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Heat kernel bounds for complex time and Schrödinger Kernel on hyperbolic spaces and Kleinian groups

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Abstract

We obtain sharp heat Kernel bounds for complex time and bounds for the Schrödinger Kernel on hyperbolic spaces and for a class of Kleinian groups as well.

1. *Introduction and statement of the main results*

The purpose of this paper is to obtain sharp upper bounds for the heat kernel $K_{n+1}(t, \rho)$ and for complex time t with $Re(t) \geqslant 0$ on hyperbolic spaces $\mathbf{H}^{n+1}$ of even dimensions. The same estimate for hyperbolic spaces of odd dimensions has been proved in [**6**, theorem 3·2 (i)], p. 192. In the above paper there exists also a worse estimate for even dimensions which blows up as $Re(t) \to 0$. Consequently, this paper closes a gap that existed for more than twenty years and answers affirmatively the conjecture raised in [**6**, p. 185]. We use this to obtain upper bounds for the Schrödinger Kernel on $\mathbf{H}^{n+1}$, and also for the heat kernel for complex time and Schrödinger Kernel on $\Gamma\backslash\mathbf{H}^{n+1}$ where Γ is a Kleinian group acting on $\mathbf{H}^{n+1}$ so chosen that the exponent of convergence $\delta(\Gamma)$ satisfies the inequality $\delta(\Gamma) < n/2$. More precisely in Section 2 we prove the following theorem and its corollary.

THEOREM 1. *If* $n \geqslant 1$ *is odd and* $Re(t) \geqslant 0$ *then*

$$|K_{n+1}(t, \rho)| \leqslant c_n |t|^{-\frac{1}{2}(n+1)} e^{-Re\left\{\frac{n^2}{4}t + \frac{\rho^2}{4t}\right\}} . (1 + \rho) . (1 + \rho + |t|)^{\frac{n}{2}-1} e^{-\frac{n}{2}\rho},$$

where ρ *is the hyperbolic distance on* $\mathbf{H}^{n+1}$.

Note. The above bound which holds in hyperbolic spaces of even dimensions is the same with the corresponding bound in spaces of odd dimensions; [**6**, theorem 3·2.(i), p. 192].

If we set $Re(t) = 0$ and $Im(t) = y$ in Theorem 1 then we have the following corollary.

COROLLARY 2. *If* $S_y(\rho)$ *is the Kernel of the Schrödinger operator* e^{iyL} *in* $\mathbf{H}^{n+1}$*, where L is the Laplace operator, then the following bound holds:*

$$|S_y(\rho)| \leqslant c_n |y|^{-\frac{1}{2}(n+1)} (1 + \rho)(1 + \rho + |y|)^{\frac{n}{2}-1} e^{-\frac{n}{2}\rho}.$$

Note. In [**13**] we have obtained a lower bound for the Schrödinger Kernel on real hyperbolic spaces of odd dimensions which is of the same order of magnitude as the above upper bound. This shows that this bound is sharp.

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Next, using the above bound of Theorem 1, we obtain in Section 3 the following bounds for the heat kernel $\widetilde{K}(t, x, y)$ and for the Schrödinger Kernel $\widetilde{S}$ on the quotient $\Gamma\backslash\mathbf{H}^{n+1}$, where Γ is a Kleinian group acting on $\mathbf{H}^{n+1}$.

THEOREM 3. *If $0 \leqslant \delta(\Gamma) < \alpha < \frac{n}{2}$ and $Re(t) \geqslant 0$ then the heat Kernel $\widetilde{K}$ on $\Gamma\backslash\mathbf{H}^{n+1}$ satisfies*

$$|\widetilde{K}(t, x, y)| \leqslant C_n|t|^{-\frac{1}{2}(n+1)}(1 + |t|)^{\frac{n}{2}-1}e^{-Re\left\{\frac{n^2}{4}t+\frac{\widetilde{\rho}^2}{4t}\right\}}.\mu_\alpha(x, y),$$

where $\mu_\alpha(x, y) := \sum_{\gamma\in\Gamma} e^{-\alpha\rho(\gamma x,y)}$, is a positive continuous function (the Poincaré series), $\widetilde{\rho}$ is the distance function on $\Gamma\backslash\mathbf{H}^{n+1}$ and $\delta(\Gamma)$ is the exponent of convergence for the Poincaré series.

COROLLARY 4. *The Schrödinger Kernel $\widetilde{S}$ on the manifold $\Gamma\backslash\mathbf{H}^{n+1}$ with $\delta(\Gamma) < \alpha < n/2$ satisfies:*

$$|\widetilde{S}_y(\rho)| \leqslant c|y|^{-\frac{1}{2}(n+1)}(1 + |y|)^{\frac{n}{2}-1}\mu_\alpha(x, y).$$

Heat kernel estimates for real time have been extensively studied in various geometric settings, as complete manifolds of non-negative Ricci curvature ([**4**] and the references therein), real hyperbolic spaces and their quotients by discrete groups of isometries [**6**], manifolds with cusps [**5**] and in particular cuspidal manifolds ([**9**], [**10**], [**14**], [**16**]), noncompact symmetric spaces ([**1**], [**3**]) and locally symmetric spaces [**19**]. Also complex time heat kernel estimates and Schrödinger Kernel estimates for real hyperbolic spaces have been studied in [**6**] and [**2**] respectively. Another topic which relates closely to this circle of ideas is the study of Riesz tranform on locally symmetric spaces [**7**, **15**]. Finally, we use the notation $f \sim g$ for functions f and g when there is a constant $c > 0$ such that $c^{-1}f(s) \leqslant g(s) \leqslant cf(s)$ for all s.

