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Diophantine approximation by orbits of expanding Markov maps

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Abstract. In 1995, Hill and Velani introduced the 'shrinking targets' theory. Given a dynamical system $([0, 1], T)$, they investigated the Hausdorff dimension of sets of points whose orbits are close to some fixed point. In this paper, we study the sets of points well approximated by orbits $\{T^n x\}_{n\geq 0}$, where T is an expanding Markov map with a finite partition supported by $[0, 1]$. The dimensions of these sets are described using the multifractal properties of invariant Gibbs measures.

1. *Introduction*

Let (X, d) be a compact metric space and $T : X \to X$ a piecewise continuous transformation. Let $\mathcal{O}(x) = \{T^n x : n \in \mathbb{N}\}$ be the orbit of $x \in X$. The distribution of $\mathcal{O}(x)$, in particular its density over X, is a historical issue, which goes back to Poincaré's results. In 1995, Hill and Velani [**20**] introduced the shrinking targets theory, which aims at investigating the Hausdorff dimensions of sets of points whose orbits are close to some fixed point. For a point $y \in X$, they studied the set

$$\{x \in X : T^n x \in B(y, r_n) \text{ for infinitely many integers } n \in \mathbb{N}\}, \tag{1}$$

where $B(x, r)$ stands for the ball of radius $r > 0$ centered at $x \in X$ and $(r_n)_{n\geq 1}$ is a sequence of positive real numbers converging to 0. In this article, we adopt a complementary point of view: we fix a point $x \in X$ and consider the set of points y that are well approximated by the orbit $\mathcal{O}(x)$ of x, i.e. we focus on

$$\{y \in X : T^n x \in B(y, r_n) \text{ for infinitely many integers } n \in \mathbb{N}\}, \tag{2}$$

which can be written as

$$\limsup_{n\to\infty} B(T^n x, r_n) = \bigcap_{N\geq 1} \bigcup_{n\geq N} B(T^n x, r_n).$$

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In fact, many questions can be asked about the set (2): for which sequence $(r_n)_{n\geq 1}$ does it cover the whole interval $[0, 1]$? When $[0, 1]$ is not fully covered, what is its Hausdorff dimension? Can the dependence on x be quantified? Answering these questions provides us with a precise description of the distribution properties of the orbit $\mathcal{O}(x)$. Such questions have been investigated in several contexts, and can be interpreted as general Diophantine approximation problems. Indeed, the classical Diophantine questions concern the dimension of the set

$$\mathcal{S}(\delta) = \left\{ y \in [0, 1] : \left| y - \frac{p}{q} \right| \leq \frac{1}{q^{2\delta}} \text{ for infinitely many couples } (p, q) = 1 \right\}. \quad (3)$$

This set can again be seen as a lim sup set

$$\limsup_{q\to+\infty} \bigcup_{p\in\mathbb{Z}} B(p/q, 1/q^{2\delta}).$$

Hill and Vilani's paper [**20**] is a precursor to this subject in the dynamical setting, and, subsequently, many people studied sets of the form (1). For instance, see [**23**] for the case where T is an irrational rotation on the torus $\mathbb{T}^1$. In the literature, one often refers to these results as shrinking targets problems or dynamical Borel–Cantelli lemma. The paper by Fan *et al* [**18**], where the doubling map on $\mathbb{T}^1$ is studied, is the first one to consider the set (2). These studies are also related to many other famous works concerned with metric theory of Diophantine approximation; see [**6, 13, 15, 16, 21, 24, 25**] and references therein.

In this work, we focus on the study of the set (2) when T is an expanding Markov map of the interval $[0, 1]$ with a finite partition—a Markov map, for short.

Definition 1.1. (Markov map) A transformation $T : [0, 1] \to [0, 1]$ is an expanding Markov map with a finite partition if there is a subdivision $\{a_i\}_{0\leq i\leq Q}$ of $[0, 1]$ (denoted by $I(k) =]a_k, a_{k+1}[$ for $0 \leq k \leq Q - 1$) such that the following hold.

(1) There is an integer n and a real number ρ such that $|(T^n)'| \geq \rho > 1$.
(2) T is strictly monotonic and can be extended to a C^2 function on each $\overline{I(i)}$.
(3) If $I(j) \cap T(I(k)) \neq \emptyset$, then $I(j) \subset T(I(k))$.
(4) There is an integer R such that $I(j) \subset \bigcup_{n=1}^{R} T^n(I(k))$ for every k, j.
(5) For every $k \in \{0, \ldots, Q - 1\}$, $\sup_{(x,y,z)\in I(k)^3} (|T''(x)|/|T'(y)||T'(z)|) < \infty$.

It appears that for Markov maps, the relevant choice for the sequence $(r_n)_{n\geq 1}$ is $r_n = 1/n^\delta$, for $\delta > 0$. We thus introduce the sets

$$\mathcal{L}^\delta(x) := \limsup_{n\to\infty} B(T^n x, n^{-\delta}),$$
$$\mathcal{F}^\delta(x) := [0, 1]\backslash\mathcal{L}^\delta(x),$$

which are, respectively, the set of points covered by infinitely many balls $B(T^n x, n^{-\delta})$ and its complement. We study the Hausdorff dimensions of $\mathcal{L}^\delta(x)$ and $\mathcal{F}^\delta(x)$.

Before stating our main theorem, we give some recalls on multifractal analysis. Denote by Leb the Lebesgue measure in $\mathbb{R}$, and by dim the Hausdorff dimension.

Definition 1.2. For any Borel probability measure μ on $[0, 1]$, define the lower (respectively upper) local dimension $\underline{d}_\mu(y)$ (respectively $\overline{d}_\mu(y)$) of μ at $y \in [0, 1]$ by

$$\underline{d}_\mu(y) := \liminf_{r\to 0} \frac{\log \mu(B(y, r))}{\log r} \quad \text{and} \quad \overline{d}_\mu(y) := \limsup_{r\to 0} \frac{\log \mu(B(y, r))}{\log r}.$$

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When $\underline{d}_\mu(y) = \overline{d}_\mu(y)$, their common value is denoted by $d_\mu(y)$, and is simply called the local dimension of μ at y. The level sets of the local dimension are

$$\mathcal{E}_\mu(\alpha) = \{y \in [0, 1] : d_\mu(y) = \alpha\}, \quad \alpha \geq 0. \tag{4}$$

Finally, the *multifractal spectrum* of μ, is defined as the map

$$D_\mu : \alpha \geq 0 \longmapsto \dim \mathcal{E}_\mu(\alpha).$$

Denote by $\mathcal{M}_{\text{inv}}$ (*respectively* $\mathcal{M}_{\text{erg}}$) the set of T-invariant (*respectively* ergodic) probability measures on [0, 1]. The dimension of a Borel probability measure μ is defined as

$$\dim \mu = \inf\{\dim E : E \text{ Borel set} \subset [0, 1] \text{ and } \mu(E) > 0\}.$$

We prove the following theorem (for precise definitions, see §2).

THEOREM 1.3. *Let $T : [0, 1] \to [0, 1]$ be an expanding Markov map. Let ϕ be a Hölder continuous potential and let μ_ϕ be the corresponding Gibbs measure. Let*

$$\alpha_+ := \max_{\mu \in \mathcal{M}_{\text{inv}}} \frac{\int_{[0,1]} (-\phi)\, d\mu}{\int_{[0,1]} \log |T'|\, d\mu} \quad \textit{and} \quad \alpha_{\max} := \frac{\int_{[0,1]} (-\phi)\, d\mu_{\max}}{\int_{[0,1]} \log |T'|\, d\mu_{\max}}, \tag{5}$$

where $\mu_{\max}$ is the Gibbs measure associated with the potential $\psi = -\log |T'|$.

(1) *For μ_ϕ-almost every $x \in [0, 1]$, the Hausdorff dimension of $\mathcal{L}^\delta(x)$ is*

$$\dim \mathcal{L}^\delta(x) = \begin{cases} 1/\delta & \text{if } 0 < 1/\delta \leq \dim \mu_\phi, \\ D_{\mu_\phi}(1/\delta) & \text{if } \dim \mu_\phi < 1/\delta \leq \alpha_{\max}, \\ 1 & \text{if } 1/\delta > \alpha_{\max}. \end{cases} \tag{6}$$

(2) *For μ_ϕ-almost every $x \in [0, 1]$, the Hausdorff dimension of $\mathcal{F}^\delta(x)$ is*

$$\dim \mathcal{F}^\delta(x) = \begin{cases} 1 & \text{if } 0 < 1/\delta \leq \alpha_{\max}, \\ D_{\mu_\phi}(1/\delta) & \text{if } 1/\delta > \alpha_{\max}. \end{cases} \tag{7}$$

(3) *Concerning the Lebesgue measure of $\mathcal{L}^\delta(x)$ and $\mathcal{F}^\delta(x)$, we have,*

$$\textit{for } \mu_\phi\text{-a.e. } x, \quad \text{Leb}(\mathcal{L}^\delta(x)) = 1 - \text{Leb}(\mathcal{F}^\delta(x)) = \begin{cases} 0 & \text{if } 0 < 1/\delta < \alpha_{\max}, \\ 1 & \text{if } 1/\delta > \alpha_{\max}. \end{cases} \tag{8}$$

(4) *If $1/\delta > \alpha_+$, then $\mathcal{F}^\delta(x) = \emptyset$ and hence $\mathcal{L}^\delta(x) = [0, 1]$.*

Remark 1.4. The dimensions of $\mathcal{L}^{1/\alpha_{\max}}(x)$ and $\mathcal{F}^{1/\alpha_{\max}}(x)$ are 1, but we do not know their Lebesgue measure. We also prove that $\text{Leb}(\mathcal{L}^{1/\alpha_+}(x)) = 1$ and $\dim \mathcal{F}^{1/\alpha_+}(x) \leq \lim_{1/\delta \to \alpha_+} D_{\mu_\phi}(1/\delta)$, but we do not know whether $\mathcal{L}^{1/\alpha_+}(x) = [0, 1]$.

The mapping $1/\delta \mapsto \dim \mathcal{L}^\delta(x)$ exhibits clearly four distinct behaviors, respectively denoted by Part I ($1/\delta \leq \dim \mu_\phi$), Part II ($\dim \mu_\phi < 1/\delta \leq \alpha_{\max}$), Part III ($\alpha_{\max} < 1/\delta \leq \alpha_+$) and finally Part IV ($1/\delta > \alpha_+$). See Figure 1 (the definition of α_- is given in Theorem 2.5).

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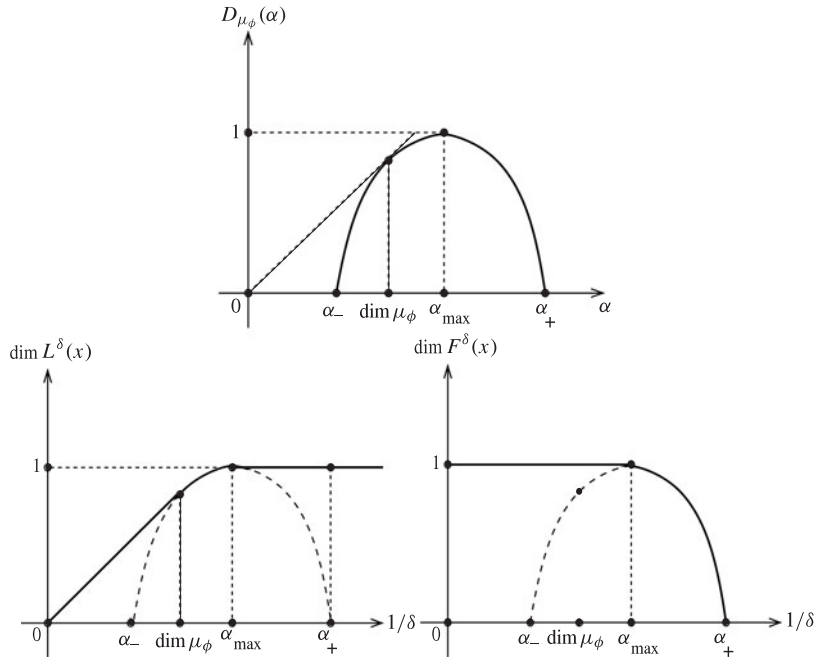


FIGURE 1. Multifractal spectrum of μ_ϕ, and the maps $1/\delta \mapsto \dim \mathcal{L}^\delta(x)$ and $1/\delta \mapsto \dim \mathcal{F}^\delta(x)$.

