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K-theory, reality, and duality

by

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Abstract

We present a new proof of Anderson's result that the real K-theory spectrum is Anderson self-dual up to a fourfold suspension shift; more strongly, we show that the Anderson dual of the complex K-theory spectrum KU is C_2-equivariantly equivalent to $\Sigma^4 KU$, where C_2 acts by complex conjugation. We give an algebro-geometric interpretation of this result in spectrally derived algebraic geometry and apply the result to calculate 2-primary Gross-Hopkins duality at height 1. From the latter we obtain a new computation of the group of exotic elements of the $K(1)$-local Picard group.

Key Words: Anderson duality, dualizing complex, K-theory, Tate spectrum, Picard group.

Mathematics Subject Classification 2010: 55N15, 19L99, 55M05, 55U30, 55P91.

1. Introduction

The purpose of this note is to establish some duality results for complex and and real K-theory and to apply them to the $K(1)$-local Picard group at the prime 2. The major ingredients we use to obtain these results can be found in the literature. However, to the best of the authors' knowledge, those ingredients have not been previously blended together in the manner presented in this paper. That perspective and approach was of great use as a guiding example for the second author in her work on duality for topological modular forms [Sto12]. Contemplating new directions, we believe that the methods of the present work can be generalized to obtain results about $K(n)$-local Picard groups for $n > 1$.

There are two main forms of duality which algebraic topologists have used in $K(n)$-local settings, namely, Spanier-Whitehead duality, which is simply the monoidal duality, and Gross-Hopkins duality, which is a $K(n)$-local analogue of Pontryagin or Brown-Comenetz duality. We will be interested in an integral version of the latter, namely Anderson duality. This is not $K(n)$-local, though upon such localization is

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closely related to Gross-Hopkins duality. Instead, it is defined on the category of spectra and makes surprising appearances in geometry and the study of quantum field theories [HS05, FMS07].

Consider the C_2-action on the periodic K-theory spectrum KU via complex conjugation. The main computational result in this paper is the following

Theorem 8.2. *The Anderson dual $I_{\mathbb{Z}}KU$ is C_2-equivariantly equivalent to $\Sigma^4 KU$.*

It is a classical fact due to Atiyah [Ati66] that the real K-theory spectrum KO is the spectrum of homotopy fixed points KU^{hC_2}; more modern references for this fact are [Dug05, Cor.7.6], [Faj95, Thm.5.2], [Kar01, Thm.3], [Rog08, Prop.5.3.1]. The duality of Theorem 8.2 is reflected in the following result as a "wrong side computation."

Theorem 3.10. *The forgetful map*

$$u_* : (KO\text{-}mod) \to (S\text{-}mod)$$

has a right adjoint $u^! = F(KO, -)$ such that for a spectrum A

$$u^! A = F(KO, A) \simeq F(I_{\mathbb{Z}} A, \Sigma^4 KO).$$

This theorem has a derived algebro-geometric interpretation which we pursue in Section 3.

The first step of the proof of Theorem 8.2 consists of a simple calculation that KU is Anderson self-dual. We then proceed to develop a descent strategy for deducing the above-stated result. The fact that KO is the homotopy fixed points of KU under the conjugation action is insufficient for descending duality. Notwithstanding, we show that KO is also equivalent to the homotopy orbits of KU under the same C_2 action by proving that the associated Tate spectrum vanishes. This allows a calculation of the Anderson dual of KO, and from there we complete the proof of Theorem 8.2.

As an application of the above theorems, we use the relationship between Anderson duality and $K(1)$-local Gross-Hopkins duality to independently conclude the well-known fact that the $K(1)$-local category contains "exotic" elements in its Picard group.

The organization of this paper is as follows. In Section 2 we define Anderson duality, and in Section 3 we give a detailed algebro-geometric interpretation of the duality in general, as well as an interpretation of Theorem 3.10 using derived algebraic geometry. Note that the proof of Theorem 3.10, while included in Section 3, depends on the results of Sections 4 through 8; these Sections build to

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the proof of Theorem 8.2. In particular, in Section 5 we use equivariant homotopy theory to deduce, from scratch, the differentials in the homotopy fixed point spectral sequence for $KO \simeq KU^{hC_2}$. Finally, in Section 9 we study the implications of Theorem 8.2 for the invertible $K(1)$-local spectra and deduce that an exotic such spectrum must exist.

Remark 1.1 Where possible we have included information that, though known to experts, is perhaps not easily or at all found in the literature, such as the use of equivariant homotopy to determine a differential in a homotopy fixed point spectral sequence, which we learned from Mike Hopkins and Mike Hill.

Conventions

We will denote homotopy fixed point and orbit spectral sequences by HFPSS and HOSS respectively. In all spectral sequence diagrams a box represents a copy of $\mathbb{Z}$ whilst a dot is a copy of $\mathbb{Z}/2$. Vertical lines represent multiplication by 2, whilst lines of slope 1 represent multiplication by η. All spectral sequences will be Adams indexed; that is, a class in cohomological degree s and internal degree t will be drawn in position $(t-s,s)$.

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2. Background on Anderson duality

The functor on spectra $X \mapsto \mathrm{Hom}(\pi_{-*}X, \mathbb{Q}/\mathbb{Z})$ is a cohomology theory since $\mathbb{Q}/\mathbb{Z}$ is injective. Let $I_{\mathbb{Q}/\mathbb{Z}}$ denote the representing spectrum. The Brown-Comenetz dual of X is then defined as the function spectrum $I_{\mathbb{Q}/\mathbb{Z}}X = F(X, I_{\mathbb{Q}/\mathbb{Z}})$. In a similar way we can define $I_{\mathbb{Q}}$ to be the spectrum representing the functor $X \mapsto \mathrm{Hom}(\pi_{-*}X, \mathbb{Q})$, and there is a natural map $I_{\mathbb{Q}} \to I_{\mathbb{Q}/\mathbb{Z}}$. We denote the homotopy fiber of this map by $I_{\mathbb{Z}}$. Of course, $I_{\mathbb{Q}}$ is the rational Eilenberg-MacLane spectrum. For a spectrum X we then define $I_{\mathbb{Z}}X$ as the function spectrum $F(X, I_{\mathbb{Z}})$. This contravariant functor $\mathrm{I}_{\mathbb{Z}}$ on the category of spectra is known as Anderson duality, first used by Anderson [And69]; see also [Yos75, Bök81, HS05]. We shall see that the representing spectrum $\mathrm{I}_{\mathbb{Z}}$ is the dualizing object in the category of spectra (see Section 3).

Example 2.1 *If A is a finite group, then the Brown-Comenetz dual of the Eilenberg-MacLane spectrum HA is again HA, via a non-canonical isomorphism between A*

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and its Pontryagin dual. Since there are no non-zero maps $A \to \mathbb{Q}$, it follows that the Anderson dual $I_{\mathbb{Z}}HA$ is equivalent to $\Sigma^{-1}HA$.

For a spectrum X, we can use a short spectral sequence to calculate the homotopy of $I_{\mathbb{Z}}X$. By considering the long exact sequence in homotopy of the fiber sequence

$$I_{\mathbb{Z}}X \to I_{\mathbb{Q}}X \to I_{\mathbb{Q}/\mathbb{Z}}X, \tag{2.a}$$

we can form an exact couple and hence a spectral sequence,

$$\mathrm{Ext}^s_{\mathbb{Z}}(\pi_t X, \mathbb{Z}) \Rightarrow \pi_{-t-s} I_{\mathbb{Z}}X, \tag{2.b}$$

where the spectral sequence is only non-zero for $s = 0, 1$.

Often this leads to very simple calculations, for example in the case of KU. Here, of course, $\pi_* KU$ is the ring of Laurent polynomials $\mathbb{Z}[u^{\pm 1}]$ on the Bott element u of degree 2. Hence we easily conclude that the homotopy groups $\pi_* I_{\mathbb{Z}}KU$ are $\mathbb{Z}$ in even degrees and 0 in odd degrees. We can do more by observing the additional available structure; note that if R is a ring spectrum, then $I_{\mathbb{Z}}R = F(R, I_{\mathbb{Z}})$ is a module over R. The following easy observation has crucial applications.

Proposition 2.2 *Let R be a homotopy ring spectrum* and let M be an R-module such that $\pi_* M$ is, as a graded module, free of rank one over $\pi_* R$, generated by an element in $\pi_t M$. Then $M \simeq \Sigma^t R$.*

Proof: The generating element is represented a map of spectra $S^t \to M$ which, by the standard adjunction can be extended to a map of R-modules $S^t \wedge R \to M$ using the R-module structure of M. The assumptions ensure that φ is an equivalence. □

Since $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[u^{\pm 1}], \mathbb{Z})$ is a free $\mathbb{Z}[u^{\pm 1}]$-module of rank one, an immediate corollary of this proposition is that $I_{\mathbb{Z}}KU$ is equivalent to KU, i.e. KU is Anderson self-dual.

Similarly, we can run the spectral sequence (2.b) for $I_{\mathbb{Z}}KO$ and conclude that $\pi_k I_{\mathbb{Z}}KO$ is $\mathbb{Z}$ when k is divisible by 4, $\mathbb{Z}/2$ when k is congruent to 5 and 6 modulo 8, and zero otherwise. This indicates that perhaps $I_{\mathbb{Z}}KO$ is a fourfold suspension of KO, which we show is true in Theorem 8.1. However, determining the KO-module structure is tricky from the direct approach that works for $I_{\mathbb{Z}}KU$ because multiplication by η on the dual non-torsion classes cannot be detected algebraically. Thus we proceed using a more sophisticated approach.†

*i.e. a ring object in the homotopy category

†The reader interested in the direct approach is referred to the appendix of [FMS07] for details.

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A natural question that arises at this point is whether we can use the self-duality of KU to deduce any information about the Anderson dual of the real K-theory spectrum KO, using (only) that KO is the homotopy fixed point spectrum KU^{hC_2}, where $C_2 = \langle c | c^2 = 1 \rangle$ acts on KU by complex conjugation. In other words, we would like to know

$$I_{\mathbb{Z}} KO \simeq I_{\mathbb{Z}} KU^{hC_2} \simeq F(KU^{hC_2}, I_{\mathbb{Z}}) \tag{2.c}$$

as a KO-module. Note that homotopy fixed points are the right, not left adjoint to the forgetful functor, hence we cannot deduce much solely from the definitions and the self-duality of KU. However, we show in Section 6 that KO is also the homotopy orbit spectrum KU_{hC_2}, and then proceed to the identification of $I_{\mathbb{Z}} KO$.