2. *Heat kernel bounds for complex time and Schrödinger Kernel on hyperbolic spaces*

Let us recall from [**6**] the following two recurrence relations for the heat kernel formulated in terms of the hyperbolic distance ρ and for complex times t

$$K_{n+1}(t, \rho) = -\frac{e^{(1-n)t}}{2\pi \sinh\rho} \quad \frac{\partial}{\partial\rho} \quad K_{n-1}(t, \rho), \tag{2·1}$$

[**6**, p. 185], and

$$K_{n+1}(t, \rho) = \int_\rho^\infty \frac{e^{\frac{1}{4}(2n+1)t}K_{n+2}(t, \mu)}{(\frac{1}{2}\cosh\mu - \frac{1}{2}\cos\rho)^{1/2}} \quad \sinh\mu\, d\mu, \tag{2·2}$$

[**6**, p. 190], with

$$K_1(t, \rho) = (4\pi t)^{-\frac{1}{2}}e^{-\rho^2/4t}.$$

From the above it is not difficult to show that if n is even then the first recurrence relation can be also written as

$$K_{n+1}(t, \rho) = -(4\pi t)^{-\frac{1}{2}}e^{-\frac{n^2}{4}t}\left(-\frac{1}{\sinh\rho}\frac{\partial}{\partial\rho}\right)^{n/2} e^{-\rho^2/4t}. \tag{2·3}$$

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Moreover if n is odd then the second reccurence relation can be written as

$$K_{n+1}(t,\rho) = (4\pi t)^{-\frac{1}{2}} \quad e^{-\frac{n^2}{4}t}.\int_{\rho}^{\infty}\left(-\frac{1}{\sinh\mu}\frac{\partial}{\partial\mu}\right)^{\frac{n+1}{2}} \times e^{-\frac{\mu^2}{4t}}.\frac{\sinh\mu}{(\frac{1}{2}\cosh\mu-\frac{1}{2}\cosh\rho)^{1/2}}\,\mathrm{d}\mu, \tag{2·4}$$

simply by substituting $K_{n+2}(t,\mu)$ with its equal from (2·3). Formulae (2·3) and (2·4) are not new but this is another way of obtaining them from the recurrence relations which were proved for complex time in [**6**, section 2]. We have the following:

PROPOSITION 2·1. *If n is odd and $Re(t) \geqslant 0$, then*

$$|K_{n+1}(t,\rho)| \leqslant c_n|t|^{-\frac{1}{2}(n+1)}e^{-Re\{\frac{n^2}{4}t+\frac{\rho^2}{4t}\}}(1+\rho).(1+\rho+|t|)^{\frac{n}{2}-1}.e^{-\frac{n}{2}\rho},$$
$$\textit{if} \quad |t| \geqslant 1+\rho.$$

Remark. If $|t| \geqslant 1+\rho$ then $(1+\rho+|t|)^{\frac{n}{2}-1} \sim |t|^{\frac{n}{2}-1}$.

Hence the above bound of Proposition 2·1 can be also written as:

$$c_n|t|^{-\frac{3}{2}}e^{-Re\{\frac{n^2}{4}t+\frac{\rho^2}{4t}\}}.(1+\rho)e^{-\frac{n}{2}\rho}.$$

Proof of Proposition 2·1. Let us recall the second recurrence relation (2·2). Since $n+2$ is odd, it holds the estimate of [**6**, theorem 3·2(i), p. 192], for the heat Kernel $K_{n+2}(t,\rho)$. Namely,

$$|K_{n+2}(t,\rho)| \leqslant c_n|t|^{-\frac{1}{2}(n+2)}e^{-Re\{\frac{1}{4}(n+1)^2t+\frac{\rho^2}{4t}\}} \times (1+\rho)(1+\rho+|t|)^{\frac{1}{2}(n+1)-1}e^{-\frac{1}{2}(n+1)\rho}.$$

(In [**6**, theorem 3·2(i) **6**] is written that $Re(t) > 0$, but the estimate clearly holds for $Re(t) \geqslant 0$, since the right-hand side remains bounded as $Re(t) \to 0$.)

Putting the above bound for $|K_{n+2}(t,\rho)|$ in (2.2), substituting $\mu = \rho + x$ and bearing in mind that

$$\cosh(\rho+x) - \cosh\rho \geqslant \frac{x(\rho+x)}{1+\rho+x} \quad e^{\rho+\frac{x}{2}}, \tag{2·5}$$

and

$$\sinh(\rho+x) \sim \frac{\rho+x}{1+\rho+x}e^{\rho+x}, \tag{2·6}$$

see [**6**, p. 190], we get

$$|K_{n+1}(t,\rho)| \leqslant C_n|t|^{-\frac{1}{2}(n+2)}e^{-Re\left\{\frac{n^2}{4}t+\frac{\rho^2}{4t}\right\}}e^{-\frac{n}{2}\rho} \times \int_0^{\infty} e^{(\frac{1}{4}-\frac{n}{2})x}e^{-Re\{(2\rho x+x^2)/4t\}}.(1+\rho+x)^{1/2}. \times (1+\rho+x+|t|)^{-\frac{1}{2}(n+1)}.(\rho+x)^{1/2}.x^{-\frac{1}{2}}\,\mathrm{d}x.$$