For μ_ϕ-typical x, the behavior of $\mathcal{L}^\delta(x)$ enjoys two remarkable characteristics compared with classical Diophantine approximation: the map $1/\delta \mapsto \dim \mathcal{L}^\delta(x)$ may have a strictly concave part (Part II), and the smallest δ for which $\mathrm{Leb}(\mathcal{L}^\delta(x)) = 1$ and the smallest δ for which $\mathcal{L}^\delta(x) = [0, 1]$ do not coincide (Part III). This contrasts with the classical Diophantine approximation, especially the approximation by rationals. In this historical context, as said above, the analog of the sets $\mathcal{L}^\delta(x)$ are the sets $S(\delta)$ defined by (3). The Dirichlet theorem ensures that $\mathcal{S}(1) = [0, 1]$, and it is well known [**7, 22**] that for every $\delta > 1$, $\dim \mathcal{S}(\delta) = 1/\delta$. In particular, the dimension of $\mathcal{S}(\delta)$ decreases linearly with respect to $1/\delta$, and as soon as the Lebesgue measure of $\mathcal{S}(\delta)$ equals 1, it instantaneously covers the whole interval $[0, 1]$. Comparable results hold for sets of numbers approximated by other families, see for instance [**3, 4, 10, 11, 33**]. The two characteristics of $\mathcal{L}^\delta(x)$ mentioned above can be interpreted by the fact that, although the orbits $\mathcal{O}(x)$ of μ_ϕ-typical points x are dense, they are not as regularly distributed when n tends to infinity as the rational numbers are. The exponents $\dim \mu_\phi$, $\alpha_{\max}$ and α_+ characterize this 'distortion'.

The paper is organized as follows. Section 2 contains some recalls on multifractal analysis and hitting time. Section 3 describes the relations between hitting time, approximation rate and local dimension of μ_ϕ. From these relations we will give a direct proof of item (3) of Theorem 1.3. In §4, two key lemmas are proved. They illustrate the fact that intervals which have a small local dimension for μ_ϕ are hit by the balls $B(T^n x, 1/n^\delta)$ with big probability, and *vice versa*. Then, §§5–8 contain the proofs of the upper and lower bounds for the Hausdorff dimensions of $\mathcal{L}^\delta(x)$ and $\mathcal{F}^\delta(x)$ for Parts I, IV, III and II, respectively.

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Theorem 1.3 is similar to the results in [**18**] for the doubling map on $\mathbb{T}^1$, but the proofs require other arguments. First, for $x \mapsto 2x$, since the Lyapunov exponents are constant, the balls of generation n have same lengths, while for the Markov maps their lengths may have different order. Second, in [**18**], the authors focus on Bowen's entropy spectrum. Third, the notions of Hölder exponent and hitting time for the doubling map involve only cylinders, while we need centered balls to define similar quantities. Finally, some arguments (Lemmas 4.2–4.4) are adapted from [**18**] in the context of Markov maps, but several others are not. The best example is the difficult lower bound for $\dim \mathcal{L}^\delta(x)$ when $0 < 1/\delta \leq \dim \mu_\phi$.

2. *First definitions, and recalls on multifractal analysis*

2.1. *Covering of* [0, 1] *by basic intervals.*

Let T be a Markov map as defined in Definition 1.1. With T are associated generations of *basic* intervals.

Definition 2.1. Let $\mathcal{A} = \{0, 1, \ldots, Q-1\}$. For every integer $n \geq 1$, we denote by $\mathcal{G}_n$ the set of *basic intervals* of generation n defined by

$$\text{if } (i_1 i_2 \cdots i_n) \in \mathcal{A}^n, \quad I_{i_1 i_2 \cdots i_n} := I(i_1) \cap T^{-1}(I(i_2)) \cap \cdots \cap T^{-n+1}(I(i_n)).$$

The following distortion property between intervals will be crucial: there is a constant $L > 1$ such that for every integer $n \geq 2$, for every $(i_1 i_2 \cdots i_n) \in \mathcal{A}^n$,

$$1 \leq \frac{|I_{i_1 i_2 \cdots i_{n-1}}|}{|I_{i_1 i_2 \cdots i_{n-1} i_n}|} \leq L \quad (|I| \text{ is the length of the interval } I). \tag{9}$$

It is obvious that the intervals of a given generation n form a covering of [0, 1]. This covering of [0, 1] is not composed of intervals of the same length. However, using (9), for every real number $0 < r < 1$, one easily shows that there is a family of basic intervals $J_1, J_2, \ldots, J_N$ (not belonging to the same $\mathcal{G}_n$) such that the following hold.

- $\bigcup_{j=1}^N J_j = [0, 1]$.
- For every $j \neq j'$, the intersection of interiors $\overset{\circ}{J}_j \cap \overset{\circ}{J}_{j'}$ is empty.
- For the same constant L as in (9), for every $j \in \{1, 2, \ldots, N\}$,

$$L^{-1} r \leq |J_j| \leq Lr. \tag{10}$$

For every $n \in \mathbb{N}$, we fix $\mathcal{C}_n$ to be one possible collection of intervals such that (10) holds for $r = 2^{-n}$. From these considerations, one deduces that there exists a number $L' > 1$ such that for all $n \in \mathbb{N}$, the generation n_J of a basic interval $J \in \mathcal{C}_n$ satisfies

$$(L')^{-1} n \leq n_J \leq L' n. \tag{11}$$

2.2. *Definition of hitting time.*

Denote $\mathcal{O}^+(x) := \mathcal{O}(x) \backslash \{x\}$.

Definition 2.2. For every $(x, y) \in [0, 1]^2$ and $r > 0$, we define the hitting time (first entrance time) of the orbit of x into the ball $B(y, r)$ by

$$\tau_r(x, y) := \inf\{n \geq 1 : T^n x \in B(y, r)\}.$$

Then we set

$$R(x, y) := \liminf_{r \to 0} \frac{\log \tau_r(x, y)}{-\log r}. \tag{12}$$

By convention, when $\mathcal{O}^+(x) \cap B(y, r) = \emptyset$, we set $\tau_r(x, y) = +\infty$ and $R(x, y) = +\infty$.

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We define the hitting time $\tau(x, C)$ of a basic interval C by a point x. Let $m \geq 1$ and let $C \in \mathcal{G}_m$. If $\mathcal{O}(x) \cap C = \emptyset$, we set $\tau(x, C) = +\infty$. Otherwise,

$$\tau(x, C) := \inf\{n \geq 0 : T^n x \in C\}. \tag{13}$$

Definition 2.3. Let $s \geq 0$ be a real number. We define the sets

$$\begin{aligned} \mathcal{R}_{\geq s}(x) &= \{y \in [0, 1] : R(x, y) \geq s\}, \quad \mathcal{R}_{\leq s}(x) = \{y \in [0, 1] : R(x, y) \leq s\}, \\ \mathcal{R}_{>s}(x) &= \{y \in [0, 1] : R(x, y) > s\}, \quad \mathcal{R}_{<s}(x) = \{y \in [0, 1] : R(x, y) < s\}. \end{aligned} \tag{14}$$

2.3. *Multifractality of Gibbs measures.* Here are some facts on Gibbs measures.

THEOREM 2.4. **[8, 35]** *Let $T : I \to I$ be a Markov map. Then for any Hölder continuous function $\phi : I \to \mathbb{R}$, there exists a unique equilibrium state μ_ϕ which satisfies the following Gibbs property: there exist constants $\gamma > 0$ and $P(\phi)$ (the topological pressure associated with ϕ), such that*

$$\text{for any basic interval } I \in \mathcal{G}_n, \quad \gamma^{-1} \leq \frac{\mu_\phi(I)}{e^{S_n\phi(x) - nP(\phi)}} \leq \gamma, \quad \text{for every } x \in I, \tag{15}$$

where $S_n\phi(x) = \phi(x) + \cdots + \phi(T^{n-1}x)$ is the nth Birkhoff sum of ϕ at x.

A potential ϕ is often assumed to be *normalized*, i.e. $P(\phi) = 0$. If it is not the case, we can replace ϕ by $\phi - P(\phi)$.

Now, let us recall some standard facts on multifractal analysis of Borel measures. By using an extensive literature **[5, 9, 14, 28, 31, 34]**, the multifractal analysis of μ_ϕ can be achieved, i.e. the multifractal spectrum D_{μ_ϕ} of μ_ϕ can be computed.

THEOREM 2.5. *Consider a Markov map $T : [0, 1] \to [0, 1]$ and a normalized Hölder continuous potential $\phi : [0, 1] \to \mathbb{R}$.*

(1) *The multifractal spectrum D_{μ_ϕ} of μ_ϕ is a concave analytic map on the interval $]\alpha_-, \alpha_+[$, where $\alpha_- := \min_{\mu \in \mathcal{M}_{\text{inv}}}[\int_{[0,1]}(-\phi)\, d\mu / \int_{[0,1]} \log|T'|\, d\mu]$ and α_+ was defined in (5).*

(2) *The spectrum D_{μ_ϕ} reaches its maximum value 1 at $\alpha_{\max}$ defined in (5).*

(3) *The graph of D_{μ_ϕ} and the first bisector intersect at a unique point which is $(\dim \mu_\phi, \dim \mu_\phi)$. Moreover, $\dim \mu_\phi$ satisfies*

$$\dim \mu_\phi = \frac{\int_{[0,1]}(-\phi)\, d\mu_\phi}{\int_{[0,1]} \log|T'|\, d\mu_\phi}.$$

For every $q \in \mathbb{R}$, there is a unique real number $\eta_\phi(q)$ such that the topological pressure $P(-\eta_\phi(q) \log|T'| + q\phi)$ associated with the Hölder potential $\phi_q := -\eta_\phi(q) \log|T'| + q\phi$ equals 0. Such a number exists since the map $P : t \mapsto (-t \log|T'| + q\phi)$ is real-analytic and decreasing in t. The resulting function $q \mapsto \eta_\phi(q)$ is real-analytic and concave. We denote by

$$\mu_q := \mu_{\phi_q} \quad \text{where } \phi_q := -\eta_\phi(q) \log|T'| + q\phi, \tag{16}$$

the Gibbs measure associated with the potential ϕ_q. Observe that $\eta_\phi(0) = 1$, and $\eta_\phi(1) = 0$. The measures μ_0 and μ_1 $(= \mu_\phi)$ are associated with the potentials $\phi_0 = -\log|T'|$ and

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$\phi_1 = \phi$ respectively. By a commonly known theorem, the Lebesgue measure Leb is equivalent to μ_0, coinciding with $\mu_{\max}$ used in Theorem 1.3.