3. Algebro-geometric meaning of Anderson duality

Before proceeding to the computational specifics for K-theory, we would like to put the above duality discussion in a broader perspective. Namely, we claim that Anderson duality naturally occurs from a categorical viewpoint, as it is related to the existence of "wrong side" adjoints, or from an algebro-geometric viewpoint as such adjoints signify the compactness of certain geometric objects.[‡]

Perhaps the first notion of duality one encounters is that for vector spaces over a field k; the dual $V^\vee$ of a vector space V is the space of linear maps from V to the base field k, and has the universal property that linear maps $W \to V^\vee$ are in bijection with bilinear pairings $W \otimes V \to k$. If we restrict our attention to finite dimensional vector spaces, we can recover V from its dual as $V \cong (V^\vee)^\vee$.

As is well known, if we try to directly imitate this situation in the more general case of (finitely generated) modules over a ring R and define a "naive" dual of M as $\mathrm{Hom}_R(M, R)$, perhaps the first problem we encounter is the inexactness of the functor Hom. For example, if $R = \mathbb{Z}$ and M is finite, $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$ is zero, so the naive dual cannot recover M. The initial obstacle is easily resolved by passing to the derived category of (complexes) of R-modules. One observes that for this to work properly, R ought to have finite injective dimension as a module over itself. If it does not, we may still be able to define good dual modules by mapping into some other object instead of R. Hence we arrive at the following definition.

Definition 3.1 An R-module[§] D is dualizing if

[‡]For example, compact manifolds have Poincaré duality, and proper smooth schemes have Serre duality.

[§]i.e. an object of the derived category of complexes of R-modules

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1. for any finitely generated module M, the natural double duality map
$$M \to \mathrm{Ext}_R(\mathrm{Ext}_R(M,D),D)$$
is an isomorphism, and
2. D has finite injective dimension; in other words, if a module M is bounded below, then its D-dual $\mathrm{Ext}_R(M,D)$ is bounded above.

Then $I_D M := \mathrm{Ext}_R(M,D)$ is called the dual of M, and the contravariant functor I_D represented by D is called a (representable) duality functor.

Note that if condition (2) holds, and the map $R \to \mathrm{Ext}_R(D,D)$ is an isomorphism, i.e. if condition (1) holds for $M = R$, then (1) holds for any M; thus checking whether an object D is dualizing requires relatively little work.

Example 3.2 *The following example may provide some motivation for the definition of the Anderson dual as the fiber of $I_{\mathbb{Q}} \to I_{\mathbb{Q}/\mathbb{Z}}$. The complex $\mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$ is an injective resolution of $\mathbb{Z}$; hence we can use it to compute $I_{\mathbb{Z}}(M) = \mathrm{Ext}_{\mathbb{Z}}(M,\mathbb{Z})$ for any abelian group M as the two-term complex $I_{\mathbb{Q}}(M) = \mathrm{Hom}_{\mathbb{Z}}(M,\mathbb{Q}) \xrightarrow{\pi} \mathrm{Hom}_{\mathbb{Z}}(M,\mathbb{Q}/\mathbb{Z}) = I_{\mathbb{Q}/\mathbb{Z}}(M)$. If M is a finite abelian group, then $I_{\mathbb{Z}}(M)$ and $I_{\mathbb{Q}/\mathbb{Z}}(M)$ are equal, up to a shift, so the double duality map $M \to I_{\mathbb{Z}}(I_{\mathbb{Z}}(M))$ is an isomorphism. For finitely generated non-torsion M, the map π is surjective, so $I_{\mathbb{Z}}(M) = \mathrm{Hom}_{\mathbb{Z}}(M,\mathbb{Z})$ and the double duality map $M \to I_{\mathbb{Z}}(I_{\mathbb{Z}}(M))$ is again an isomorphism. Hence $\mathbb{Z}$ is a dualizing $\mathbb{Z}$-module.*

A dualizing R-module in the non-derived setting need not exist; for example the category of $\mathbb{Z}$-modules (i.e. abelian groups) does not have a dualizing module. Nevertheless, its derived category does has a dualizing module: $\mathbb{Z}$ itself serves that role (as in Example 3.2). This is a significant example of duality in algebra as $\mathbb{Z}$ is an initial object in rings, and dualities for (derived) categories of R-modules can be obtained by studying the map $u : \mathbb{Z} \to R$ (or rather, the induced map on the corresponding categories.)

To be more precise, consider the forgetful functor u_* between the derived categories of R-modules and $\mathbb{Z}$-modules. It always has a left adjoint u^* given by tensoring up with R. However, suppose u_* also has a right adjoint $u^!$. Then for any abelian group A and R-module M, there is an isomorphism
$$\mathrm{Ext}_{\mathbb{Z}}(u_* M, A) \cong \mathrm{Ext}_R(M, u^! A);$$
which immediately gives that $u^!\mathbb{Z}$ is a good candidate for a dualizing R-module. Indeed, we have that
$$I_{u^!\mathbb{Z}}(I_{u^!\mathbb{Z}} M) = \mathrm{Ext}_R(\mathrm{Ext}_R(M,u^!\mathbb{Z}),u^!\mathbb{Z}) \cong \mathrm{Ext}_{\mathbb{Z}}(\mathrm{Ext}_{\mathbb{Z}}(M,\mathbb{Z}),\mathbb{Z}) \cong M.$$

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In the above discussion, there was no need to restrict ourselves to modules over commutative rings. Indeed, considering quasi-coherent modules over the structure sheaf of a scheme gave rise to Grothendieck-Serre duality; in the algebro-geometric world, a dualizing module is defined precisely as in the algebraic Definition 3.1. Moreover, given a map of schemes $f : X \to Y$, such that Y has a dualizing module D_Y and the pushforward functor f_* is faithful and has a right adjoint $f^!$, we get that $f^! D_Y$ is a dualizing module over X. For details, the reader is referred to [Har66, Nee96, FHM03].

Dualizing objects can also be constructed in the category of ring spectra. Here the sphere is an initial object, and the category of S-modules is the category of spectra. (Note that this is already derived.) The following definition is due to Lurie [Lur12].

Definition 3.3 Let A be a connective commutative ring spectrum, and let K be an A-module (for example, as in [EKMM97]). Then K is a dualizing A-module if

1. Each homotopy group $\pi_n K$ is a finitely generated module over $\pi_0 A$ and $\pi_n K$ vanishes for $n \gg 0$.

2. [Dualizing property] The canonical map $A \to F_A(K, K)$ is an equivalence.

3. The module K has finite injective dimension: there is an integer n such that if M is an A-module with $\pi_i M = 0$ for $i > n$, then $\pi_i F_A(M, K) = 0$ for $i < 0$.

Example 3.4 *We immediately see that the sphere is not a dualizing module over itself (in particular it does not satisfy the vanishing condition), but the Anderson spectrum $I_{\mathbb{Z}}$ plays this role. The proof is easy and can be found in [Lur12, Example 4.3.9]. Indeed,* (1) *and* (3) *are obvious from the definition, and* (2) *follows from the fact that $\mathbb{Z}$ is the dualizing object in the derived category of $\mathbb{Z}$-modules. More precisely, duality in $\mathbb{Z}$-modules tells us that the homotopy groups $\pi_* F(I_{\mathbb{Z}}, I_{\mathbb{Z}}) \cong \mathrm{Ext}_{\mathbb{Z}}(\mathrm{Ext}_{\mathbb{Z}}(\pi_* S, \mathbb{Z}))$ are isomorphic to $\pi_* S$ as a $\pi_* S$-module. Similarly, the category of p-local spectra, i.e. modules over the p-local sphere spectrum, has a dualizing object which is given by the spectrum $I_{\mathbb{Z}_{(p)}}$, the homotopy fiber of the natural map $I_{\mathbb{Q}} \to I_{\mathbb{Q}/\mathbb{Z}_{(p)}}$.*

Remark 3.5 The above definition is suitable only for modules over *connective* ring spectra, and therefore cannot be applied as is to KU or KO-modules. However, if R is *any* ring spectrum, we can study dualizing R-modules relative to the unit map $u : S \to R$. Namely, the forgetful functor u_* from R-modules to spectra has both a left and a right adjoint, the right being given by taking the function spectrum $F(R, -)$. Thus by the above formal reasoning, $u^! I_{\mathbb{Z}} = F(R, I_{\mathbb{Z}})$ will have the dualizing property for R-modules.

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3.1. Homotopical duality for stacks

We would like to patch together the geometric and homotopical notions of duality, thus we need a notion of a derived scheme which is a good notion of a locally ringed-by-spectra space. In fact, we shall not restrict ourselves to schemes, but consider (very simple) stacks, as we would like to have an object BG for a finite group G. For our purposes, the following definition (which is in fact a theorem of Lurie's) will suffice.

Definition 3.6 A derived stack is an ordinary stack $(\mathcal{X},\mathcal{O}_0)$ equipped with a sheaf of commutative ring spectra $\mathcal{O}_\mathcal{X}$ on its small étale site such that

1. The pair $(\mathcal{X},\pi_0\mathcal{O}_\mathcal{X})$ is equivalent to $(\mathcal{X},\mathcal{O}_0)$, and
2. $\pi_k\mathcal{O}_\mathcal{X}$ is a quasi-coherent sheaf of $\pi_0\mathcal{O}_\mathcal{X}$-modules.

Here $\pi_k\mathcal{O}_\mathcal{X}$ is the sheafification of the presheaf $U \mapsto \pi_k(\mathcal{O}_\mathcal{X}(U))$. In this case say that $(\mathcal{X},\mathcal{O}_\mathcal{X})$ or, abusively, $\mathcal{X}$, is a derived stack.