But

$$(1+\rho+x+|t|) \leqslant (1+\rho+|t|)(1+x),$$
$$(1+\rho+x) \leqslant (1+\rho)(1+x)) \tag{2·7}$$

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and

$$e^{(\frac{1}{4}-\frac{n}{2})x}(1+x)^{n/2} \leqslant c.e^{-\frac{x}{8}}.$$

Hence we obtain

$$|K_{n+1}(t,\rho)| \leqslant c_n|t|^{-\frac{1}{2}(n+2)}e^{-Re\left\{\frac{n^2}{4}t+\frac{\rho^2}{4t}\right\}}e^{-\frac{n}{2}\rho}.(1+\rho)^{1/2}(1+\rho+|t|)^{\frac{1}{2}(n-1)}.I,$$

where

$$I := \int_0^\infty \left(1+\frac{\rho^{1/2}}{x^{1/2}}\right).e^{-Re\{(2\rho x+x)/4t\}-\frac{x}{8}}.\,\mathrm{d}x.$$

We need the following lemma.

LEMMA 2·1. *If* $|t| \geqslant 1+\rho$ *then*

$$I \leqslant c|t|^{1/2}(1+\rho)^{1/2}(1+\rho+|t|)^{-\frac{1}{2}}.$$

Proof. Obviously

$$\begin{aligned} I &\leqslant \int_0^\infty \left(1+\frac{\rho^{1/2}}{x^{1/2}}\right).e^{-\frac{x}{8}}\,\mathrm{d}x \\ &\sim 1+\rho^{1/2} \\ &\sim (1+\rho)^{1/2} \\ &\sim |t|^{1/2}(1+\rho)^{1/2}(1+\rho+|t|)^{-\frac{1}{2}}. \end{aligned}$$

since $\frac{1}{\sqrt{2}} \leqslant (\frac{|t|}{1+\rho+|t|})^{1/2} \leqslant 1$.

This completes the proof of the lemma and also the proof of Proposition 2·1.

PROPOSITION 2·2. *If* n *is odd,* $Re(t) \geqslant 0$ *and* $|t| \leqslant 1+\rho$ *then*

$$|K_{n+1}(t,\rho)| \leqslant c|t|^{-\frac{1}{2}(n+1)}e^{-Re\{\frac{n^2}{4}t+\frac{\rho^2}{4t}\}}e^{-\frac{n}{2}\rho}(1+\rho)^{n/2}.$$

We need the following lemma.

LEMMA 2·2. *The following expansion holds*:

$$\begin{aligned} \left(-\frac{1}{\sinh\rho}\frac{\partial}{\partial\rho}\right)^m e^{-\rho^2/4t} &= \sum_{0<j<m} t^{-j} f_j(\rho)e^{-\rho^2/4t} \\ &\quad + t^{-(m-1)}\left(-\frac{\rho}{2\sinh\rho}\right)^{m-1}\left(-\frac{1}{\sinh\rho}\frac{\partial}{\partial\rho}\right)e^{-\rho^2/4t}, \end{aligned}$$

where the functions $f_j(\rho)$ *are linear combination of products* $\phi_{l_1}(\rho)\ldots\phi_{l_j}(\rho)$, *where* $\phi_l(\rho) = (\frac{1}{\sinh\rho}\frac{\partial}{\partial\rho})^l\rho^2$, *and* $l_1,\ldots,l_j \in N^*$ *are such that* $l_1+\cdots+l_j = m$.

Note. A similar expansion is mentioned in [**2**, p. 1860], but stated without proof. We give the proof of the above lemma for reasons of completeness.

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Proof of Lemma 2·2. We use induction. For m = 2 the expansion holds. We assume that it holds for m and we prove that it also holds for m + 1. We have

$$\begin{aligned}\left(-\frac{1}{\sinh\rho}\frac{\partial}{\partial\rho}\right)^{m+1} e^{-\rho^2/4t} &= \left(-\frac{1}{\sinh\rho}\frac{\partial}{\partial\rho}\right)\left(-\frac{1}{\sinh\rho}\frac{\partial}{\partial\rho}\right)^{m} e^{-\rho^2/4t}\\ &= \left(-\frac{1}{\sinh\rho}\frac{\partial}{\partial\rho}\right)\left\{\sum_{0<j<m} t^{-j} f_j(\rho) e^{-\rho^2/4t}\right.\\ &\quad \left. + t^{-(m-1)}\left(-\frac{\rho}{2\sinh\rho}\right)^{m-1}\left(-\frac{1}{\sinh\rho}\frac{\partial}{\partial\rho}\right) e^{-\rho^2/4t}\right\}\\ &\equiv A + B.\end{aligned}$$

We examine each term separately.

$$A:\ \sum_{0<j<m} t^{-j}\left[\left(-\frac{1}{\sinh\rho}\frac{\partial}{\partial\rho}\right) f_j(\rho).e^{-\rho^2/4t} + f_j(\rho)\left(-\frac{1}{\sinh\rho}\frac{\partial}{\partial\rho}\right) e^{-\rho^2/4t}\right].$$

