alpha,1}(r(\alpha)) + \tilde{S}_{\pi(\alpha),1}(r(\alpha) - j + 1). \end{aligned}$$

Therefore,

$$\begin{aligned} \max\{S_{\alpha,1}(j) : 0 \le j \le r(\alpha)\} &= S_{\alpha,1}(r(\alpha)) + \max\{\tilde{S}_{\pi(\alpha),1}(r(\alpha) - j + 1) : 0 \le j \le r(\alpha)\} \\ &= S_{\alpha,1}(r(\alpha)) + \tilde{S}_{\pi(\alpha),1}(p) = S_{\alpha,1}(r(\alpha) + 1 - p) \end{aligned}$$

where $0 < r(\alpha) + 1 - p < r(\alpha)$, a contradiction of assumptions (a) and (b).

Let us prove that conditions (a) and (b) also imply:

(a′) if $\zeta_\alpha < 0$, $\max\{S_{\alpha,n}(l) : 1 \le l \le r(\alpha, n)\} = S_{\alpha,n}(0) = 0$; and
(b′) if $\zeta_\alpha \ge 0$, $\max\{S_{\alpha,n}(l) : 1 \le l \le r(\alpha, n)\} = S_{\alpha,n}(r(\alpha, n)) = \eta^n \zeta_\alpha$.

We prove these statements by induction on n. The conclusion is true for $n = 1$ by hypotheses. Suppose that the conclusion is true for $n = k$ and let us prove that it is also true for $n = k + 1$. Fix $1 \le \alpha \le m$ and write $\theta(\alpha) = a_1 a_2 \dots a_{r(\alpha)}$. Let

$$M_{\alpha,n} = \max\{S_{\alpha,n}(l) : 1 \le l \le r(\alpha, n)\}, \quad n \ge 1.$$

Set $r_0 = 0$ and $r_j = |\theta^k(a_1 \dots a_j)|$, $1 \le j \le r(\alpha)$. Since

$$\theta^{k+1}(\alpha) = \theta^k(a_1 \dots a_j)\theta^k(a_{j+1})\theta^k(a_{j+2} \dots a_{r(\alpha)})$$

and by (44), $\phi(\theta^k(a_1 \dots a_j)) = \eta^k \cdot (\zeta_{a_1} + \dots + \zeta_{a_j}) = \eta^k S_{\alpha,1}(j)$, the value of $S_{\alpha,k+1}$ at $q = r_j + p$ in the interval $[r_j, r_{j+1}]$, is given by

$$S_{\alpha,k+1}(q) = \eta^k S_{\alpha,1}(j) + S_{a_{j+1},k}(p)$$

for each $0 \le j \le r(\alpha) - 1$. Therefore

$$M_{\alpha,k+1} = \max_{0 \le j \le r(\alpha)-1} \{\eta^k S_{\alpha,1}(j) + M_{a_{j+1},k}\}. \tag{48}$$

and using our induction hypotheses on $M_{a_{j+1},k}$ we can write

$$\begin{aligned} M_{\alpha,k+1} &= \max_{0 \le j \le r(\alpha)-1} \{\eta^k S_{\alpha,1}(j) + \eta^k(\max(\zeta_{a_{j+1}}, 0))\} \\ &= \eta^k \Big(\max_{0 \le j \le r(\alpha)} S_{\alpha,1(j)} \Big). \end{aligned}$$

https://doi.org/10.1017/S0143385702000196 Published online by Cambridge University Press

Thus, conditions (a) and (b) imply conditions (c′) and (b′). In particular, observe that if $\zeta_\alpha \geq 0$,

$$\begin{aligned} M_{\alpha,k+1} &= \max_{1\leq j\leq r(\alpha)} \eta^k S_{\alpha,1}(j) \\ &= \eta^k S_{\alpha,1}(r(\alpha)) = \eta^{k+1}\zeta_\alpha. \end{aligned}$$

We claim that, for $n \geq 0$,

$$\max\{S_{\alpha,n+1}(l) : l \in [r_1, r(\alpha, n+1)]\} = \max\{\eta^n S_{\alpha,1}(j) : 1 \leq j \leq r(\alpha)\}. \tag{49}$$

Indeed, (49) is true for $n = 0$ by definition. Suppose that it is true for $n = k$. If it were not true for $n = k+1$, then for some $1 \leq j < r(\alpha)$ and $q = r_j + p \in (r_j, r_{j+1})$ with $0 < p < r(a_{j+1}, k)$,

$$\max\{S_{\alpha,k+1}(l) : l \in [r_1, r(\alpha, k+1)]\} = S_{\alpha,k+1}(q) = \eta^k S_{\alpha,1}(j) + S_{a_{j+1},k}(p)$$

which implies that $M_{a_{j+1},k} = S_{a_{j+1},k}(p)$, a contradiction of (a′) or (b′). Therefore (49) is true for all $n \geq 0$.

Let q be the maximum of the numbers $\{r(\alpha) : \alpha \in \mathcal{A}\}$. We clearly have $\rho \leq q$ and $r(\alpha, n) \leq q^n$, $\alpha \in \mathcal{A}$. In the same way, $r_j \leq jq^k$, $\alpha \in \mathcal{A}$, $1 \leq j \leq r(\alpha)$.

