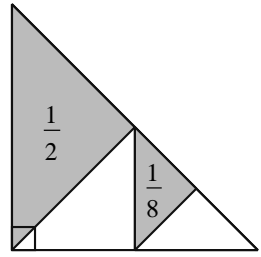
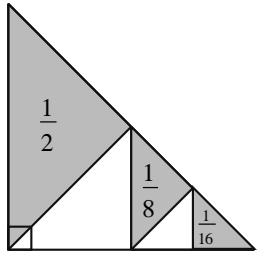


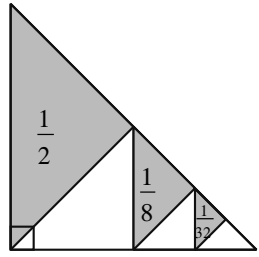
$$1 - \frac{1}{2} + \frac{1}{4}$$



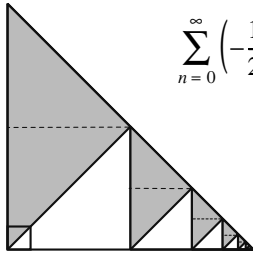
$$1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8}$$



$$1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16}$$



$$1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \frac{1}{32}$$



$$\sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^n = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \ \ldots = \frac{2}{3}$$

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102.43 A comparative study of relative rates of change

To visualise the rate of change among different sizes of quantities, we look at the relative rates of change. One common definition of relative rate of change is

$$\frac{\dfrac{y_2 - y_1}{t_2 - t_1}}{y_1}. \tag{1}$$

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Assuming an investment has values of $y_1 > 0$ at time t_1 and $y_2 > 0$ at time t_2, one can calculate the simple interest rate r during the time period t_1 and t_2 by solving $y_2 = y_1(1 + (t_2 - t_1)r)$ which produces the common relative rate of change formula (1). We shall refer to r as *arithmetic* (*linear*) *relative rate of change* and denote it by r_A.

We next naturally extend the definition of relative rate of change based on interest rate models. Without loss of generality, we assume $t_1 = 0$ and $t_2 = t > 0$.

- We define *geometric relative rate of change*, r_G, by solving the n-fold compound interest rate model $y_2 = y_1\left(1 + \dfrac{r}{n}\right)^{nt}$, where n is the number of compounding periods per year, for r to obtain

$$r_G = n\sqrt[nt]{\frac{y_2}{y_1}} - n; \tag{2}$$

- We define *exponential relative rate of change*, r_E, by solving the continuously compounding relation $y_2 = y_1 e^{rt}$ for r to arrive at

$$r_E = \frac{\ln\left(\dfrac{y_2}{y_1}\right)}{t} = \frac{\ln y_2 - \ln y_1}{t}. \tag{3}$$

The following are some natural questions.

1. For given y_1 and y_2, how can we rank the three relative rates of change based on their numerical values?
2. For a given rate, which of the following investment options yield a highest value: simple interest, discrete compounding, or continuously compounding?

To compare r_A, r_G and r_E, we consider (without loss of generality) $y_1 = 1, y_2 = y$. Hence the three models become:

$$y = 1 + r_A t, \tag{4}$$

$$y = \left(1 + \frac{r_G}{n}\right)^{nt}, \tag{5}$$

$$y = e^{r_E t}. \tag{6}$$

Since both r_A and r_E are in $(-\infty, \infty)$, we first consider r_A and r_E. It is easy to show that $f_1(t) = e^{rt}$ is convex (namely, $f''(t) > 0$) and $f_2(t) = 1 + rt$ is a tangent line of $f_1(t)$ at $t = 0$. Convex functions are always above their tangent lines [1], therefore

$$1 + r_A t = e^{r_E t} \geqslant 1 + r_E t \qquad \Rightarrow \qquad r_A \geqslant r_E. \tag{7}$$

Equality holds if, and only if, $r_A = r_E = 0$.

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The relation $\left(1 + \frac{r_G}{n}\right)^{nt} = e^{r_E t}$ implies $1 + \frac{r_G}{n} = e^{r_E/n} \geqslant 1 + \frac{r_E}{n}$. So $r_G \geqslant r_E$. Here we use the fact that $f(x) = 1 + x$ is the tangent line of the convex function $f(x) = e^x$ at $x = 0$. Equality holds if, and only if, $r_G = r_E = 0$.