For every $q \in \mathbb{R}$, we introduce the exponent

$$\alpha(q) = \frac{\int_{[0,1]} (-\phi)\, d\mu_q}{\int_{[0,1]} \log |T'|\, d\mu_q}. \tag{17}$$

By the Gibbs property of μ_ϕ and the ergodicity of μ_q, the measure μ_q is supported by the level set $\mathcal{E}_{\mu_\phi}(\alpha(q))$ (see (4)). Hence $D_{\mu_\phi}(\alpha(q)) = \dim \mu_q = \eta_\phi(q) + q\alpha(q)$. The map $q \mapsto \alpha(q)$ is decreasing, and

$$\begin{gathered} \lim_{q \to +\infty} \alpha(q) = \alpha_-, \quad \lim_{q \to -\infty} \alpha(q) = \alpha_+, \\ \alpha(1) = \dim \mu_\phi, \quad \alpha(0) = \alpha_{\max}. \end{gathered} \tag{18}$$

For $\alpha \in]\alpha_-, \alpha_+[$, we write the inverse function of $q \mapsto \alpha(q)$ as $\alpha \mapsto q(\alpha)$.

Remark 2.6. By a standard result, we have

$$\sup_{y\,:d_{\mu_\phi}(y) \text{ exists}} d_{\mu_\phi}(y) = \sup_{\nu \in \mathcal{M}_{\text{erg}}} \frac{-\int_{[0,1]} \phi\, d_\nu}{\int_{[0,1]} \log |T'|\, d\nu} = \alpha_+.$$

Definition 2.7. Let $s \in \mathbb{R}^+$. We define the sets

$$\mathcal{E}_{\geq s} = \{y \in [0,1] : \underline{d}_{\mu_\phi}(y) \geq s\} \quad \text{and} \quad \mathcal{E}_{\leq s} = \{y \in [0,1] : \underline{d}_{\mu_\phi}(y) \leq s\}, \tag{19}$$

$$\mathcal{E}_{> s} = \{y \in [0,1] : \underline{d}_{\mu_\phi}(y) > s\} \quad \text{and} \quad \mathcal{E}_{< s} = \{y \in [0,1] : \underline{d}_{\mu_\phi}(y) < s\}. \tag{20}$$

For sake of simplicity, we omit the dependence on ϕ of the quantities $\alpha(q)$, $q(\alpha)$, and $\mathcal{E}$. Indeed, ϕ will be fixed along the proof.

From large deviations theory [**9**], we get the values for the Hausdorff dimensions of the sets $\mathcal{E}$ in Definition 2.7. These values depend on whether s is located in the increasing or in the decreasing part of the multifractal spectrum D_{μ_ϕ}.

PROPOSITION 2.8. *Let μ_ϕ be a Gibbs measure. Then we have the following.*

(1) *For every $s < \alpha_{\max}$, $\dim(\mathcal{E}_{<s}) = \dim(\mathcal{E}_{\leq s}) = D_{\mu_\phi}(s)$.*

(2) *For every $s > \alpha_{\max}$, $\dim(\mathcal{E}_{>s}) = \dim(\mathcal{E}_{\geq s}) = D_{\mu_\phi}(s)$.*

3. *Results about hitting time*

3.1. *Orbits and hitting time.*

The proof of Lemma 3.1 is left to the reader.

LEMMA 3.1. *The following three assertions are equivalent.*

(1) *There exists an integer $n_0 \geq 1$ such that $y = T^{n_0}x$ (i.e. $y \in \mathcal{O}^+(x)$).*

(2) *The hitting time $\tau_r(x, y)$ is bounded for all $r > 0$.*

(3) *There is a sequence $r_i \to 0$ such that $\tau_{r_i}(x, y)$ is bounded.*

Lemmas 3.2–3.4 investigate the relationship between $\mathcal{L}^\delta(x)$ and hitting times.

LEMMA 3.2. *For every $\delta > 0$, we have the two embedding properties,*

$$\mathcal{R}_{<1/\delta}(x) \backslash \mathcal{O}^+(x) \subset \mathcal{L}^\delta(x) \subset \mathcal{R}_{\leq 1/\delta}(x), \tag{21}$$

$$\mathcal{R}_{>1/\delta}(x) \subset \mathcal{F}^\delta(x) \subset \mathcal{R}_{\geq 1/\delta}(x) \cup \mathcal{O}^+(x). \tag{22}$$

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Proof. We prove (21), and (22) is deduced by taking complements.

For the first inclusion, consider y such that $R(x, y) < 1/\delta$ and $y \notin \mathcal{O}^+(x)$. Choose $\varepsilon > 0$ such that $R(x, y) < 1/\delta - \varepsilon$. Then, by the definition of $R(x, y)$, there is a positive real sequence (r_i) converging to zero such that $\tau_{r_i}(x, y) < (1/r_i)^{1/\delta-\varepsilon}$. Consider the sequence of integers $n_i := \tau_{r_i}(x, y)$, for all $i \geq 1$. By construction,

$$r_i < (n_i)^{-1/(1/\delta-\varepsilon)} < n_i^{-\delta}.$$

Thus $T^{n_i}x \in B(y, n_i^{-\delta})$. Since $y \notin \mathcal{O}^+(x)$, Lemma 3.1 yields that $(n_i)_{i\geq 1}$ is not bounded. We deduce that $y \in B(T^{n_i}x, n_i^{-\delta})$ for infinitely many increasing integers n_i. This proves that $y \in \mathcal{L}^\delta(x)$.

For the second inclusion of (21), consider $y \in \mathcal{L}^\delta(x)$. By definition, $T^{n_i}x \in B(y, n_i^{-\delta})$ for infinitely many integers $(n_i)_{i\geq 1}$. Hence, for these n_i, we have $\tau_{1/n_i^\delta}(x, y) \leq n_i$, which implies that

$$R(x, y) \leq \liminf_{i\to\infty} \frac{\log \tau_{1/n_i^\delta}(x, y)}{-\log(1/n_i^\delta)} \leq \liminf_{n_i\to\infty} \frac{\log n_i}{\delta \log n_i} = \frac{1}{\delta}.$$

This completes the proof. □

LEMMA 3.3. *Suppose that x is not eventually periodic. If $y \in \mathcal{O}^+(x)$, then*

$$y \in \mathcal{L}^\delta(x) \quad \text{if } R(y, y) < 1/\delta \quad \text{and} \quad y \in \mathcal{F}^\delta(x) \quad \text{if } R(y, y) > 1/\delta.$$

Observe that the case where $R(y, y) = 1/\delta$ is not determined yet.

Proof. Suppose that $y \in \mathcal{O}^+(x)$. Since x is not eventually periodic, there exists a unique positive integer n_0 such that $T^{n_0}x = y$ and $y \notin \mathcal{O}^+(y)$. The rest of the proof is the same as that of Lemma 3.2. □

LEMMA 3.4. *Suppose that $\mu \in \mathcal{M}_{inv}$ has no atom. Then for μ-almost every x, we have*

$$\mathcal{O}^+(x) \subset \mathcal{L}^\delta(x) \quad \text{if } 1/\delta > \dim \mu \quad \text{and} \quad \mathcal{O}^+(x) \subset \mathcal{F}^\delta(x) \quad \text{if } 1/\delta < \dim \mu.$$

Proof. We remark that the set of eventually periodic points is a countable set, hence it has a μ-measure equal to zero. By the Ornstein–Weiss theorem [**27**],

$$\text{for } \mu\text{-almost all } x, \text{ for every } n \geq 1, \quad R(T^n x, T^n x) = \dim \mu. \tag{23}$$

Hence, for a μ-typical x (which is not eventually periodic), consider $y \in \mathcal{O}^+(x)$. By the same argument as above, $y = T^{n_0}x$ for some unique integer $n_0 \geq 0$. By (23), $R(y, y) = \dim \mu$. Applying Lemma 3.3, we find that if $1/\delta > \dim \mu$ (*respectively* $1/\delta < \dim \mu$), then $y \in \mathcal{L}^\delta(x)$ (*respectively* $y \in \mathcal{F}^\delta(x)$). □

3.2. *Local dimension and hitting time.* Gibbs measures enjoy exponential decay of correlations [**1, 26, 29, 32**]. More precisely, we have the following theorem.

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THEOREM 3.5. *Suppose that* $f : [0, 1] \to [0, 1]$ *has bounded variation and* $g : [0, 1] \to [0, 1]$ *is integrable. Then there exist constants* $0 < \beta < 1$ *and* $\Theta > 0$*, such that for every integer* $n \geq 1$*,*

$$\left| \int fg \circ T^n \, d\mu_\phi - \int f \, d\mu_\phi \int g \, d\mu_\phi \right| \leq \Theta \beta^n \left(\int |f| \, d\mu_\phi + \mathrm{var}(f) \right) \int |g| \, d\mu_\phi,$$

where $\mathrm{var}(f)$ *stands for the total variation of* f *on* $[0, 1]$*. In particular, if* $f = \mathbf{1}_A$ *and* $g = \mathbf{1}_B$ *where* A *is an interval and* B *is a measurable set, then, for every* n*,*

$$|\mu_\phi(A \cap T^{-n} B) - \mu_\phi(A)\mu_\phi(B)| \leq \Theta \beta^n (\mu_\phi(A) + 2)\mu_\phi(B). \tag{24}$$

Theorem 3.5 allows us to use the following theorem which describes the relationship between hitting time and local dimension of invariant measures.

THEOREM 3.6. [**19**] *If* (X, T, μ) *has superpolynomial decay of correlations and if* $d_\mu(y)$ *exists, then for* μ*-almost every* x *we have*

$$R(x, y) = d_\mu(y).$$

We return to the study of the Markov map T on the interval $[0, 1]$.

COROLLARY 3.7. *Let* μ_ϕ, μ_ψ *be two* T*-invariant Gibbs probability measures on* $[0, 1]$ *associated with normalized Hölder potentials* ϕ *and* ψ*. We have*

$$\text{for } \mu_\phi\text{-a.e. } x, \text{ for } \mu_\psi\text{-a.e. } y, \quad R(x, y) = d_{\mu_\phi}(y) = \frac{-\int_{[0,1]} \phi \, d\mu_\psi}{\int_{[0,1]} \log |T'| \, d\mu_\psi}.$$

Proof. For μ_ψ-almost every y, $(1/n)S_n\phi(y)$ tends to $\int_{[0,1]} \phi \, d\mu_\psi$. Hence, for μ_ψ-almost every y, using the Gibbs property of μ_ϕ and μ_{ϕ_0}, the Hölder exponent $\tilde{d}_{\mu_\phi}(y)$ computed on the basic intervals defined as

$$\tilde{d}_{\mu_\phi}(y) := \lim_{n \to +\infty} \frac{\log \mu(I_n(y))}{\log |I_n(y)|}$$

exists and is equal to $((-\int_{[0,1]} \phi \, d\mu_\psi)/(\int_{[0,1]} \log |T'| \, d\mu_\psi))$. By [**2**, Theorem 5.1], for quasi-Bernoulli measures, and in particular for Gibbs measures associated with Markov maps, $\tilde{d}_{\mu_\phi}(y)$ coincides with $d_{\mu_\phi}(y)$, for μ_ψ-almost every y. The lemma then follows by Theorem 3.6 and the Fubini theorem. □

COROLLARY 3.8. *Let* μ_ϕ, μ_ψ *be two* T*-invariant Gibbs probability measures on* $[0, 1]$ *associated with normalized Hölder potentials* ϕ *and* ψ*. Then*

$$\text{for } \mu_\phi \times \mu_\psi\text{-almost every } (x, y), \quad R(x, y) = d_{\mu_\phi}(y) = \frac{\int_{[0,1]} (-\phi) \, d\mu_\psi}{\int_{[0,1]} \log |T'| \, d\mu_\psi}.$$

3.3. *First results on covering.* We introduce the real number

$$\delta(\phi, \psi) := \sup\{\delta \geq 0 : \mu_\psi(\mathcal{L}^\delta(x)) = 1 \text{ for } \mu_\phi\text{-almost every } x\}.$$

The following proposition, which summarizes the results of the previous sections, will be useful when proving the lower bound for item (1) of Theorem 1.3. We can also use it to give a direct proof for the item (3) of Theorem 1.3.

