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Towards Metric Temporal Answer Set Programming

PEDRO CABALAR∗
University of Corunna, Spain

MARTÍN DIÉGUEZ
LERIA, Université d'Angers, France

TORSTEN SCHAUB and ANNA SCHUHMANN
University of Potsdam, Germany

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Abstract

We elaborate upon the theoretical foundations of a metric temporal extension of Answer Set Programming. In analogy to previous extensions of ASP with constructs from Linear Temporal and Dynamic Logic, we accomplish this in the setting of the logic of Here-and-There and its non-monotonic extension, called Equilibrium Logic. More precisely, we develop our logic on the same semantic underpinnings as its predecessors and thus use a simple time domain of bounded time steps. This allows us to compare all variants in a uniform framework and ultimately combine them in a common implementation.

1 Introduction

Reasoning about action and change, or more generally reasoning about dynamic systems, is not only central to knowledge representation and reasoning but at the heart of computer science. We addressed this over the last years by combining traditional approaches, like Dynamic and Linear Temporal Logic (DL (Harel et al. 2000) and LTL (Pnueli 1977)), with the base logic of Answer Set Programming (ASP (Lifschitz 1999)), namely, the logic of Here-and-There (HT (Heyting 1930)) and its non-monotonic extension, called Equilibrium Logic (EL (Pearce 1997). This resulted in non-monotonic linear dynamic and temporal equilibrium logics (DEL (Bosser et al. 2018; Cabalar et al. 2019) and TEL (Aguado et al. 2013; Cabalar et al. 2018)) that gave rise to the temporal ASP system *telingo* (Cabalar et al. 2019; Cabalar et al. 2020) extending the full-featured ASP system *clingo* (Gebser et al. 2016). A key design decision has been to base both logics on the same semantic structures so that language constructs from both can be jointly used in an implementation. Another commonality of dynamic and temporal logics is that they abstract from specific time points and rather focus on capturing temporal relationships. For instance, we can express in a temporal logic that a machine has to be eventually cleaned after being used with the formula $\Box(use \rightarrow \Diamond clean)$. However, sometimes this is not enough to capture the desired relation. That is, we might also want to quantify the time difference between both

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events. For instance, whenever the machine is used, it has to be cleaned within less than 5 time units. This can be expressed by means of metric temporal operators as follows:

$$\Box(use \rightarrow \Diamond_5 clean)\ . \tag{1}$$

In this paper, we address this issue and elaborate upon a combination of Metric Temporal Logic (MTL (Alur and Henzinger 1992; Ouaknine and Worrell 2005) [1]) with HT and EL. Our development of *Metric Equilibrium Logic* (MEL) not only parallels the one of TEL and DEL mentioned above but, moreover, builds on the same semantic foundations. This allows us to relate all three systems in a uniform semantic setting and, ultimately, to integrate the corresponding language constructs in a common implementation.

A full version of this paper including proofs of results can be found at `http://arxiv.org/abs/2008.02038`.

2 Metric Equilibrium Logic

Given a set $\mathcal{A}$ of atoms, or *alphabet*, we define a *(metric) formula* φ by the grammar:

$$\varphi ::= a \mid \bot \mid \varphi_1 \bowtie \varphi_2 \mid \bullet\varphi \mid \varphi_1 \mathbf{S}_n \varphi_2 \mid \varphi_1 \mathbf{T}_n \varphi_2 \mid \circ\varphi \mid \varphi_1 \mathbb{U}_n \varphi_2 \mid \varphi_1 \mathbb{R}_n \varphi_2$$

where $a \in \mathcal{A}$ is an atom and $\bowtie \in \{\rightarrow, \wedge, \vee\}$ is a binary Boolean connective; n is a numeral constant (referring to some integer number) or the symbolic constant $\boldsymbol{\ell}$ (standing for the length of a trace; see below). The last six cases of φ correspond to the metric past connectives *previous*, *since*, *trigger*, and their future counterparts *next*, *until*, and *release*, where $n > 0$ restricts the scope of each operator to the last (resp. next) n time points, including the current state.[2]

We also define several derived operators like the Boolean connectives $\top \stackrel{def}{=} \neg\bot$, $\neg\varphi \stackrel{def}{=} \varphi \rightarrow \bot$, $\varphi \leftrightarrow \psi \stackrel{def}{=} (\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$, and the following derived metric operators:

$$\begin{array}{rcl@{\qquad}rcl} \mathbf{I} & \stackrel{def}{=} & \neg\bullet\top & \mathbb{F} & \stackrel{def}{=} & \neg\circ\top \\ \widehat{\bullet}\varphi & \stackrel{def}{=} & \bullet\varphi \vee \mathbf{I} & \widehat{\circ}\varphi & \stackrel{def}{=} & \circ\varphi \vee \mathbb{F} \\ \blacksquare_n\varphi & \stackrel{def}{=} & \bot \mathbf{T}_n \varphi & \Box_n\varphi & \stackrel{def}{=} & \bot \mathbb{R}_n \varphi \\ \blacklozenge_n\varphi & \stackrel{def}{=} & \top \mathbf{S}_n \varphi & \Diamond_n\varphi & \stackrel{def}{=} & \top \mathbb{U}_n \varphi \end{array}$$

On the left, we give past operators, namely, *initial*, *weak previous*, *always before*, *eventually before*, while the right column lists their future counterparts *final*, *weak next*, *always afterward*, *eventually afterward*. We define the iterated application of the one step operators as $\otimes^0\varphi \stackrel{def}{=} \varphi$ and $\otimes^n\varphi \stackrel{def}{=} \otimes \otimes^{n-1} \varphi$ for $n > 0$ and $\otimes \in \{\bullet, \circ, \widehat{\bullet}, \widehat{\circ}\}$. For instance, $\circ^2 p$ corresponds to $\circ\circ p$. Here, the use of n with temporal operators captures a number of iterations of some one step expression so that, as we see below, for instance, $\Diamond_3\varphi$ amounts to $\varphi \vee \circ\varphi \vee \circ^2\varphi$.

An example of metric formulas is the modeling of traffic lights. While the light is red by default, it changes to green within less than 3 time units whenever the button is pushed; and it stays green for other 3 time units. This can be represented by

$$\Box(red \wedge green \rightarrow \bot) \tag{2}$$

[1] Unlike traditional approaches usually having continuous time domains (Alur and Henzinger 1992), we deal with point-based semantics based on discrete linear time, similar to the work of Ouaknine and Worrell in 2005 (Ouaknine and Worrell 2005).

[2] Values $n \leq 0$ are tolerated but trivialize the subformula at hand, as made precise in Proposition 6.

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$$\Box(\neg \mathit{green} \rightarrow \mathit{red}) \tag{3}$$

$$\Box(\mathit{push} \rightarrow \Diamond_3 \Box_4 \mathit{green}) \tag{4}$$

Note that this example combines a default rule (3) with a metric rule (4), describing the initiation and duration period of events. This nicely illustrates the interest in non-monotonic metric representation and reasoning methods.

Given $a \in \mathbb{N}$ and $b \in \mathbb{N} \cup \{\omega\}$, we let $[a..b]$ stand for the set $\{i \in \mathbb{N} \mid a \leq i \leq b\}$, $[a..b)$ for $\{i \in \mathbb{N} \mid a \leq i < b\}$ and $(a..b]$ for $\{i \in \mathbb{N} \mid a < i \leq b\}$. For the semantics, we start by defining a *trace* of length λ over alphabet $\mathcal{A}$ as a sequence $(H_i)_{i\in[0..\lambda)}$ of sets $H_i \subseteq \mathcal{A}$. A trace is *infinite* if $\lambda = \omega$ and *finite* if $\lambda = n$ for some natural number $n \in \mathbb{N}$. Given traces $\mathbf{H} = (H_i)_{i\in[0..\lambda)}$ and $\mathbf{H}' = (H'_i)_{i\in[0..\lambda)}$ both of length λ, we write $\mathbf{H} \leq \mathbf{H}'$ if $H_i \subseteq H'_i$ for each $i \in [0..\lambda)$; accordingly, $\mathbf{H} < \mathbf{H}'$ iff both $\mathbf{H} \leq \mathbf{H}'$ and $\mathbf{H} \neq \mathbf{H}'$.

Our semantics is based on *Here-and-There traces* (for short HT-*traces* (Cabalar et al. 2018)) of length λ over alphabet $\mathcal{A}$ being sequences of pairs $(\langle H_i, T_i\rangle)_{i\in[0..\lambda)}$ such that $H_i \subseteq T_i \subseteq \mathcal{A}$ for any $i \in [0..\lambda)$. We often represent an HT-trace as a pair of traces $\langle \mathbf{H}, \mathbf{T}\rangle$ of length λ where $\mathbf{H} = (H_i)_{i\in[0..\lambda)}$ and $\mathbf{T} = (T_i)_{i\in[0..\lambda)}$ such that $\mathbf{H} \leq \mathbf{T}$. When an HT-trace $\langle \mathbf{H}, \mathbf{T}\rangle$ satisfies $\mathbf{H} = \mathbf{T}$, it is called *total*.