To compare r_A and r_G, we consider $f(r) = \left(1 + \frac{r}{n}\right)^{nt}$. Then

$$f'(r) = t\left(1 + \frac{r}{n}\right)^{nt-1}, \qquad f''(r) = \frac{t(nt-1)}{n}\left(1 + \frac{r}{n}\right)^{nt-2}.$$

It follows from $f(0) = 1$ and $f'(0) = t$ that $f(r) = 1 + rt$ is the tangent line of $f(r) = \left(1 + \frac{r}{n}\right)^{nt}$ at $r = 0$. Since $f''(r) = \frac{t(nt-1)}{n}\left(1 + \frac{r}{n}\right)^{nt-2} > 0$ for $t > \frac{1}{n}$ and $f''(r) = \frac{t(nt-1)}{n}\left(1 + \frac{r}{n}\right)^{nt-2} < 0$ for $0 < t < \frac{1}{n}$, we obtain $1 + r_A t = \left(1 + \frac{r_G}{n}\right)^{nt} < 1 + r_G t$, that is, $r_A < r_G$ for $0 < t < \frac{1}{n}$. For $t > \frac{1}{n}$, $1 + r_A t = \left(1 + \frac{r_G}{n}\right)^{nt} > 1 + r_G t$, that is, $r_A > r_G$.

We want to point out that Farmer [2, p.152] found that it is easier to apply Taylor series to the power function $f(r) = (1 + r)^t$ than the exponential function $f(t) = (1 + r)^t$ to argue the sign of nonlinear Taylor expansion terms. Here we give a reason why Farmer's selection [1] of power function $f(r) = (1 + r)^t$ is a good choice. Suppose that $f(x)$ has the following Taylor expansion about x_0

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(\xi)}{2!}(x - x_0)^2$$

where ξ is a number between x_0 and x. The remainder term $R_1 = \frac{f''(\xi)}{2!}(x - x_0)^2$ is known as the Lagrange remainder [3]. Notice that $f_1(x) = f(x_0) + f'(x_0)(x - x_0)$ is the tangent line of $f(x)$ at x_0. If a linear function of interest happens to be the tangent line of $f(x)$ at $x = a$, one can construct a Taylor formula about $x = a$, then compare the Taylor formula about $x = a$ with the linear function of interest by simply finding the sign of the Lagrange remainder $R_1 = \frac{f''(\xi)}{2!}(x - x_0)^2$, which only depends on the sign of the second derivative $f''(\xi)$. It is easy to see that $f(r) = 1 + rt$ is the tangent line of power function $f(r) = (1 + r)^t$ at $r = 0$ while $f(t) = 1 + rt$ is a secant line of exponential function $f(t) = (1 + r)^t$. If the linear function of interest is just a secant line on $f(x)$, then the difference between $f(x)$ and the linear function of interest does not solely depend on $R_1 = \frac{f''(\xi)}{2!}(x - x_0)^2$.

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Let us summarise our results of relative rates for given $y_1 > 0$ and $y_2 > 0$.

1. $r_A > r_E$ when $\dfrac{y_2}{y_1} \neq 1$ and $r_A = r_E = 0$ when $\dfrac{y_2}{y_1} = 1$.
2. $r_G \geqslant r_E$. Equality holds if, and only if, $r_G = r_E = 0$.
3. $\dfrac{y_2}{y_1} = 1 \Leftrightarrow r_A = r_G = 0$. Otherwise, $r_A > r_G$ if $t > \frac{1}{n}$, $r_A = r_G$ if $t = \frac{1}{n}$, and $r_A < r_G$ if $0 < t < \frac{1}{n}$.

If a lower rate can achieve the same investment value as a higher rate, then when both methods have the same rates, the former will outperform the latter. Just as in interest rate markets, as bond prices increase, bond yields fall.

Using this logic, we can rank the three interest rate models. For fixed rates, namely, $r_A = r_G = r_E \neq 0$,

1. continuously compounding achieves the best investment value.
2. simple interest gives the largest investment return when $0 < t < \frac{1}{n}$ and compound interest when $t > \frac{1}{n}$. We want to point out that Farmer [2] proved similar results for the case $n = 1$.

The above definitions of relative rates have been used in many areas with some modifications. We will next consider the *arithmetic relative rate of change* proposed in a well-known demographic textbook [4, p.258], namely

$$r_A^* = \frac{\frac{y_2 - y_1}{t}}{\frac{1}{2}(y_1 + y_2)}. \tag{8}$$

The authors provided the following rationale: "Arithmetic rates of increase are comparable only when the periods are of equal length. When the periods are of unequal duration, the longer period tends to have the higher arithmetic rate. We must therefore choose a base for the rate that offsets the effects of unequal periods. To eliminate the bias due to unequal periods, an obvious choice for the base is the mean of the populations at the beginning and end of the periods." [4, p.258].