We will prove that if $\zeta_\alpha < 0$ and for each $1 \leq l \leq r(\alpha, n)$,

$$S_{\alpha,n}(l) \leq Cl^\tau, \tag{50}$$

where

$$C := \frac{1}{q}\max\{S_{\alpha,1}(j) : 1 \leq j \leq r(\alpha)\} < 0 \quad \text{and} \quad \tau = \log\eta/\log q,$$

therefore,

$$\begin{aligned} \sum_{l=0}^{r(\alpha,n)} \Phi_\zeta(l, x) &= \sum_{l=0}^{r(\alpha,n)} \exp(S_{\alpha,n}(l)) \\ &\leq \sum_{l\geq 0} \exp(Cl^\tau) = M_1 < \infty, \quad n \geq 1. \end{aligned}$$

A similar argument, applied to the substitution $\tilde{\theta}$, shows that there is a constant M_2 such that $\sum_{0\leq l\leq r(\alpha,n)} \exp(\tilde{S}_{\alpha,n}(l)) < M_2, n \geq 1$ if $\zeta_\alpha > 0$. Thus, the proposition holds for $M = \max\{M_1, M_2\}$.

Equation (50) is proved by induction on n. The statement is true for $n = 1$ by the definition of the constant C. Suppose that the statement is true for $n = k$ and let us prove that it is true for $n = k+1$. Let $\theta(\alpha) = a_1a_2\ldots a_{r(\alpha)}$ and $Q(l) = Ck^\tau, l \geq 1$. Set $r_0 = 0$ and $r_j = |\theta^k(a_1\ldots a_j)1 \leq j \leq r(\alpha)$.

Observe that, in particular, the graph of $S_{\alpha,k+1}$ in the interval $[0, r_1]$ is given by the graph of $S_{a_1,k}$ where necessarily a_1 satisfies $\zeta_{a_1} < 0$. By induction hypotheses, the graph of the function Q is above the graph of $S_{\alpha,k+1}$ in the interval $[0, r_1]$.

On one hand, by (49), the number $\max\{\eta^k S_{\alpha,1}(j) : 1 \leq j \leq r(\alpha)\}$ is the maximum of the values $S_{\alpha,k+1}(l)$ with $l \in [r_1, r(\alpha, k+1)]$; on the other hand, the minimum value of

https://doi.org/10.1017/S0143385702000196 Published online by Cambridge University Press

the function Q in $[r_1, r(\alpha, k+1)]$ is given by

$$\begin{aligned}\min\{Q(l) : r_1 \le l \le r(\alpha, k+1)\} &= Q(r(\alpha, k+1)) \\ &= Cr(\alpha, k+1)^\tau \ge C(r(\alpha)q^k)^\tau \\ &= Cr(\alpha)^\tau \eta^k > Cq\eta^k \\ &= \Big(\max_{1\le j\le r(\alpha)} S_{\alpha,1}(j)\Big)\eta^k \\ &= \max\{S_{\alpha,k+1}(l) : r_1 \le l \le r(\alpha, k+1)\}\end{aligned}$$

and therefore, the graph of Q is always above the graph of $S_{\alpha,k+1}$ in the interval $[0, r(\alpha, k+1)]$. □

Remark. The constant $\tau = \log\eta/\log q$ is used in the proof of the previous proposition just to make it shorter. By slightly changing the proof it is possible to use the more natural constant $\tau = \log\eta/\log\rho$.

COROLLARY 5.6. *Under the hypotheses of Proposition 5.5, there are constants $C < 0$ and $0 < \tau < 1$ such that*

$$\Phi_\zeta(l, x_0) \le e^{C|l|^\tau},$$

for all $l \in \mathbb{Z}$. In particular the series

$$\sum_{l\in\mathbb{Z}} \Phi_\zeta(l, x_0)$$

is finite.

COROLLARY 5.7. *For all $a > 0$ the series $\sum_{l\in\mathbb{Z}} \Phi_{a\zeta}(l, x_0) < \infty$.*

Proof. As $\Phi_{a\zeta}(l, x_0) = \Phi^a_\zeta(l, x_0)$, we have

$$\sum_{l\in\mathbb{Z}} \Phi_{a\zeta}(l, x_0) < \sum_{l\in\mathbb{Z}} e^{aC|l|^\tau} < \infty.$$ □

PROPOSITION 5.8. *Let $\xi \in \mathbb{R}^m$ be an eigenvector of ${}^tB$ associated to an eigenvalue $\nu < 1$. Consider the functions $\tilde{S}_{\pi(\alpha),n}$ as in (46) and (47) (associated to ξ in place of ζ). Then there exists a constant $R_0 > 0$ such that for each $\alpha \in \mathcal{A}$ and each $n \ge 1$,*

$$-R_0 \le S_{\alpha,n}(l) \le R_0 \quad \text{and} \quad -R_0 \le \tilde{S}_{\pi(\alpha),n}(l) \le R_0, \quad 1 \le l \le r(\alpha, n).$$

Proof. We will prove the statement only for $S_{\alpha,n}$; the proof for the functions $\tilde{S}_{\pi(\alpha),n}$ is completely similar. Let

$$M = \max_{\alpha\in\mathcal{A}} \Big\{ \max_{1\le j\le r(\alpha)} |S_{\alpha,1}(j)| \Big\}.$$