Example 3.7 *A commutative ring spectrum R defines a derived stack* $\operatorname{Spec} R$ *whose underlying ordinary "stack" is the affine scheme* $\operatorname{Spec}\pi_0 R$. *In particular, the sphere spectrum* $\operatorname{Spec} S$ *derives* $\operatorname{Spec}\mathbb{Z}$, *as does the Eilenberg-MacLane spectrum* $H\mathbb{Z}$. *In fact,* $\operatorname{Spec} H\mathbb{Z}$ *is the usual* $\operatorname{Spec}\mathbb{Z}$, *but it is no longer terminal; now* $\operatorname{Spec} S$ *is the terminal object.*

By an $\mathcal{O}_\mathcal{X}$-module we will mean a quasi-coherent sheaf $\mathcal{F}$ of $\mathcal{O}_\mathcal{X}$-module spectra, i.e. a sheaf of $\mathcal{O}_\mathcal{X}$-modules such that for any diagram

$$\begin{array}{ccc} U & \longrightarrow & V \\ & \searrow \quad \swarrow & \\ & \mathcal{X}, & \end{array}$$

in which U and V are derived affine schemes étale over $\mathcal{X}$, we get an equivalence

$$\mathcal{F}(V) \underset{\mathcal{O}_\mathcal{X}(V)}{\wedge} \mathcal{O}_\mathcal{X}(U) \xrightarrow{\sim} \mathcal{F}(U).$$

We will consider the following situation. Let $(\mathcal{X},\mathcal{O}_\mathcal{X})$ be a derived stack over the sphere spectrum S and let

$$f : \mathcal{X} \to \operatorname{Spec} S$$

be its structure map. The push-forward or global sections functor $f_* : (\mathcal{O}_\mathcal{X}\text{-mod}) \to (S\text{-mod})$ always has a left adjoint, f^*, the constant sheaf functor. However, in some

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situations, it also has a right adjoint, usually denoted $f^!$. When such $f^!$ exists, $f^!\mathrm{I}_{\mathbb{Z}}$ is a perfect candidate for a dualizing module for $\mathcal{O}_{\mathcal{X}}$-modules.¶

Example 3.8 *Let R be a commutative ring spectrum, and let $u : \operatorname{Spec} R \to \operatorname{Spec} S$ be the structure map. Then u_*, which is the forgetful functor from R-modules to spectra has both a left adjoint $u^* = - \wedge R$ and a right adjoint $u^! = F(R, -)$. Hence $u^! I_{\mathbb{Z}} = F(R, I_{\mathbb{Z}})$ has the dualizing property for R-modules.*

Example 3.9 *Let R be a commutative ring spectrum with an action by a finite group G. Then (BG, R) is an example of a derived stack, whereby modules over the structure sheaf are R-modules with a compatible G-action. If M is such a module, then $f_* M = M^{hG}$ has a left adjoint given by smashing with R. However, f_* may or may not have a right adjoint for non-trivial G. The existence of such is related to the Tate spectra that we recall in the next section. Note that in a $K(n)$-local setting the right adjoint of such an f_* always exists by a theorem of Greenlees-Sadofsky [GS96] recently generalized by Hopkins-Lurie [HL13].*

3.2. Duality for K-theory, geometrically

We now describe how K-theory fits into this perspective; to begin with, we recall the setup from [LN12, Appendix A]. The reader is referred to loc.cit. for more details and proofs.

Let $\mathcal{M}_{\mathbb{G}_m}/\operatorname{Spec}\mathbb{Z}$ denote the stack classifying group schemes which become isomorphic to $\mathbb{G}_m$ after a faithfully flat extension. These are also called forms of the multiplicative group. Since the automorphism group of $\mathbb{G}_m$ over $\mathbb{Z}$ is C_2, we get that $\mathcal{M}_{\mathbb{G}_m}$ is equivalent to the stack BC_2 classifying principal C_2-torsors; explicit details can be found in loc.cit.

There exists a sheaf $\mathcal{O}^{\mathrm{top}}$ of commutative ring spectra on the stack $\mathcal{M}_{\mathbb{G}_m}$ such that to each étale map $\operatorname{Spec} R \to \mathcal{M}_{\mathbb{G}_m}$, the sheaf $\mathcal{O}^{\mathrm{top}}$ assigns a complex orientable weakly even periodic commutative ring spectrum $\mathcal{O}^{\mathrm{top}}(R)$ such that

(a) $\pi_0\mathcal{O}^{\mathrm{top}}(R)$=R, and

(b) the formal completion of the form of $\mathbb{G}_m$ classified by $\operatorname{Spec} R \to \mathcal{M}_{\mathbb{G}_m}$ is isomorphic to the formal group law of $\mathcal{O}^{\mathrm{top}}(R)$.

Lawson-Nauman show that the spectrally ringed stack $(\mathcal{M}_{\mathbb{G}_m}, \mathcal{O}^{\mathrm{top}})$ (over the sphere spectrum $\operatorname{Spec} S$) is equivalent to (BC_2, KU), where C_2 acts on KU (by commutative ring maps) via complex conjugation. In particular, the category of $\mathcal{O}^{\mathrm{top}}$-modules is equivalent to the category of C_2-equivariant KU modules.

¶Compare Remark 3.5

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More specifically, let $\operatorname{Spec} R \to \mathcal{M}_{\mathbb{G}_m}$ be an étale map; and let $\tilde{R}$ denote the ring which fits in a pullback square

$$\begin{array}{ccc} \operatorname{Spec}\tilde{R} & \longrightarrow & \operatorname{Spec} R \\ \downarrow & & \downarrow \\ \operatorname{Spec}\mathbb{Z} & \longrightarrow & \mathcal{M}_{\mathbb{G}_m} \cong BC_2, \end{array}$$

where the bottom map is the usual covering map of BC_2 (which, in particular, is an affine map). The ring $\tilde{R}$ constructed in this way has a free C_2-action and as such has a homotopically unique C_2-equivariant realization $S(\tilde{R})$ according to [BR07], with

$$\pi_* S(\tilde{R}) \cong \pi_* S \otimes_{\mathbb{Z}} \tilde{R}.$$

The value that the sheaf $\mathcal{O}^{\text{top}}$ takes on the affine $\operatorname{Spec} R \to \mathcal{M}_{\mathbb{G}_m}$ is then given by $(KU \wedge S(\tilde{R}))^{hC_2}$, where C_2 acts diagonally on the smash product. We have

$$\pi_* \mathcal{O}^{top}(R) = H^0(C_2, \pi_* KU \otimes \tilde{R}).$$

For example, if $R = \mathbb{Z}$ and the map is the covering $\operatorname{Spec}\mathbb{Z} \to \mathcal{M}_{\mathbb{G}_m}$ (i.e. the map classifying $\mathbb{G}_m$ itself), $\tilde{R} \cong \mathbb{Z}[C_2]$, and $\mathcal{O}^{top}(\mathbb{Z}) = (KU[C_2])^{hC_2} \simeq KU$.

Thus the global sections of $\mathcal{O}^{\text{top}}$ over $\mathcal{M}_{\mathbb{G}_m}$ are the homotopy fixed points KU^{hC_2}, i.e. the real K-theory spectrum KO. In fact, using Galois descent or the general affineness machinery of Mathew-Meier [MM13, Ex. 6.1], one shows that the category of $\mathcal{O}^{\text{top}}$-modules, which we saw is equivalent to C_2-KU-modules, is further equivalent to the category of KO modules as, by loc.cit., the derived stack (BC_2, KU) is equivalent to $\operatorname{Spec} KO$. Note that in the category of C_2-KU-modules, the internal Hom is given by the function spectrum of non-equivariant KU-module maps equipped with the C_2-action by conjugation; we will denote this function spectrum by $F_{C_2\text{-}KU}$.

Theorem 3.10 *The following three equivalent statements are true.*

(i) Let u be the structure map $\operatorname{Spec} KO \to \operatorname{Spec} S$. The push forward (i.e. forgetful) map

$$u_* : (KO\text{-}mod) \to (S\text{-}mod)$$

has a right adjoint $u^! = F(KO, -)$ such that for a spectrum A

$$u^! A = F(KO, A) \simeq F(I_{\mathbb{Z}} A, \Sigma^4 KO).$$

In particular, $u^! I_{\mathbb{Z}} \simeq \Sigma^4 KO$ is the relative dualizing KO-module.

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(ii) The push forward g_* along the structure map $g : (BC_2, KU) \to \operatorname{Spec} S$ has a right adjoint $g^!$ given by

$$g^! A = F_{C_2\text{-}KU}(g^*(I_{\mathbb{Z}} A), \Sigma^4 KU).$$

(iii) The push forward f_* along $(\mathcal{M}_{\mathbb{G}_m}, \mathcal{O}^{top}) \to \operatorname{Spec} S$ has a right adjoint $f^!$ such that

$$f^! A = \operatorname{Hom}_{\mathcal{O}^{top}}(f^*(I_{\mathbb{Z}} A), \Sigma^4 \mathcal{O}^{top}).$$

In particular, $f^! I_{\mathbb{Z}} \simeq \Sigma^4 \mathcal{O}^{top}$ is the relative dualizing $\mathcal{O}^{top}$-module.

Remark 3.11 The proof of this theorem depends on Theorem 8.2, which is the main result of this paper and to whose proof Section 4 through Section 8 are devoted. Thus the logical placement of the proof of this theorem is at the end of Section 8, but we have decided to include it here for better readability. The reader is advised to come back to the proof after consulting the statements of Theorem 8.2 and Corollary 6.3.

Proof: That the statements are equivalent is immediate from the above discussion. We will prove Item ii. What we need to show is that for a spectrum A and a C_2-KU module M, there is a natural equivalence

$$g_* F_{C_2\text{-}KU}(M, F_{C_2\text{-}KU}(g^*(I_{\mathbb{Z}} A), \Sigma^4 KU)) \simeq F(g_* M, A). \tag{3.d}$$

We split the proof in two parts.

(a) When A is the Anderson spectrum $I_{\mathbb{Z}}$, (3.d) reduces to showing that

$$F_{KU}(M, \Sigma^4 KU)^{hC_2} \simeq F(M^{hC_2}, I_{\mathbb{Z}}).$$

However, $\Sigma^4 KU$ is equivariantly equivalent to $I_{\mathbb{Z}} KU$ by Theorem 8.2, and M^{hC_2} is equivalent to M_{hC_2} by Corollary 6.3. The latter gives an isomorphism $F(M^{hC_2}, I_{\mathbb{Z}}) \simeq F(M_{hC_2}, I_{\mathbb{Z}}) \simeq F(M, I_{\mathbb{Z}})^{hC_2}$ which in turn implies that we have a chain of equivalences

$$\begin{aligned} F_{KU}(M, \Sigma^4 KU)^{hC_2} &\simeq F_{KU}(M, F(KU, I_{\mathbb{Z}}))^{hC_2} \\ &\simeq F(M, I_{\mathbb{Z}})^{hC_2} \simeq F(M^{hC_2}, I_{\mathbb{Z}}). \end{aligned}$$

(b) For a general A, we note that the map $A \to F(F(A, I_{\mathbb{Z}}), I_{\mathbb{Z}})$ is an equivalence, so we substitute A by its double dual. For a C_2-KU module M, since $M_{hC_2} \simeq M^{hC_2}$ (by Corollary 6.3) we have that

$$M^{hC_2} \wedge F(A, I_{\mathbb{Z}}) \simeq (M \wedge F(A, I_{\mathbb{Z}}))^{hC_2},$$

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implying

$$F(M^{hC_2}, A) \simeq F\left(M^{hC_2}, F(F(A, I_{\mathbb{Z}}), I_{\mathbb{Z}})\right) \simeq F(M^{hC_2} \wedge F(A, I_{\mathbb{Z}}), I_{\mathbb{Z}}) \\ \simeq F\left((M \wedge F(A, I_{\mathbb{Z}}))^{hC_2}, I_{\mathbb{Z}}\right).$$

By part (a), $F\left((M \wedge F(A, I_{\mathbb{Z}}))^{hC_2}, I_{\mathbb{Z}}\right)$ is equivalent to

$$F_{KU}\left(M \wedge F(A, I_{\mathbb{Z}}), \Sigma^4 KU\right)^{hC_2},$$

proving our result.