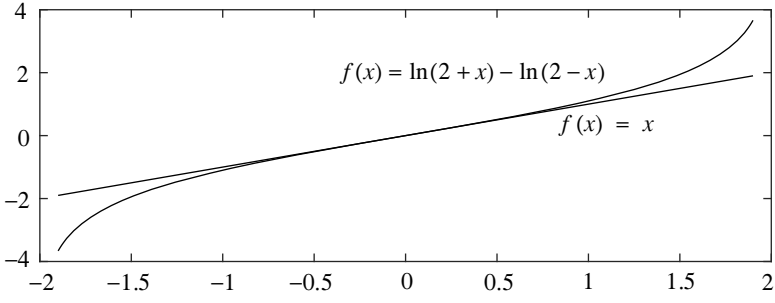


FIGURE 1: $f(x) = x$ is a tangent of $f(x) = \ln(2 + x) - \ln(2 - x)$ at $x = 0$

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In the following we will prove the observation in [4, p. 259]: "The geometric method tends to yield the highest estimates of growth, followed by the exponential method, then the arithmetic approximation."

Because geometric and exponential models proposed in the textbook [4] are the same as (2) when $n = 1$ and (3), we will focus on comparing r_A^* and r_E.

The expression $r_A^* = \frac{\frac{y_2 - y_1}{t}}{\frac{1}{2}(y_1 + y_2)}$ implies

$$-2 < tr_A^* = \frac{2(y_2 - y_1)}{y_1 + y_2} < 2 \tag{9}$$

for $y_1 > 0$ and $y_2 > 0$. Solving $r_A^* = \frac{\frac{y_2 - y_1}{t}}{\frac{1}{2}(y_1 + y_2)}$ for y_2 gives $y_2 = y_1 \frac{2 + r_A^* t}{2 - r_A^* t}$. To compare r_A^* and r_E, we set

$$\frac{2 + r_A^* t}{2 - r_A^* t} = e^{r_E t} \qquad \Rightarrow \qquad r_E t = \ln \frac{2 + r_A^* t}{2 - r_A^* t} = \ln(2 + r_A^* t) - \ln(2 - r_A^* t).$$

We will use the tangent line method to prove the inequality. Let $f(x) = \ln(2 + x) - \ln(2 - x)$. Then

$$f'(x) = \frac{1}{2 + x} + \frac{1}{2 - x} \qquad \text{and} \qquad f''(x) = -\frac{1}{(2 + x)^2} + \frac{1}{(2 - x)^2}.$$

Notice that $f'(0) = 1$ and $f''(x) > 0$ for $0 < x < 2$ and $f''(x) < 0$ for $-2 < x < 0$. Because $f(x)$ is convex for $0 < x < 2$ and $y = x$ is the tangent line at $x = 0$, $f(x)$ is above the 45° line $y = x$ for $0 < x < 2$ (see Figure 1). This implies that $r_E t = \ln \frac{2 + r_A^* t}{2 - r_A^* t} > r_A^* t$, for $0 < tr_A^* < 2$. It follows from the inequality condition (9) that $r_E > r_A^*$ for all increasing populations with $y_2 > y_1 > 0$, which verifies this interesting observation.

We proceed to show the relation between r_E and r_A^* for the case where the population decreases. Since $f(x) = \ln(2 + x) - \ln(2 - x)$ is concave for $-2 < x < 0$ and $y = x$ is still the tangent line at $x = 0$, the graph of $f(x)$ is below the line $y = x$ for $-2 < x < 0$ (see Figure 1). This implies that $r_E t = \ln \frac{2 + r_A^* t}{2 - r_A^* t}$, for $-2 < tr_A^* < 0$, which is equivalent to $r_A^* < 0$ by (9). Hence the inequality $r_E < r_A^*$ is established when the population decreases.

Since r_G and r_A are also greater than r_E when $0 < y_2 < y_1$, an immediate question one can ask is where r_A^* should be located when the population declines. By definition of r_A and r_A^*, we have

$$0 < y_2 < y_1 \Rightarrow 0 < \frac{y_2 + y_1}{2} < y_1$$

$$\Rightarrow 0 > r_A = \frac{\frac{y_2 - y_1}{t}}{y_1} > \frac{\frac{y_2 - y_1}{t}}{\frac{1}{2}(y_1 + y_2)} = r_A^*.$$

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The geometric rate r_G and arithmetic rate r_A^* do not have a clear inequality relation when $0 < y_2 < y_1$. (See Figure 2 where $t = 4.5$ is used.)