We know that for each $\alpha \in \mathcal{A}$, $l \ge 1$ and $0 \le j \le r(\alpha) - 1$,

$$\max\{S_{\alpha,k+1}(l) : l \in [r_j, r_{j+1}]\} = \phi(\theta^k(a_1 \dots a_j)) + \max\{S_{a_{j+1},k}(l) : 1 \le l \le r(a_{j+1}, k)\}.$$

From this equation we obtain that, for each $\alpha \in \mathcal{A}$,

$$\max_{1\le l\le r(\alpha,k+1)} |S_{\alpha,k+1}(l)| \le \nu^k M + \max_\alpha \max_{1\le l\le r(\alpha,k)} |S_{\alpha,k}(l)|. \tag{51}$$

https://doi.org/10.1017/S0143385702000196 Published online by Cambridge University Press

Then if $k = 1$ we get

$$\max_{1\le l\le r(\alpha,2)} |S_{\alpha,2}(l)| \le \nu M + M,$$

and for $k = 2$,

$$\max_{1\le l\le r(\alpha,3)} |S_{\alpha,3}(l)| \le \nu^2 M + \max_{\alpha} \max_{1\le i\le r(\alpha,2)} |S_{\alpha,2}(i)| \le \nu^2 M + \nu M + M.$$

Therefore, recursively using (51) we obtain that

$$\max_{1\le l\le r(\alpha,k+1)} |S_{\alpha,k+1}(l)| \le M(1 + \nu + \nu^2 + \cdots) < \infty$$

for $k \ge 1$. Hence that proposition is true taking $R_0 = M \sum_{l\ge 0} \nu^l$. □

COROLLARY 5.9. *Let $\xi \in \mathbb{R}^m$ be an eigenvector of ${}^tB$ associated to an eigenvalue $\nu < 1$. Then there is a constant $R_0 > 0$ such that*

$$e^{-R_0} \le \phi_\xi(l, x_0) \le e^{R_0},$$

for all $l \in \mathbb{Z}$.

COROLLARY 5.10. *For all $x \in X$ and every vector $\gamma' \in H^s$, there is a constant $R = R(\gamma') > 0$ such that*

$$e^{-2R} \le \Phi_{\gamma'}(l, x) \le e^{2R} \tag{52}$$

for all $l \in \mathbb{Z}$.

Proof. It follows from Corollary 5.9 that for all $\gamma' \in H^s$ there exists a constant $R(\gamma') > 0$ such that for all $l \in \mathbb{Z}$, $e^{-R(\gamma')} \le \Phi_{\gamma'}(l, x_0) \le e^{R(\gamma')}$. In fact, if $\gamma' \in H^s$, then γ' is a linear combination $\gamma' = \sum_{i=1}^{l} a_i \xi_i$ where the ξ_is are eigenvectors of ${}^tB^\theta$ associated to eigenvalues <1. Therefore as $\log \Phi_{\gamma'} = \sum_{i=1}^{l} a_i \log \Phi_\xi$, we obtain that $\exp\left(-R_0 \sum_{i=1}^{l} a_i\right) \le \Phi_{\gamma'}(l, x_0) < \exp\left(R_0 \sum_{i=1}^{l} a_i\right)$, for all $l \ge 1$. Set $R(\gamma') = R_0 \sum_i a_i$.

We will prove now that (52), with $R = 2R(\gamma')$, holds for every $x \in X$ and $\gamma' \in H^s$. Remember that the T_0-orbit of x_0 is dense in the interval X. Note that

$$e^{-R(\gamma')} \le \Phi(k + l, x_0) = \Phi(k, x_0)\Phi(l, T^k(x_0)) \le e^{R(\gamma')}$$

and as $e^{-R(\gamma')} \le \Phi(k, x_0) \le e^{R(\gamma')}$ we obtain that

$$e^{-2R(\gamma')} \le \Phi(l, T^k(x_0)) \le e^{-2R(\gamma')}$$

for all $k, l \in \mathbb{Z}$. By continuity this inequality holds, for all $l \in \mathbb{Z}$, when replacing $T^k(x_0)$ by any $x \in X$.

Proof of Theorem 5.3. First of all, it is clear that the hypotheses (i) and (ii) of Theorem 5.3 are equivalent to the hypotheses (a) and (b) of Proposition 5.5. Observe that if $\gamma = a\zeta + \xi, \xi \in H^s$ then

$$\log \Phi_\gamma = a \log \Phi_\zeta + \Phi_\xi,$$

https://doi.org/10.1017/S0143385702000196 Published online by Cambridge University Press

and since the hypotheses of Proposition 5.5 are also valid if we change ζ to $a\zeta$, $a > 0$, the series $\sum_{l\in\mathbb{Z}} \Phi_{a\zeta}(l, x_0)$ is finite and so

$$\sum_{l\in\mathbb{Z}} \Phi_\gamma(l, x_0) = \sum_{l\in\mathbb{Z}} \Phi_{a\zeta}(l, x_0)\Phi_\xi(l, x_0) \leq e^{R(\xi)} \sum_{l\in\mathbb{Z}} \Phi_{a\zeta}(l, x_0)$$