□

4. The norm cofibration sequence

Suppose a finite group G acts on a module M. The map on M given by

$$m \mapsto \sum_{g \in G} gm$$

factors through the orbits M_G and lands in the invariants M^G, giving the norm $N_G : M_G \to M^G$. The Tate cohomology groups are then defined as

$$\hat{H}^n(G; M) = \begin{cases} H^n(G; M) & \text{for } n \geq 1 \\ \operatorname{coker}(N_G) & \text{for } n = 0 \\ \ker(N_G) & \text{for } n = -1 \\ H_{-n-1}(G; M) & \text{for } n \leq -2. \end{cases} \tag{4.e}$$

In the same vein, Greenlees and May [GM95] have assigned a Tate spectrum to a G-spectrum X. The story starts with the cofiber sequence

$$EG_+ \to S^0 \to \tilde{E}G,$$

which gives rise to the following commutative diagram of equivariant maps

$$\begin{array}{ccccc} EG_+ \wedge X & \longrightarrow & X & \longrightarrow & \tilde{E}G \wedge X \\ \downarrow & & \downarrow & & \downarrow \\ EG_+ \wedge F(EG_+, X) & \longrightarrow & F(EG_+, X) & \longrightarrow & \tilde{E}G \wedge F(EG_+, X), \end{array}$$

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in which the rows are cofiber sequences. The left vertical arrow is always a G-equivalence. Upon taking G-fixed points and using the Adams isomorphism [May96, XVI.5.4], the diagram becomes

$$\begin{array}{ccccc} X_{hG} & \longrightarrow & X^G & \longrightarrow & \Phi^G X \\ \| & & \downarrow & & \downarrow \\ X_{hG} & \longrightarrow & X^{hG} & \longrightarrow & X^{tG}, \end{array} \tag{4.f}$$

where $\Phi^G X$, which we define as $(\tilde{E}G \wedge X)^G$, is the spectrum of geometric fixed points, and $X^{tG} = (\tilde{E}G \wedge F(EG_+, X))^G$ is the Tate spectrum of X.

The map $X_{hG} \to X^{hG}$ is often called the "norm map", and can be thought of as the spectrum level version of the algebraic norm map considered above. Thus the Tate spectrum is the cofiber of the norm map.

Associated to the homotopy orbit, homotopy fixed point, and Tate spectrum are spectral sequences

$$\begin{aligned} H_s(G, \pi_t(X)) &\Rightarrow \pi_{t-s} X_{hG} \\ H^s(G, \pi_t(X)) &\Rightarrow \pi_{t-s} X^{hG} \\ \hat{H}^s(G, \pi_t(X)) &\Rightarrow \pi_{t-s} X^{tG}. \end{aligned} \tag{4.g}$$

If R is a ring spectrum, then so are R^{hG} and R^{tG}; moreover, the homotopy fixed point and Tate spectral sequences are spectral sequences of differential algebras and there is a natural map between them that is compatible with the differential algebra structure.

From the definition it is clear that X^{tG} is contractible if and only if the norm map is an equivalence $X_{hG} \simeq X^{hG}$ between homotopy orbits and homotopy fixed points. We will show that this is the case for $X = KU$ with the C_2-action by complex conjugation.

5. The homotopy fixed point spectral sequence for $KO \simeq KU^{hC_2}$

In the equivalence $KO \simeq KU^{hC_2}$, C_2 acts on KU by complex conjugation. This action has been made "genuinely" equivariant by Atiyah [Ati66], who constructed a C_2-equivariant version of KU often denoted by $K\mathbb{R}$ and indexed by representation spheres. It follows from Atiyah's work that the underlying non-equivariant spectrum of $K\mathbb{R}$ is KU, and that the fixed points $K\mathbb{R}^{C_2}$ are equivalent to the homotopy

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fixed points $K\mathbb{R}^{hC_2} \simeq KU^{hC_2}$;** for a modern reference, see [Faj95]. In view of this equivalence, we will abandon the notation $K\mathbb{R}$ and refer to this "genuine" C_2-spectrum by the name KU as well.

The E_2-term of the homotopy fixed point spectral sequence for KU^{hC_2} is given by $H^*(C_2; KU_*)$, with the generator c acting on $KU_* = \mathbb{Z}[u^{\pm 1}]$ via $c(u) = -u$. The standard projective resolution

$$\cdots \xrightarrow{1-c} \mathbb{Z}[C_2] \xrightarrow{1+c} \mathbb{Z}[C_2] \xrightarrow{1-c} \mathbb{Z}$$

of $\mathbb{Z}$ immediately gives that the E_2-term of the HFPSS can be described as

$$E_2^{s,t} = \mathbb{Z}[\eta, t^{\pm 1}]/(2\eta),$$

with $|\eta| = (1,1)$ and $|t| = (4,0)$, where $t = u^2$ is the square of the Bott class and η is the class that ends up representing the Hopf element $S^1 \to KO$.

It has become somewhat standard to compute the differentials in the HFPSS as follows. Via Bott periodicity we know that KO is an 8 periodic ring spectrum, hence, an analysis of the spectral sequence forces there to be a differential $d_3(t) = \eta^3$, and the algebra structure determines the rest of the differentials. Even though this is a famous differential, we include here a proof from scratch that we learned from Mike Hopkins and Mike Hill and which only uses the fact that $KO \simeq KU^{hC_2} \simeq KU^{C_2}$, as well as the equivariant form of Bott periodicity. The importance of the argument lies in its potential to be generalized to higher chromatic analogues of KO, such as TMF or EO_n's, where less is known a priori.

The HFPSS is the Bousfield-Kan spectral sequence for the cosimplicial spectrum $F((EC_2)_+, KU)$, i.e.

$$KU \underset{1}{\overset{c}{\rightrightarrows}} F((C_2)_+, KU) \;\substack{\rightarrow\\ \rightarrow\\ \rightarrow}\; F((C_2)_+^2, KU) \;\substack{\rightarrow\\ \rightarrow\\ \rightarrow\\ \rightarrow}\; \cdots$$

Its associated normalized complex (or E_1 page of the spectral sequence) is

$$KU \xrightarrow{1-c} KU \xrightarrow{1+c} KU \xrightarrow{1-c} \cdots$$

Let ρ denote the regular complex representation of C_2; then $\rho = 1 \oplus \sigma$, with σ the sign representation. The one-point compactification S^ρ of ρ is C_2-equivalent to $\mathbb{C}P^1$. As such, it gives a C_2-map $S^\rho \hookrightarrow BU(1) \to BU \to KU$ which we will

**We remark that this is an equivalence of non-equivariant spectra and as such defines an equivalence of cohomology theories on spaces *without* an action.

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denote by v_1. This v_1 defines an element in $\pi_\rho KU$ which corresponds to a KU-module map

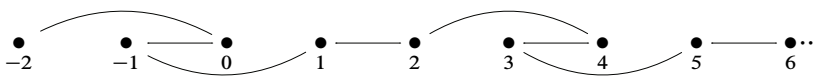
$$S^\rho \wedge KU \to KU$$

which is the C_2-equivalence known as Bott periodicity [May96, XIV.3.2].

An equivalent way to construct v_1 is by observing that the composition $S^1 \xrightarrow{\eta} KO \to KU$ is null and C_2-equivariant (where, of course, S^1 and KO have trivial actions). Thus it extends over the interior D^2 of S^1 in two ways; if $\tau : D^2 \to KU$ is one of them, then $c \circ \tau : D^2 \to KU$ is the other. By construction, τ and $c \circ \tau$ agree on the boundary S^1 of D^2, hence can be glued to give a map $S^\rho \to KU$ which is precisely v_1, as $cu = -u$.

Our goal is to show that in the HFPSS for KU, $d_3(v_1^2) = \eta^3$, and our method is to look at the map of HFPSSs induced by $v_1^2 : S^{2\rho} \to KU$. This will be fruitful because the HFPSS for $S^{2\rho}$ is determined by the cell decomposition of $(S^{2\rho})^{hC_2}$.

For any integer n, we have that the homotopy orbits $(S^{-n\sigma})_{hC_2}$ are equivalent to the Thom spectrum $(\mathbb{R}P^\infty)^{-n\xi}$, where ξ is the tautological line bundle of $\mathbb{R}P^\infty$. This Thom spectrum is briefly denoted $\mathbb{R}P^\infty_{-n}$, is called a stunted projective space, and has been extensively studied. In particular, its cell structure is well known. The schematic of the cell structure for $n = 2$ is as follows



(5.h)

where the bullets denote cells in dimension given by the corresponding number, straight short lines between two bullets indicate that the cells are attached via multiplication by 2, and the curved lines indicate attaching maps η. The pattern is four-periodic.

Let DX denote the Spanier-Whitehead dual of X. We have a chain of equivalences

$$(S^{2\rho})^{hC_2} \simeq (DS^{-2\rho})^{hC_2} \simeq D((S^{-2\rho})_{hC_2}) \simeq D(\Sigma^{-2}(S^{-2\sigma})_{hC_2}) \simeq \Sigma^2 D(\mathbb{R}P^\infty_{-2}).$$

Hence the cell diagram of $(S^{2\rho})^{hC_2}$ is



(5.i)

i.e. it is the diagram in (5.h) flipped and shifted by 2.

This determines the E_1-page of the HFPSS for $S^{2\rho}$ and all differentials in HFPSS for KU. Let us describe how that works. Figure 1 shows the relevant parts of the E_1-terms for the HFPSS for $S^{2\rho}$ and KU, the first following from the cell

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decomposition above. In particular, the d_1 differential is non-zero where shown in which case it is simply given by multiplication by 2. The map v_1^2 relates the spectral sequences, and in particular, as noted, the generator labelled e_4 maps to the generator labeled $t := v_1^2$.