But

$$\left(-\frac{1}{\sinh\rho}\frac{\partial}{\partial\rho}\right) f_j(\rho) = \left(-\frac{1}{\sinh\rho}\frac{\partial}{\partial\rho}\right)\sum a_{l_1\dots l_j}\left(\frac{1}{\sinh\rho}\frac{\partial}{\partial\rho}\right)^{l_1}\rho^2\cdots\left(\frac{1}{\sinh\rho}\frac{\partial}{\partial\rho}\right)^{l_j}\rho^2.$$

If at this point, put, for simplicity $D_p \equiv 1/\sinh\rho\,\partial/\partial\rho$, then the last expression gives

$$\begin{aligned}&-\sum a_{l_1\dots l_j} D_\rho\left[D_\rho^{l_1}\rho^2.D_\rho^{l_2}\rho^2\cdots D_\rho^{l_j}\rho^2\right]\\ &= -\sum a_{l_1\dots l_j}\left\{D_\rho^{l_1+1}\rho^2.D_\rho^{l_2}\rho^2\cdots D_\rho^{l_j}\rho^2 + D_\rho^{l_1}\rho^2.D_\rho^{l_2+1}\rho^2\cdots\right.\\ &\quad \left.\cdots D_\rho^{l_j}\rho^2 + \cdots + D_\rho^{l_1}\rho^2.D_\rho^{l_2}\rho^2\cdots D_\rho^{l_j+1}\rho^2\right\},\end{aligned}$$

and this is of the form we want since

$$\begin{aligned}l_1 + 1 + l_2 + \cdots + l_j &= l_1 + l_2 + 1 + \cdots + l_j = \cdots\\ &= l_1 + l_2 + \cdots + l_j + 1 = m + 1.\end{aligned}$$

Moreover $(-\frac{1}{\sinh\rho}\frac{\partial}{\partial\rho})e^{-\rho^2/4t} = e^{-\rho^2/4t}.\frac{t^{-1}}{4}(-\frac{1}{\sinh\rho}\frac{\partial}{\partial\rho})\rho^2$.

Thus $A = \sum_{0<j<m} t^{-j} f_j(\rho)e^{-\rho^2/4t} + \sum_{0<j<m} t^{-j-1} f_{j+1}(\rho)e^{-\rho^2/4t}$.

Next we examine term B. We have

$$\begin{aligned}&-\frac{1}{\sinh\rho}\frac{\partial}{\partial\rho}\left\{\left(-\frac{\rho}{2\sinh\rho}\frac{\partial}{\partial\rho}\right)^{m-1}\left(-\frac{1}{\sinh\rho}\frac{\partial}{\partial\rho}\right) e^{-\rho^2/4t}\right\}\\ &= (m-1)\left(-\frac{\rho}{2\sinh\rho}\right)^{m-2}\left(-\frac{1}{\sinh\rho}\frac{\partial}{\partial\rho}\right)\left(-\frac{\rho}{2\sinh\rho}\right)\left(-\frac{1}{\sinh\rho}\frac{\partial}{\partial\rho}\right) e^{-\rho^2/4t}\\ &\quad + \left(-\frac{\rho}{2\sinh\rho}\right)^{m-1}\left(-\frac{1}{\sinh\rho}\frac{\partial}{\partial\rho}\right)^2 e^{-\rho^2/4t} \equiv \alpha + \beta.\end{aligned}$$

We examine term α.

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Since $-\frac{\rho}{2\sinh\rho} = \frac{1}{4}(-\frac{1}{\sinh\rho}\frac{\partial}{\partial\rho})\rho^2$, we get

$$(m-1)4^{2-m}\left[\left(-\frac{1}{\sinh\rho}\frac{\partial}{\partial\rho}\right)\rho^2\right]^{m-2}.\frac{1}{4}\left(-\frac{1}{\sinh\rho}\frac{\partial}{\partial\rho}\right)^2\rho^2.e^{-\rho^2/4t}\frac{t^{-1}}{4}\left(-\frac{1}{\sinh\rho}\frac{\partial}{\partial\rho}\right)\rho^2,$$

which is again of the form we want.

Term β:

$$\left(-\frac{1}{\sinh\rho}\frac{\partial}{\partial\rho}\right)^2 e^{-\rho^2/4t} = \left(-\frac{1}{\sinh\rho}\frac{\partial}{\partial\rho}\right)\left\{e^{-\rho^2/4t}\frac{t^{-1}}{4}\left(-\frac{1}{\sinh\rho}\frac{\partial}{\partial\rho}\right)\rho^2\right\}$$
$$= \left(-\frac{1}{\sinh\rho}\frac{\partial}{\partial\rho}\right)e^{-\rho^2/4t}.t^{-1}\left(-\frac{\rho}{2\sinh\rho}\right) + e^{-\rho^2/4t}\frac{t^{-1}}{4}\left(-\frac{1}{\sinh\rho}\frac{\partial}{\partial\rho}\right)^2\rho^2.$$

Now putting, in the factor $(-\frac{\rho}{2\sinh\rho})^{m-1}$, that we had left out, we deduce that the first term of the last expression will give the factor $(-\frac{\rho}{2\sinh\rho})^m$ and for the second term we get

$$e^{-\rho^2/4t}\frac{t^{-1}}{4}4^{1-m}\left\{\left(-\frac{1}{\sinh\rho}\frac{\partial}{\partial\rho}\right)\rho^2\right\}^{m-1}\left(-\frac{1}{\sinh\rho}\frac{\partial}{\partial\rho}\right)^2\rho^2,$$

simply by using the identity

$$-\frac{\rho}{2\sinh\rho} = \frac{1}{4}\left(-\frac{1}{\sinh\rho}\frac{\partial}{\partial\rho}\right)\rho^2.$$

This term is again of the form we want. This completes the proof of Lemma 2·2.