is also finite. Theorem 5.3 is proved. □

Proof of Corollary 5.4. Observe that the hypotheses $X^\pm \subset J_1$ of Corollary 5.4 implies conditions (i) and (ii) of Theorem 5.3. In fact, if one of these conditions does not hold, then for some $\alpha \in \{1, 2, \ldots, m\}$ there is a $0 < p < r(\alpha)$ such that

$$S_{\alpha,1}(p) = \phi_\zeta(a_1 a_2 \ldots a_p) = \max\{S_{\alpha,1}(j) : 1 \leq j \leq r(\alpha)\}$$

where $\theta(\alpha) = a_1 a_2 \ldots a_{r(\alpha)}$. In particular,

$$S_{\alpha,1}(p-1) = S_{\alpha,1}(p) - \zeta_{a_p} < S_{\alpha,1}(p),$$

and

$$S_{\alpha,1}(p+1) = S_{\alpha,1}(p) + \zeta_{a_{p+1}} < S_{\alpha,1}(p).$$

Hence $\zeta_{a_p} > 0$ and $\zeta_{a_{p+1}} < 0$. Recall that $\theta(\alpha) = a_1 a_2 \ldots a_{r(\alpha)}$ is the itinerary of J_1^α before it returns to J_1. From this we see that $T^p(J_1^\alpha) \subset X^+$ and $T^{p+1}(J_1^\alpha) \subset X^-$, that is, $T^{p+1}(J_1^\alpha) \subset X^\pm \subset J_1$. Therefore $p = r(\alpha)$, a contradiction. □

5.2. *Proof of Theorem 5.2.* Remember that λ_0 is an eigenvector associated to the maximum eigenvalue $\rho > 1$ of the matrix A defined in equation (37). It is not hard to check that T_0 is self-induced in the last sub-interval of continuity $[a_3, 1)$. In particular, the length of this sub-interval is equal to ρ^{-1}.

Consider the eigenvalue $\eta > 1$ ($\eta \approx 1.837\,85$) of ${}^tA$ and the associated eigenvector $\zeta = (\zeta_1, \zeta_2, \zeta_3, \zeta_4)$ which is approximately equal to

$$\zeta \approx (-0.33, -0.73, 0.54, 0.24).$$

Recall that we are assuming that $\zeta \in H_1^u$. Let us put $X_i = [a_{i-1}, a_i)$, $i = 1, 2, 3, 4$. First of all we can check that the T_0 is self-induced in the last interval of continuity $X_4 = [a_3, 1)$. As $J_1 = [0, \rho^{-1}) = T_0(X_4) \subset X_1$, T_0 is also self-induced in J_1 (indeed, T_0 is a periodic point for the map $\mathcal{T}$ defined in (16)). Let T_1 be the Poincaré induced map of T_0 in J_1. The corresponding substitution θ, associated to T_1, is given by (see [**2**])

$$\begin{aligned} \theta(1) &= 12134 \\ \theta(2) &= 122134 \\ \theta(3) &= 1334 \\ \theta(4) &= 134 \end{aligned} \tag{53}$$

and the matrix associated to θ is exactly the matrix A defined in (37).

https://doi.org/10.1017/S0143385702000196 Published online by Cambridge University Press

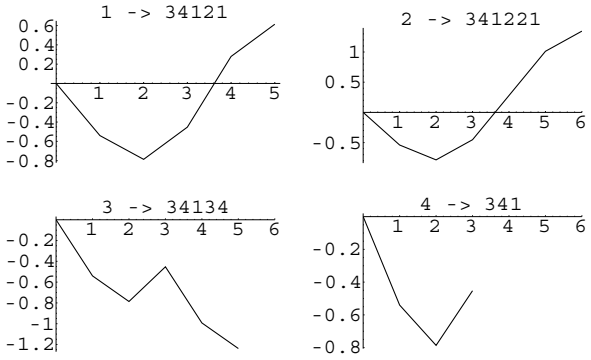


FIGURE 2. $S_{\alpha,1}$ associated to θ and $-\zeta$.

To prove that $\sum_{k=-\infty}^{\infty} \Phi_\gamma(k,0)$ is finite for each $\gamma \in a\zeta + H^s, a > 0$, we only have to check that J_1 satisfies the hypotheses of Corollary 5.4. Indeed as $\zeta \approx (-0.33, -0.73, 0.54, 0.24)$, $X^+ = X_3 \cup X_4$ and $X^- = X_1 \cup X_2$. Then we can check that $X^\pm = X^- \cap T_0(X^+) = T_0(X_4)$ is exactly the interval J_1. Let us prove now that if $a < 0$, the series

$$\sum_{k=-\infty}^{\infty} \Phi_\gamma(k, z_0)$$

is finite for all $\gamma \in a\zeta + H^s$. Let $J_1 = T_0^{-1}(X_4) \subset X_3$. As T_0 is self-induced in X_4 then it is also self-induced in J_1. The corresponding substitution is

$$\begin{aligned} \theta(1) &= 34121 \\ \theta(2) &= 341221 \\ \theta(3) &= 34134 \\ \theta(4) &= 341. \end{aligned} \tag{54}$$

We consider the eigenvector $-\zeta \approx (0.33, 0.73, -0.54, -0.24)$ of ${}^t B^\theta$ associated to $\eta > 1$. Observe that J_1 does not satisfy the hypotheses of Corollary 5.4. In fact, $X^+ = X_1 \cup X_2$, and $X^- = X_3 \cup X_4$ and $X^\pm = X_3 \cap T(X_1)$ is not contained in J_1. However, it satisfies the hypotheses of Theorem 5.3. In Figure 2, we can see the graphs of the functions $S_{\alpha,1}, 1 \le \alpha \le 4$, associated to the substitution θ, defined in (54), and the vector $-\zeta$, and we can check that they satisfy conditions (a) and (b) of Proposition 5.5. Therefore, from Theorem 5.3 we conclude that

$$\sum_{k=-\infty}^{\infty} \Phi_\gamma(k, z_0)$$

is finite, for all $\gamma \in a\zeta + H^s, a < 0$.

