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SIMULTANEOUS RUIN PROBABILITY FOR TWO-DIMENSIONAL BROWNIAN RISK MODEL

KRZYSZTOF DĘBICKI,* *University of Wrocław*
ENKELEJD HASHORVA,** *University of Lausanne*
ZBIGNIEW MICHNA,*** *Wrocław University of Economics and Business*

Abstract

The ruin probability in the classical Brownian risk model can be explicitly calculated for both finite and infinite time horizon. This is not the case for the simultaneous ruin probability in the two-dimensional Brownian risk model. Relying on asymptotic theory, we derive in this contribution approximations for both simultaneous ruin probability and simultaneous ruin time for the two-dimensional Brownian risk model when the initial capital increases to infinity.

Keywords: two-dimensional Brownian risk model; Brownian motion; simultaneous ruin probability; simultaneous ruin time; ruin time approximation

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1. Introduction

The classical Brownian risk model (BRM) of an insurance portfolio,

$$R_1(t) = u + c_1 t - \sigma_1 W_1(t), \qquad t \geq 0,$$

with W_1 a standard Brownian motion, $\sigma_1 > 0$, the initial capital $u > 0$, and the premium rate $c_1 > 0$, is a key benchmark model in risk theory; see, e.g., [15] and other extensions of the model [12].

The ruin probability in the time horizon $[0, T]$ for some finite positive T is given by (see, e.g., [8])

$$\begin{aligned}\widetilde{\psi}_T(u) = \mathbb{P}\left\{\inf_{t\in[0,T]} R_1(t) < 0\right\} &= \mathbb{P}\left\{\sup_{t\in[0,T]} (\sigma_1 W_1(t) - c_1 t) > u\right\} \\ &= \Phi\left(-\frac{u}{\sigma_1\sqrt{T}} - \frac{c_1\sqrt{T}}{\sigma_1}\right) \\ &\quad + \mathrm{e}^{-2c_1 u/\sigma_1^2}\Phi\left(-\frac{u}{\sigma_1\sqrt{T}} + \frac{c_1\sqrt{T}}{\sigma_1}\right)\end{aligned} \tag{1}$$

for any $u \geq 0$, with Φ the distribution function of an $N(0,1)$ random variable.

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* Postal address: Mathematical Institute, University of Wrocław, pl. Grunwaldzki 2/4, 50-384 Wrocław, Poland. Email: Krzysztof.Debicki@math.uni.wroc.pl
** Postal address: Department of Actuarial Science, University of Lausanne, UNIL-Dorigny, 1015 Lausanne, Switzerland. Email: Enkelejd.Hashorva@unil.ch
*** Postal address: Department of Logistics, Wrocław University of Economics and Business, Komandorska 118/120, 53-345 Wrocław, Poland. Email: zbigniew.michna@ue.wroc.pl

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In the infinite time horizon, i.e. for $T=\infty$, the corresponding ruin probability for this risk model is

$$\widetilde{\psi}_{\infty}(u)=\mathbb{P}\left\{\inf_{t\geq 0} R_1(t)<0\right\}=\mathrm{e}^{-2c_1u/\sigma_1^2}.$$

Clearly, $\widetilde{\psi}_T(u)$ is smaller than $\widetilde{\psi}_{\infty}(u)$ for any value of the initial capital $u>0$. In fact, from the above formulas, for any $p\in(1,2)$,

$$\lim_{u\to\infty}\frac{(-\ln\widetilde{\psi}_{\infty}(u))^p}{-\ln\widetilde{\psi}_T(u)}=0, \tag{2}$$

which shows that the finite time horizon ruin probability is much smaller than the infinite time horizon ruin probability for large u. Since in practice an insurance company runs multiple portfolios simultaneously, it is of interest to calculate the simultaneous ruin probability for the classical benchmark BRM. In this contribution we shall consider only the two-dimensional setup, where for the second portfolio the risk process is modelled by

$$R_2(t)=v+c_2t-\sigma_2W_2(t),\qquad t\geq 0,$$

with W_2 another standard Brownian motion, v the initial capital, and $c_2>0$.

In order to construct a valid two-dimensional risk model we need to define the law of the vector-valued process $(W_1(t),W_2(t))$, $t\geq 0$. A recent contribution [9] shows that a natural limiting model for a bivariate risk model is obtained if we specify

$$(W_1(t),W_2(t))=(d_{11}B_1(t)+d_{12}B_2(t),d_{21}B_1(t)+d_{22}B_2(t)),\qquad t\geq 0, \tag{3}$$

where B_1, B_2 are two independent standard Brownian motions and the d_{ij} are given constants. Necessarily, we have that $d_{i1}^2+d_{i2}^2=1$, $i=1,2$. It follows easily that the law of (W_1,W_2) specified by (3) depends only on $\rho=d_{11}d_{21}+d_{12}d_{22}\in[-1,1]$. Hence, for simplicity, in the following we set

$$d_{11}=1,\quad d_{12}=0,\quad d_{21}=\rho\in[-1,1],\quad d_{22}=\sqrt{1-\rho^2};$$

that is, we shall consider

$$(W_1(t),W_2(t))=(B_1(t),\rho B_1(t)+\sqrt{1-\rho^2}B_2(t)),\qquad t\geq 0. \tag{4}$$

The analysis of the simultaneous ruin probability in the special case $\rho=1$ is easy and can be carried out based on the findings of [3] or [21]; it is therefore not discussed here.