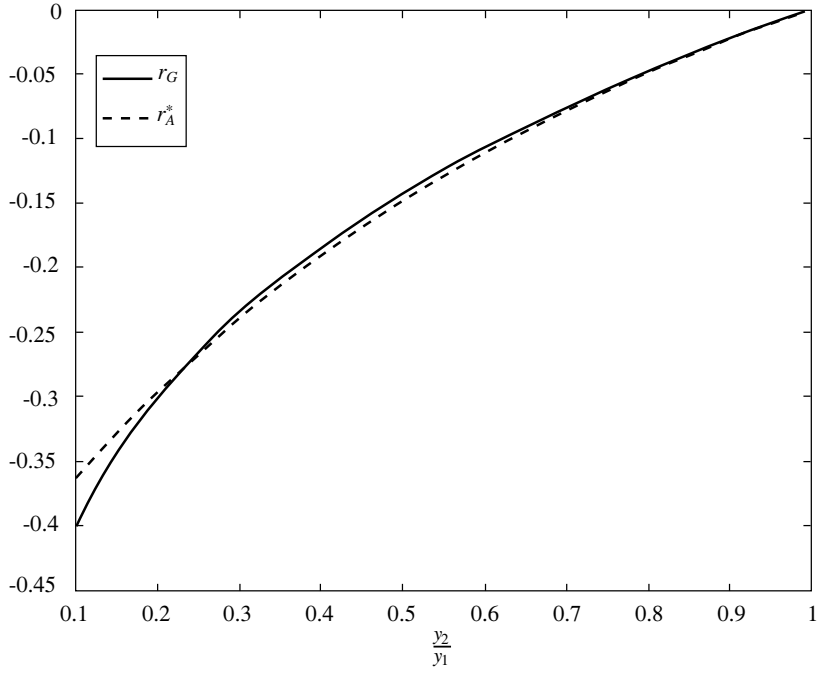
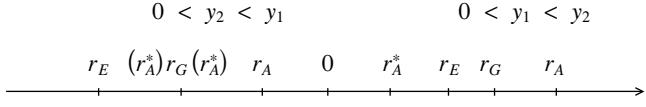


FIGURE 2: The relation between r_G and r_A^* depends on both $\frac{y_2}{y_1}$ and t when $0 < y_2 < y_1$.

We summarise the relations of the four relative rates of change in Figure 3.

For $t > \frac{1}{n}$



For $0 < t < \frac{1}{n}$

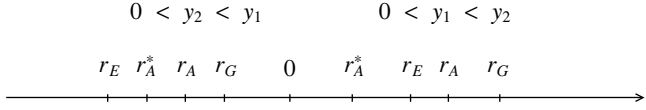


FIGURE 3: The diagram of the four relative rates of change

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Students with knowledge of time series may also define the *regression relative rate of change* by finding the slope of the linear trend model $\ln y_i = \beta_0 + \beta_1 t_i + \varepsilon_t$

$$r_R = \beta_1 = \frac{\operatorname{cov}(\ln y_i, t_i)}{\operatorname{var}(t_i)}.$$

If both y_1 and y_2 satisfy $\ln y_i = \beta_0 + \beta_1 t_i$, (i.e., both (t_1, y_1) and (t_2, y_2) on the regression line), then $r_E = r_R$.

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102.44 Another proof of the irrationality of square roots

Let k be a positive integer greater than 2 and suppose that the quadratic $x^2 - kx + 1 = 0$ has a rational root $\frac{n}{m}$ where, necessarily, $m \neq n$. The other root is then the rational number $\frac{m}{n}$, so we may suppose that $m > n$. Since $n^2 - kmn + m^2 = 0$, we have $\frac{n}{m} = \frac{kn - m}{n}$. But we can see that leads to a contradiction in two ways.

1. If m is taken as minimal in the representation of the fraction $\frac{n}{m}$, then $\frac{kn - m}{n}$ is a representation with smaller denominator.
2. In Figure 1, O is the midpoint of AC with triangle BCD similar to, but smaller than, triangle ABD. Iterating the construction inside triangle BCD leads to a contradictory sequence of smaller and smaller triangles with integer side-lengths.

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