We assume a one-to-one correspondence between numeral constants and integers and let $\overline{n}$ stand for the number corresponding to numeral n. For the symbolic constant $\boldsymbol{\ell}$, we fix $\overline{\boldsymbol{\ell}} = \lambda$ to the length λ of the trace. For simplicity, we let expressions like $n - m$, formed with numeral constants n and m, stand for the numeral representing the difference between $\overline{n}$ and $\overline{m}$.

We define the semantics of metric formulas in terms of HT-traces.

Definition 1 (*Satisfaction*)
Let $\mathbf{M} = \langle \mathbf{H}, \mathbf{T}\rangle$ be an HT-trace of length λ over alphabet $\mathcal{A}$, and let φ be a metric formula over $\mathcal{A}$. The trace $\mathbf{M}$ satisfies φ at time point $k \in [0..\lambda)$, written $\mathbf{M}, k \models \varphi$, if

1. $\mathbf{M}, k \not\models \bot$
2. $\mathbf{M}, k \models a$ iff $a \in H_k$, for any atom $a \in \mathcal{A}$
3. $\mathbf{M}, k \models \varphi \wedge \psi$ iff $\mathbf{M}, k \models \varphi$ and $\mathbf{M}, k \models \psi$
4. $\mathbf{M}, k \models \varphi \vee \psi$ iff $\mathbf{M}, k \models \varphi$ or $\mathbf{M}, k \models \psi$
5. $\mathbf{M}, k \models \varphi \rightarrow \psi$ iff $\langle \mathbf{H}', \mathbf{T}\rangle, k \not\models \varphi$ or $\langle \mathbf{H}', \mathbf{T}\rangle, k \models \psi$, for all $\mathbf{H}' \in \{\mathbf{H}, \mathbf{T}\}$
6. $\mathbf{M}, k \models \bullet\varphi$ iff $k > 0$ and $\mathbf{M}, k{-}1 \models \varphi$
7. $\mathbf{M}, k \models \circ\varphi$ iff $k + 1 < \lambda$ and $\mathbf{M}, k{+}1 \models \varphi$
8. $\mathbf{M}, k \models \varphi \mathbb{U}_n \psi$ iff for some $j \in [0..\overline{n})$ such that $k{+}j \in [0..\lambda)$ we have $\mathbf{M}, k{+}j \models \psi$ and $\mathbf{M}, k{+}i \models \varphi$ for all $i \in [0..j)$
9. $\mathbf{M}, k \models \varphi \mathbb{R}_n \psi$ iff for all $j \in [0..\overline{n})$ such that $k{+}j \in [0..\lambda)$ we have $\mathbf{M}, k{+}j \models \psi$ or $\mathbf{M}, k{+}i \models \varphi$ for some $i \in [0..j)$
10. $\mathbf{M}, k \models \varphi \mathbf{S}_n \psi$ iff for some $j \in [0..\overline{n})$ such that $k{-}j \in [0..\lambda)$ we have $\mathbf{M}, k{-}j \models \psi$ and $\mathbf{M}, k{-}i \models \varphi$ for all $i \in [0..j)$
11. $\mathbf{M}, k \models \varphi \mathbf{T}_n \psi$ iff for all $j \in [0..\overline{n})$ such that $k{-}j \in [0..\lambda)$ we have $\mathbf{M}, k{-}j \models \psi$ or $\mathbf{M}, k{-}i \models \varphi$ for some $i \in [0..j)$

The fundamental difference to standard temporal logics is clearly the satisfaction of implication '$\rightarrow$' that is inherited from the (non-temporal) logic HT, an intermediate logic dealing with exactly two worlds h, t with the reflexive accessibility relation $h \leq t$. When traces are total $\langle \mathbf{T}, \mathbf{T}\rangle$, we get $h = t$ and this distinction disappears, so implication

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becomes classical. From the perspective of metric temporal logic, the definition of release and trigger in *9* and *11*, respectively, are additionally conditioned by the trace's limits and can thus be seen as weak variants of the standard counterparts. Similarly, the satisfaction of until and since formulas in *8* and *10*, respectively, is restricted to the time points within a trace. Clearly, for infinite traces, the restriction of future operators vanishes.

An HT-trace $\mathbf{M}$ is a *model* of a metric theory Γ if $\mathbf{M}, 0 \models \varphi$ for all $\varphi \in \Gamma$. A formula φ is a *tautology* (or is *valid*), written $\models \varphi$, iff $\mathbf{M}, k \models \varphi$ for any HT-trace $\mathbf{M}$ and any $k \in [0..\lambda)$. We call the logic induced by the set of all tautologies *Metric logic of Here and There* (MHT for short). We say that an HT-trace $\mathbf{M}$ is a *model* of a set of formulas (or *theory*) Γ iff $\mathbf{M}, 0 \models \varphi$ for any $\varphi \in \Gamma$. Two formulas φ, ψ are *equivalent* if $\models \varphi \leftrightarrow \psi$. Whenever two formulas φ and ψ are equivalent, they are completely interchangeable in any theory without altering the theory's semantics.

We write $\mathrm{MHT}(\Gamma, \lambda)$ to stand for the set of models of length λ of a theory Γ, and define $\mathrm{MHT}(\Gamma) \stackrel{def}{=} \mathrm{MHT}(\Gamma, \omega) \cup \bigcup_{\lambda \geq 0} \mathrm{MHT}(\Gamma, \lambda)$, that is, the whole set of models of Γ of any length. An interesting subset of $\mathrm{MHT}(\Gamma, \lambda)$ is the one formed by total traces $\langle \mathbf{T}, \mathbf{T} \rangle$, we denote as $\mathrm{MTL}(\Gamma, \lambda)$. We also use $\mathrm{MTL}(\Gamma)$ to stand for $\mathrm{MHT}(\Gamma, \omega) \cup \bigcup_{\lambda \geq 0} \mathrm{MTL}(\Gamma, \lambda)$. In the non-metric version of temporal HT, the restriction to total models turns out to correspond to Linear Temporal Logic (LTL). In our case, it defines a metric version of LTL that we call *Metric Temporal Logic* (MTL for short). It can be proved that $\mathrm{MTL}(\Gamma, \lambda)$ are those models of $\mathrm{MHT}(\Gamma, \lambda)$ satisfying the *excluded middle axiom* schema:

$$\Box_{\boldsymbol{\ell}}(p \vee \neg p) \qquad \text{(for any atom } p \in \mathcal{A}\text{)} \tag{5}$$

The semantics of the derived operators in MHT can be easily deduced.

Proposition 1 (*Satisfaction*)
Let $\mathbf{M} = \langle \mathbf{H}, \mathbf{T} \rangle$ be an HT-trace of length λ over $\mathcal{A}$. Given the respective definitions of derived operators, we get the following satisfaction conditions:

12. $\mathbf{M}, k \models \blacksquare_n \varphi$ iff for all $j \in [0..\overline{n})$ such that $k-j \in [0..\lambda)$ we have $\mathbf{M}, k-j \models \psi$
13. $\mathbf{M}, k \models \blacklozenge_n \varphi$ iff for some $j \in [0..\overline{n})$ such that $k-j \in [0..\lambda)$ we have $\mathbf{M}, k-j \models \psi$
14. $\mathbf{M}, k \models \mathbf{I}$ iff $k = 0$
15. $\mathbf{M}, k \models \Box_n \varphi$ iff for all $j \in [0..\overline{n})$ such that $k+j \in [0..\lambda)$ we have $\mathbf{M}, k+j \models \psi$
16. $\mathbf{M}, k \models \Diamond_n \varphi$ iff for some $j \in [0..\overline{n})$ such that $k+j \in [0..\lambda)$ we have $\mathbf{M}, k+j \models \psi$
17. $\mathbf{M}, k \models \mathbb{F}$ iff $k + 1 = \lambda$
18. $\mathbf{M}, k \models \widehat{\bullet} \varphi$ iff $k = 0$ or $\mathbf{M}, k-1 \models \varphi$
19. $\mathbf{M}, k \models \widehat{\circ} \varphi$ iff $k + 1 = \lambda$ or $\mathbf{M}, k+1 \models \varphi$

As in the temporal logic of HT (Cabalar et al. 2018), referred to as THT, the operators $\mathbf{I}$ and $\mathbb{F}$ exclusively depend on the value of time point k, and are thus independent of $\mathbf{M}$. In fact, operator $\mathbb{F}$ allows us to influence the length of models. The inclusion of the axiom $\Diamond_n \mathbb{F}$ for example forces its models to have length $\lambda \leq \overline{n}$ with $\overline{n} \in \mathbb{N}$. On the other hand, the inclusion of the axiom $\neg \Diamond_{\boldsymbol{\ell}} \mathbb{F}$ forces models to be of infinite length. As well, we distinguish MHT on finite and infinite traces, and refer to the respective logics as MHT_f and MHT_ω.