We will prove now that if the series $\sum_{k=-\infty}^{\infty} \Phi_\gamma(k, x_0)$ is finite for some $x_0 \in X$ and $\gamma \in H^u$ then x_0 is in the orbit of $x = 0$ if $\gamma \in H_1^u$ and x_0 is in the orbit of $x = z_0$ if $\gamma \in H_2^u$.

https://doi.org/10.1017/S0143385702000196 Published online by Cambridge University Press

We will prove only the case when $\gamma \in H_1^u$; the proof of the case $\gamma \in H_2^u$ is completely similar.

For each $(x, \gamma) \in X \times H_1^u$, let us put $S(x, \gamma) = \sum_{k=-\infty}^{\infty} \Phi_\gamma(k, x)$. Let $\gamma = a\zeta + \xi$ where $a > 0$ and $\xi \in H^s$. Observe that if $S(x, \gamma)$ is finite then $S(x, a\zeta)$ is finite. In fact, as $\Phi_{a\zeta+\xi} = \Phi_{a\zeta}\Phi_\xi$ and, by Corollary 5.10, there is a constant $R(\xi)$ such that

$$e^{-R(\xi)} < \Phi_\xi(k, y) < e^{R(\xi)}$$

for all $(k, y) \in \mathbb{Z} \times X$, we have that

$$e^{-R(\xi)} \sum_{k=-\infty}^{\infty} \Phi_{a\zeta}(k, x) < \sum_{k=-\infty}^{\infty} \Phi_{a\zeta+\xi}(k, x) < \infty,$$

and $S(x, a\zeta)$ is finite. Therefore, let us assume, without loss of generality, that $x_0 \in X$ is a point such that $S(x_0, a\zeta)$ is finite. Then $\sup_{k\in\mathbb{Z}} \Phi_{a\zeta}(k, x_0)$ is finite. Set $\Phi_{a\zeta}(n_0, x_0)$ for this supremum and write $w_0 = T_0^{n_0}(x_0)$. We have that

$$\Phi_{a\zeta}(n_0 + l, x_0) = \Phi_{a\zeta}(n_0, x_0)\Phi_{a\zeta}(l, w_0) \leq \Phi_{a\zeta}(n_0, x_0), \quad l \in \{-1, 1\}$$

which implies that

$$\Phi_{a\zeta}(1, w_0) \leq 1 \quad \text{and} \quad \Phi_{a\zeta}(-1, w_0) = \frac{1}{\Phi_{a\zeta}(1, w_0)} \leq 1,$$

or, which is the same, $w_0 \in X^-$ and $T_0^{-1}(w_0) \in X^+$. Then, necessarily, $w_0 \in X_1 \cap T_0(X_4) = J_1$ (because the signs of the entries of $a\zeta$ are the same as those of ζ).

Let us prove that if $w_0 \in J_n$ then $w_0 \in J_{n+1}$ which implies that $w_0 \in \cap_n J_n = \{0\}$ and then x_0 is in the orbit of $x = 0$.

First of all, it is clear that $S(w_0, a\zeta)$ is finite. Let T_n be the Poincaré induced map of T in J_n. Let $W_0 = \ldots a_{-1}a_0a_1a_2\ldots$ be the T_n-itinerary of w_0 and let $r(a_i, n)$ be the length of the word $\theta^n(a_i)$ if $i \geq 0$ or the length of the word $\tilde{\theta}^n(a_i)$ if $i < 0$. The times $r(a_i, n), i > 0$ (respectively $i < 0$) are the successive forward (respectively backward) returns of x to the sub-interval J_n (setting $r(a_0, n) = 0$). Let us define a new co-cycle $\hat{\Phi}(k, x)$ relative to the map T_n and the vector $a\eta^n\zeta$. If $x \in J_n$ and $\gamma \in \mathbb{R}^m$, we define $\hat{\phi}_\gamma(x) = e^{\gamma_i}$ if $x \in J_n^i$. For $(k, x) \in \mathbb{Z} \times J_n$ we define:

$$\hat{\Phi}_{a\eta^n\zeta}(k, x) = \begin{cases} \prod_{j=0}^{k-1} \hat{\phi}_{a\eta^n\zeta}(T_n^j(x)), & k \geq 0 \\ \prod_{j=1}^{k} \hat{\phi}^{-1}_{a\eta^n\zeta^\pi}(T_n^{-j}(x)), & k < 0. \end{cases} \tag{55}$$