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PROPOSITION 3.9. *For any normalized Hölder potentials ϕ and ψ, we have*

$$\delta(\phi, \psi) = \frac{\int_{[0,1]} \log |T'| \, d\mu_\psi}{\int_{[0,1]} (-\phi) \, d\mu_\psi}.$$

In particular, for every $\alpha \in]\alpha_-, \alpha_+[$,

$$\delta(\phi, \phi_{q(\alpha)}) = \sup\{\delta \geq 0 : \mu_{q(\alpha)}(\mathcal{L}^\delta(x)) = 1 \text{ for } \mu_\phi\text{-a.e. } x\} = 1/\alpha. \tag{25}$$

Proof. Combine Lemma 3.2, Corollary 3.8 and the definition (17) of $\alpha(q)$. □

We are now able to give a direct proof of the item (3) of Theorem 1.3.

Proof. (Direct proof of Theorem 1.3(3)) Take the potential $\psi := \phi_0 = -\log|T'|$. As already observed, the corresponding Gibbs measure μ_0 is an invariant measure equivalent to the Lebesgue measure. Thus, 'μ_0-almost everywhere' is equivalent to 'Lebesgue-almost everywhere'. Hence, $\delta(\phi, \psi)$ is also equal to

$$\sup\{\delta \geq 0 : \text{Leb}(\mathcal{L}^\delta(x)) = 1 \text{ for } \mu_\phi\text{-a.e. } x\}.$$

From Proposition 3.9, applying (25) with the measure μ_0, the exponent $\delta(\phi, \psi)$ coincides with $1/\alpha_{\max}$ (defined by (5)). This concludes the proof. □

We investigate other exponents (recall that $q(\alpha)$ is the inverse function of $\alpha(q)$).

- By Proposition 3.9, for any $\varepsilon > 0$ and $\delta \in]1/\alpha_+, 1/\alpha_-[$, for μ_ϕ-almost every x, we have
$$\mu_{q(1/\delta)}(\mathcal{L}^{\delta-\varepsilon}(x)) = 1. \tag{26}$$
- For $1/\delta = \alpha(1) = \dim \mu_\phi$, we have $q(1/\delta) = 1$ and $\mu_q = \mu_1 = \mu_\phi$. Hence, applying Proposition 3.9 and (18), we get
$$\sup\{\delta \geq 0 : \mu_\phi(\mathcal{L}^\delta(x)) = 1 \text{ for } \mu_\phi\text{-a.e. } x\} = \frac{1}{\dim \mu_\phi}.$$
Thus for every $0 < \delta < 1/\dim \mu_\phi$, we have $\mu_\phi(\mathcal{L}^\delta(x)) = 1$ for μ_ϕ-almost every x.

4. *Multiple-quasi-Bernoulli inequalities and hitting lemmas*

4.1. *Multiple-quasi-Bernoulli inequalities.* Let ϕ be a normalized potential, i.e. $P(\phi) = 0$. The Gibbs property (15) of μ_ϕ can be written as

$$\text{for all } x \in [0, 1], \text{ for all } n \geq 1, \quad \frac{1}{\gamma} e^{S_n\phi(x)} \leq \mu_\phi(I_n(x)) \leq \gamma e^{S_n\phi(x)}. \tag{27}$$

It is a classical result that (27) implies the quasi-Bernoulli property of μ_ϕ.

LEMMA 4.1. *For any basic intervals A and B of generation n_A and n_B respectively,*

$$\frac{1}{\gamma^3}\mu_\phi(A)\mu_\phi(B) \leq \mu_\phi(A \cap T^{-n_A}B) \leq \gamma^3 \mu_\phi(A)\mu_\phi(B). \tag{28}$$

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Proof. Consider any $x \in A \cap T^{-n_A} B$. Applying (27) three times, we get

$$\mu_\phi(A \cap T^{-n_A} B) \geq \gamma^{-1} e^{S_{n_A}\phi(x)+S_{n_B}(T^{n_A}x)} \geq \gamma^{-3}\mu_\phi(A)\mu_\phi(B),$$
$$\mu_\phi(A \cap T^{-n_A} B) \leq \gamma e^{S_{n_A}\phi(x)+S_{n_B}(T^{n_A}x)} \leq \gamma^{3}\mu_\phi(A)\mu_\phi(B).$$ □

Moreover, by (24), the following multiple-quasi-Bernoulli inequalities hold. The same inequalities are referred to as a *multi-relation* for the doubling map in [**18**].

LEMMA 4.2. *Let μ_ϕ be the Gibbs measure associated with a normalized potential ϕ. Let $n \in \mathbb{N}$, $n \geq 1$ and let $C_0, C_1, \ldots, C_k$ be $(k+1)$ basic intervals in $\mathcal{C}_n$. Then there exist a constant $M > 0$ independent of the choice of n such that, for any integer ω large enough (β is the constant appearing in (24)),*

$$\frac{1}{\gamma^3}(1 - M\beta^{\omega n})^{k-1} \leq \frac{\mu_\phi(C_0 \cap \bigcap_{j=1}^{k} T^{-2j\omega n} C_j)}{\prod_{j=0}^{k} \mu_\phi(C_j)} \leq \gamma^3(1 + M\beta^{\omega n})^{k-1}. \tag{29}$$

Proof. Let n_j be the generation of C_j, and let ω be an integer large enough that $n_j - 2\omega n \leq -1$ for all $0 \leq j \leq k$. Observe that

$$C_0 \cap \bigcap_{j=1}^{k} T^{-2j\omega n} C_j = C_0 \cap T^{-n_0}\mathcal{B} \quad \text{where } \mathcal{B} = \bigcap_{j=1}^{k} B_j \text{ and } B_j := T^{n_0 - 2j\omega n} C_j.$$

Applying (28) to $A = C_0$ and to each $B = B_j$, we obtain

$$\gamma^{-3}\mu_\phi(C_0)\mu_\phi(B_j) \leq \mu_\phi(C_0 \cap T^{-n_0} B_j) \leq \gamma^3 \mu_\phi(C_0)\mu_\phi(B_j).$$

Then, summing over all the B_j, we get

$$\gamma^{-3}\mu_\phi(C_0)\mu_\phi(\mathcal{B}) \leq \mu_\phi(C_0 \cap T^{-n_0}\mathcal{B}) \leq \gamma^3 \mu_\phi(C_0)\mu_\phi(\mathcal{B}). \tag{30}$$

The invariance of μ_ϕ implies that

$$\mu_\phi(\mathcal{B}) = \mu_\phi\left(\bigcap_{j=1}^{k} T^{-2j\omega n} C_j \right).$$

Thus, in order to get (29), we need only to prove that, for some constant M,

$$(1 - M\beta^{\omega n})^{k-1} \leq \frac{\mu_\phi(\bigcap_{j=1}^{k} T^{-2j\omega n} C_j)}{\prod_{j=1}^{k} \mu_\phi(C_j)} \leq (1 + M\beta^{\omega n})^{k-1}. \tag{31}$$

Recalling the exponential decay of correlation (24), for every choice of two basic intervals A, B, and for every integer m, we have

$$|\mu_\phi(A \cap T^{-m} B) - \mu_\phi(A)\mu_\phi(B)| \leq \Theta\beta^m(\mu_\phi(A) + 2)\mu_\phi(B),$$

which can be rewritten as

$$\left(1 - \Theta\beta^m \frac{\mu_\phi(A) + 2}{\mu_\phi(A)}\right) \leq \frac{\mu_\phi(A \cap T^{-m} B)}{\mu_\phi(A)\mu_\phi(B)} \leq \left(1 + \Theta\beta^m \frac{\mu_\phi(A) + 2}{\mu_\phi(A)}\right). \tag{32}$$

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Consider the intervals $C_1, \dots, C_k$ and observe that

$$\bigcap_{j=1}^{k} T^{-2j\omega n} C_j = (T^{-2\omega n} C_1) \cap \left(T^{-2\omega n} \left(\bigcap_{j=2}^{k} T^{-2(j-1)\omega n} C_j \right) \right). \tag{33}$$

Iterating (33), we apply the double-sided inequality (32) inductively to obtain

$$\begin{aligned} &\prod_{j=1}^{k-1} \left(1 - \Theta \beta^{2\omega n} \frac{\mu_\phi(C_j) + 2}{\mu_\phi(C_j)} \right) \\ &\quad \le \frac{\mu_\phi(\bigcap_{j=1}^{k} T^{-2j\omega n} C_j)}{\prod_{j=1}^{k} \mu_\phi(C_j)} \le \prod_{j=1}^{k-1} \left(1 + \Theta \beta^{2\omega n} \frac{\mu_\phi(C_j) + 2}{\mu_\phi(C_j)} \right). \end{aligned} \tag{34}$$

By the Gibbs property (27), we have, for $1 \le j \le k$,

$$\frac{\mu_\phi(C_j) + 2}{\mu_\phi(C_j)} \le \frac{3}{\mu_\phi(C_j)} \le 3\gamma \cdot e^{-S_{n_j}\phi(x)} \le 3\gamma \cdot e^{-n_j(\min_{x\in[0,1]}(\phi(x)))}.$$

Since ϕ is normalized, $\min_{x\in[0,1]}(\phi(x))$ is negative. Recalling (11), we find that

$$\frac{\mu_\phi(C_j) + 2}{\mu_\phi(C_j)} \le 3\gamma \cdot e^{-L' n(\min_{x\in[0,1]}(\phi(x)))}.$$

For ω large enough, $e^{-L'(\min_{x\in[0,1]}(\phi(x)))} < \beta^{-\omega}$. Then each term in the product on the right side of (34) can be bounded from above by

$$1 + \Theta \beta^{2\omega n} \cdot 3\gamma \beta^{-n\omega} \le 1 + M\beta^{\omega n},$$

where M is some constant depending on γ, Θ. Then (31) is obtained by multiplying $(k-1)$ identical terms. One gets the lower bound similarly. □

4.2. *Big hitting probability lemma.* Lemma 4.3 illustrates the fact that intervals with small local dimension for μ_ϕ are hit by the balls $B(T^n x, 1/n^\delta)$ with big probability.