Following the definitions of TEL (Cabalar et al. 2018) and DEL (Cabalar et al. 2019), we now introduce non-monotonicity by selecting a particular set of traces that we call *temporal equilibrium models*. First, given an arbitrary set $\mathfrak{S}$ of HT-traces, we define the ones in equilibrium as follows.

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Definition 2 (*Temporal Equilibrium/Stable models*)
Let $\mathfrak{S}$ be some set of HT-traces. A total HT-trace $\langle \mathbf{T}, \mathbf{T} \rangle \in \mathfrak{S}$ is an *equilibrium trace* of $\mathfrak{S}$ iff there is no other $\langle \mathbf{H}, \mathbf{T} \rangle \in \mathfrak{S}$ such that $\mathbf{H} < \mathbf{T}$.

If $\langle \mathbf{T}, \mathbf{T} \rangle$ is such an equilibrium trace, we also say that trace $\mathbf{T}$ is a *stable trace* of $\mathfrak{S}$. We further talk about *temporal equilibrium* or *temporal stable models* of a theory Γ when $\mathfrak{S} = \mathrm{MHT}(\Gamma)$, respectively.

We write $\mathrm{MEL}(\Gamma, \lambda)$ and $\mathrm{MEL}(\Gamma)$ to stand for the temporal equilibrium models of $\mathrm{MHT}(\Gamma, \lambda)$ and $\mathrm{MHT}(\Gamma)$, respectively. Besides, as the ordering relation among traces is only defined for a fixed λ, it is easy to see the following result:

Proposition 2
The set of temporal equilibrium models of Γ can be partitioned by the trace length λ, that is, $\bigcup_{\lambda=0}^{\omega} \mathrm{MEL}(\Gamma, \lambda) = \mathrm{MEL}(\Gamma)$.

Metric Equilibrium Logic (MEL) is the non-monotonic logic induced by temporal equilibrium models of metric theories. We obtain the variants MEL_f and MEL_ω by applying the respective restriction to finite or infinite traces, respectively.

Let us illustrate this by using the example of the pedestrian traffic light introduced above. Consider the models of the theory $\Gamma = \{(2), (3), (4)\}$ for length $\lambda = 1$. In this case, we only have time point $k = 0$, and the metric or temporal aspect is less interesting, since HT-traces amount to pairs $\langle H_0, T_0 \rangle$. Still, this helps to illustrate the difference among the different sets of models defined above. In the example, we abbreviate a set of atoms as a string formed by their initials: for instance *gp* stands for $\{green, push\}$. Then, we obtain the following sets:

$$\begin{aligned}
\mathrm{MTL}(\Gamma, 1) &= \{\ \langle r \rangle,\ \langle g \rangle,\ \langle gp \rangle\ \} \\
\mathrm{MHT}(\Gamma, 1) &= \mathrm{MTL}(\Gamma, 1) \cup \{\ \langle \emptyset, g \rangle,\ \langle \emptyset, gp \rangle,\ \langle g, gp \rangle\ \} \\
\mathrm{MEL}(\Gamma, 1) &= \{\ \langle r \rangle\ \}
\end{aligned}$$

As we can see, total models $\mathrm{MTL}(\Gamma, 1)$ allow for choosing either *red* or *green* but, if we include *push*, then *green* is mandatory because it is the only way to satisfy $\Diamond_3 \Box_4 green$ with $\lambda = 1$. Note how the unique equilibrium model in $\mathrm{MEL}(\Gamma, 1)$ is the only total model $\langle \mathbf{T}, \mathbf{T} \rangle$ with $\mathbf{T} = \langle r \rangle$ in $\mathrm{MTL}(\Gamma, 1)$ for which there is no $\langle \mathbf{H}, \mathbf{T} \rangle \in \mathrm{MHT}(\Gamma, 1)$ with smaller $\mathbf{H} < \mathbf{T}$. Informally, this is because (3) suffices to justify *red* by default, while the other two total models in $\mathrm{MTL}(\Gamma, 1)$, which assume *green* or *push* in T_0, admit weaker H_0's where these atoms are not justified. As a result, $\mathrm{MEL}(\Gamma, \lambda) = \langle T_i \rangle_{i \in [0..\lambda)}$ with $T_i = \{red\}$ for all $i \in [0..\lambda)$. To illustrate non-monotonicity, suppose we add the formula

$$\circ push \tag{6}$$

that ensures the button is pushed in the second state of the trace (i.e. at $k = 1$) and take $\lambda = 3$. For readability sake, we represent traces (T_0, T_1, T_2) as $T_0 \cdot T_1 \cdot T_2$. For length $\lambda = 3$, formula (4) amounts in MTL to requiring *green* at $k = 2$ whenever *push* holds at any point. Thus, given $\Gamma' = \Gamma \cup \{(6)\}$ and $\lambda = 3$, we get

$$\mathrm{MTL}(\Gamma', 3) = \{\langle \mathbf{T} \rangle = \langle T_0 \cdot T_1 \cdot T_2 \rangle \mid T_0 \in \{r, g, rp, gp\}, T_1 \in \{rp, gp\}, T_2 \in \{g, gp\}\}$$

Now, for those $\langle \mathbf{T}, \mathbf{T} \rangle$ with $T_0 \neq r$ and $T_2 = gp$, we always have, among others, a smaller model $\langle \emptyset \cdot T_1 \cdot g, \mathbf{T} \rangle$ in $\mathrm{MHT}(\Gamma', 3)$. This means that, in MEL, we conclude by default that

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we do not push in other situations $k \neq 1$ and the traffic light is red at the initial state. From the remaining possible total models $\langle r \cdot rp \cdot g \rangle$ and $\langle r \cdot gp \cdot g \rangle$, the latter is not in equilibrium, since $\langle r \cdot p \cdot g, \quad r \cdot gp \cdot g \rangle$ is an model in MHT that reveals that *green* at $k = 0$ is not justified. As a result MEL$(\Gamma', 3)$ contains the unique temporal equilibrium model $\langle r \cdot rp \cdot g \rangle$ and the MEL conclusion $\bigcirc^2 r$, we could obtain from Γ alone, is not derived any more once (6) is added to the theory. In this example, we obtain one equilibrium model because $\Box_4 green$ becomes trivially true on traces shorter than $\lambda = 4$. When the trace is long enough ($\lambda \geq 7$), Γ' generates the three expected temporal equilibrium models:

$$\begin{array}{l} \langle r \cdot gp \cdot g \cdot g \cdot g \cdot r \cdot r \cdot r \ \ \ldots \rangle \\ \langle r \cdot rp \cdot g \cdot g \cdot g \cdot g \cdot r \cdot r \ \ \ldots \rangle \\ \langle r \cdot rp \cdot r \cdot g \cdot g \cdot g \cdot g \cdot r \ \ \ldots \rangle \end{array}$$

In the following, we elaborate on the formal characteristics of our approach. At first, we show that a basic property of HT is maintained in MHT:

Proposition 3 (*Persistence*)
Let $\langle \mathbf{H}, \mathbf{T} \rangle$ be an HT-trace of length λ and φ be a metric formula. Then, for any $k \in [0..\lambda)$,

1. if $\langle \mathbf{H}, \mathbf{T} \rangle, k \models \varphi$ then $\langle \mathbf{T}, \mathbf{T} \rangle, k \models \varphi$.
2. $\langle \mathbf{H}, \mathbf{T} \rangle, k \models \neg\varphi$ iff $\langle \mathbf{T}, \mathbf{T} \rangle, k \not\models \varphi$

All MHT tautologies are MTL tautologies but not vice versa (cf. (5) above). However, they coincide for some types of equivalences, as stated below.

Proposition 4
Let φ and ψ be metric formulas without implications (and so, without negations either). Then, $\varphi \equiv \psi$ in MTL iff $\varphi \equiv \psi$ in MHT.

Another useful tool that can save some effort when proving groups of equivalences in MHT is the use of duality properties. A first type of duality has to do with the temporal direction (future or past) of the modal operators. Let $\bigcirc/\bullet$, $\widehat{\bigcirc}/\widehat{\bullet}$, $\mathbb{U}_n/\mathbf{S}_n$, $\mathbb{R}_n/\mathbf{T}_n$, $\Box_n/\blacksquare_n$ and $\Diamond_n/\blacklozenge_n$ denote all pairs of swapped-time connectives and let $\sigma(\varphi)$ denote the replacement in φ of each connective by its swapped-time version. If we restrict ourselves to finite traces, we get the following result.

Lemma 1
Let $\mathbf{M}$ be an HT-trace of length λ and φ be a metric formula. Then, there exists an HT-trace $\mathbf{M}'$ of length λ such that $\mathbf{M}, k \models \varphi$ iff $\mathbf{M}', n-k \models \sigma(\varphi)$ for any $k \in [0..\lambda)$.

Theorem 1 (*Temporal Duality Theorem*)
A metric formula φ is a tautology in MHT$_f$ iff $\sigma(\varphi)$ is a tautology in MHT$_f$.