Observe that equation (44) tells us that if $x \in J_n^\alpha$ then

$$\Phi_{a\zeta}(r(\alpha, n), x) = e^{a\eta^n\zeta_\alpha} \tag{56}$$

which implies that

$$\Phi_{a\zeta}(r(a_k, n), x) = \hat{\Phi}_{a\eta^n\zeta}(k, x), \quad k \in \mathbb{Z}. \tag{57}$$

https://doi.org/10.1017/S0143385702000196 Published online by Cambridge University Press

From this we obtain

$$\sum_{k=-\infty}^{\infty} \hat{\Phi}_{a\eta^n\zeta}(k, w_0) < S(w_0, \zeta) < \infty.$$

Observe that

$$\sup_{k\in\mathbb{Z}} \hat{\Phi}_{a\eta^n\zeta}(k, w_0) = \sup_{k\in\mathbb{Z}} \Phi_{a\zeta}(r(a_k, n), w_0) = \Phi_{a\zeta}(r(a_0, n), w_0) = \hat{\Phi}_{a\eta^n\zeta}(0, w_0) = 1.$$

Therefore,

$$\hat{\Phi}_{a\eta^n\zeta}(0, w_0) \le 1 \quad \text{and} \quad \Phi_{a\eta^n\zeta}(-1, w_0) \le 1,$$

which implies that w_0 belongs to J_n^α with $\zeta_\alpha < 0$ and $T_n^{-1}(w_0)$ belongs to some J_n^β with $\zeta_\beta > 0$. Therefore $\alpha = 1$ and $\beta = 4$, i.e. $w_0 \in J_n^1 \cap T_n(J_n^4) = J_{n+1}$. This concludes the proof of Theorem 5.2. □

PROPOSITION 5.11. *If $\gamma \in H^u$, then there are no affine interval exchanges in $C_{\text{aff}}(T_0, e^\gamma)$.*

Proof. Suppose by contradiction that $h : X \to X$ is a continuous conjugacy between T_0 and some affine interval exchange f. To be more precise, suppose that the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{T_0} & X \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h} \\ X & \xrightarrow[f]{} & X \end{array}$$

Observe that for each $n \ge 1$ the intervals $T^j(J_n^k)$ with $1 \le k \le m$, $0 \le j \le r(k, n)$, form a partition of X whose diameter goes to zero when $n \to \infty$. As h is continuous, the intervals $h(T^j(J_n^k))$ with $1 \le k \le m$, $0 \le j \le r(k, n)$, also form a partition of X whose diameter goes to zero when $n \to \infty$. Observe that the derivative of f^j in J_n^k is constant and that

$$m(h(T^j(J_n^k))) = (f^j)'(x) \cdot m(h(J_n^k)), \quad x \in J_n^k,$$

where m represents the Lebesgue measure in X. However, $(f^j)'(x) = \Phi_\gamma(j, x)$, $x \in J_n^k$, and therefore, for any fixed k and n,

$$m\left(\bigcup_{j=0}^{r(k,n)} h(T^j(J_n^k))\right) = \sum_{j=0}^{r(k,n)} \Phi(j, x) \cdot m(h(J_n^k)), \quad x \in J_n^k.$$

In the same way, the Lebesgue measure of the union $\bigcup_{j=1}^{r(k,n)+1} h(T^{-j}(\tilde{J}_n^k))$ is given by

$$\sum_{j=1}^{r(k,n)+1} \Phi(-j, x) \cdot m(h(\tilde{J}_n^k)), \quad x \in \tilde{J}_n^k,$$

where $\tilde{J}_n^k = T_n(J_n^k)$. Note that X is the disjoint union of the sets

$$\bigcup_{j=0}^{r(k,n)} h(T^j(J_n^k)),\ \zeta_k > 0 \quad \text{and} \quad \bigcup_{j=1}^{r(k,n)+1} h(T^{-j}(\tilde{J}_n^k)),\ \zeta_k < 0.$$

https://doi.org/10.1017/S0143385702000196 Published online by Cambridge University Press

Let us put

$$S_n^k = \sum_{j=0}^{r(k,n)} \Phi_\zeta(j,x),\ x \in J_n^k, \quad \tilde{S}_n^{\pi(k)} = \sum_{j=1}^{r(k,n)+1} \Phi_\zeta(j,x),\ x \in \tilde{J}_n^{\pi(k)}$$

and

$$\mu_n^k = m(h(J_n^k)), \quad \tilde{\mu}_n^k = m(h(\tilde{J}_n^k)).$$

Then we have that

$$1 = m(X) = \sum_{\zeta_k<0} \mu_n^k \cdot S_n^k + \sum_{\zeta_k>0} \tilde{\mu}_n^k \cdot \tilde{S}_n^{\pi(k)}.$$

However, this equality is impossible for all $n \geq 1$ if h is a continuous function since then both μ_n^k and $\tilde{\mu}_n^k$ go to zero when $n \to \infty$ whereas by Proposition 5.5, both sums S_n^k and $S_n^{\pi(k)}$ remain bounded for all $n \geq 1$. Therefore there are no elements in $C(T_0, e^\gamma)$. □

COROLLARY 5.12. *For each $\gamma \in H^u$ there is a unique map $f \in S_{\text{aff}}(T_0, e^\gamma)$.*