It is a simple matter now to determine the relevant parts of the E_2-page. Note that there is a possible d_2 differential on the $S^{2\rho}$ spectral sequence, whilst sparsity precludes such a differential on KU^{hC_2}. In fact, the attaching map η in the cell decomposition above determines the d_2 differential $d_2(e_4) = \eta h_1^2$ on the $S^{2\rho}$ spectral sequence, as shown in Figure 2.

Under the map of spectral sequences induced by v_1^2, the classes e_4 and h_1^2 map to the classes $v_1^2 = t$ and η^2, respectively. Moreover, multiplication by η remains multiplication by η under v_1^2, but it is a map of cohomological degree 1 in the HFPSS for KU. In particular, we get the differential $d_3(v_1^2) = \eta^3$ as expected; multiplicativity now gives that

$$d_3(v_1^{2+4k}\eta^m) = v_1^{4k}\eta^m d_3(v_1^2) = v_1^{4k}\eta^{m+3},$$

i.e. the usual differential pattern depicted in Figure 3.

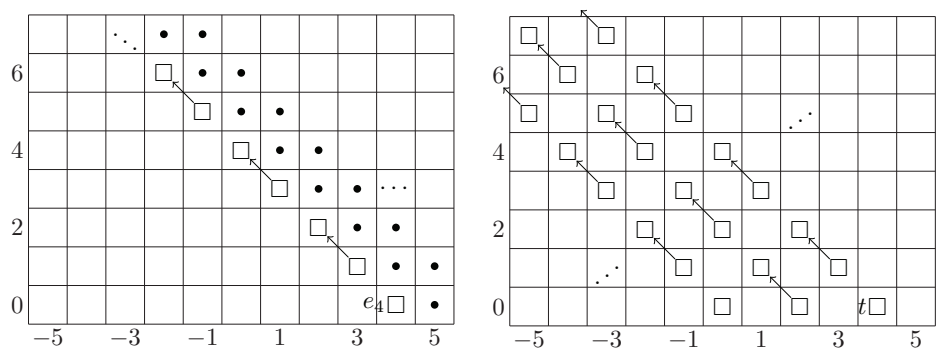


Figure 1:The E_1-pages for the HFPSS for $S^{2\rho}$ and KU. By equivariant Bott periodicity, v_1^2 induces an isomorphism in degree (4,0).

6. The Tate spectrum and the Euler class

In addition to the spectral sequences (4.g), there is a conceptual way to compute the homotopy of a Tate spectrum from that of the homotopy fixed points and it involves a (co)homology class called the Euler class, described below. The calculations in this section bear some similarity to those of Fajstrup [Faj95] who calculates that the

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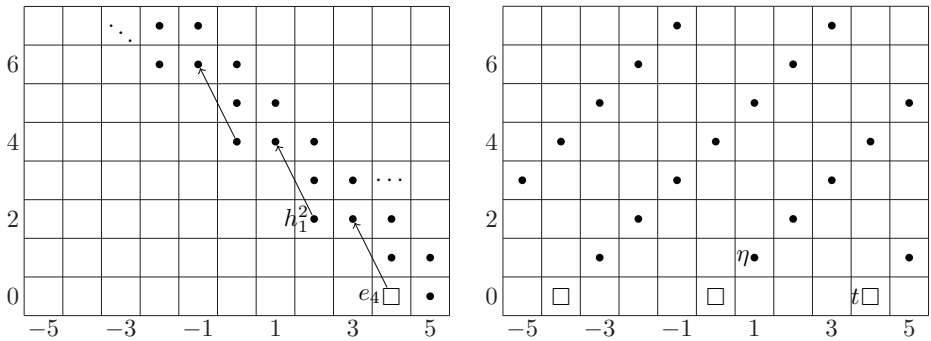


Figure 2:The E_2-pages for the HFPSS for $S^{2\rho}$ and KU. Note that there is a d_2-differential on the E_2-page for $S^{2\rho}$ but there is no room for a d_2 differential for KU.

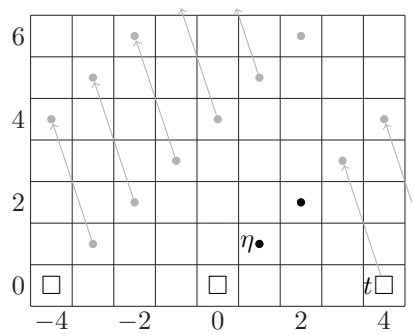


Figure 3: The E_3-page for the HFPSS for KU.

generalized Tate spectrum associated to $K\mathbb{R}$ is equivariantly contractible. The key to both calculations is the fact that the generator $\eta \in \pi_1 KO$ is nilpotent.

We now define the Euler class. Think of the sign representation σ as a C_2-bundle over a point. Then the Euler class is a characteristic class

$$e_\sigma \in H^1_{C_2}(*) = H^1(\mathbb{R}P^\infty) \simeq \mathbb{Z}/2.$$

Alternatively, the Euler class e_σ is the element of

$$\pi^{C_2}_{-\sigma} KU = [S^0, \Sigma^\sigma KU]^{C_2} \cong [S^1, \Sigma^\rho KU]^{C_2} \cong [S^1, KU]^{C_2} \cong \pi_1 KO = \mathbb{Z}/2,$$

which is defined as the C_2-fixed points of composition of the inclusion of fixed points $S^0 \to S^\sigma$ with the σ-suspension of the unit map $S^0 \to KU$. Note that in the

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chain of equivalences above, we have used the equivariant form of Bott periodicity. We claim that e_σ is the non-trivial element of $\pi_{-\sigma}^{C_2} KU$; indeed, the inclusion $S^0 \to S^\sigma$ is (unstably) the identity on C_2-fixed points, and the fixed points of the unit map $S^0 \to KU$ contain the unit map for $KU^{C_2} \simeq KO$, which is non-trivial.

Thus we conclude that $e_\sigma = \eta \in \pi_1 KO$.

Proposition 6.1 *The Tate spectrum KU^{tC_2} as well as the geometric fixed point spectrum $\Phi^{C_2} KU$ are both contractible.*

Proof: By [May96, XVI.6.8], $\Phi^{C_2} KU \simeq KU^{C_2}[e_\sigma^{-1}]$ and $KU^{tC_2} \simeq KU^{hC_2}[e_\sigma^{-1}]$. But $e_\sigma = \eta$ is nilpotent in $KO \simeq KU^{C_2} \simeq KU^{hC_2}$, whence the claim. □

We can also show that the Tate spectrum KU^{tC_2} vanishes by a direct computation. This will have the advantage of making the homotopy orbit spectral sequence easy to calculate. From the above discussion, the E_2-term of the Tate spectral sequence for KU can be described as

$$\hat{E}_2^{s,t} = E_2^{s,t}[\eta^{-1}] = \mathbb{Z}/2[\eta^{\pm 1}, t^{\pm 1}].$$

We proved above that $\eta^3 = 0$ in the homotopy of KO. Since the map between the homotopy fixed point and Tate spectral sequence is compatible with the differential algebra structure, we also have that $\eta^3 = 0$ in the Tate spectral sequence. Thus we must have $d_3(t) = \eta^3$ and the differentials then follow the pattern shown in Figure 4 because of multiplicativity. In particular the E_4-page is 0, hence the Tate spectrum is contractible as the Tate spectral sequence is conditionally convergent.

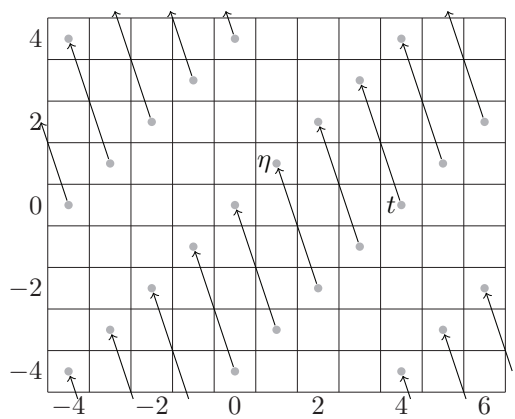


Figure 4: Tate spectral sequence for KU

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The vanishing of the Tate spectrum allows us to prove the following two corollaries which we used in Section 3.

Proposition 6.2 *Let M be a C_2-equivariant KU module. Then M^{tG} and $\Phi^G M$ are contractible.*

Proof: The spectra M^{tG} and $\Phi^G M$ are modules over KU^{tG} and $\Phi^G KU$, respectively, and these are contractible by Proposition 6.1. □

Corollary 6.3 *Let M be a C_2-equivariant KU module. Then the fixed points M^{C_2} and the homotopy fixed points M^{hC_2} are equivalent.*

Proof: Follows immediately from the diagram (4.f), in which the two rightmost spectra become contractible by the above proposition. □

7. The homotopy orbit spectrum for KO

To calculate the homotopy orbit spectral sequence we can use the Tate cohomology groups as given in Equation (4.e) as well as the calculations shown in Figure 4. We just need to calculate the co-invariants; for the trivial action $H_0(C_2;\mathbb{Z}) \simeq \mathbb{Z}$, whilst for the sign-representation $H_0(C_2,\mathbb{Z}_{sgn}) = \mathbb{Z}/2$.

We draw the homotopy orbit spectral sequence with cohomological grading, i.e. set $E_2^{s,t} = H_{-s}(C_2,\pi_t KU)$. The differentials can be inferred from those in the Tate spectral sequence. The resulting spectral sequence is shown in Figure 5. Note that there has to be an additive extension as shown with the dashed line, since $\pi_0 KO \simeq \mathbb{Z}$ is torsion free.

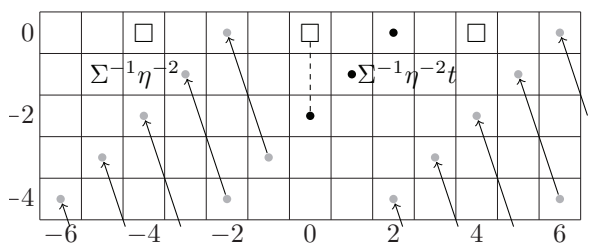


Figure 5: The homotopy orbit spectral sequence for KU.