For instance, suppose we obtain a proof for

$$\Diamond_n \, p \leftrightarrow p \vee \bigcirc \Diamond_{n-1} \, p \tag{7}$$

Then, we can immediately apply Theorem 1 to guarantee that $\blacklozenge_n p \leftrightarrow p \vee \bullet \blacklozenge_{n-1} \, p$ is a tautology too. A second kind of duality has to do the analogy between Boolean disjunction and conjunction. Let us define all the pairs of dual connectives as follows: $\wedge/\vee$, $\top/\bot$, $\mathbb{U}_n/\mathbb{R}_n$, $\bigcirc/\widehat{\bigcirc}$, $\Diamond_n/\Box_n$, $\mathbf{S}_n/\mathbf{T}_n$, $\bullet/\widehat{\bullet}$, $\blacklozenge_n/\blacksquare_n$. For a formula φ without implications, we define $\delta(\varphi)$ as the result of replacing each connective by its dual operator. Then, we get the following result.

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Proposition 5 (*Boolean Duality*)
Let φ and ψ be formulas without implication.[3] Then, $\varphi \leftrightarrow \psi$ is a tautology in MHT iff $\delta(\varphi) \leftrightarrow \delta(\psi)$ is a tautology in MHT.

Following with our example of equivalence (7), we can now apply Theorem 5 to conclude:

$$\Box_n\, p \leftrightarrow p \wedge \widehat{\circ}\Box_{n-1}\, p$$

Next, we show how metric operators on formulas can be characterized inductively.

Proposition 6
The following formulas are valid in MHT. For any numeral n with $\overline{n} \leq 0$:

$$\varphi \mathbb{U}_n \psi \leftrightarrow \bot \quad \varphi \mathbb{R}_n \psi \leftrightarrow \top \quad \varphi \mathbf{S}_n \psi \leftrightarrow \bot \quad \varphi \mathbf{T}_n \psi \leftrightarrow \top$$

For any numeral n with $\overline{n} > 0$, we have

$$\varphi \mathbb{U}_n \psi \leftrightarrow \psi \vee (\varphi \wedge \circ(\varphi \mathbb{U}_{n-1} \psi)) \quad \varphi \mathbb{R}_n \psi \leftrightarrow \psi \wedge \big(\varphi \vee \widehat{\circ}(\varphi \mathbb{R}_{n-1} \psi)\big)$$
$$\varphi \mathbf{S}_n \psi \leftrightarrow \psi \vee (\varphi \wedge \bullet(\varphi \mathbf{S}_{n-1} \psi)) \quad \varphi \mathbf{T}_n \psi \leftrightarrow \psi \wedge \big(\varphi \vee \widehat{\bullet}(\varphi \mathbf{T}_{n-1} \psi)\big)$$

The propositions above allow us to unfold metric operators containing numerals. It is easy to see that, for $n = 1$, the four operators collapse to the formula ψ on their right. For instance, $\Diamond_5 \mathit{clean} \leftrightarrow \mathit{clean} \vee \circ\Diamond_4 \mathit{clean}$ whereas $\Diamond_1 \mathit{clean} \leftrightarrow \mathit{clean}$ and $\Diamond_0 \mathit{clean} \leftrightarrow \bot$.

For metric operators depending on the trace length, the value of which is not necessarily known, Proposition 6 cannot be applied. Instead, we have the following tautologies.

Proposition 7
The following formulas are valid in MHT:

$$\varphi \mathbb{U}_{\boldsymbol{\ell}} \psi \leftrightarrow \psi \vee (\varphi \wedge \circ(\varphi \mathbb{U}_{\boldsymbol{\ell}} \psi)) \quad \varphi \mathbb{R}_{\boldsymbol{\ell}} \psi \leftrightarrow \psi \wedge \big(\varphi \vee \widehat{\circ}(\varphi \mathbb{R}_{\boldsymbol{\ell}} \psi)\big)$$
$$\varphi \mathbf{S}_{\boldsymbol{\ell}} \psi \leftrightarrow \psi \vee (\varphi \wedge \bullet(\varphi \mathbf{S}_{\boldsymbol{\ell}} \psi)) \quad \varphi \mathbf{T}_{\boldsymbol{\ell}} \psi \leftrightarrow \psi \wedge \big(\varphi \vee \widehat{\bullet}(\varphi \mathbf{T}_{\boldsymbol{\ell}} \psi)\big)$$

That is, when the limit is the trace length $\boldsymbol{\ell}$, the unfolding contains the same operator, it is not altered. As an example, we consider the machine that has to be cleaned eventually before the end of the trace. We then have $\Diamond_{\boldsymbol{\ell}} \mathit{clean} \leftrightarrow \mathit{clean} \vee \circ\Diamond_{\boldsymbol{\ell}} \mathit{clean}$.

Alternatively, metric operators may also be parametrized by intervals[4] rather than a mere upper bound. In our setting, this is however no restriction since such metric operators can also be expressed, as we show next.

The definition of our interval operators then depends on the type of numeral expression involved. For numeral constants n, m where $\overline{n}, \overline{m} \in [0..\lambda)$, we define:

$$\blacksquare_{[m;n)}\, \varphi \stackrel{def}{=} \widehat{\bullet}^{\overline{m}}\, \blacksquare_{n-m} \varphi \qquad \Box_{[m;n)}\, \varphi \stackrel{def}{=} \widehat{\circ}^{\overline{m}}\, \Box_{n-m} \varphi$$
$$\blacklozenge_{[m;n)}\, \varphi \stackrel{def}{=} \bullet^{\overline{m}}\, \blacklozenge_{n-m} \varphi \qquad \Diamond_{[m;n)}\, \varphi \stackrel{def}{=} \circ^{\overline{m}}\, \Diamond_{n-m} \varphi$$
$$\varphi\, \mathbf{S}_{[m;n)}\, \psi \stackrel{def}{=} \bullet^{\overline{m}}\, (\varphi \mathbf{S}_{n-m} \psi) \qquad \varphi\, \mathbb{U}_{[m;n)}\, \psi \stackrel{def}{=} \circ^{\overline{m}}\, (\varphi \mathbb{U}_{n-m} \psi)$$
$$\varphi\, \mathbf{T}_{[m;n)}\, \psi \stackrel{def}{=} \widehat{\bullet}^{\overline{m}}\, (\varphi \mathbf{T}_{n-m} \psi) \qquad \varphi\, \mathbb{R}_{[m;n)}\, \psi \stackrel{def}{=} \widehat{\circ}^{\overline{m}}\, (\varphi \mathbb{R}_{n-m} \psi)$$

[3] Note that this also means without negation.
[4] This concept of interval operators should not be confounded with the ones of Allen's interval algebra (Allen 1983).

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For intervals spanning to the end (or beginning) of the trace, we have:

$$\blacksquare_{[m;\boldsymbol{\ell})}\,\varphi \stackrel{def}{=} \widehat{\bullet}^{\overline{m}}\,\blacksquare_{\boldsymbol{\ell}}\varphi \qquad \blacklozenge_{[m;\boldsymbol{\ell})}\,\varphi \stackrel{def}{=} \bullet^{\overline{m}}\,\blacklozenge_{\boldsymbol{\ell}}\varphi$$
$$\varphi\,\mathbf{S}_{[m;\boldsymbol{\ell})}\,\psi \stackrel{def}{=} \bullet^{\overline{m}}\,(\varphi\mathbf{S}_{\boldsymbol{\ell}}\psi) \qquad \varphi\,\mathbf{T}_{[m;\boldsymbol{\ell})}\,\psi \stackrel{def}{=} \widehat{\bullet}^{\overline{m}}\,(\varphi\mathbf{T}_{\boldsymbol{\ell}}\psi)$$

$$\Box_{[m;\boldsymbol{\ell})}\,\varphi \stackrel{def}{=} \widehat{\circ}^{\overline{m}}\,\Box_{\boldsymbol{\ell}}\varphi \qquad \Diamond_{[m;\boldsymbol{\ell})}\,\varphi \stackrel{def}{=} \circ^{\overline{m}}\,\Diamond_{\boldsymbol{\ell}}\varphi$$
$$\varphi\,\mathbb{U}_{[m;\boldsymbol{\ell})}\,\psi \stackrel{def}{=} \circ^{\overline{m}}\,(\varphi\mathbb{U}_{\boldsymbol{\ell}}\psi) \qquad \varphi\,\mathbb{R}_{[m;\boldsymbol{\ell})}\,\psi \stackrel{def}{=} \widehat{\circ}^{\overline{m}}\,(\varphi\mathbb{R}_{\boldsymbol{\ell}}\psi)$$

As an example of an interval formula, reconsider the machine, discussed in the beginning, and assume that it cannot be cleaned immediately but only within 3 to 5 time steps after usage. This can be expressed as $\Box(use \rightarrow \Diamond_{[3;5)}\, clean)$. In the definitions, note that when $m \geq n$ the interval $[m;n)$ is empty and the operator is always reducible to a truth constant. For instance, $\Diamond_{[5;3)}\varphi$ becomes $\circ^5\Diamond_{-2}\varphi$ which amounts to $\circ^5\bot$ or simply $\bot$. For this reason, we do not consider intervals with $\boldsymbol{\ell}$ as lower bound, since they are always empty by definition.