Proof. By the previous proposition there are only atomic measures in $\Delta_\gamma(T_0)$. However, by Theorem 5.2, the only atom of the measure is equal to either $x_0 = 0$ if $\gamma \in H_1^u$ or $x_0 = z_0$ if $\gamma \in H_2^u$. Therefore the only measure in $\Delta_\gamma(T_0)$ is the atomic measure defined in (39). □

PROPOSITION 5.13. *Let $\gamma, \gamma' \in H_1^u$ (respectively H_2^u) and let $f_1 \in S_{\text{aff}}(T_0, e^\gamma)$ and $f_2 \in S_{\text{aff}}(T_0, e^{\gamma'})$. Then f_1 and f_2 are C^0-conjugated. Moreover, there is a C^1-conjugacy between f_1 and f_2 iff $\gamma - \gamma'$ belongs to $H^s(t)$.*

Proof. We will prove the proposition only for H_1^u; the proof for H_2^u is completely similar. The proof takes two steps. First we suppose that γ and γ' are of the particular form $\gamma = \zeta$ and $\gamma' = t\zeta, t > 0$. In this case, f_1 and f_2 are C^0-conjugated but not C^1-conjugated. After this, we will prove that if $\gamma - \gamma' \in H^s$, then there is a C^1-conjugacy between f_1 and f_2. These two facts imply the proposition.

Consider $\gamma = \zeta$ and $\gamma' = t\zeta, t > 0$ and take $f_1 \in S_{\text{aff}}(T_0, e^\zeta)$ and $f_2 \in S_{\text{aff}}(T_0, e^{t\zeta})$. It is easy to see that f_1 is C^0-conjugated to f_2. In fact, let $I_0 = [0, c_1]$ and $I_0' = [0, c_2]$ be the maximal wandering intervals of f_1 and f_2 respectively. Set

$$I_n = f_1^n(I_0) \quad \text{and} \quad I_n' = (f_2)^n(I_0'), \quad n \in \mathbb{Z}.$$

Let $\mathcal{R} = \bigcup_n I_n$ and $\mathcal{R}' = \bigcup_n I_n'$. Recall that $\mathcal{R}$ and $\mathcal{R}'$ are sets of total measure in X and there are a semi-conjugacy h_1 between f_1 and T, and a semi-conjugacy h_2 between f_2 and T,

$$\begin{array}{ccc} X & \xrightarrow{F_1} & X \\ {\scriptstyle h_1}\downarrow & & \downarrow{\scriptstyle h_1} \\ X & \xrightarrow{T_0} & X \\ {\scriptstyle h_2}\downarrow & & \downarrow{\scriptstyle h_2} \\ X & \xrightarrow{f_2} & X \end{array}$$

https://doi.org/10.1017/S0143385702000196 Published online by Cambridge University Press

Also h_1 and h_2 are bijective out of the sets $\mathcal{R}$ and $\mathcal{R}'$, and necessarily,

$$h_1(I_n) = h_2(I'_n) = T^n(0), n \in \mathbb{Z}.$$

We define a conjugacy $h : X \to X$ between f_1 and f_2 in the following way. h restricted to $\mathcal{R}$ sends each I_n onto I'_n in a linear way, and if $z \notin \mathcal{R}$ then there exists a unique $y \in X$ such that

$$h_1(z) = h_2(y).$$

We define $h(z) = y$. It is easy to check that h is an increasing bijective function. Let us prove that it is continuous. In fact, if it were not, then for some point $z_0 \in X$ there would exist a sequence of points z_n converging to z_0 and such that $|h(z_0) - h(z_n)| \geq \epsilon$ for fixed $\epsilon > 0$. But then there would be some (fixed) interval I'_{n_0} between $h(z_0)$ and $h(z_n), n \geq 1$. This situation is not possible since $h^{-1}(I'_{n_0}) = I_{n_0}$ could not be inside the interval $[z_0, z_n]$ if z_n is very close to z_0, contradicting the fact that h is increasing.

Let us prove that f_1 and f_2 cannot be C^1-conjugated. Suppose by contradiction that h is a C^1-conjugacy between them. Then necessarily, $h(I_n) = I'_n, n \in \mathbb{Z}$ and also h' must be a continuous function which is bounded and bounded away from zero. Therefore, for all $n \geq 1$ and some $\xi_n \in I_n$,

$$h'(\xi_n) = \frac{|I'_n|}{|I_n|}.$$

Check that

$$|I_n| = |I_0|\Phi_\zeta(n, 0) \quad \text{and} \quad |I'_n| = |I'_0|\Phi_{t\zeta}(n, 0), \quad n \in \mathbb{Z}.$$

Therefore,

$$\begin{aligned} h'(\xi_n) = \frac{|I'_n|}{|I_n|} &= \frac{|I'_0|\Phi_{t\zeta}(n, 0)}{|I_0|\Phi_\zeta(n, 0)} \\ &= \frac{|I'_0|}{|I_0|}\Phi_\zeta(n, 0)^{t-1}, \end{aligned}$$

because $\Phi_{t\zeta}(n, 0) = \Phi_\zeta(n, 0)^t$.

However, $\lim_{n\to\pm\infty} \Phi_\zeta(n, 0) = 0$ (because $\sum_{n\in\mathbb{Z}} \Phi_\zeta(n, 0) < \infty$), which contradicts the fact that h' is bounded and bounded away from zero.