Proof of Proposition 2·2. Let us recall relation (2·4). We set $(n+1)/2 = m$ and expand $((-1/\sinh\mu)(\partial/\partial\mu))^m e^{-\frac{\mu^2}{4t}}$ as in Lemma 2·2.

The first term of this expansion will give

$$\sum_{0<j<m} t^{-j} f_j(\mu) e^{-\frac{\mu^2}{4t}}.$$

Using the elementary global estimate as in [**2**],

$$\phi_l(\mu) = O((1+\mu)e^{-l\mu})$$

which implies;

$$f_j(\mu) = O((1+\mu)^j e^{-m\mu}),$$

we get that the contribution of this term to the heat kernel is dominated by

$$\leqslant C|t|^{-\frac{1}{2}} e^{-Re\left(\frac{n^2}{4}t\right)}\int_\rho^\infty \left\{\frac{1+\mu}{|t|} + \left(\frac{1+\mu}{|t|}\right)^{m-1}\right\} e^{-m\mu}$$
$$\times e^{-Re\left(\frac{\mu^2}{4t}\right)}\frac{\sinh\mu}{\left(\frac{1}{2}\cosh\mu - \frac{1}{2}\cosh\rho\right)^{1/2}}\,\mathrm{d}\mu.$$

Substituting in the last integral $\mu = \rho + x$ and making use of relations (2·5), (2·6), (2·7) and $(1+\rho+x)/|t| \geqslant 1$, we get that the last expression is dominated by

$$c|t|^{-m+\frac{1}{2}}(1+\rho)^{m-\frac{3}{2}}\int_0^\infty (1+x)^{m-\frac{3}{2}}\left(1+\frac{\rho}{x}\right)^{1/2} e^{-(m-\frac{3}{4})x}\,\mathrm{d}x$$
$$\times e^{-\frac{n}{2}\rho}e^{-Re\left\{\frac{n^2}{4}t+\frac{\rho^2}{4t}\right\}}.$$

But $(1+\frac{\rho}{x})^{1/2} \sim (1+\frac{\rho^{1/2}}{x^{1/2}})$ and the last integral is $\sim (1+\rho)^{1/2}$.

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Putting everything together we obtain that this term is dominated by

$$c|t|^{-m+\frac{1}{2}}(1+\rho)^{m-1}e^{-Re\left\{\frac{n^2}{4}t+\frac{\rho^2}{4t}\right\}}.e^{-\frac{n}{2}\rho} \leqslant c|t|^{-\frac{n+1}{2}}(1+\rho)^{\frac{n}{2}}e^{-Re\left\{\frac{n^2}{4}t+\frac{\rho^2}{4t}\right\}}e^{-\frac{n}{2}\rho},$$

since $|t|^{-m+\frac{1}{2}} = |t|^{-\frac{n}{2}} \leqslant (1+\rho)^{1/2}|t|^{-\frac{n+1}{2}}$ as $|t| \leqslant 1+\rho$.

Next we examine the contribution of the second term of the original expansion to the heat kernel.

We need to estimate the integral

$$I(t,\rho) = \int_\rho^\infty \left(\frac{\mu}{\sinh\mu}\right)^{m-1}\left(\frac{\partial}{\partial\mu}e^{-\mu^2/4t}\right)\frac{\mathrm{d}\mu}{(\cosh\mu-\cosh\rho)^{1/2}}.$$

Following [**2**, p. 1861], we split this integral as follows:

$$\int_\rho^\infty = \int_\rho^{\sqrt{\rho^2+|t|}} + \int_{\sqrt{\rho^2+|t|}}^{\rho+1} + \int_{\rho+1}^\infty \equiv I_1(|t|,\rho) + I_2(|t|,\rho) + I_3(|t|,\rho)$$

First examine $I_1(|t|,\rho)$. Since:

$$\frac{\mu}{\sinh\mu} \sim \frac{1+\mu}{e^\mu},$$

$$\cosh\mu-\cosh\rho \geqslant \frac{(\mu-\rho)\mu}{1+\mu}.e^{\frac{1}{2}(\mu+\rho)}, \quad [\mathbf{6}, \text{p. 190}],$$

$$e^{-Re\left(\frac{\mu^2}{4t}\right)} \leqslant e^{-Re(\rho^2/4t)},\ e^{-\frac{1}{4}(\mu+\rho)} \leqslant e^{-\rho/2},$$

$$\text{and } [(\mu-\rho)\mu]^{-1} \leqslant 2(\mu^2-\rho^2)^{-1},$$

we get

$$|I_1| \leqslant c(1+\rho)^{m-\frac{1}{2}}e^{-Re(\frac{\rho^2}{4t})}e^{-\frac{n}{2}\rho}|t|^{-1}\int_\rho^{\sqrt{\rho^2+|t|}}\frac{\mu\,\mathrm{d}\mu}{(\mu^2-\rho^2)^{1/2}}$$