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8. The Anderson dual of KO

From Equation (2.c) and Proposition 6.1, we get the following sequence of equivalences

$$I_{\mathbb{Z}}KO \simeq F(KU^{hC_2}, I_{\mathbb{Z}}) \simeq F(KU_{hC_2}, I_{\mathbb{Z}}) \simeq F(KU, I_{\mathbb{Z}})^{hC_2}.$$

Note that even though $I_{\mathbb{Z}}KU \simeq KU$, this does not imply that $(I_{\mathbb{Z}}KU)^{hC_2} \simeq KO$, as the action may have changed. However, the action on the homotopy groups of $I_{\mathbb{Z}}KU$ is unchanged as both the trivial representation and the sign representation of C_2 are self-dual. Hence the E_2-terms of the HFPSS associated to KU and $I_{\mathbb{Z}}KU$ are isomorphic, and a twist in the action would alter the differential pattern. Since the E_2-term of the HFPSS's is 4-periodic, and the HFPSS of $I_{\mathbb{Z}}KU$ is a module over the one for KU, we have that $I_{\mathbb{Z}}KO$ is either KO or $\Sigma^4 KO$. We will show that it is the latter by determining the pattern of differentials.

Theorem 8.1 *The Anderson dual of KO is $\Sigma^4 KO$.*

Proof: The argument here will follow the proof of Theorem 13.1 of [Sto12]. Our goal is to show that the spectral sequence computing $F(KU_{hC_2}, I_{\mathbb{Z}})$ is the linear dual of the homotopy orbit spectral sequence for $KO = KU_{hC_2}$.

As in the proof of [Sto12, Thm.13.1] (which in turn is modeled after Deligne's "Lemma of two filtrations" [Del71]), we can construct a commuting square of spectral sequences

$$\begin{array}{ccc} \mathrm{Ext}^v_{\mathbb{Z}}(H_h(C_2, \pi_t KU), \mathbb{Z}) & \xRightarrow{B} & \mathrm{Ext}^v_{\mathbb{Z}}(\pi_{t+h}(KU)_{hC_2}, \mathbb{Z}) \\ \Big\Downarrow A & & \Big\Downarrow D \\ H^{h+v}(C_2, \pi_{-t} I_{\mathbb{Z}} KU) & \xRightarrow[C]{} & \pi_{-t-h-v} I_{\mathbb{Z}}(KU_{hC_2}). \end{array}$$

Here B is dual to the homotopy orbit spectral sequence, A is obtained via the Grothendieck composite functor spectral sequence associated to the functors Hom and coinvariants, and D is the Anderson duality spectral sequence given by Equation (2.b). We want to show that the spectral sequences B and C are isomorphic. As explained in [Sto12], the differentials in B are compatible with the filtration giving C if and only if A collapses, which holds in our case. Consequently, B and C are isomorphic. Thus we apply $\mathrm{Ext}_{\mathbb{Z}}(-, \mathbb{Z})$ to homotopy orbit spectral sequence to obtain the homotopy fixed point spectral sequence for computing the Anderson dual of KO, and we can read the differentials straight from the homotopy orbit spectral sequence.

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A diagram of the spectral sequence can be seen in Figure 6. This gives an isomorphism on homotopy $\pi_*(I_{\mathbb{Z}}KO) \simeq \pi_*(\Sigma^4 KO)$. Invoking Proposition 2.2 now produces an equivalence $\Sigma^4 KO \simeq I_{\mathbb{Z}}KO$. □

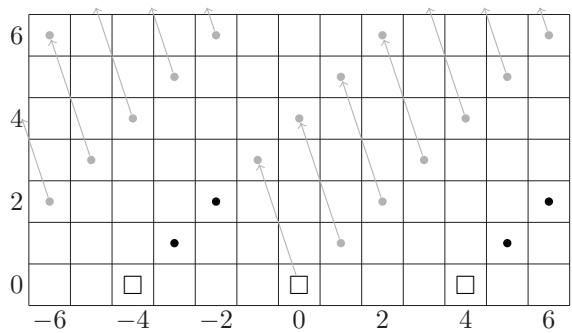


Figure 6: The homotopy fixed point spectral sequence for $I_{\mathbb{Z}}KU$

We are finally in a position to prove Theorem 8.2.

Theorem 8.2 *The Anderson dual $F(KU, I_{\mathbb{Z}})$ is C_2-equivariantly equivalent to $\Sigma^4 KU$.*

Proof: By Proposition 2.2, we can construct a non-equivariant equivalence $\Sigma^4 KU \to I_{\mathbb{Z}}KU$ which is nevertheless a KU-module map. We claim that this map has an equivariant refinement, or equivalently, that the spectrum map $d : S^4 \to I_{\mathbb{Z}}KU$ generating the dualizing class has an equivariant refinement. To see that, consider the C_2-cofiber sequence

$$C_{2+} \to S^0 \to S^\sigma,$$

which after mapping to $I_{\mathbb{Z}}KU$ on $\pi_4^{C_2}$ gives an exact sequence

$$[S^4, I_{\mathbb{Z}}KU]_{C_2} \to [S^4, I_{\mathbb{Z}}KU] \to [S^{\sigma+3}, I_{\mathbb{Z}}KU]_{C_2}.$$

It suffices to show that $d \in [S^4, I_{\mathbb{Z}}KU]$ maps to zero in $[S^{\sigma+3}, I_{\mathbb{Z}}KU]_{C_2}$. But using equivariant Bott periodicity and Corollary 6.3, the latter is equivalent to

$$[S^2, I_{\mathbb{Z}}KU]_{C_2} \cong \pi_2(I_{\mathbb{Z}}KU)^{C_2} \cong \pi_2(I_{\mathbb{Z}}KU)^{hC_2},$$

which by Theorem 8.1 is zero. □

9. Applications to the $K(1)$-local Picard group

Fix a prime p; in this section we use the usual notation for the chromatic players. So, $K(n)$ and E_n denote the Morava K and E-theory spectra at height n; $\mathbb{G}_n$ is the

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(big) Morava stabilizer group, L_n denotes localization with respect to E_n[††], and M_n is the monochromatic layer functor, i.e. the homotopy fiber of the map $L_n \to L_{n-1}$.

As currently defined, even if X is a $K(n)$-local spectrum, $I_{\mathbb{Q}/\mathbb{Z}}X$ need not be. The relevant details to properly extend Brown-Comenetz duality to the $K(n)$-local category have been worked out by Gross and Hopkins [HG94], and this form of duality also goes by the name Gross-Hopkins duality. For $K(n)$-local X, the Gross-Hopkins dual of X is defined as

$$I_n X = F(M_n X, I_{\mathbb{Q}/\mathbb{Z}});$$

This spectrum is $K(n)$-local; indeed by [Sto12, Proposition 2.2] it is in fact isomorphic to $L_{K(n)}F(L_n X, I_{\mathbb{Q}/\mathbb{Z}})$.

Since $I_{\mathbb{Q}}X$ becomes contractible after $K(n)$-localisation (for $n \geq 1$) we get that, if X is E_n-local, the fiber sequence in Equation (2.a) gives

$$I_n X = L_{K(n)}\Sigma I_{\mathbb{Z}} X. \tag{9.j}$$

Since KO is E_1-local we immediately obtain the following statement.

Corollary 9.1 *The $K(1)$-local Brown-Comenetz dual of KO is given by*

$$I_1 KO \simeq \Sigma^5 KO.$$

The other main type of duality in the $K(n)$-local category is $K(n)$-local Spanier-Whitehead duality, defined by $D_n X = F(X, L_{K(n)}S^0)$, and now we relate it to Gross-Hopkins duality. To do so we need the Picard group of the $K(n)$-local homotopy category, introduced by Hopkins [HMS94]. We call a $K(n)$-local spectrum X invertible if there exists a spectrum X^{-1} such that $L_{K(n)}(X \wedge X^{-1}) \simeq L_{K(n)}S^0$; the Picard group Pic_n is the group of isomorphism classes of invertible objects. There is a map $\epsilon : \mathrm{Pic}_n \to H^1(\mathbb{G}_n, (E_n)_0^\times)$, and if p is large compared to n then this map is injective [HMS94, Prop.7.5].

Hopkins, Mahowald and Sadofsky further show that a $K(n)$-local X being invertible is equivalent to $(E_n)_*X$ being a free $(E_n)_*$-module of rank 1.[§§] Noting that there is an equivalence $M_n X \simeq M_n L_{K(n)}X$, let I_n denote $I_n L_{K(n)}S^0$. By work of Gross and Hopkins [HG94], $(E_n)_* I_n$ is a free $(E_n)_*$-module of rank 1 and hence I_n defines an element of Pic_n. In particular I_n is dualisable, so there is a $K(n)$-local equivalence

$$I_n X \simeq F(X, I_n) \simeq D_n X \wedge I_n. \tag{9.k}$$

[††]or equivalently with respect to the Johnson-Wilson spectrum $E(n)$

[§§]Note that we follow the usual convention in that $(E_n)_*X := \pi_* L_{K(n)}(E_n \wedge X)$.

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In fact more can be said by incorporating the action of the Morava stabilizer group $\mathbb{G}_n$. We recall briefly that there is a reduced determinant map $\mathbb{G}_n \to \mathbb{Z}_p^\times$, and if M is a $(E_n)_*$-module with a compatible action of $\mathbb{G}_n$, then we write $M\langle\det\rangle$ for the module with $\mathbb{G}_n$ action twisted by the determinant map. Then Hopkins and Gross in fact prove the stronger statement (see also [Str00]) that

$$(E_n)_* I_n \simeq \Sigma^{n^2-n}(E_n)_*\langle\det\rangle.$$

There is a 'twisted sphere' spectrum $S^0\langle\det\rangle$ (see [GHMR12, Remark 2.7]) such that $(E_n)_* S\langle\det\rangle \simeq (E_n)_*\langle\det\rangle$ and thus whenever $\epsilon : \mathrm{Pic}_n \to H^1(\mathbb{G}_n, (E_n)_0^\times)$ is injective there is an equivalence

$$I_n \simeq \Sigma^{n^2-n} S^0\langle\det\rangle. \tag{9.1}$$

We will show that at $n = 1, p = 2$ this does *not* hold, and hence that the map ϵ has non-trivial kernel traditionally denoted κ_1 and called the 'exotic' $K(1)$-local Picard group. We note that at $n = 1, S^0\langle\det\rangle \simeq S^2$, so trivial kernel would imply that $I_1 \simeq S^2$.

From now on, we set $p = 2$ and omit $K(1)$-localization from the notation with the understanding that everything is in the $K(1)$-local category. Note that the Morava E-theory spectrum E_1 is now the 2-completion[¶¶] of KU, the Morava stabilizer group $\mathbb{G}_1$ is $\mathbb{Z}_2^\times$ with its elements corresponding to 2-adic Adams operations. The subgroup $C_2 = \{\pm 1\} \in \mathbb{Z}_2^\times$ is a maximal finite subgroup whose non-trivial element acts on KU by ψ^{-1} which is complex conjugation.