Suppose now that $\gamma - \gamma' = \xi \in H^s$, and let $f_1 \in S_{\text{aff}}(T_0, e^\gamma)$ and $f_2 \in S_{\text{aff}}(T_0, e^{\gamma'})$. We will prove that f_1 and f_2 are C^1-conjugated. Let h_0 be a solution to the equation

$$h_0(T_0(x)) = \Phi_\xi(x) \cdot h_0(x), \quad x \in X,$$

where $\Phi_\xi(x) = e^{\xi_i}$ if $x \in X_i$. Set $\tilde{f} = h \circ f_2 \circ h^{-1}$ where $h(x) = \int_0^x h_0(s)\, ds$. By Proposition 2.1, $\tilde{f}$ is an affine interval exchange with an associated slope-vector $e^{\xi+\gamma'} = e^\gamma$. However, $h^{-1} \circ h_2$ is a semi-conjugacy between $\tilde{f}$ and T_0. But from Corollary 5.12 we know that $S_{\text{aff}}(T_0, e^\gamma) = \{f_1\}$, therefore, $f_1 = \tilde{f}$ and f_1 is C^1-conjugated to f_2. □

PROPOSITION 5.14. *For all $\gamma \in H^s$, there is a unique $f \in S_{\text{aff}}(T_0, e^\gamma)$ which is C^1-conjugated to T_0. In particular, f has a unique absolutely continuous invariant measure.*

https://doi.org/10.1017/S0143385702000196 Published online by Cambridge University Press

Proof. By Corollary 5.10 the function

$$f(x) = \sup_{k\in\mathbb{Z}} \Phi_\gamma(k, x)^{-1},$$

is well defined and satisfies $f(T(x)) = \Phi_\gamma(x) \cdot f(x)$, $x \in X$. Normalizing the measure $\mu = \int f\, dm$ we obtain an absolutely continuous probability measure in $\Delta_\gamma(T_0)$. By Proposition 2.1, there is an affine interval exchange f_μ in $C_{\text{aff}}(T_0, e^\gamma)$ associated with this μ which is C^1-conjugated to T_0. An argument completely similar to that in Proposition 4.3 shows that every measure in $\Delta_\gamma(T_0)$ is absolutely continuous with respect to the Lebesgue measure and then, by Lemma 2.2, μ is the unique measure in $\Delta_\gamma(T_0)$. □

Acknowledgements. I wish to express my gratitude to Professor Carlos Gutierrez for all the encouragement and support during the preparation of this work. I thank IMPA and IMECC-Unicamp for their hospitality during the preparation of this paper. This research was supported by FAPESP-Brazil, grant number 99/01442-8.

References

[1] R. Barreto. On C^r-class of conjugacy between affine interval exchange maps and its isometric model. *PhD Thesis*, Instituto de Matemática Pura e Aplicada (IMPA), 1999.

[2] R. Camelier and C. Gutierrez. Affine interval exchange transformations with wandering intervals. *Ergod. Th. & Dynam. Sys.* **17** (1997), 1315–1338.

[3] M. Cobo. Affine transformations semi-conjugate to interval exchanges. *PhD Thesis*, Instituto de Matemática Pura e Aplicada (IMPA), 1999.

[4] Z. Coelho, A. Lopes and L. F. da Rocha. Absolutely continuous invariant measures for a class of affine interval exchange maps. *Proc. Amer. Math. Soc.* **123**(11) (1995), 3533–3542.

[5] M. Keane. Interval exchange transformations. *Math. Z.* **141** (1975), 25–31.

[6] G. Levitt. La décomposition dynamique et la différentiabilité des feuilletages des surfaces. *Ann. Inst. Fourier* **37** (1987), 85–116.

[7] I. Liousse and H. Marzougui. Échanges d'intervalles affines conjugueés à des linéaires. *Preprint*.

[8] H. Masur. Interval exchange transformations and measured foliations. *Ann. Math.* **115** (1982), 169–200.

[9] G. Rauzy. Une generalization du developpement en fractions continues. Seminaire Delonge–Pisot–Poitou (1976–1977).

[10] W. Veech. Gauss measures for transformations on the space of interval exchange maps. *Ann. Math.* **115** (1982), 201–242.

[11] W. Veech. Metric theory of interval exchange transformations I, generic spectral properties. *Amer. J. Math.* **106** (1984), 1331–1359.

[12] W. Veech. Metric theory of interval exchange transformations II, approximations by primitive interval exchanges. *Amer. J. Math.* **106** (1984), 1361–1387.

[13] W. Veech. Interval exchange transformations. *J. D'Analyse Math.* **33** (1978), 222–278.

[14] A. Zorich. Deviation for interval exchange transformations. *Ergod. Th. & Dynam. Sys.* **17**(6) (1997), 1477–1499.

[15] A. Zorich. Gauss map on the space of interval exchange transformations. Finiteness of the measure. Lyapunov exponents. *Ann. l'Institute Fourier* **46** (1996), 325–370.

[16] G. Forni. Deviation of ergodic averages for area-preserving vector fields on higher genus surfaces. *Ann. Math.* to appear.

https://doi.org/10.1017/S0143385702000196 Published online by Cambridge University Press