LEMMA 4.3. *Let h and ε be two positive real numbers. Consider N distinct basic intervals $C_1, \dots, C_N$ in $\mathcal{C}_n$ satisfying $\mu_\phi(C_i) \ge |C_i|^{h-\varepsilon}$. Set*

$$\mathcal{C}_{n,N,h} = \{x \in [0,1] : \exists C \in \{C_i\}_{i=1,\dots,N} \text{ such that } \tau(x, C) > |C|^{-h}\}.$$

Then there exists an integer $n_h \in \mathbb{N}$ independent of N such that

$$\text{for every } n \ge n_h, \quad \mu_\phi(\mathcal{C}_{n,N,h}) \le 2^{-n}.$$

Proof. Fix one interval $\widetilde{C}$ among the N basic intervals $C_1, \dots, C_N$. Let

$$\mathcal{X}_{\widetilde{C}} := \{x \in [0,1] : \forall n \le |\widetilde{C}|^{-h}, T^n x \notin \widetilde{C}\}.$$

Obviously we have the embedding property $\mathcal{C}_{n,N,h} \subset \bigcup_{i=1}^{N} \mathcal{X}_{C_i}$, so we are going to bound from above each $\mu_\phi(\mathcal{X}_{\widetilde{C}})$. Pick an integer ω such that $2\omega > L$ (L appears in (10)), and set $m_{\widetilde{C}} = \lfloor |\widetilde{C}|^{-h}/(2\omega n) \rfloor$. Then, by definition of $m_{\widetilde{C}}$, we have

$$\mathcal{X}_{\widetilde{C}} \subset \bigcap_{j=0}^{m_{\widetilde{C}}} \{x \in [0,1] : T^{2j\omega n} x \notin \widetilde{C}\} = \bigcap_{j=0}^{m_{\widetilde{C}}} ([0,1] \setminus T^{-2j\omega n}(\widetilde{C})).$$

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We know that the union of the intervals belonging to $\mathcal{C}_n$ is the whole interval $[0, 1]$. Observe also that the cardinality of $\mathcal{C}_n$ is of order 2^n. Let us denote by $\mathcal{C}_n(\widetilde{C})$ the subset of $\mathcal{C}_n$ consisting of the basic intervals disjoint from $\widetilde{C}$.

Since $2\omega > L$, the definition of $\mathcal{X}_{\widetilde{C}}$ implies that, for any point $x \in \mathcal{X}_{\widetilde{C}}$, there is a choice of $m_{\widetilde{C}} + 1$ basic intervals $(D_0, \dots, D_{m_{\widetilde{C}}})$ all belonging to $\mathcal{C}_n(\widetilde{C})$, such that $x \in D_0 \cap T^{-2\omega n} D_1 \cap \cdots \cap T^{-2m_{\widetilde{C}}\omega n} D_{m_{\widetilde{C}}}$. From this, we deduce that

$$\mu_\phi(\mathcal{X}_{\widetilde{C}}) \le \sum_{(D_0,\dots,D_{m_{\widetilde{C}}})\in(\mathcal{C}_n(\widetilde{C}))^n} \mu_\phi(D_0 \cap T^{-2\omega n} D_1 \cap \cdots \cap T^{-2m_{\widetilde{C}}\omega n} D_{m_{\widetilde{C}}}).$$

We choose ω large enough that Lemma 4.2 can be applied. Inequality (29) yields

$$\begin{aligned}\mu_\phi(\mathcal{X}_{\widetilde{C}}) &\le \gamma^3(1 + M\beta^{\omega n})^{m_{\widetilde{C}}} \sum_{(D_0,\dots,D_{m_{\widetilde{C}}})\in(\mathcal{C}_n(\widetilde{C}))^n} \prod_{j=0}^{m_{\widetilde{C}}} \mu_\phi(T^{-2j\omega n} D_i)\\ &= \gamma^3(1 + M\beta^{\omega n})^{m_{\widetilde{C}}} \Big(\sum_{D\in\mathcal{C}_n(\widetilde{C})} \mu_\phi(D)\Big)^{m_{\widetilde{C}}+1}.\end{aligned}$$

Since the intervals of $\mathcal{C}_n$ have disjoint interiors, $\sum_{D\in\mathcal{C}_n(\widetilde{C})} \mu_\phi(D) \le 1 - \mu_\phi(\widetilde{C})$. Hence

$$\begin{aligned}\mu_\phi(\mathcal{X}_{\widetilde{C}}) &\le \gamma^3(1 + M\beta^{\omega n})^{m_{\widetilde{C}}}(1 - \mu_\phi(\widetilde{C}))^{m_{\widetilde{C}}+1}\\ &\le \frac{\gamma^3}{1 + M\beta^{\omega n}}((1 + M\beta^{\omega n})(1 - \mu_\phi(\widetilde{C})))^{m_{\widetilde{C}}+1}.\end{aligned}$$

If ω is chosen large enough,

$$(1 + M\beta^{\omega n})(1 - \mu_\phi(\widetilde{C})) \le 1 - \mu_\phi(\widetilde{C})/2. \tag{35}$$

Thus we finally obtain

$$\mu_\phi(\mathcal{X}_{\widetilde{C}}) \le \frac{\gamma^3}{1 + M\beta^{\omega n}}(1 - \mu_\phi(\widetilde{C})/2)^{m_{\widetilde{C}}+1}. \tag{36}$$

Here, we emphasize that ω can be chosen large enough that (35), and thus (36), can be realized simultaneously for all $\widetilde{C}$ and for all n. In fact, from the Gibbs property (15) of μ_ϕ, there exists a maximal exponent $H > 0$ such that

$$\text{for every basic interval } \widetilde{C} \text{ of any generation } n, \quad \mu_\phi(\widetilde{C}) \ge |\widetilde{C}|^H \ge L^{-H} 2^{-nH}.$$

Thus, for all $\widetilde{C} \in \mathcal{C}_n$,

$$\frac{1 - \frac{1}{2}\mu_\phi(\widetilde{C})}{1 - \mu_\phi(\widetilde{C})} \ge \frac{1 - \frac{1}{2}L^{-H}2^{-nH}}{1 - L^{-H}2^{-nH}}.$$

Hence we can choose ω so that

$$1 + M\beta^{\omega n} \le \frac{1 - \frac{1}{2}L^{-H}2^{-nH}}{1 - L^{-H}2^{-nH}} \le \frac{1 - \frac{1}{2}\mu_\phi(\widetilde{C})}{1 - \mu_\phi(\widetilde{C})},$$

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which implies (35) for all $\widetilde{C}$. Summing over all $\widetilde{C} \in \{C_1, \dots, C_N\}$, by (36) and the definition of $m_{\widetilde{C}}$, we have

$$\begin{aligned}\mu_\phi(\mathcal{C}_{n,N,h}) &\le \frac{\gamma^3}{1+M\beta^{\omega n}} \sum_{\widetilde{C}} (1-\mu_\phi(\widetilde{C})/2)^{|\widetilde{C}|^{-h}/(2\omega n)} \\ &\le \frac{\gamma^3}{1+M\beta^{\omega n}} \sum_{\widetilde{C}} (1-\mu_\phi(\widetilde{C})/2)^{|\widetilde{C}|^{-\varepsilon}/(2\omega n \mu_\phi(\widetilde{C}))} \\ &= \frac{\gamma^3}{1+M\beta^{\omega n}} \sum_{\widetilde{C}} \exp\left(\frac{|\widetilde{C}|^{-\varepsilon}}{2\omega n \mu_\phi(\widetilde{C})} \log\left(1-\mu_\phi(\widetilde{C})/2\right)\right) \\ &\le \frac{\gamma^3}{1+M\beta^{\omega n}} \sum_{\widetilde{C}} \exp\left(\frac{-|\widetilde{C}|^{-\varepsilon}}{4\omega n}\right).\end{aligned}$$

Now, recalling (10), we have $|\widetilde{C}|^{-\varepsilon} \ge L^{-\varepsilon} 2^{\varepsilon n}$. Since the number N of possible choices for $\widetilde{C}$ is less than $L \cdot 2^n$, we have

$$\mu_\phi(\mathcal{C}_{n,N,h}) \le \frac{\gamma^3}{1+M\beta^{\omega n}} \sum_{\widetilde{C}} \exp\left(\frac{-2^{\varepsilon n}}{4\omega n L^\varepsilon}\right) \le \frac{L\gamma^3}{1+M\beta^{\omega n}} 2^{(1/\log 2)(n \log 2 - (2^{\varepsilon n}/4\omega n L^\varepsilon))}.$$

This last term is independent of N and less than 2^{-n} for sufficiently large n. □

4.3. *Small hitting probability lemma.* We now study the probability of hitting points with high local dimension for μ_ϕ. The arguments are close to those of [**18**].

LEMMA 4.4. *Let $0 < a < 1$, $0 < c < b < 1$ and $\eta > b - c$. Consider 2^{bn} different basic intervals $C_1, \dots, C_{2^{bn}}$ in $\mathcal{C}_n$. Assume that for every $j \in \{1, \dots, 2^{bn}\}$,*

$$\mu_\phi(C_j) \le 2^{-(a+\eta)n}.$$

Set

$$\mathcal{X}_{a,b,c} := \{x : \tau(x, C_i) \le 2^{an} \text{ for } 2^{cn} \text{ distinct intervals among the } \{C_i\}_{i=1,\dots,2^{bn}}\}.$$

Then there exists an integer $n_{a,b,c} \in \mathbb{N}$ such that as soon as $n \ge n_{a,b,c}$,

$$\mu_\phi(\mathcal{X}_{a,b,c}) \le 2^{-n}.$$

Proof. Let us write $K := 2^{an}$, $P := 2^{bn}$, $N := 2^{cn}$. When $x \in \mathcal{X}_{a,b,c}$, there exist N integers $0 < \ell_1 < \ell_2 < \cdots < \ell_N \le K$ and N different basic intervals $C_{i_1}, C_{i_2}, \dots, C_{i_N}$ such that

$$T^{\ell_1} x \in C_{i_1}, \quad T^{\ell_2} x \in C_{i_2}, \dots, T^{\ell_N} x \in C_{i_N}. \tag{37}$$

Let $N' := \lfloor N/(2\omega n) \rfloor$ and let $(t_p)_{p=1}^{N'}$ be a subset of $(\ell_j)_{j\in\{1,\dots,N\}}$ defined by $t_p = \ell_{2\omega n p}$. Let j_p be the unique index i such that $T^{t_p} x \in C_i$ in (37). If $x \in \mathcal{X}_{a,b,c}$,

$$T^{t_1} x \in C_{j_1}, \quad T^{t_2} x \in C_{j_2}, \dots, T^{t_{N'}} x \in C_{j_{N'}}, \tag{38}$$

where $C_{j_1}, \dots, C_{j_{N'}}$ are N' different basic intervals among the intervals $C_1, \dots, C_P$.