Let l be a topological generator of $\mathbb{Z}_2^\times/\{\pm 1\}$; it is well known (eg. [Bou79]) that the $K(1)$ local sphere sits in a fiber sequence

$$S \to KO \xrightarrow{\psi^l - 1} KO.$$

Taking the Anderson dual of this sequence gives

$$\Sigma^4 KO \xrightarrow{I_{\mathbb{Z}}\psi^l - 1} \Sigma^4 KO \to \Sigma^{-1} I_1.$$

Let P denote $\Sigma^{-2} I_1$; hence P is an element of Pic_1 such that $P \neq S$ if and only if $I_1 \neq S^2$.

Lemma 9.2 *The Anderson dual of the Adams operation ψ^l is*

$$I_{\mathbb{Z}}\psi^l = \Sigma^4(l^{-2} \quad {}^{1/l}) = \Sigma^{-4}(l^2 \quad {}^{1/l}).$$

[¶¶] equivalently, $K(1)$-localization

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Proof: The idea of the proof is that the self-maps of KO are detected by their effect on homotopy. To be more precise, we recall that there is an equivalence

$$KO[[\mathbb{Z}_2^\times/\{\pm 1\}]] \to F(KO, KO),$$

where the left hand side is defined as $\operatorname{holim}_H (KO \wedge H)$, where H runs over the finite quotients of the profinite group $\mathbb{Z}_2^\times/\{\pm 1\}$ (a reference is, for example, [GHMR05, Prop.2.6]). In particular, this describes the homotopy classes of maps $KO \to KO$ as the completed group ring $\mathbb{Z}_2[[\mathbb{Z}_2^\times/\{\pm 1\}]]$, i.e. power series on the 2-adic Adams operations of KO, and these can be distinguished by what they do on the non-torsion elements of KO_*.

Let $t \in KO_{4n}$ be an arbitrary non-torsion element of KO_*. Then the Adams operation ψ^l acts on t as

$$^l(t) = l^{2n}t.$$

Note that therefore, $\Sigma^{8m}\psi^l : KO \to KO$ is $l^{-4m}\psi^l$.

Since the linear dual of multiplication by m is again multiplication by m, we see that

$$I_{\mathbb{Z}}\psi^l : I_{\mathbb{Z}}KO \to I_{\mathbb{Z}}KO$$

is given on the non-torsion homotopy by

$$I_{\mathbb{Z}}\psi^l(t^\vee) = l^{2n}t^\vee.$$

Identifying $\Sigma^{-4}I_{\mathbb{Z}}KO$ with KO, we get that

$$\Sigma^{-4}I_{\mathbb{Z}}\psi^l : KO \to KO$$

takes an element $t \in KO_{4n} \cong (I_{\mathbb{Z}}KO)_{4n+4}$ and maps it to $l^{-2n-2}t = l^{-2}\psi^{1/l}(t)$. □

Note that, in particular, this lemma provides resolutions of P as

$$\begin{aligned} P \to \Sigma^4 KO \xrightarrow{\Sigma^4(l^{-2}\psi^{1/l})-1} \Sigma^4 KO, \text{ or} \\ P \to \Sigma^{-4} KO \xrightarrow{\Sigma^{-4}(l^{2}\psi^{1/l})-1} \Sigma^{-4} KO, \end{aligned} \tag{9.m}$$

which are a $K(1)$-local analogue (us to a suspension shift) of the resolution of the $K(2)$-local Brown-Comenetz spectrum of Goerss-Henn [GH12]. We smash the second resolution with $\Sigma^4 KO$ to deduce the following result.

Proposition 9.3 *The smash product $P \wedge KO$ is $\Sigma^4 KO$. Consequently, P defines a non-trivial element of κ_1.*

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Proof: We consider the fiber sequence of KO-modules

$$\Sigma^4 P \wedge KO \to KO \wedge KO \xrightarrow{\alpha} KO \wedge KO,$$

from which we will determine $\pi_* \Sigma^4 P \wedge KO$, where for brevity we have denoted by α the map $(l^2\psi^{1/l} - 1) \wedge 1$. The homotopy groups of $KO \wedge KO$ are (see [Hop, Prop.1] or [GHMR05, Prop.2.4])

$$\pi_*(KO \wedge KO) \cong KO_0 KO \otimes_{KO_0} KO_* \cong \mathrm{Map}^c(\mathbb{Z}_2^\times/\{\pm 1\}, \mathbb{Z}_2) \otimes_{\mathbb{Z}_2} KO_*,$$

and the map α sends $f \otimes x \in \mathrm{Map}^c(\mathbb{Z}_2^\times/\{\pm 1\}, \mathbb{Z}_2) \otimes_{\mathbb{Z}_2} KO_*$ to $g \otimes x$, where $g \in \mathrm{Map}^c(\mathbb{Z}_2^\times/\{\pm 1\}, \mathbb{Z}_2)$ is the function determined by

$$g(k) = l^2 f(k/l) - f(k).$$

Consequently, $f \otimes x$ is in the kernel of $\pi_*\alpha$ if and only if for every $k \in \mathbb{Z}_2^\times/\{\pm 1\}$, we have $f(k) = l^2 f(k/l)$. In particular, the kernel of $\pi_0\alpha$ is $\mathbb{Z}_2$, where $a \in \mathbb{Z}_2$ corresponds to the continuous function f_a determined by $f_a(1) = a$ and the relation $f_a(k) = l^2 f_a(k/l)$; thus the kernel of $\pi_*\alpha$ is $\mathrm{Ker}\pi_0\alpha \otimes KO_*$.

Next, we claim that $\pi_0\alpha$, and therefore $\pi_*\alpha$, is surjective. Indeed, let g be a function $\mathbb{Z}_2^\times/\{\pm 1\} \to \mathbb{Z}_2$; then the function defined by

$$f(k) = \sum_{n=1}^{\infty} \frac{1}{l^{2n}} g(l^n k)$$

is such that $\pi_0(\alpha)(f) = g$.

We conclude that $\pi_* \Sigma^4 P \wedge KO \cong \mathrm{Ker}\pi_0\alpha \otimes KO_* \cong KO_*$ as a KO_*-module. Appealing to Proposition 2.2 now gives us that $\Sigma^4 P \wedge KO \simeq KO$. □

Proposition 9.4 *The element P of κ_1 has order two, i.e. $P \wedge P \simeq S$.*

Proof: The proof is similar to the one for Proposition 9.3; we smash the resolution of P in Equation (9.m) with P and use the description of $P \wedge KO$ from the proof of Proposition 9.3. Denote again by α the map $(l^2\psi^{1/l} - 1) \wedge 1 : KO \wedge KO$, and let β denote the map $1 \wedge (l^2\psi^{1/l} - 1) : KO \wedge KO$. Consider the commutative diagram

$$\begin{array}{ccc} \Sigma^{-4}KO \wedge \Sigma^{-4}KO & \xrightarrow{\Sigma^{-8}\beta} & \Sigma^{-4}KO \wedge \Sigma^{-4}KO \\ \downarrow{\scriptstyle \Sigma^{-8}\alpha} & & \downarrow{\scriptstyle \Sigma^{-8}\alpha} \\ \Sigma^{-4}KO \wedge \Sigma^{-4}KO & \xrightarrow{\Sigma^{-8}\beta} & \Sigma^{-4}KO \wedge \Sigma^{-4}KO; \end{array}$$

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we know that the vertical fibers are $P \wedge \Sigma^{-4} KO \simeq KO$, and the fiber of the induced map between them is $P \wedge P$.

Let us again identify the homotopy groups of $KO \wedge KO$ with $\mathrm{Map}^c(\mathbb{Z}_2^\times/\{\pm 1\}, \mathbb{Z}_2) \otimes KO_*$; then (as in [Hop]) β takes a function f to the function βf defined by

$$\beta f(k) = l^2 \quad {}^{1/l}(f(kl)) - f(k).$$

Now suppose that f is in the kernel of α, i.e. it is in the homotopy of the fiber of α; then for every k, $f(kl) = l^2 f(k)$, whence

$$\beta f(k) = l^4 \quad {}^{1/l}(f(k)) - f(k).$$

This implies that the restriction of $\Sigma^{-8}\beta$ to the fibers of $\Sigma^{-8}\alpha$ is

$$\Sigma^{-8}(l^4 \psi^{1/l} - 1) : P \wedge \Sigma^{-4} KO \simeq KO \to P \wedge \Sigma^{-4} KO \simeq KO.$$

But we have that, as in the proof of Lemma 9.2,

$$\Sigma^{-8}(l^4 \quad {}^{1/l}) = \frac{1}{l^4} l^4 \quad {}^{1/l} = \quad {}^{1/l},$$

so our fiber sequence becomes

$$P \wedge P \to KO \xrightarrow{\psi^{1/l} - 1} KO,$$

showing that $P \wedge P \simeq S$, as $1/l$ is also a topological generator of $\mathbb{Z}_2^\times/\{\pm 1\}$. □

The effect of Propositions 9.3 and 9.4 is that we have produced a subgroup $\mathbb{Z}/2$ of κ_1 generated by P. It is known by the work of Hopkins-Mahowald-Sadofsky [HMS94] that in fact this is all of κ_1; for completeness we include here what is essentially their construction of a surjective map $\kappa_1 \to \mathbb{Z}/2$, thus recovering the following result.