Proposition 8 (*Satisfaction*)
Let $\mathbf{M} = \langle \mathbf{H}, \mathbf{T}\rangle$ be an HT-trace of length λ over $\mathcal{A}$. Given the respective definitions of derived operators, we get the following satisfaction conditions:

20. $\mathbf{M}, k \models \varphi\mathbb{U}_{[m;n)}\psi$ iff for some $j \in [\overline{m}..\overline{n})$ such that $k{+}j \in [0..\lambda)$ we have $\mathbf{M}, k{+}j \models \psi$ and $\mathbf{M}, k{+}i \models \varphi$ for all $i \in [0..j)$
21. $\mathbf{M}, k \models \varphi\mathbb{R}_{[m;n)}\psi$ iff for all $j \in [\overline{m}..\overline{n})$ such that $k{+}j \in [0..\lambda)$, we have $\mathbf{M}, k{+}j \models \psi$ or $\mathbf{M}, k{+}i \models \varphi$ for some $i \in [0..j)$
22. $\mathbf{M}, k \models \varphi\mathbf{S}_{[m;n)}\psi$ iff for some $j \in [\overline{m}..\overline{n})$ such that $k{-}j \in [0..\lambda)$ we have $\mathbf{M}, k{-}j \models \psi$ and $\mathbf{M}, k{-}i \models \varphi$ for all $i \in [0..j)$
23. $\mathbf{M}, k \models \varphi\mathbf{T}_{[m;n)}\psi$ iff for all $j \in [\overline{m}..\overline{n})$ such that $k{-}j \in [0..\lambda)$, we have $\mathbf{M}, k{-}j \models \psi$ or $\mathbf{M}, k{-}i \models \varphi$ for some $i \in [0..j)$
24. $\mathbf{M}, k \models \Diamond_{[m;n)}\varphi$ iff for some $j \in [\overline{m}..\overline{n})$ such that $k{+}j \in [0..\lambda)$ we have $\mathbf{M}, k{+}j \models \varphi$
25. $\mathbf{M}, k \models \Box_{[m;n)}\varphi$ iff for all $j \in [\overline{m}..\overline{n})$ such that $k{+}j \in [0..\lambda)$, we have $\mathbf{M}, k{+}j \models \varphi$
26. $\mathbf{M}, k \models \blacklozenge_{[m;n)}\varphi$ iff for some $j \in [\overline{m}..\overline{n})$ such that $k{-}j \in [0..\lambda)$ we have $\mathbf{M}, k{-}j \models \varphi$
27. $\mathbf{M}, k \models \blacksquare_{[m;n)}\varphi$ iff for all $j \in [\overline{m}..\overline{n})$ such that $k{-}j \in [0..\lambda)$, we have $\mathbf{M}, k{-}j \models \varphi$

Note that one-step operators can be represented in terms of intervals, since we have

$$\circ\varphi = \circ\Diamond_1\varphi = \Diamond_{[1;2)}\varphi \qquad \bullet\varphi = \bullet\blacklozenge_1\varphi = \blacklozenge_{[1;2)}\varphi$$
$$\widehat{\circ}\varphi = \widehat{\circ}\Box_1\varphi = \Box_{[1;2)}\varphi \qquad \widehat{\bullet}\varphi = \widehat{\bullet}\blacksquare_1\varphi = \blacksquare_{[1;2)}\varphi$$

Next, we present a three-valued semantics for MHT which turns out to be particularly useful for formal elaborations. In particular, this three-valued interpretation has an important advantage: it allows the interchange of subformulas in a larger formula provided that the interchanged subformulas have the same three-valued interpretations. This characterization relies on temporal three-valued interpretation (Cabalar 2010) and is inspired, in its turn, by the characterization of HT in terms of Gödel's logic G_3 (Gödel 1932). Under this orientation, we deal with three truth values $\{0, 1, 2\}$ standing for: 2 (or proved true) meaning satisfaction "here"; 0 (or assumed false) meaning falsity "there"; and 1 (potentially true) for formulas assumed but not proved to be true. Given an HT-trace $\langle \mathbf{H}, \mathbf{T}\rangle$, we define its associated *truth valuation* as a function $\boldsymbol{m}(k, \varphi)$ that assigns a truth value in the set $\{0, 1, 2\}$ to a metric formula φ at time point $k \in [0..\lambda)$ as follows. For propositional connectives, $\boldsymbol{m}(k, \varphi)$ directly corresponds to G_3, that is,

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conjunction is the minimum, disjunction the maximum and implication $\boldsymbol{m}(k, \varphi \rightarrow \psi)$ is 2 if $\boldsymbol{m}(k, \varphi) \leq \boldsymbol{m}(k, \psi)$, or is $\boldsymbol{m}(k, \psi)$ otherwise. For the rest of cases, we have:

$$
\begin{array}{rcl}
\boldsymbol{m}(k, \bot) & \stackrel{def}{=} & 0 \\
\boldsymbol{m}(k, \top) & \stackrel{def}{=} & 2 \\
\boldsymbol{m}(k, p) & \stackrel{def}{=} & \begin{cases} 0 & \text{if } p \notin T_k \\ 1 & \text{if } p \in T_k \setminus H_k \\ 2 & \text{if } p \in H_k \end{cases} \quad \text{for any atom } p \in \mathcal{A} \\
\boldsymbol{m}(k, \circ\varphi) & \stackrel{def}{=} & \begin{cases} 0 & \text{if } k+1 = \lambda; \\ \boldsymbol{m}(k+1, \varphi) & \text{otherwise} \end{cases} \\
\boldsymbol{m}(k, \widehat{\circ}\varphi) & \stackrel{def}{=} & \begin{cases} 2 & \text{if } k+1 = \lambda; \\ \boldsymbol{m}(k+1, \varphi) & \text{otherwise} \end{cases} \\
\boldsymbol{m}(k, \bullet\varphi) & \stackrel{def}{=} & \begin{cases} 0 & \text{if } k = 0; \\ \boldsymbol{m}(k-1, \varphi) & \text{otherwise} \end{cases} \\
\boldsymbol{m}(k, \widehat{\bullet}\varphi) & \stackrel{def}{=} & \begin{cases} 2 & \text{if } k = 0; \\ \boldsymbol{m}(k-1, \varphi) & \text{otherwise} \end{cases}
\end{array}
$$

For n being a numeral constant or the constant $\boldsymbol{\ell}$, we have[5]

$$
\begin{array}{rcl}
\boldsymbol{m}(k, \varphi \mathbb{U}_n \psi) & \stackrel{def}{=} & \max\{\min\{\boldsymbol{m}(k+i, \psi), \boldsymbol{m}(k+j, \varphi) \mid j \in [0..i),\ k+i < \lambda\} \mid 0 \leq i < \overline{n}\} \\
\boldsymbol{m}(k, \varphi \mathbb{R}_n \psi) & \stackrel{def}{=} & \min\{\max\{\boldsymbol{m}(k+i, \psi), \boldsymbol{m}(k+j, \varphi) \mid j \in [0..i),\ k+i < \lambda\} \mid 0 \leq i < \overline{n}\} \\
\boldsymbol{m}(k, \varphi \mathbf{S}_n \psi) & \stackrel{def}{=} & \max\{\min\{\boldsymbol{m}(k-i, \psi), \boldsymbol{m}(k-j, \varphi) \mid j \in [0..i),\ k-i \geq 0\} \mid 0 \leq i < \overline{n}\} \\
\boldsymbol{m}(k, \varphi \mathbf{T}_n \psi) & \stackrel{def}{=} & \min\{\max\{\boldsymbol{m}(k-i, \psi), \boldsymbol{m}(k-j, \varphi) \mid j \in [0..i),\ k-i \geq 0\} \mid 0 \leq i < \overline{n}\}
\end{array}
$$

Proposition 9
Let $\langle \mathbf{H}, \mathbf{T} \rangle$ be a HT-trace of length λ, $\boldsymbol{m}$ its associated valuation and $k \in [0..\lambda)$. Then, for any formula φ, we have

- $\langle \mathbf{H}, \mathbf{T} \rangle, k \models \varphi$ iff $\boldsymbol{m}(k, \varphi) = 2$ and
- $\langle \mathbf{T}, \mathbf{T} \rangle, k \models \varphi$ iff $\boldsymbol{m}(k, \varphi) \neq 0$.