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Now fix N' basic intervals $C_{j_1}, \dots, C_{j_{N'}}$ among the intervals $C_1, \dots, C_P$ and fix also the integers $t_1 < \cdots < t_{N'} \le K$. Consider the set $\widetilde{\mathcal{X}}$ of points x such that (38) is satisfied. This set $\widetilde{\mathcal{X}}$ depends on a, b, c, and on the intervals and the integers we have chosen. As said above, $\mathcal{X}_{a,b,c} \subset \bigcup \widetilde{\mathcal{X}}$, where the union is taken over all possible choices of parameters $C_{j_1}, \dots, C_{j_{N'}}$ and $t_1 < \cdots < t_{N'}$. In order to bound from above the μ_ϕ-measure of $\mathcal{X}_{a,b,c}$, we will first study the μ_ϕ-measure of one set $\widetilde{\mathcal{X}}$. Applying (31) again and using the same arguments as in Lemma 4.3, we see that the μ_ϕ-measure of $\widetilde{\mathcal{X}}$ is bounded from above by

$$\mu_\phi(\widetilde{\mathcal{X}}) \le \max_{1\le i\le L} \mu_\phi(C_i)^{N'}(1 + M\beta^{2\omega n})^{N'}. \tag{39}$$

It remains for us to estimate the maximal number of choices for the associated intervals $C_{j_1}, \dots, C_{j_{N'}}$ and integers $(t_1, \dots, t_{N'})$. We have $\binom{P}{N'}$ possible choices for the N' different basic intervals among the list of P intervals $C_1, \dots, C_P$, and there are at most $\binom{K}{N'}$ choices for the integers $t_1 < t_2 < \cdots < t_{N'} < K$. Finally there are $N'!$ ways to arrange the N' intervals. Combining this and (39), we find

$$\mu_\phi(\mathcal{X}_{a,b,c}) \le \sum_{\widetilde{\mathcal{X}}} \mu_\phi(\widetilde{\mathcal{X}}) \le \binom{P}{N'}\binom{K}{N'} \cdot N'! \cdot \max_{C_i} \mu_\phi(C_i)^{N'} \cdot (1 + M\beta^{\omega n})^{N'}.$$

Since

$$\binom{P}{N'}\binom{K}{N'} \cdot N'! = \frac{P!}{(P-N')!} \cdot \frac{K!}{(K-N')!} \cdot \frac{1}{N'!},$$

using the estimates

$$\frac{P!}{(P-N')!} \le P^{N'}, \quad \frac{K!}{(K-N')!} \le K^{N'}, \quad \frac{1}{N'!} \le \xi \cdot \frac{e^{N'}}{N'^{N'}}$$

for some universal constant ξ, we conclude that

$$\mu_\phi(\mathcal{X}_{a,b,c}) \le \xi \cdot P^{N'} \cdot K^{N'} \cdot e^{N'} \cdot N'^{-N'} \cdot (\max_{C_i} \mu_\phi(C_i))^{N'} \cdot (1 + M\beta^{\omega n})^{N'}.$$

Replacing all constants K, P, N by their values, we get

$$\mu_\phi(\mathcal{X}_{a,b,c}) \le \xi \cdot (2^{bn} \cdot 2^{an} \cdot e \cdot (N')^{-1} \cdot 2^{-(a+\eta)n} \cdot (1 + M\beta^{\omega n}))^{N'}.$$

By the definition of N', we have $(N')^{-1} \le 2\omega n/N = 2\omega n 2^{-cn}$ when ω is large enough. Consequently, the last inequality yields

$$\mu_\phi(\mathcal{X}_{a,b,c}) \le \xi \cdot (e \cdot 2\omega n \cdot (1 + M\beta^{\omega n}) \cdot 2^{(b-c-\eta)n})^{N'}.$$

By assumption, $\eta > b - c$, so the quantity between brackets tends to 0 exponentially fast. In particular, it is less than $1/2$. Using the fact that $N' \ge 2^{cn}/2\omega n$, we get

$$\mu_\phi(\mathcal{X}_{a,b,c}) \le \xi \cdot 2^{-(2^{cn}/2\omega n)}.$$

The right-hand term in the above inequality is less than 2^{-n} when n becomes large. □

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5. *Part I of the spectrum:* $1/\delta < \alpha(1) = \dim \mu_\phi$

5.1. *Upper bound for* $\dim \mathcal{L}^\delta(x)$. By (21), we need only to show the following.

PROPOSITION 5.1. *For every* $0 < s \leq \dim \mu_\phi$, *for every* $x \in [0, 1]$, *we have*

$$\dim(\mathcal{R}_{\leq s}(x)) \leq s. \tag{40}$$

Proof. Notice that in the definition of $R(x, y)$, one can replace the limit process of $r \to 0$ by the sequence 2^{-n} with $n \to \infty$. Then, for $x \in [0, 1]$ and any $a > s$,

$$\mathcal{R}_{\leq s}(x) \subset \limsup_{n\to\infty} \{y : \tau_{2^{-n}}(x, y) \leq 2^{an}\}.$$

In other words, given $y \in \mathcal{R}_{\leq s}(x)$, there exists an integer $1 \leq k_n \leq 2^{an}$ such that $y \in B(T^{k_n}x, 2^{-n})$, for infinitely many n. Assume that the sequence of integers (k_n) tends to infinity. Using the fact that $2^{-n} \leq (k_n)^{-1/a}$ for such a couple of integers (k_n, n), we have $y \in B(T^{k_n}x, (k_n)^{-1/a})$ for infinitely many integers k_n. Hence

$$\mathcal{R}_{\leq s}(x) \subset \limsup_{n\to\infty} B(T^k x, k^{-1/a}).$$

For each integer n, we deduce that the set of balls $\{B(T^k x, k^{-1/a})\}_{k\geq n}$ forms a covering of $\mathcal{R}_{\leq s}(x)$ by intervals of length smaller than $n^{-1/a}$. Let $\mathcal{H}^a_\varepsilon$ stand for the a-Hausdorff measure obtained by using coverings by balls of size less than ε. Using $\{B(T^k x, k^{-1/a})\}_{k\geq n}$ as a covering, we see that, for any $a' > a$,

$$\mathcal{H}^{a'}_{n^{-1/a}}(\mathcal{R}_{\leq s}(x)) \leq \sum_{k\geq n} |B(T^k x, k^{-1/a})|^{a'} \leq 2^{a'/a} \sum_{k\geq n} k^{-a'/a} \leq \xi' n^{1-a'/a},$$

which tends to 0 when n tends to infinity. Here ξ' is a universal constant. We deduce that the a'-Hausdorff measure of $\mathcal{R}_{\leq s}(x)$ is necessarily 0. Thus $\dim \mathcal{R}_{\leq s}(x) \leq a'$. Since this holds for any $a' > a$, and then for any $a > s$, (40) follows. □

5.2. *Lower bound for* $\dim \mathcal{L}^\delta(x)$. Let $(x_n)_{n\geq 1}$ be a sequence in $[0, 1]$, and let $(l_n)_{n\geq 1}$ be a positive decreasing sequence. Consider the limsup sets of the form

$$\mathcal{L}_\zeta := \bigcap_{N\geq 1} \bigcup_{n\geq N} B(x_n, (l_n)^\zeta).$$

Provided that $\mu_\phi(\mathcal{L}_{\zeta_0}) = 1$ for some $\zeta_0 > 0$, the dimensions of $\mathcal{L}_\zeta$ can be bounded from below using the heterogeneous ubiquity theorems developed in [**3**]. To apply such theorems, some assumptions need to be checked for μ_ϕ. We refer to [**3**, Definition 2] for the precise description of these assumptions. We explain now why these assumptions are fulfilled in our framework. From the work of Baladi [**1**, Theorem 1.11(2), Theorem 7.1] and of Philipp and Stout [**30**], we deduce the following properties for μ_ϕ.

THEOREM 5.2. *Assume that the potential* ϕ *associated with* μ_ϕ *is Hölderian. There exists a non-decreasing continuous function* χ *defined on* $\mathbb{R}_+$ *with the following properties.*

- $\chi(0) = 0$, *and* $r \mapsto r^{-\chi(r)}$ *is non-increasing near* 0^+.
- $\lim_{r\to 0^+} r^{-\chi(r)} = +\infty$, *and, for all* $\varepsilon > 0$, $r \mapsto r^{\varepsilon-\chi(r)}$ *is non-decreasing near 0.*

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Furthermore, the function χ is such that for μ_ϕ-almost every $y \in [0,1]^d$, there exists $r(y) > 0$, such that

$$\text{for all } 0 < r \le r(y), \quad r^{\dim \mu_\phi + \chi(r)} \le \mu_\phi(B(y,r)) \le r^{\dim \mu_\phi - \chi(r)}. \tag{41}$$

Property (41) shall be viewed as an illustration of the iterated logarithm law for invariant measures. By the theorems of [**1, 30**], one can take χ equal to

$$\chi(0) = 0 \quad \text{and} \quad \chi : r \mapsto \left(\frac{\log\log|\log(r)|}{|\log r|} \right)^{1/2} \text{ if } r > 0. \tag{42}$$

In the previous section, we proved the following: for any δ such that $1/\delta > \dim \mu_\phi = \alpha(1)$, for μ_ϕ-almost every $x \in [0,1]$, $\mu_\phi(\mathcal{L}^\delta(x)) = 1$. Theorem 5.2 and the quasi-Bernoulli property of μ_ϕ and μ_q imply that the conditions of [**3**, Definition 2] are fulfilled for μ_ϕ-almost every $x \in [0,1]$. We can then apply the heterogeneous ubiquity [**3**, Theorem 4] to get the following lower bound.

THEOREM 5.3. *For any δ such that $1/\delta > \dim \mu_\phi$, for μ_ϕ-almost every $x \in [0,1]$, we have,*

$$\text{for every } \zeta > 1, \quad \dim(\mathcal{L}^{\zeta\cdot\delta}(x)) \ge (\dim \mu_\phi)/\zeta. \tag{43}$$

By considering an increasing countable sequence (δ_n) tending to $\delta_0 = 1/\dim \mu_\phi$ and applying Theorem 5.3 to each δ_n, we get immediately the following corollary.

COROLLARY 5.4. *For μ_ϕ-almost every x, for every $\zeta > 1$,*

$$\dim(\mathcal{L}^{\zeta\cdot\delta_0}(x)) \ge (\dim \mu_\phi)/\zeta = 1/(\zeta\cdot\delta_0).$$

In other words, for every δ such that $1/\delta < \dim \mu_\phi$, we have the lower bound

$$\dim(\mathcal{L}^\delta(x)) \ge 1/\delta.$$

6. *Part IV of the spectrum: $1/\delta > \alpha_+$*

PROPOSITION 6.1. *Let $s \ge 0$. For μ_ϕ-almost every x,*

$$\mathcal{R}_{\ge s}(x) \subset \mathcal{E}_{\ge s}. \tag{44}$$

Moreover, for any Gibbs measure μ_ψ on $[0,1]$,

$$\text{for } \mu_\phi\text{-almost every } x \in [0,1], \quad \mathcal{R}_{\ge s}(x) \overset{\mu_\psi}{=} \mathcal{E}_{\ge s},$$

where the equality means that the two sets differ by a set of μ_ψ-measure zero.

Remark 6.2. The full μ_ϕ-measure set satisfying the first assertion of Proposition 6.1 depends on s.

Proof. The case $s = 0$ is obvious. We therefore assume that $s > 0$. For any integer $n \ge 1$, let $I_n(y)$ be the basic interval in $\mathcal{C}_n$ containing y. Observe that *a priori* the generation of $I_n(y)$ is *not* n. For any real number $\varepsilon > 0$, we introduce the sets

$$\mathcal{R}_{n,s,\varepsilon}(x) = \{y : \tau(x, I_n(y)) \ge |I_n(y)|^{s-\varepsilon}\}, \quad \mathcal{E}_{n,s,\varepsilon} = \{y : \mu_\phi(I_n(y)) \le |I_n(y)|^{s-2\varepsilon}\}.$$

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By the definition of $R(x, y)$ and $\underline{d}_{\mu_\phi}(y)$, we have

$$\mathcal{R}_{\geq s}(x) = \bigcap_{\varepsilon>0} \liminf_{n\to\infty} \mathcal{R}_{n,s,\varepsilon}(x) \quad \text{and} \quad \mathcal{E}_{\geq s} = \bigcap_{\varepsilon>0} \liminf_{n\to\infty} \mathcal{E}_{n,s,\varepsilon}.$$

In order to prove (44), it is sufficient to prove that, for μ_ϕ-almost every x, there exists some integer $n(x)$ such that

$$\text{for all } n \geq n(x), \quad \mathcal{R}_{n,s,\varepsilon}(x) \subset \mathcal{E}_{n,s,\varepsilon}. \tag{45}$$

Notice that $\mathcal{E}^c_{n,s,\varepsilon}$ is the union of basic intervals C in $\mathcal{C}_n$ such that $\mu_\phi(C) > |C|^{s-2\varepsilon}$. Let $\mathcal{D}_{n,s,\varepsilon} := \{C_1, \ldots, C_N\}$ be the set of these basic intervals. Applying Lemma 4.3 to the basic intervals $\mathcal{D}_{n,s,\varepsilon}$ and to $h = s - \varepsilon$, we see that, for n larger than some $n_{s,\varepsilon}$,

$$P_n := \mu_\phi(\{x : \exists C \in \mathcal{D}_{n,s,\varepsilon} \text{ such that } \tau(x, C) \geq |C|^{s-\varepsilon}\}) \leq 2^{-n}.$$

The sum over $n \geq n_{s,\varepsilon}$ of the P_n is finite. Applying the Borel–Cantelli lemma, the following holds for μ_ϕ-almost every x: there exists an integer $n(x)$ such that

$$\text{for all } n \geq n(x), \text{ for all } C \in \mathcal{D}_{n,s,\varepsilon}, \quad \tau(x, C) < |C|^{s-\varepsilon}.$$

Hence, if $n \geq n(x)$, then every $C \in \mathcal{D}_{n,s,\varepsilon}$ is included in $\mathcal{R}_{n,s,\varepsilon}(x)^c$. This yields the fact that $\mathcal{E}^c_{n,s,\varepsilon} \subset \mathcal{R}_{n,s,\varepsilon}(x)^c$, which is clearly equivalent to (45). Then the first assertion (44) of Proposition 6.1 follows.