But the last integral is

$$\sim |t|^{1/2}.$$

Consequently the total contribution of this term to the heat kernel is

$$\leqslant c|t|^{-\frac{n+1}{2}}(1+\rho)^{n/2}e^{-\frac{n}{2}\rho}e^{-Re\left\{\frac{\rho^2}{4t}+\frac{n^2}{4}t\right\}}.$$

Next we examine the two remaining integrals I_2 and I_3 that we integrate by parts

$$\begin{aligned} I_2+I_3 = \Bigg[&\left(\frac{\mu}{\sinh\mu}\right)^{m-1}\cdot\frac{1}{(\cosh\mu-\cosh\rho)^{1/2}}.e^{-\mu^2/4t} \\ &\times\left\{\Big]_{\mu=\sqrt{\rho^2+4t}}^{\mu=\rho+1} + \Big]_{\mu=\rho+1}^{\mu=+\infty}\right\} + \left\{\int_{\sqrt{\rho^2+|t|}}^{\rho+1} + \int_{\rho+1}^\infty\right\}e^{-\mu^2/4t} \\ &\times\Bigg\{(m-1)\left(\frac{\mu}{\sinh\mu}\right)^{m-2}\frac{\mu\cosh\mu-\sinh\mu}{\sinh^2\mu}\cdot\frac{1}{(\cosh\mu-\cosh\rho)^{1/2}} \\ &+\left(\frac{\mu}{\sinh\mu}\right)^{m-1}\cdot\frac{1}{2}.(\cosh\mu-\cosh\rho)^{-\frac{3}{2}}\sinh\mu\Bigg\}\,\mathrm{d}\mu. \end{aligned}$$

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We estimate first the boundary terms. Using the same estimates as the case of I_1 we get after some easy manipulations that

$$|I_2 + I_3|_{boundary} \leqslant c|t|^{-\frac{1}{2}} e^{-\rho\left(m-\frac{1}{2}\right)} (1+\rho)^{m-\frac{1}{2}} e^{-Re(\rho^2/4t)}.$$

Hence the total contribution of the boundary terms is

$$\leqslant c|t|^{-\frac{n+1}{2}} (1+\rho)^{n/2} e^{-Re\left(\frac{\rho^2}{4t}+\frac{n^2}{4}t\right)}.e^{-\frac{n}{2}\rho}.$$

Now we estimate the integral terms using the elementary relation

$$\frac{\mu\coth\mu - 1}{\sinh\mu} \sim \mu e^{-\mu}.$$

The first integral term is bounded by

$$\begin{aligned}
&e^{-Re\left(\frac{\rho^2}{4t}\right)} \int_{\sqrt{\rho^2+|t|}}^{\rho+1} \left\{ (m-1)\frac{(1+\mu)^{m-2}}{e^{\mu(m-2)}} . \frac{(1+\mu)^{1/2}}{[(\mu-\rho)\mu]^{1/2}} . e^{-\frac{1}{4}(\mu+\rho)} \mu e^{-\mu} \right. \\
&\quad \left. + \frac{1}{2}\frac{(1+\mu)^{m-1}}{e^{\mu(m-1)}} \frac{(1+\mu)}{[(\mu-\rho)\mu]^{1/2}} e^{-\frac{3}{4}(\mu+\rho)} . \frac{\mu}{1+\mu} e^{\mu} \right\} \mathrm{d}\mu \\
&\quad \leqslant c e^{-Re\left(\frac{\rho^2}{4t}\right)} \left\{ \int_{\sqrt{\rho^2+|t|}}^{\rho+1} \frac{(1+\rho)^{m-2}(1+\rho)^{1/2}}{e^{\rho\left(m-\frac{1}{2}\right)}} . \frac{\mu}{(\mu^2-\rho^2)^{1/2}} \mathrm{d}\mu \right. \\
&\quad \left. + \int_{\sqrt{\rho^2+|t|}}^{\rho+1} \frac{(1+\rho)^{m-2}}{e^{\rho\left(m-\frac{1}{2}\right)}} \left(\frac{1+\rho}{\mu^2-\rho^2}\right)^{3/2} \mu \,\mathrm{d}\mu \right\} \\
&\quad \leqslant c e^{-Re\left(\frac{\rho^2}{4t}\right)} . (1+\rho)^{m-2} e^{-\rho\left(m-\frac{1}{2}\right)} \int_{\sqrt{\rho^2+|t|}}^{\rho+1} \left\{ \left(\frac{1+\rho}{\mu^2-\rho^2}\right)^{1/2} \right. \\
&\quad \left. + \left(\frac{1+\rho}{\mu^2-\rho^2}\right)^{3/2} \right\} \mu \,\mathrm{d}\mu \equiv I_{1st}
\end{aligned}$$

But

$$\left(\frac{1+\rho}{\mu^2-\rho^2}\right)^{1/2} \leqslant 2\left(\frac{1+\rho}{\mu^2-\rho^2}\right)^{3/2}.$$