3 Metric and Temporal Equilibrium Logic

In this section, we study the relation between metric and temporal (equilibrium) logics. In fact, temporal formulas constitute a subclass of metric formulas.

$$
\begin{array}{cccc}
\Diamond\varphi \stackrel{def}{=} \Diamond_{\boldsymbol{\ell}}\varphi & \Box\varphi \stackrel{def}{=} \Box_{\boldsymbol{\ell}}\varphi & \blacklozenge\varphi \stackrel{def}{=} \blacklozenge_{\boldsymbol{\ell}}\varphi & \blacksquare\varphi \stackrel{def}{=} \blacksquare_{\boldsymbol{\ell}}\varphi \\
\varphi \,\mathcal{U}\, \psi \stackrel{def}{=} \varphi \mathbb{U}_{\boldsymbol{\ell}} \psi & \varphi \,\mathcal{R}\, \psi \stackrel{def}{=} \varphi \mathbb{R}_{\boldsymbol{\ell}} \psi & \varphi \,\mathbf{S}\, \psi \stackrel{def}{=} \varphi \mathbf{S}_{\boldsymbol{\ell}} \psi & \varphi \,\mathbf{T}\, \psi \stackrel{def}{=} \varphi \mathbf{T}_{\boldsymbol{\ell}} \psi
\end{array}
$$

The next result guarantees that the original semantics of the temporal operators in THT (Cabalar et al. 2018) is preserved.

[5] Here, we assume that $\max(\emptyset) = 0$ and $\min(\emptyset) = 2$.

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Proposition 10 (*Satisfaction*)
Let $\mathbf{M} = \langle \mathbf{H}, \mathbf{T} \rangle$ be an HT-trace of length λ over $\mathcal{A}$. Given the respective definitions of derived operators, we get the following satisfaction conditions:

28. $\mathbf{M}, k \models \Diamond\varphi$ iff $\mathbf{M}, i \models \varphi$ for some $i \in [k..\lambda)$
29. $\mathbf{M}, k \models \Box\varphi$ iff $\mathbf{M}, i \models \varphi$ for all $i \in [k..\lambda)$
30. $\mathbf{M}, k \models \varphi \,\mathbb{U}\, \psi$ iff for some $j \in [k..\lambda)$, we have $\mathbf{M}, j \models \psi$ and $\mathbf{M}, i \models \varphi$ for all $i \in [k..j)$
31. $\mathbf{M}, k \models \varphi \,\mathbb{R}\, \psi$ iff for all $j \in [k..\lambda)$, we have $\mathbf{M}, j \models \psi$ or $\mathbf{M}, i \models \varphi$ for some $i \in [k..j)$
32. $\mathbf{M}, k \models \blacklozenge\varphi$ iff $\mathbf{M}, i \models \varphi$ for some $i \in [0..k]$
33. $\mathbf{M}, k \models \blacksquare\varphi$ iff $\mathbf{M}, i \models \varphi$ for all $i \in [0..k]$
34. $\mathbf{M}, k \models \varphi \,\mathbf{S}\, \psi$ iff for some $j \in [0..k]$, we have $\mathbf{M}, j \models \psi$ and $\mathbf{M}, i \models \varphi$ for all $i \in (j..k]$
35. $\mathbf{M}, k \models \varphi \,\mathbf{T}\, \psi$ iff for all $j \in [0..k]$, we have $\mathbf{M}, j \models \psi$ or $\mathbf{M}, i \models \varphi$ for some $i \in (j..k]$

Interestingly, it turns out that metric formulas can also be translated into temporal formulas. This is due to the discrete time domain of MHT and the resulting semantic structure common to MHT and THT. In fact, we provide two alternative translations possessing complementary properties.

Language-preserving translation. Our first translation refrains from extending the original language and is independent of the length of the trace (and so the specific value of $\boldsymbol{\ell}$). Although this allows us to search for models of varying length without recompiling a formula, when using temporal ASP solvers such as *telingo* (Cabalar et al. 2019), the translation suffers from an exponential blowup in the worst case.

We define the translation recursively as follows:

$$
\begin{aligned}
\tau(a) &\stackrel{def}{=} a \text{ for } a \in \mathcal{A} \\
\tau(\oplus\varphi) &\stackrel{def}{=} \oplus\tau(\varphi) \text{ for } \oplus \in \{\neg, \bullet, \circ, \widehat{\bullet}, \widehat{\circ}\} \\
\tau(\varphi \otimes \psi) &\stackrel{def}{=} \tau(\varphi) \otimes \tau(\psi) \text{ for } \otimes \in \{\wedge, \vee, \rightarrow\} \\
\tau(\varphi \otimes_{\boldsymbol{\ell}} \psi) &\stackrel{def}{=} \tau(\varphi) \otimes \tau(\psi) \text{ for } \otimes \in \{\mathbb{U}, \mathbb{R}, \mathbf{S}, \mathbf{T}\} \\
\tau(\varphi \otimes_1 \psi) &\stackrel{def}{=} \tau(\psi) \text{ for } \otimes \in \{\mathbb{U}, \mathbb{R}, \mathbf{S}, \mathbf{T}\} \\
\tau(\varphi \,\mathbb{U}_n\, \psi) &\stackrel{def}{=} \tau(\psi \vee (\varphi \wedge \circ(\varphi \,\mathbb{U}_{n-1}\, \psi))) \\
\tau(\varphi \,\mathbb{R}_n\, \psi) &\stackrel{def}{=} \tau(\psi \wedge (\varphi \vee \widehat{\circ}(\varphi \,\mathbb{R}_{n-1}\, \psi))) \\
\tau(\varphi \,\mathbf{S}_n\, \psi) &\stackrel{def}{=} \tau(\psi \vee (\varphi \wedge \bullet(\varphi \,\mathbf{S}_{n-1}\, \psi))) \\
\tau(\varphi \,\mathbf{T}_n\, \psi) &\stackrel{def}{=} \tau(\psi \wedge (\varphi \vee \widehat{\bullet}(\varphi \,\mathbf{T}_{n-1}\, \psi)))
\end{aligned}
$$

Translating formula (4) from the traffic light example into a temporal formula yields

$$
\begin{aligned}
&\tau(\Box\big(push \rightarrow \Diamond_3 \Box_4 green\big)) \\
&= \Box(push \rightarrow ((green \wedge \widehat{\circ} green \wedge \widehat{\circ}\widehat{\circ} green \wedge \widehat{\circ}\widehat{\circ}\widehat{\circ} green) \\
&\qquad\qquad \vee (\circ(green \wedge \widehat{\circ} green \wedge \widehat{\circ}\widehat{\circ} green \wedge \widehat{\circ}\widehat{\circ}\widehat{\circ} green)) \\
&\qquad\qquad \vee (\circ\circ(green \wedge \widehat{\circ} green \wedge \widehat{\circ}\widehat{\circ} green \wedge \widehat{\circ}\widehat{\circ}\widehat{\circ} green))))
\end{aligned}
\tag{8}
$$

This illustrates the benefit of metric operators. While we are also able to express the

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same formula with temporal operators, it is much more concise and more readable with metric operators.

Proposition 11
The translation $\tau(\varphi)$ terminates for any metric formula φ.

Proposition 12
For any HT-trace $\mathbf{M}$ and any time point $k \in [0..\lambda)$, we have
$\mathbf{M}, k \models \varphi$ in MHT iff $\mathbf{M}, k \models \tau(\varphi)$ in THT.

Corollary 1
For any metric formula φ over $\mathcal{A}$, there exists a temporal formula ψ over $\mathcal{A}$ such that an HT-trace $\mathbf{M}$ is a model for ψ in THT iff $\mathbf{M}$ is a model for φ in MHT.

Language-extending translation. For a complement, we provide an alternative translation using an extended alphabet. This translation has the advantage of avoiding applications of distributivity, a source of an exponential increase in size.

To this end, we adapt the notion of closure from Cabalar et al., 2020 (Cabalar et al. 2020) to provide a translation of metric into temporal formulas. The original definition of closure is due to Fischer and Ladner in 1979 (Fischer and Ladner 1979).

Definition 3 (*Closure*)
The closure $\mathrm{cl}(\gamma)$ of a metric formula γ is the subset minimal set of formulas satisfying the inductive conditions:

1. $\gamma \in \mathrm{cl}(\gamma)$
2. $(\varphi \otimes \psi) \in \mathrm{cl}(\gamma)$ implies $\varphi \in \mathrm{cl}(\gamma)$ and $\psi \in \mathrm{cl}(\gamma)$
   for $\otimes \in \{\wedge, \vee, \rightarrow, \mathbb{U}_n, \mathbb{R}_n, \mathbf{S}_n, \mathbf{T}_n\}$ with $n \in \mathbb{N} \cup \boldsymbol{\ell}$
3. If $\otimes\psi \in \mathrm{cl}(\gamma)$ then $\psi \in \mathrm{cl}(\gamma)$ for $\otimes \in \{\bigcirc, \widehat{\bigcirc}, \bullet, \widehat{\bullet}\}$
4. If $\varphi\mathbb{U}_n\psi \in \mathrm{cl}(\gamma)$ and $n > 1$ then $\bigcirc(\varphi\mathbb{U}_{n-1}\psi) \in \mathrm{cl}(\gamma)$
5. If $\varphi\mathbb{R}_n\psi \in \mathrm{cl}(\gamma)$ and $n > 1$ then $\widehat{\bigcirc}(\varphi\mathbb{R}_{n-1}\psi) \in \mathrm{cl}(\gamma)$
6. If $\varphi\mathbf{S}_n\psi \in \mathrm{cl}(\gamma)$ and $n > 1$ then $\bullet(\varphi\mathbf{S}_{n-1}\psi) \in \mathrm{cl}(\gamma)$
7. If $\varphi\mathbf{T}_n\psi \in \mathrm{cl}(\gamma)$ and $n > 1$ then $\widehat{\bullet}(\varphi\mathbf{T}_{n-1}\psi) \in \mathrm{cl}(\gamma)$
8. If $\varphi\mathbb{U}_{\boldsymbol{\ell}}\psi \in \mathrm{cl}(\gamma)$ then $\bigcirc(\varphi\mathbb{U}_{\boldsymbol{\ell}}\psi) \in \mathrm{cl}(\gamma)$
9. If $\varphi\mathbb{R}_{\boldsymbol{\ell}}\psi \in \mathrm{cl}(\gamma)$ then $\widehat{\bigcirc}(\varphi\mathbb{R}_{\boldsymbol{\ell}}\psi) \in \mathrm{cl}(\gamma)$
10. If $\varphi\mathbf{S}_{\boldsymbol{\ell}}\psi \in \mathrm{cl}(\gamma)$ then $\bullet(\varphi\mathbf{S}_{\boldsymbol{\ell}}\psi) \in \mathrm{cl}(\gamma)$
11. If $\varphi\mathbf{T}_{\boldsymbol{\ell}}\psi \in \mathrm{cl}(\gamma)$ then $\widehat{\bullet}(\varphi\mathbf{T}_{\boldsymbol{\ell}}\psi) \in \mathrm{cl}(\gamma)$

Any set satisfying these conditions is called *closed*.

Proposition 13
For any metric formula φ, $\mathrm{cl}(\varphi)$ is finite. Moreover, the total size of all the formulas in $\mathrm{cl}(\varphi)$, $|\mathrm{cl}(\varphi)|$ is bound by $|\mathrm{cl}(\varphi)| \leq 2k_\varphi|\varphi|$, where $k_\varphi = \max\{1, n_\varphi\}$ and n_φ is the maximum (metric) integer subindex occurring in φ.

Thus, given a metric formula φ over alphabet $\mathcal{A}$ at hand, we define the extended alphabet $\mathcal{A}^\varphi \stackrel{def}{=} \mathcal{A} \cup \{\mathbf{L}_\mu \mid \mu \in \mathrm{cl}(\varphi)\}$. For convenience, we simply use $\mathbf{L}_\varphi \stackrel{def}{=} \varphi$ if φ is $\top, \bot$ or an atom $a \in \mathcal{A}$.

As happened with the normal form reduction for TEL_f (Cabalar et al. 2018), the

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translation is done in two phases: we first obtain a temporal theory containing double implications, and then we unfold them into temporal rules. We start by defining the temporal theory $\upsilon(\varphi)$ that introduces new labels $\mathbf{L}_\mu$ for each formula $\mu \in \mathrm{cl}(\varphi)$. This theory contains the formula $\mathbf{L}_\varphi$ and, per each label $\mathbf{L}_\mu$, a set of formulas $df(\mu)$ fixing the label's truth value. Formally, we define that

$$\upsilon(\varphi) = \{\mathbf{L}_\varphi\} \cup \{df(\mu) \mid \mu \in \mathrm{cl}(\varphi)\} \qquad \text{and} \qquad \upsilon(\Gamma) = \{\upsilon(\varphi) \mid \varphi \in \Gamma\}.$$

Table 1 shows the definitions $df(\mu)$ for each μ in the closure $\mathrm{cl}(\varphi)$ depending on the outer modality in the formula.

$\mu \in \mathrm{cl}(\varphi)$	$df(\mu)$	
$\circ\alpha$	$\widehat{\circ}\Box(\bullet\mathbf{L}_\mu \leftrightarrow \mathbf{L}_\alpha)$	$\Box(\mathbb{F} \rightarrow \neg\mathbf{L}_\mu)$
$\bullet\alpha$	$\widehat{\circ}\Box(\mathbf{L}_\mu \leftrightarrow \bullet\mathbf{L}_\alpha)$	$\neg\mathbf{L}_\mu$
$\widehat{\circ}\alpha$	$\widehat{\circ}\Box(\bullet\mathbf{L}_\mu \leftrightarrow \mathbf{L}_\alpha)$	$\Box(\mathbb{F} \rightarrow \mathbf{L}_\mu)$
$\widehat{\bullet}\alpha$	$\widehat{\circ}\Box(\mathbf{L}_\mu \leftrightarrow \bullet\mathbf{L}_\alpha)$	$\mathbf{L}_\mu$
$\alpha\mathbb{U}_1\beta$	$\Box(\mathbf{L}_\mu \leftrightarrow \mathbf{L}_\beta)$	
$\alpha\mathbb{R}_1\beta$	$\Box(\mathbf{L}_\mu \leftrightarrow \mathbf{L}_\beta)$	
$\alpha\mathbf{S}_1\beta$	$\Box(\mathbf{L}_\mu \leftrightarrow \mathbf{L}_\beta)$	
$\alpha\mathbf{T}_1\beta$	$\Box(\mathbf{L}_\mu \leftrightarrow \mathbf{L}_\beta)$	
$\alpha\mathbb{U}_n\beta$	$\Box(\mathbf{L}_\mu \leftrightarrow \mathbf{L}_\beta \vee (\mathbf{L}_\alpha \wedge \mathbf{L}_{\alpha'}))$	with $\alpha' = \circ(\alpha\mathbb{U}_{n-1}\beta)$
$\alpha\mathbb{R}_n\beta$	$\Box(\mathbf{L}_\mu \leftrightarrow \mathbf{L}_\beta \wedge (\mathbf{L}_\alpha \vee \mathbf{L}_{\alpha'}))$	with $\alpha' = \widehat{\circ}(\alpha\mathbb{R}_{n-1}\beta)$
$\alpha\mathbf{S}_n\beta$	$\Box(\mathbf{L}_\mu \leftrightarrow \mathbf{L}_\beta \vee (\mathbf{L}_\alpha \wedge \mathbf{L}_{\alpha'}))$	with $\alpha' = \bullet(\alpha\mathbf{S}_{n-1}\beta)$
$\alpha\mathbf{T}_n\beta$	$\Box(\mathbf{L}_\mu \leftrightarrow \mathbf{L}_\beta \wedge (\mathbf{L}_\alpha \vee \mathbf{L}_{\alpha'}))$	with $\alpha' = \widehat{\bullet}(\alpha\mathbf{T}_{n-1}\beta)$
$\alpha\mathbb{U}_{\boldsymbol{\ell}}\beta$	$\Box(\mathbf{L}_\mu \leftrightarrow \mathbf{L}_\beta \vee (\mathbf{L}_\alpha \wedge \mathbf{L}_{\alpha'}))$	with $\alpha' = \circ(\alpha\mathbb{U}_{\boldsymbol{\ell}}\beta)$
$\alpha\mathbb{R}_{\boldsymbol{\ell}}\beta$	$\Box(\mathbf{L}_\mu \leftrightarrow \mathbf{L}_\beta \wedge (\mathbf{L}_\alpha \vee \mathbf{L}_{\alpha'}))$	with $\alpha' = \widehat{\circ}(\alpha\mathbb{R}_{\boldsymbol{\ell}}\beta)$
$\alpha\mathbf{S}_{\boldsymbol{\ell}}\beta$	$\Box(\mathbf{L}_\mu \leftrightarrow \mathbf{L}_\beta \vee (\mathbf{L}_\alpha \wedge \mathbf{L}_{\alpha'}))$	with $\alpha' = \bullet(\alpha\mathbf{S}_{\boldsymbol{\ell}}\beta)$
$\alpha\mathbf{T}_{\boldsymbol{\ell}}\beta$	$\Box(\mathbf{L}_\mu \leftrightarrow \mathbf{L}_\beta \wedge (\mathbf{L}_\alpha \vee \mathbf{L}_{\alpha'}))$	with $\alpha' = \widehat{\bullet}(\alpha\mathbf{T}_{\boldsymbol{\ell}}\beta)$

Table 1. *Translation of metric modal operators*

As we have seen, in the general case, formulas in $df(\mu)$ are not temporal rules, since they sometimes contain double implications. However, they all have the forms φ, $\Box\varphi$, $\widehat{\circ}\Box\varphi$ or $\Box(\mathbb{F} \rightarrow \varphi)$, for some inner propositional formula φ formed with temporal literals.

Given an HT-trace $\langle \mathbf{H}, \mathbf{T}\rangle = \langle H_i, T_i\rangle_{i\in[0..\lambda)}$, we define its restriction to alphabet $\mathcal{A}$ as $\langle \mathbf{H}, \mathbf{T}\rangle|_{\mathcal{A}} \stackrel{def}{=} \langle H_i \cap \mathcal{A}, T_i \cap \mathcal{A}\rangle_{i\in[0..\lambda)}$. Similarly, for any set $\mathfrak{S}$ of HT-traces, we write $\mathfrak{S}|_{\mathcal{A}}$ to stand for $\{\langle \mathbf{H}, \mathbf{T}\rangle|_{\mathcal{A}} \mid \langle \mathbf{H}, \mathbf{T}\rangle \in \mathfrak{S}\}$.