To prove the second assertion, using the ergodicity of μ_ψ, it suffices to show that

$$\text{for } \mu_\phi\text{-almost every } x, \quad \mu_\psi(\{y \in [0, 1] : \underline{d}_{\mu_\phi}(y) \geq s \text{ and } R(x, y) < s\}) = 0.$$

This last statement is directly deduced from Corollary 3.7. □

We are now ready to prove some of the statements of Theorem 1.3.

Proof. (Part IV of the spectrum: Theorem 1.3(4)) By Remark 2.6 and Proposition 6.1, for each $s > \alpha_+$ for μ_ϕ-almost every $x \in [0, 1]$, we have

$$\mathcal{R}_{\geq s}(x) = \{y \in [0, 1] : R(x, y) \geq s\} = \emptyset,$$

i.e. there is no point with hitting times larger than $s > \alpha_+$. Then, applying formula (22) and Lemma 3.4, we deduce that when $1/\delta > \alpha_+$, for μ_ϕ-almost every $x \in [0, 1]$, $\mathcal{F}^\delta(x) = \emptyset$ and thus $\mathcal{L}^\delta(x) = [0, 1]$. However, as mentioned in Remark 6.2, the full μ_ϕ-measure set depends on δ. To solve this problem, i.e. to get $\mathcal{F}^\delta(x) = \emptyset$ for every δ satisfying $1/\delta > \alpha_+$, we take a sequence $(\delta_n)_{n\geq 1}$ such that $(1/\delta_n)$ is dense in $]\alpha_+, \infty[$. By taking the intersection of countable full μ_ϕ-measure sets, we obtain the fact that for μ_ϕ-almost every $x \in [0, 1]$, for all n, $\mathcal{F}^{\delta_n}(x) = \emptyset$ and $\mathcal{L}^{\delta_n}(x) = [0, 1]$. Finally, the case of an arbitrary δ such that $1/\delta > \alpha_+$ is obtained by using the monotonicity of the sets $\mathcal{F}^\delta(x)$ and $\mathcal{L}^\delta(x)$ with respect to δ. □

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7. *Part III of the spectrum:* $\alpha_{\max} < 1/\delta \le \alpha_+$

In this short section, we gather the previous results to obtain Part III of the spectrum and item (3) of Theorem 1.3. We adopt the notations of §4. Let δ be such that $\alpha_{\max} < 1/\delta \le \alpha_+$, and consider the unique real number $q(1/\delta)$. Then the associated Gibbs measure $\mu_{q(1/\delta)}$ is supported on the level set $\mathcal{E}_{\mu_\phi}(1/\delta)$, which has Hausdorff dimension $D_{\mu_\phi}(1/\delta)$.

Further, we apply the second part of Proposition 6.1 to the measure $\mu_\psi = \mu_{q(1/\delta)}$. Then for μ_ϕ-almost every x, the measure $\mu_{q(1/\delta)}$ is also supported on the set $\mathcal{R}_{\ge 1/\delta}(x)$. In particular, we conclude that $\dim \mathcal{R}_{\ge 1/\delta}(x) \ge \dim \mu_{q(1/\delta)}$.

Now, consider a countable sequence $(\delta_n)_{n\ge 1}$ such that $1/\delta_n$ is dense in the interval $[\alpha_{\max}, \alpha_+]$. The above argument applies to each δ_n. Taking a countable intersection of full μ_ϕ-measure sets, we find a set of full μ_ϕ-measure of points x such that for all $n \ge 1$, $\mu_{q(1/\delta_n)}$ is also supported on the set $\mathcal{R}_{\ge 1/\delta_n}(x)$.

Let us fix δ_0 such that $\alpha_{\max} < 1/\delta_0 \le \alpha_+$, and consider a subsequence $(\delta_{\varphi(n)})_{n\ge 1}$ decreasing to δ_0. By (22), for every integer n,

$$\dim(\mathcal{F}^{\delta_0}) \ge \dim \mathcal{R}_{\ge 1/\delta_{\varphi(n)}}(x) \ge \dim(\mu_{q(1/\delta_{\varphi(n)})}) = D_{\mu_\phi}(1/\delta_{\varphi(n)}).$$

Using the continuity of D_{μ_ϕ} on its support, we see that for μ_ϕ-almost every $x \in [0, 1]$,

$$\dim(\mathcal{F}^{\delta_0}) \ge \dim(\mu_{q(1/\delta_0)}) = D_{\mu_\phi}(1/\delta_0).$$

Conversely, by choosing an increasing subsequence $(\delta_{\varphi(n)})_{n\ge 1}$ converging to δ_0, by (22) and Proposition 6.1, we have for μ_ϕ-almost every x,

$$\dim(\mathcal{F}^{\delta_0}) \le \inf_n \dim(\mathcal{E}_{\ge 1/\delta_{\varphi(n)}}) = \inf_n D_{\mu_\phi}(1/\delta_{\varphi(n)}) = D_{\mu_\phi}(1/\delta_0).$$

This completes the proofs for the Part III and for item (3) of Theorem 1.3.

8. *Part II of the spectrum:* $\dim \mu_\phi < 1/\delta \le \alpha_{\max}$

PROPOSITION 8.1. *If* $\dim \mu_\phi < s < \alpha_{\max}$ *then for* μ_ϕ*-almost every* x *we have*

$$\dim \mathcal{R}_{\le s}(x) \ge D_{\mu_\phi}(s). \tag{46}$$

Proof. For $\dim \mu_\phi < s < \alpha_{\max}$, there exists a real number $q_s > 0$ such that

$$\frac{\int(-\phi)\, d\mu_{q_s}}{\int \log|T'|\, d\mu_{q_s}} = s.$$

By the Gibbs property of μ_ϕ and the ergodicity of μ_{q_s}, the measure μ_{q_s} is supported on $\mathcal{E}_{\mu_\phi}(s)$. Then by Corollary 3.8 applied to μ_ϕ and μ_{q_s}, for μ_ϕ-almost every x we have

$$\dim\ \mathcal{R}_{\le s}(x) \ge \dim \mu_{q_s} = D_{\mu_\phi}(s). \qquad \square$$

We finish by bounding from above the spectrum $\dim \mathcal{L}^\delta(x)$.

PROPOSITION 8.2. *If* $\dim \mu_\phi < s < \alpha_{max}$, *then for* μ_ϕ*-almost every* x *we have*

$$\dim \mathcal{R}_{\le s}(x) \le D_{\mu_\phi}(s). \tag{47}$$

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Proof. Fix $s \in (\dim \mu_\phi, \alpha_{\max})$, and let us decompose $\mathcal{R}_{\leq s}(x)$ into

$$\mathcal{R}_{\leq s}(x) = (\mathcal{R}_{\leq s}(x) \cap \mathcal{E}_{\leq s}) \cup (\mathcal{R}_{\leq s}(x) \cap \mathcal{E}_{>s}).$$

Since s lies in the increasing part of the spectrum, by Proposition 2.8, we have the upper bound $\dim \mathcal{E}_{\leq s} \leq D_{\mu_\phi}(s)$. Thus, to obtain (47), it suffices to prove that

$$\dim(\mathcal{R}_{\leq s}(x) \cap \mathcal{E}_{>s}) \leq D_{\mu_\phi}(s).$$

Recall that $\mathcal{C}_n$ forms a covering of $[0, 1]$ by basic intervals of size $\sim 2^{-n}$, these intervals having pairwise disjoint interiors.

Let $0 < h' < h''$. We define the subsets $\mathcal{C}_n(h', h'')$ of $\mathcal{C}_n$ and $\mathcal{Y}_n(h', h'')$ of $[0, 1]$:

$$\mathcal{C}_n(h', h'') = \{C \in \mathcal{C}_n : |C|^{h''} \leq \mu_\phi(C) \leq |C|^{h'}\} \subset \mathcal{C}_n,$$
$$\mathcal{Y}_n(h', h'') = \{y \in [0, 1] : \exists C \in \mathcal{C}_n(h', h'') \text{ such that } y \in C\} \subset [0, 1].$$

LEMMA 8.3. *For every $\varepsilon > 0$, there exists an integer $n_{h',h'',\varepsilon}$ large enough so that as soon as $n \geq n_{h',h'',\varepsilon}$ holds,*

$$\text{Card } \mathcal{C}_n(h', h'') \leq 2^{n(D_{\mu_\phi}(h'')+\varepsilon)} \quad \text{if } h'' < \alpha_{\max}, \tag{48}$$
$$\text{Card } \mathcal{C}_n(h', h'') \leq 2^{n(D_{\mu_\phi}(h')+\varepsilon)} \quad \text{if } h' > \alpha_{\max}. \tag{49}$$

These properties follow again from standard large deviations properties (see [**9**]).

Let $\zeta > 0$ be a positive real number that we will soon choose in a suitable manner. Set $h'_1 = s$ and $h''_1 = s + \frac{1}{2}\zeta$. It is possible to cover the interval $[s + \frac{1}{2}\zeta, \alpha_+]$ by a finite number of open intervals $\{(h'_i, h''_i)\}_{2 \leq i \leq \ell}$ with length less than ζ. We have the inclusion

$$\mathcal{E}_{>s} \subset \bigcup_{N=1}^{+\infty} \bigcap_{n \geq N} \bigcup_{i=1}^{\ell} \mathcal{Y}_n(h'_i, h''_i). \tag{50}$$

Recall that $I_n(y)$ is the unique basic interval contained in $\mathcal{C}_n$ containing y. The embedding property (50) emphasizes that when $\underline{d}_{\mu_\phi}(y) > s$ for a point $y \in [0, 1]$ then necessarily $\mu_\phi(I_n(y)) < |I_n(y)|^s$ for *every* integer n large enough, not only for an infinite number of integers.

We introduce the subset $\mathcal{C}_{n,a}(x)$ of $\mathcal{C}_n$:

$$\mathcal{C}_{n,a}(x) := \{C \in \mathcal{C}_n : \tau(x, C) < 2^{an}\}.$$

By the definition of $\mathcal{R}_{\leq s}(x)$, for any real number $a > s$, $\mathcal{R}_{\leq s}(x) \subset \{y \in [0, 1] : R(x, y) < a\}$. The distortion property (9) guarantees that $I_n(y)$ tends to zero very regularly when n tends to infinity. Hence, if $y \in \mathcal{R}_{\leq s}(x)$, there is an infinite number of integers n such that $\tau_{2^{-n}}(x, y) \leq 2^{an}$, which means that $T^p x \in B(y, 2^{-n})$ for some $p \leq 2^{an}$.