Hence

$$\begin{aligned}
I_{1st} &\leqslant c . e^{-Re\left(\frac{\rho^2}{4t}\right)} . (1+\rho)^{m-2} e^{-\rho\left(m-\frac{1}{2}\right)} \int_{\sqrt{\rho^2+|t|}}^{\rho+1} \left(\frac{1+\rho}{\mu^2-\rho^2}\right)^{3/2} \mu \,\mathrm{d}\mu \\
&\leqslant c e^{-Re\left(\frac{\rho^2}{4t}\right)} |t|^{-1/2} (1+\rho)^{n/2} e^{-\frac{n}{2}\rho}.
\end{aligned}$$

Finally let us estimate the second integral term, say I_{2nd}. We have

$$\begin{aligned}
|I_{2nd}| &\leqslant e^{-Re\left(\frac{\rho^2}{4t}\right)} \int_{\rho+1}^{\infty} \left\{ (m-1)\left(\frac{\mu}{\sinh\mu}\right)^{m-2} \frac{\mu\cosh\mu - \sinh\mu}{\sinh^2\mu} \right. \\
&\quad \left. \times \frac{1}{(\cosh\mu - \cosh\rho)^{1/2}} + \left(\frac{\mu}{\sinh\mu}\right)^{m-1} . \frac{1}{2} . (\cosh\mu - \cosh\rho)^{-3/2} . \sinh\mu \right\} \mathrm{d}\mu \\
&\equiv e^{-Re\left(\frac{\rho^2}{4t}\right)} (I I_1 + I I_2).
\end{aligned}$$

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Here we shall use the fact that for $\mu \geqslant \rho + 1$ we have

$$\cosh \mu - \cosh \rho \sim e^{\mu}, \text{ [2, p. 1860].}$$

Hence we get, after some elementary calculations,

$$II_1 + II_2 \leqslant c \int_{\rho+1}^{\infty} (1+\mu)^{m-1} e^{-\mu\left(m-\frac{1}{2}\right)} \, d\mu.$$

We substitute in the above integral $\mu = \rho + x$ and we get

$$\begin{aligned} &\int_1^{\infty} (1+\rho+x)^{m-1} e^{-(\rho+x)\left(m-\frac{1}{2}\right)} \, dx \\ &\leqslant (1+\rho)^{m-1} e^{-\rho\left(m-\frac{1}{2}\right)} \int_1^{\infty} (1+x)^{m-1} e^{-x\left(m-\frac{1}{2}\right)} \, dx \\ &\leqslant c(1+\rho)^{m-1} e^{-\rho\left(m-\frac{1}{2}\right)} \leqslant c|t|^{-1/2} (1+\rho)^{n/2} e^{-\frac{n}{2}\rho}, \end{aligned}$$

since $|t| \leqslant 1 + \rho$ and $m = (n+1)/2$.

This completes the proof of Proposition 2·2.

Proof of Theorem 1. Using the fact that if $|t| \geqslant 1 + \rho$ then

$$(1 + \rho + |t|)^{\frac{n}{2}-1} \sim |t|^{n/2-1}$$

and if $|t| \leqslant 1 + \rho$ then

$$(1 + \rho + |t|)^{\frac{n}{2}-1} \sim (1+\rho)^{n/2-1}$$

and using Propositions 2·1 and 2·2 as well, we see that Theorem 1 follows.

Proof of Corollary 2. If we set $Re(t) = 0$ in Theorem 1 above and in [**6**, theorem 3·2(i), p. 192], the result follows at once. An equivalent way at writing the above bound is, as in the case of the heat kernel, the following:

$$|S_y(\rho)| \leqslant c_n \begin{cases} |y|^{-\frac{3}{2}} . (1+\rho) e^{-\frac{n}{2}\rho}, & \text{if} \quad |y| \geqslant 1 + \rho \\ |y|^{-\frac{1}{2}(n+1)} e^{-\frac{n}{2}\rho} (1+\rho)^{\frac{n}{2}}, & \text{if} \quad |y| \leqslant 1 + \rho \end{cases}$$

Thus we see that this bound is the same with the bound of [**2**, p. 1859].

3. *Heat kernel bounds for complex time and Schrödinger Kernel for Kleinian groups*

In what follows, Γ denotes a discrete group of isometries of the structure which is geometrically finite and torsion free, with $\delta(\Gamma) < n/2$. Here $\delta(\Gamma)$ denotes the exponent of convergence of the Poincaré series

$$\sum_{\gamma \in \Gamma} exp(-s\rho(x, \gamma x)).$$

As it is well known the assumption $\delta(\Gamma) < n/2$ implies that the Laplacian has only absolutely continuous spectrum which is described by Eisenstein series [**8**, **11**, **12**], and allows us to construct the heat and Schrödinger Kernels by periodizing the corresponding kernels on the covering space $\mathbf{H}^{n+1}$. It is worth mentioning that $\delta(\Gamma)$ is for geometrically finite Kleinian groups, equal to the Hausdorff dimension of the limit set of Γ. This is the Patterson-Sullivan theorem (see for example [**18**], ([**17**, p. 158])).

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Proof of Theorem 3. Theorem 1, together with the Note attached to it, implies that

$$|K_{n+1}(t,\rho)| \leqslant c_n|t|^{-\frac{1}{2}(n+1)}(1+|t|)^{\frac{n}{2}-1}e^{-Re\left\{\frac{n^2}{4}t+\frac{\widetilde{\rho}^2}{4t}\right\}}.(1+\rho)^{\frac{n}{2}}e^{-\frac{n}{2}\rho},$$

since $\widetilde{\rho}(x,y)=\min_{\gamma\in\Gamma}\rho(\gamma x,y)\leqslant\rho(x,y)$.