Proposition 9.5 *The exotic Picard group κ_1 is $\mathbb{Z}/2$ generated by P.*

Proof: Let Z be an arbitrary element of κ_1; the key point is that $KU_* Z$ and KU_* are isomorphic as KU_*-modules, and the isomorphism respects the $\mathbb{Z}_2^\times$-action. (Recall that we are working $K(1)$-locally, so KU really means the 2-completion of KU.) Therefore the E_2-term for the $K(1)$-local Adams-Novikov spectral sequence for Z coincides with that for the sphere, although the differentials may be different. By [GHMR12, Rem.3.4] (see also the work of [HS99, KS07]) there is a group homomorphism

$$\tau : \kappa_1 \to H^3(\mathbb{Z}_2^\times, (KU)_2) \simeq \mathbb{Z}/2,$$

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defined in the following way. Let $\iota_Z \in H^0(\mathbb{Z}_2^\times, (KU)_0 Z) \simeq \mathbb{Z}_2$ be the identity element. The first (and indeed the only) possible differential in the $K(1)$-local ANSS for Z is a d_3. Given a choice of a $\mathbb{Z}_2^\times$-equivariant isomorphism $f : KU_* \xrightarrow{\simeq} KU_* Z$, we have a diagram

$$\begin{array}{ccc} H^0(\mathbb{Z}_2^\times, (KU)_0) & \xrightarrow{\phi} & H^3(\mathbb{Z}_2^\times, (KU)_2) \\ {\scriptstyle f_*}\downarrow{\scriptstyle \simeq} & & {\scriptstyle \simeq}\downarrow{\scriptstyle f_*} \\ H^0(\mathbb{Z}_2^\times, (KU)_0 Z) & \xrightarrow[d_3]{} & H^3(\mathbb{Z}_2^\times, (KU)_2 Z) \end{array}$$

and we define $\tau(Z) = \phi(\iota_Z) = f_*^{-1} d_3 f_*(\iota_Z)$. This does not depend on the choice of f and defines the claimed group homomorphism $\kappa_1 \to H^3(\mathbb{Z}_2^\times, (KU)_2) \simeq \mathbb{Z}/2$.

Note that $Z \simeq L_{K(1)} S^0$ if and only if ι_Z survives the spectral sequence. The E_2-term of the spectral sequence for Z is a free module of rank one over the E_2-term of that for the sphere, generated by the class ι_Z, and this fully determines the d_3 differential for Z. Standard calculations (for example [HS99]) show that $E_4 = E_\infty$ and therefore the only differential possible is a d_3. This implies that P maps to the non-trivial element of $\mathbb{Z}/2$ under the map τ, and so τ is a surjection. It is also injective for if $d_3(\iota_Z) = 0$, then ι_Z is a permanent cycle and the resulting map extends to an equivalence $Z \simeq L_{K(1)} S^0$. □

As an additional application, we can use Propositions 9.3 and 9.4 to compute the $K(1)$-local Spanier-Whitehead dual of KO, thus recovering the result of [HM07, Lemma 8.16].

Corollary 9.6 *The $K(1)$-local Spanier-Whitehead dual of KO is given by $D_1(KO) \simeq \Sigma^{-1} KO$.*

Proof: We have the series of equivalences

$$\begin{aligned} D_1 KO &\simeq F(KO, S) \simeq F(KO, P) \wedge P \\ &\simeq F(KO, \Sigma^{-1} I_{\mathbb{Z}}) \wedge P \simeq \Sigma^{-1} \Sigma^4 KO \wedge P \\ &\simeq \Sigma^{-1} KO. \end{aligned}$$

□

References

And69. D.W. Anderson, *Universal coefficient theorems for K-theory, mimeographed notes*, Univ. California, Berkeley, Calif, 1969.

https://doi.org/10.1017/is014007001jkt275 Published online by Cambridge University Press

Ati66. M. F. Atiyah, *K-theory and reality*, Quart. J. Math. Oxford Ser. (2) **17** (1966), 367–386. MR 0206940 (34 #6756)

Bök81. Marcel Bökstedt, *Universal coefficient theorems for equivariant K and KO-theory*, PhD Thesis, 1981.

Bou79. A. K. Bousfield, *The localization of spectra with respect to homology*, Topology **18** (1979), no. 4, 257–281. MR 551009 (80m:55006)

BR07. Andrew Baker and Birgit Richter, *Realizability of algebraic Galois extensions by strictly commutative ring spectra*, Trans. Amer. Math. Soc. **359** (2007), no. 2, 827–857 (electronic).

Del71. Pierre Deligne, *Théorie de Hodge. II*, Inst. Hautes Études Sci. Publ. Math. (1971), no. 40, 5–57. MR 0498551 (58 #16653a)

Dug05. Daniel Dugger, *An Atiyah-Hirzebruch spectral sequence for KR-theory*, *K*-Theory **35** (2005), no. 3-4, 213–256 (2006). MR MR2240234 (2007g:19004)

EKMM97. A. D. Elmendorf, I. Kriz, M. A. Mandell, and J. P. May, *Rings, modules, and algebras in stable homotopy theory*, Mathematical Surveys and Monographs, vol. 47, American Mathematical Society, Providence, RI, 1997, With an appendix by M. Cole. MR 1417719 (97h:55006)

Faj95. Lisbeth Fajstrup, *Tate cohomology of periodic K-theory with reality is trivial*, Trans. Amer. Math. Soc. **347** (1995), no. 5, 1841–1846.

FHM03. H. Fausk, P. Hu, and J. P. May, *Isomorphisms between left and right adjoints*, Theory Appl. Categ. **11** (2003), No. 4, 107–131. MR 1988072 (2004d:18006)

FMS07. Daniel S. Freed, Gregory W. Moore, and Graeme Segal, *The uncertainty of fluxes*, Comm. Math. Phys. **271** (2007), no. 1, 247–274. MR 2283960 (2008j:58051b)

GH12. Paul G Goerss and Hans-Werner Henn, *The brown-comenetz dual of the $k(2)$-local sphere at the prime 3*, arXiv preprint arXiv:1212.2836, 2012.

GHMR05. P. Goerss, H.-W. Henn, M. Mahowald, and C. Rezk, *A resolution of the $K(2)$-local sphere at the prime 3*, Ann. of Math. (2) **162** (2005), no. 2, 777–822. MR 2183282 (2006j:55016)

GHMR12. Paul Goerss, Hans-Werner Henn, Mark Mahowald, and Charles Rezk, *On Hopkins' Picard groups for the prime 3 and chromatic level 2*, arXiv preprint arXiv:1210.7033, 2012.

GM95. J. P. C. Greenlees and J. P. May, *Generalized Tate cohomology*, Mem. Amer. Math. Soc. **113** (1995), no. 543, viii+178. MR 1230773 (96e:55006)

GS96. J. P. C. Greenlees and Hal Sadofsky, *The Tate spectrum of v_n-periodic complex oriented theories*, Math. Z. **222** (1996), no. 3, 391–405. MR 1400199 (97d:55010)

https://doi.org/10.1017/is014007001jkt275 Published online by Cambridge University Press

Har66. Robin Hartshorne, *Residues and duality*, Lecture notes of a seminar on the work of A. Grothendieck, given at Harvard 1963/64. With an appendix by P. Deligne. Lecture Notes in Mathematics, No. 20, Springer-Verlag, Berlin, 1966. MR 0222093 (36 #5145)

HG94. M. J. Hopkins and B. H. Gross, *The rigid analytic period mapping, Lubin-Tate space, and stable homotopy theory*, Bull. Amer. Math. Soc. (N.S.) **30** (1994), no. 1, 76–86. MR 1217353 (94k:55009)

HL13. Michael Hopkins and Jacob Lurie, *Ambidexterity in $K(n)$-local stable homotopy theory*, Available at http://www.math.harvard.edu/~lurie/papers/Ambidexterity.pdf, 2013.

HM07. Rebekah Hahn and Stephen Mitchell, *Iwasawa theory for $K(1)$-local spectra*, Trans. Amer. Math. Soc. **359** (2007), no. 11, 5207–5238. MR 2327028 (2008h:55005)

HMS94. Michael J. Hopkins, Mark Mahowald, and Hal Sadofsky, *Constructions of elements in Picard groups*, Topology and representation theory (Evanston, IL, 1992), Contemp. Math., vol. 158, Amer. Math. Soc., Providence, RI, 1994, pp. 89–126. MR 1263713 (95a:55020)

Hop. Michael J Hopkins, *$K(1)$-local E_∞-ring spectra*, Available at http://www.math.rochester.edu/people/faculty/doug/otherpapers/knlocal.pdf.

HS99. Mark Hovey and Hal Sadofsky, *Invertible spectra in the $E(n)$-local stable homotopy category*, J. London Math. Soc. (2) **60** (1999), no. 1, 284–302. MR 1722151 (2000h:55017)

HS05. M. J. Hopkins and I. M. Singer, *Quadratic functions in geometry, topology, and M-theory*, J. Differential Geom. **70** (2005), no. 3, 329–452. MR 2192936 (2007b:53052)

Kar01. Max Karoubi, *A descent theorem in topological K-theory*, K-Theory **24** (2001), no. 2, 109–114. MR 1869624 (2002m:19005)

KS07. Yousuke Kamiya and Katsumi Shimomura, *Picard groups of some local categories*, Publ. Res. Inst. Math. Sci. **43** (2007), no. 2, 303–314. MR 2341012 (2008e:55011)

LN12. Tyler Lawson and Niko Naumann, *Strictly commutative realizations of diagrams over the Steenrod algebra and topological modular forms at the prime 2*, arXiv preprint arXiv:1203.1696, 2012.

Lur12. Jacob Lurie, *Derived Algebraic Geometry XIV: Representability Theorems*, Available at http://http://www.math.harvard.edu/~lurie/papers/DAG-XIV.pdf, 2012.

May96. J. P. May, *Equivariant homotopy and cohomology theory*, CBMS Regional Conference Series in Mathematics, vol. 91, Published for the Conference

https://doi.org/10.1017/is014007001jkt275 Published online by Cambridge University Press

Board of the Mathematical Sciences, Washington, DC, 1996, With contributions by M. Cole, G. Comezaña, S. Costenoble, A. D. Elmendorf, J. P. C. Greenlees, L. G. Lewis, Jr., R. J. Piacenza, G. Triantafillou, and S. Waner. MR 1413302 (97k:55016)

MM13. Akhil Mathew and Lennart Meier, *Affineness and chromatic homotopy theory*, arXiv preprint arXiv:1311.0514, 2013.

Nee96. Amnon Neeman, *The Grothendieck duality theorem via Bousfield's techniques and Brown representability*, J. Amer. Math. Soc. **9** (1996), no. 1, 205–236. MR 1308405 (96c:18006)

Rog08. John Rognes, *Galois extensions of structured ring spectra. Stably dualizable groups*, Mem. Amer. Math. Soc. **192** (2008), no. 898, viii+137.

Sto12. Vesna Stojanoska, *Duality for topological modular forms*, Doc. Math. **17** (2012), 271–311. MR 2946825

Str00. N. P. Strickland, *Gross-Hopkins duality*, Topology **39** (2000), no. 5, 1021–1033. MR 1763961 (2001d:55006)

Yos75. Zen-ichi Yosimura, *Universal coefficient sequences for cohomology theories of* CW*-spectra*, Osaka J. Math. **12** (1975), no. 2, 305–323. MR 0388375 (52 #9212)

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