Denote by $d(y, C)$ the distance from y to the set C. We introduce the sets

$$\widetilde{\mathcal{Y}}_{n,a}(x) = \{y \in [0, 1] : \exists C \in \mathcal{C}_{n,a}(x) \text{ such that } d(y, C) \leq 2^{-n}\}.$$

Since $T^p x \in B(y, 2^{-n})$ implies that $d(y, I_n(T^p x)) \leq 2^{-n}$, we have

$$\mathcal{R}_{\leq s}(x) \subset \bigcap_{N=1}^{+\infty} \bigcup_{n \geq N} \widetilde{\mathcal{Y}}_{n,a}(x). \tag{51}$$

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Thus combining (50) and (51), we find that $\mathcal{R}_{\le s}(x) \cap \mathcal{E}_{>s}$ is included in

$$\bigcap_{N=1}^{+\infty} \bigcup_{n\ge N} \left(\widetilde{\mathcal{Y}}_{n,a}(x) \cap \bigcup_{i=1}^{\ell} \mathcal{Y}_n(h_i', h_i'') \right) \subset \bigcup_{i=1}^{\ell} \bigcap_{N=1}^{+\infty} \bigcup_{n\ge N} (\widetilde{\mathcal{Y}}_{n,a}(x) \cap \mathcal{Y}_n(h_i', h_i'')).$$

The above inversion of $\cap$ and $\cup$ follows from the fact that there is a finite number of intervals $[h_i', h_i'']$. Thus, we need only to show that, for all $1 \le i \le \ell$,

$$\text{for all } \varepsilon > 0, \quad \dim(\limsup_{n\to\infty} (\widetilde{\mathcal{Y}}_{n,a}(x) \cap \mathcal{Y}_n(h_i', h_i''))) \le D_{\mu_\phi}(s) + \varepsilon.$$

Let $\mathcal{C}_{n,a,h_i',h_i''}(x)$ be the subset of the basic intervals of $\mathcal{C}_n$ belonging to both $\mathcal{C}_{n,a}(x)$ and $\mathcal{C}_n(h_i', h_i'')$.

LEMMA 8.4. *For every $a \in (s, \alpha_{\max})$, for every $\varepsilon > 0$, and for each $2 \le i \le \ell$,*

$$\sum_n \mu_\phi(\{x : \operatorname{Card} \mathcal{C}_{n,a,h_i',h_i''}(x) > 2^{n(D_{\mu_\phi}(a)+\varepsilon)}\}) < \infty. \tag{52}$$

Observe that in Lemma 8.4 we do not consider the first interval $[h_1', h_1'']$. For this interval, (52) follows from (48), if ζ is small enough. Indeed, if $\zeta \ll 1$, $h_1'' = s + \zeta/2$ is very close to s, so that $D_{\mu_\phi}(h_1'')$ is close to $D_{\mu_\phi}(s)$.

Let us assume for a while that Lemma 8.4 holds true. Then, the Borel–Cantelli lemma yields the fact that, for μ_ϕ-almost every x, there exists an integer $n(x)$ such that

$$\text{for } n \ge n(x), \quad \operatorname{Card} \mathcal{C}_{n,a,h_i',h_i''}(x) \le 2^{n(D_{\mu_\phi}(a)+\varepsilon)}.$$

In order to obtain a covering of the set $\limsup_{n\to\infty}(\widetilde{\mathcal{Y}}_{n,a}(x) \cap \mathcal{Y}_n(h_i', h_i''))$, by construction one considers, for any $N \ge 1$, the union

$$\bigcup_{n\ge N} \bigcup_{C\in\mathcal{C}_{n,a,h_i',h_i''}(x)} \{y \in [0,1] : d(y, C) \le 2^{-n}\}.$$

Using this family of coverings, if $N \ge n(x)$, then, for any $\varepsilon > 0$, the $(D_{\mu_\phi}(a) + 2\varepsilon)$-Hausdorff measure of the above limsup set is bounded by

$$\begin{aligned} \sum_{n\ge N} \sum_{C\in\mathcal{C}_{n,a,h_i',h_i''}(x)} ((L+2)\cdot 2^{-n})^{D_{\mu_\phi}(a)+2\varepsilon} &\le \Theta' \sum_{n\ge N} 2^{-n(D_{\mu_\phi}(a)+2\varepsilon)} \cdot 2^{n(D_{\mu_\phi}(a)+\varepsilon)} \\ &\le \Theta' \sum_{n\ge N} 2^{-n\varepsilon} < \infty, \end{aligned}$$

where Θ' is some constant depending on L, a, μ_ϕ and ε. Hence, letting N tend to infinity, we see that the $(D_{\mu_\phi}(a) + 2\varepsilon)$-Hausdorff measure of the limsup set $\limsup_{n\to\infty}(\widetilde{\mathcal{Y}}_{n,a}(x) \cap \mathcal{Y}_n(h_i', h_i''))$ is necessarily 0. This implies that

$$\dim(\limsup_{n\to\infty} (\widetilde{\mathcal{Y}}_{n,a}(x) \cap \mathcal{Y}_n(h_i', h_i''))) \le D_{\mu_\phi}(a) + 2\varepsilon.$$

We finish the proof of Proposition 8.2 by letting first $\varepsilon \downarrow 0$ and then $a \downarrow s$.

It remains for us to prove Lemma 8.4. Let $a \in (s, \alpha_{\max})$. It is enough to prove Lemma 8.4 for a close to s. Hence we suppose that $a < h_2'$.

We assume that the intervals $[h_i', h_i'']$ are chosen so that, except for at most one of them, either $h_i' > \alpha_{\max}$ or $h_i'' < \alpha_{\max}$. In other words, we suppose that there is only one integer $i \in \{2, \dots, l\}$ such that $\alpha_{\max} \in (h_i', h_i'')$. We use two key properties.

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- The multifractal spectrum D_{μ_ϕ} is real-analytic and concave on $]\alpha_-, \alpha_+[$.
- For every $h \geq a > s > \dim \mu_\phi$, the derivative of D_{μ_ϕ} at s is strictly less than 1, and the derivative $(D_{\mu_\phi})'(s)$ is decreasing. Hence, there is a real number $0 < \xi_a = (D_{\mu_\phi})'(a) < 1$ such that for every h in every interval $[h_i', h_i'']$ $(i \geq 2)$,

$$\text{for every } h \geq a, \quad (D_{\mu_\phi})'(h) \leq \xi_a.$$

We distinguish three cases.

• *Suppose* $h_i'' < \alpha_{\max}$. Take $b = D_{\mu_\phi}(h_i'') + \varepsilon$, $c = D_{\mu_\phi}(a) + \varepsilon$, and $\eta = h_i' - a$. Then on the one hand, by (48), for n large enough there are at most 2^{bn} basic intervals C in $\mathcal{C}_n(h_i', h_i'') = \mathcal{C}_n(a + \eta, h_i'')$. On the other hand, by the mean value theorem and the fact that $D_{\mu_\phi}(\cdot)$ is increasing on $(\alpha_-, \alpha_{\max})$,

$$b - c = D_{\mu_\phi}(h_i'') - D_{\mu_\phi}(a) < \xi_a(h_i'' - a) = \xi_a(h_i' - a) + \xi_a(h_i'' - h_i').$$

Since $\xi_a < 1$ and $h_i'' - h_i' < \zeta$, we can choose ζ small enough such that

$$b - c < h_i' - a = \eta.$$

This choice of ζ can be uniform, i.e. valid for every index i such that $h_i'' < \alpha_{\max}$.

By Lemma 4.4, for sufficiently large n,

$$\mu_\phi\left(\left\{x : \begin{cases} \tau(x, C) \leq 2^{an} \text{ for } 2^{cn} \text{ distinct} \\ \text{intervals } C \text{ among the } 2^{bn} \text{ intervals} \end{cases}\right\}\right) \leq 2^{-n}. \tag{53}$$

This is equivalent to saying that

$$\mu_\phi(\{x : \operatorname{Card} \mathcal{C}_{n,a,h_i',h_i''}(x) > 2^{n(D_{\mu_\phi}(a)+\varepsilon)}\}) \leq 2^{-n}.$$

Then (52) follows.

• *Suppose that* i *is the unique integer such that* $h_i' \leq \alpha_{\max} \leq h_i''$. This occurs for one and only one interval $[h_i', h_i'']$. Recall that the cardinality of $\mathcal{C}_n$ is less than $L \cdot 2^n$. Take $b = 1$, $c = D_{\mu_\phi}(a)$, and $\eta = h_i' - a$. Then

$$b - c = 1 - D_{\mu_\phi}(a) < \xi_a(\alpha_{\max} - a) = \xi_a(h_i' - a) + \xi_a(\alpha_{\max} - h_i').$$

Since $D'_{\mu_\phi}(a) < 1$ and $\alpha_{\max} - h_i' \leq h_i'' - h_i' < \zeta$, we can choose ζ small enough that

$$b - c < h_i' - a = \eta.$$

By Lemma 4.4 and applying the same arguments as above, for large n, we have

$$\mu_\phi\left(\left\{x : \begin{cases} \tau(x, C) \leq 2^{an} \text{ for } 2^{cn} \text{ distinct} \\ \text{intervals } C \text{ among } 2^{bn} \text{ intervals of } \mathcal{C}_n \end{cases}\right\}\right) \leq 2^{-n}.$$

It is not difficult to prove that we can replace 2^{bn} by $L \cdot 2^{bn}$, since constants do not infer in the proofs of Lemma 4.4. In other words,

$$\mu_\phi(\{x : \operatorname{Card} \mathcal{C}_{n,a,h_i',h_i''}(x) > 2^{nD_{\mu_\phi}(a)}\}) \leq 2^{-n},$$

and (52) is proved.

• *Suppose* $\alpha_{\max} < h_i'$. Take $b = D_{\mu_\phi}(h_i') + \varepsilon$, $c = D_{\mu_\phi}(a) + \varepsilon$, and $\eta = h_i' - a$. On the one hand, by (49), for n large enough there are at most 2^{bn} basic intervals in $\mathcal{C}_n(h_i', h_i'') = \mathcal{C}_n(a + \eta, h_i'')$. On the other hand,

$$b - c = D_{\mu_\phi}(h_i') - D_{\mu_\phi}(a) < \xi_a(h_i' - a) < (h_i' - a).$$

Thus by Lemma 4.4, for sufficiently large n, (53) follows, and (52) is proved. □

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We can now compute the value of the spectrum in Part II. Combining Propositions 8.1 and 8.2, we have that for every $s \in (\dim \mu_\phi, \alpha_{\max})$,

$$\text{for } \mu_\phi\text{-almost every } x, \quad \dim_H \mathcal{R}_{\leq s}(x) = D_{\mu_\phi}(s). \tag{54}$$

Then by Lemma 3.2, we have for every δ such that $\dim \mu_\phi < 1/\delta \leq \alpha_{\max}$,

$$\text{for } \mu_\phi\text{-almost every } x, \quad \dim_H \mathcal{L}^\delta(x) = D_{\mu_\phi}(1/\delta). \tag{55}$$

As we did in proving Part III and Part IV, by noticing the monotonicity of sets $\mathcal{L}^\delta(x)$ with respect to δ and applying (55) to a dense countable set, we can obtain that for μ_ϕ-almost every $x \in [0, 1]$ for every δ, (55) holds. This concludes the proof.

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