The above bound is dominated by:

$$\leqslant c_n'|t|^{-\frac{1}{2}(n+1)}(1+|t|)^{\frac{n}{2}-1}e^{-Re\left\{\frac{n^2}{4}t+\frac{\widetilde{\rho}^2}{4t}\right\}}.e^{-\alpha\rho},$$

since $(1+\rho)^{\frac{n}{2}}e^{-\varepsilon\rho}\leqslant c$ for any $\varepsilon>0$ and

$$e^{-\left(\frac{n}{2}-\varepsilon\right)\rho}\leqslant e^{-\alpha\rho}$$

since $\alpha<\frac{n}{2}-\varepsilon$ for ε sufficiently small.

Bearing in mind that

$$\widetilde{K}(t,x,y)=\sum_{\gamma\in\Gamma}K(t,x,\gamma y),$$

the result now follows at once.

Proof of Corollary 4. If we set $Re(t)=0$ in Theorem 3, the result follows at once.

Final comment. There are other interesting papers in this area that deserve some mention. One in particular, which was pointed out to me by the referee, is Giles Carron's recent paper [3] on bounds of the Green and heat kernels on symmetric spaces. It would be interesting to know whether his results on estimates of the heat kernel extend also to complex t. Althought this is not an easy problem, one should observe that the parabolic estimate, [**3**, proposition 4·2], which is one of the main ingredients, does not extend to complex t.

Hence the question is raised whether one can obtain the heat kernel estimates without using the parabolic estimate and this is something that requires further research.

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REFERENCES

[**1**] J. P. ANKER and L. JI. Heat kernel and Green function estimates on noncompact symmetric spaces. *Geom. Funct. Anal*. **9** (1999), 1035–1091.
[**2**] J. P. ANKER and V. PIERFELICE. Nonlinear Schrödinger equation on real hyperbolic spaces. *Anal. Non Linéaire* **26** (2009), 1853–1869.
[**3**] G. CARRON. Estimées du noyau de Green et de la chaleur sur les espaces symétriques. *Analysis & PDE Journal* **3** (2010), no. 2, 197–205.
[**4**] I. CHAVEL. *Eigenvalues in Riemannian Geometry* (Academic Press, New York, 1984).
[**5**] E. B. DAVIES and N. MANDOUVALOS. Heat kernel bounds on manifolds with cusps. *J. Funct. Anal.* **75** (1987), 311–322.
[**6**] E. B. DAVIES and N. MANDOUVALOS. Heat Kernel bounds on hyperbolic space and Kleinian groups. *Proc. London Math. Soc*. (3) **57** (1988), 182–208.
[**7**] L. JI, P. KUNSTMANN and A. WEBER. Riesz transform on locally symmetric spaces and Riemannian manifolds with a spectral gap. *Bull. Sci. math*. **134** (2010), 37–43.
[**8**] P. D. LAX and R. S. PHILLIPS. Translation representations for automorphic solutions of the wave equation in non-Euclidean spaces; I. *Comm. Pure Appl. Math*. **37** (1984), 303–328; II. *Comm. Pure Appl. Math*. **37** (1984), 779–813.
[**9**] H. Q. LI. Analyse sur les varietés cuspidale. *Math. Ann*. **326** (2003), 625–647.
[**10**] H. Q. LI. Estimations optimales du noyau de la chaleur sur les varietés cuspidales. *Potential Anal*. **27** (2007), 225–249.

https://doi.org/10.1017/S0305004112000035 Published online by Cambridge University Press

[**11**] N. MANDOUVALOS. The theory of Eisenstein series for Kleinian groups. *Contemp. Math.* **53** (1986), 357–370.
[**12**] N. MANDOUVALOS. Spectral theory and Eistenstein series for Kleinian groups. *Proc. London Math. Soc*. (3) **57** (1988), 209–238.
[**13**] N. MANDOUVALOS. Lower bounds for the Schrödinger kernel on real hyperbolic spaces. (Preprint).
[**14**] N. MANDOUVALOS and M. MARIAS. L^p spectral theory for manifolds of hyperbolic type. *Bull. London Math. Soc*. **28** (1996), 533–539.
[**15**] N. MANDOUVALOS and M. MARIAS. Spectrum of the Laplacian and Riesz transform on locally symmetric spaces. *Bull. Sci. Math*. **133** (2009), 134–144.
[**16**] W. MÜLER. Spectral theory for Riemannian manifolds with cusps and a related trace formula. *Math. Nachr*. **111** (1983), 197–288.
[**17**] P. J. NICHOLLS. *The Ergotic theory of discrete groups*. London Math. Soc. Lecture Note Series 143 (Cambridge University Press, 1989).
[**18**] S. J. PATTERSON. Lectures on measures on limit sets of Kleinian groups. *Analytical and geometrical aspects of hyperbolic space*, ed. D.B.A. Epstein. London Math. Soc. Lecture Notes, vol. 111 (1987), pp. 281–323 (Cambridge University Press).
[**19**] A. WEBER. Heat kernel bounds, Poincaré series and L^2 spectrum for locally symmetric spaces. *Bull. Austral. Math. Soc*. **78** (2008), 73–86.

https://doi.org/10.1017/S0305004112000035 Published online by Cambridge University Press