The following lemma shows that $\mathbf{L}_\mu$ and μ are equivalent:

Proposition 14
Let γ be a metric formula over $\mathcal{A}$ and let $\langle \mathbf{H}, \mathbf{T}\rangle$ be a model in MHT_f of $\upsilon(\gamma)$ being associated with the three-valuation $\boldsymbol{m}$.

Then, for any $\mu \in \mathrm{cl}(\gamma)$ and any $k \in [0..\lambda)$, we have $\boldsymbol{m}(k, \mathbf{L}_\mu) = \boldsymbol{m}(k, \mu)$.

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Theorem 2
For any metric formula φ and any length λ, we have

$$\mathrm{MHT}_f(\varphi,\lambda) = \mathrm{MHT}_f(\upsilon(\varphi),\lambda)|_{\mathcal{A}} = \mathrm{THT}_f(\upsilon(\varphi),\lambda)|_{\mathcal{A}}.$$

Corollary 2
For any temporal formula φ and any length λ, we have

$$\mathrm{THT}(\varphi,\lambda) = \mathrm{MHT}(\varphi,\lambda)$$

For example, the formula $\Box(\neg \mathit{green} \rightarrow \mathit{red})$ in (3) has the same models whether it is interpreted as a temporal or a metric formula.

Corollary 3
Let φ be a metric formula over $\mathcal{A}$. Then, translation $\upsilon(\varphi)$ is *strongly faithful*, that is:

$$\mathrm{MEL}_f(\varphi \wedge \varphi') = \mathrm{MEL}_f(\upsilon(\varphi) \wedge \varphi')|_{\mathcal{A}}.$$

for any arbitrary metric formula φ' over $\mathcal{A}$.

The translation of formula (4) from the traffic light example yields the resulting set of rules in Table 2 computed from cl(4). Although in this case, cl(4) is larger than the

$\mathbf{L}_\mu$	$\mu \in \mathrm{cl}(4)$	$df(\mu)$
$\mathbf{L}_1$	$\Box\,(\mathit{push} \rightarrow \Diamond_3\Box_4 \mathit{green})$	$\Box\,(\mathbf{L}_1 \leftrightarrow \mathbf{L}_3 \wedge \mathbf{L}_2)$
$\mathbf{L}_2$	$\widehat{\bigcirc}\Box\,(\mathit{push} \rightarrow \Diamond_3\Box_4 \mathit{green})$	$\widehat{\bigcirc}\Box\,(\bullet\mathbf{L}_2 \leftrightarrow \mathbf{L}_1) \quad \Box\,(\mathbb{F} \rightarrow \mathbf{L}_2)$
$\mathbf{L}_3$	$\mathit{push} \rightarrow \Diamond_3\Box_4 \mathit{green}$	$\Box\,(\mathbf{L}_3 \leftrightarrow (\mathit{push} \rightarrow \mathbf{L}_4))$
$\mathbf{L}_4$	$\Diamond_3\Box_4 \mathit{green}$	$\Box\,(\mathbf{L}_4 \leftrightarrow \mathbf{L}_9 \vee \mathbf{L}_5)$
$\mathbf{L}_5$	$\bigcirc\Diamond_2\Box_4 \mathit{green}$	$\widehat{\bigcirc}\Box\,(\bullet\mathbf{L}_5 \leftrightarrow \mathbf{L}_6) \quad \Box\,(\mathbb{F} \rightarrow \neg\mathbf{L}_5)$
$\mathbf{L}_6$	$\Diamond_2\Box_4 \mathit{green}$	$\Box\,(\mathbf{L}_6 \leftrightarrow \mathbf{L}_9 \vee \mathbf{L}_7)$
$\mathbf{L}_7$	$\bigcirc\Diamond_1\Box_4 \mathit{green}$	$\widehat{\bigcirc}\Box\,(\bullet\mathbf{L}_7 \leftrightarrow \mathbf{L}_8) \quad \Box\,(\mathbb{F} \rightarrow \neg\mathbf{L}_7)$
$\mathbf{L}_8$	$\Diamond_1\Box_4 \mathit{green}$	$\Box\,(\mathbf{L}_8 \leftrightarrow \mathbf{L}_9)$
$\mathbf{L}_9$	$\Box_4 \mathit{green}$	$\Box\,(\mathbf{L}_9 \leftrightarrow \mathit{green} \wedge \mathbf{L}_{10})$
$\mathbf{L}_{10}$	$\widehat{\bigcirc}\Box_3 \mathit{green}$	$\widehat{\bigcirc}\Box\,(\bullet\mathbf{L}_{10} \leftrightarrow \mathbf{L}_{11}) \quad \Box\,(\mathbb{F} \rightarrow \mathbf{L}_{10})$
$\mathbf{L}_{11}$	$\Box_3 \mathit{green}$	$\Box\,(\mathbf{L}_{11} \leftrightarrow \mathit{green} \wedge \mathbf{L}_{12})$
$\mathbf{L}_{12}$	$\widehat{\bigcirc}\Box_2 \mathit{green}$	$\widehat{\bigcirc}\Box\,(\bullet\mathbf{L}_{12} \leftrightarrow \mathbf{L}_{13}) \quad \Box\,(\mathbb{F} \rightarrow \mathbf{L}_{12})$
$\mathbf{L}_{13}$	$\Box_2 \mathit{green}$	$\Box\,(\mathbf{L}_{13} \leftrightarrow \mathit{green} \wedge \mathbf{L}_{14})$
$\mathbf{L}_{14}$	$\widehat{\bigcirc}\Box_1 \mathit{green}$	$\widehat{\bigcirc}\Box\,(\bullet\mathbf{L}_{14} \leftrightarrow \mathbf{L}_{15}) \quad \Box\,(\mathbb{F} \rightarrow \mathbf{L}_{14})$
$\mathbf{L}_{15}$	$\Box_1 \mathit{green}$	$\Box\,(\mathbf{L}_{15} \leftrightarrow \mathit{green})$

Table 2. *Translating formula* (4) *from the traffic light example.*

language-preserving translation $\tau(4) = (8)$, note that the latter is not in the form of a logic program yet, while double implications in Table 2 can be linearly reduced to a logic program. In general, reducing $\tau(\gamma)$ to a (language-preserving) logic program induces an exponential size increase due to the application of distributivity laws. For instance, the formula $\tau(p\,\mathbb{U}_n\,q)$ amounts to a combination of conjunctions and disjunctions that, to become a logic program, must be previously reduced to conjunctive normal form. On the other hand, $\mathrm{cl}(\gamma)$ preserves a polynomial size, although at the cost of introducing auxiliary atoms, one per formula in $\mathrm{cl}(\gamma)$.

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4 Discussion

We have defined and elaborated upon a metric temporal extension of the logic of Here-and-There in order to lay the theoretical foundations of metric Answer Set Programming. The resulting logics, MHT and its non-monotonic extension MEL, have a point-based semantics based on discrete linear time. The choice of such a simple time domain was motivated by the desire to base the approach on the same semantic structures as used in previous extensions of HT and ASP with constructs from Linear Temporal and Dynamic Logic, namely, linear HT-traces. As a result, we were able to inter-translate our approach and the temporal logics of THT and TEL. This is of practical relevance since it allows us to use temporal ASP solvers such as *telingo* (Cabalar et al. 2019) for implementation.

There exist other approaches that introduce metric temporal operators in logic programming. For instance, a first metric extension for Horn Prolog was presented by Brzoska in 1995 (Brzoska 1995). This approach is more limited than MEL for several reasons: it does not consider default negation, metric operators can only be used under a restricted syntax and only for unary temporal operators, and the interpretation of programs relies on resolution rather than on a purely model-theoretic description. Other more recent related approaches introduce temporal metrics for stream reasoning for ASP (Beck et al. 2015; Beck et al. 2016) and DATALOG (Brandt et al. 2018; Walega et al. 2019). The former is not only closely related due to its ASP-based approach but also because it can be characterized in terms of EL (Beck et al. 2016), although extra conditions are needed to guarantee the persistence property of HT. While the DATALOG-based approach lacks the rich language of ASP, they rely on more expressive metric operators. For instance, Brandt et al. deal with intervals over $\mathbb{Q} \cup \{-\infty, +\infty\}$ (Brandt et al. 2018). It will be interesting to investigate the formal connection to such stream-oriented approaches and metric temporal logic programming formalisms in general.

Another aspect to explore has to do with the fact that traditional metric logics rely on continuous time domains (cf. Alur and Henzinger, 1992 (Alur and Henzinger 1992)), which often brings about undecidability. Although we do not strive for such expressiveness, it will be interesting future work to generalize our approach to more fine-grained time domains using integer or rational numbers and to explore implementations with hybrid ASP systems. However, the presented approach should already furnish a host system for action languages with durative actions (Son et al. 2004) or even traditional event calculus (Kowalski and Sergot 1986), although this remains to be worked out.

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