The study of the simultaneous ruin probability for the d-dimensional BRM with $d>2$ requires more technical details, namely other proofs and different notation (suitable for working with vectors and matrices); it will therefore be the topic of a forthcoming project.

In the two-dimensional BRM, the probability of simultaneous ruin of both portfolios in the time period $[0,T]$ is given by

$$\begin{aligned}&\mathbb{P}\left\{\text{there exists } t\in[0,T]: R_1(t)<0, R_2(t)<0\right\}\\&\qquad=\mathbb{P}\left\{\text{there exists } t\in[0,T]: \sigma_1W_1(t)-c_1t>u, \sigma_2W_2(t)-c_2t>v\right\},\end{aligned}$$

which by self-similarity (the time-scaling property) of Brownian motion reduces to

$$\mathbb{P}\left\{\text{there exists } t\in[0,1]: W_1(t)-\frac{c_1\sqrt{T}}{\sigma_1}t>\frac{u}{\sigma_1\sqrt{T}}, W_2(t)-\frac{c_2\sqrt{T}}{\sigma_2}t>\frac{v}{\sigma_2\sqrt{T}}\right\}.$$

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Consequently, in order to simplify the presentation we shall consider in the following $T = 1$, $\sigma_1 = \sigma_2 = 1$, and define, for any $u, v \in \mathbb{R}$, the simultaneous ruin probability as

$$\psi(u, v) = \mathbb{P}\{\text{there exists } t \in [0, 1] : W_1(t) - c_1 t > u,\ W_2(t) - c_2 t > v\}.$$

In the bivariate risk model, tractable expressions for the simultaneous ruin probability are not available for both finite and infinite time horizon. The main findings of this contribution concern the approximation of

$$\psi(u, au) = \mathbb{P}\{\text{there exists } t \in [0, 1] : W_1(t) - c_1 t > u,\ W_2(t) - c_2 t > au\}$$

as $u \to \infty$, for any given constant $a \in (-\infty, 1]$. Note that there is no restriction to consider only $a \leq 1$, and in our model it is possible to also deal with a, c_1, c_2 being negative. This reflects the fact that, depending on the correlations between two portfolios, the need for initial capital u and v can be different.

Clearly, the simplest possible model is when W_1 and W_2 are independent, i.e. $\rho = 0$. Even in that model, it is not possible to calculate $\psi(u, au)$ explicitly. Since the main tools of asymptotic theory for independent Gaussian processes are still available, the asymptotic behaviour of $\psi(u, au)$ can be established by modifying the classical approach (i.e. using the Gordon inequality, see [4, Proposition 3.6], instead of the well-known Slepian inequality, see, e.g., [19, 23]).

If $\rho \neq 0$ and $\rho \in (-1, 1)$, then the calculation of the simultaneous ruin probability is much more difficult to deal with, since there is no substitute for the Gordon inequality and the current methodology cannot cover the approximation of extremes of vector-valued dependent risk processes; see also the discussion in Section 2. In order to understand the asymptotic behaviour of the simultaneous ruin probability as the initial capital u tends to infinity, we present next a sharp bound for $\psi(u, v)$, which also gives some insights on the asymptotic approximation of the simultaneous ruin probability when u tends to infinity.

First, observe that, for any u, c_1, c_2, we have

$$\begin{aligned}\psi(u, au) &\leq \min\left(\mathbb{P}\left\{\sup_{t\in[0,1]}(W_1(t) - c_1 t) > u\right\}, \mathbb{P}\left\{\sup_{t\in[0,1]}(W_2(t) - c_2 t) > au\right\}\right)\\ &=: g(u, au).\end{aligned}$$

In view of (1), the upper bound $g(u, au)$ can be calculated explicitly. However, if $a \in (\rho, 1]$ this upper bound is too rough, as the next result shows. Throughout the following, $\mathbf{1}(\cdot)$ is the indicator function and $\rho^* = \sqrt{1 - \rho^2} \in [0, 1]$. Further, $\Psi = 1 - \Phi$ with Φ the standard normal distribution on $\mathbb{R}$.

Proposition 1.1 *For $c_1, c_2 \in \mathbb{R}$, $(u, v) \in \mathbb{R}^2 \setminus (-\infty, 0]^2$, and $\rho \in (-1, 1]$ we have*

$$\begin{aligned}\mathbb{P}\{W_1(1) > u + c_1,\ W_2(1) > v + c_2\} &\leq \psi(u, v)\\ &\leq \frac{\mathbb{P}\{W_1(1) > u + c_1,\ W_2(1) > v + c_2\}}{\mathbb{P}\{W_1(1) > \max(0, c_1),\ W_2(1) > \max(0, c_2)\}}.\end{aligned}$$

The main result of this contribution, given in Theorem 2.1 below, shows that a precise asymptotic approximation of $\psi(u, au)$ as $u \to \infty$ can be obtained by using more advanced techniques. Theorem 2.1 presents an interesting insight on the simultaneous probability of ruin given the correlation ρ that governs the risk processes R_1 and R_2. For this case, we have that if

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the proportion of initial capitals between the second and first risk processes is larger than the correlation function, i.e. $a > \rho$, then the ruin probability is much smaller.

Related results for the infinite time horizon are obtained in [2, 3, 5, 7, 16, 18, 22]. The first three papers consider the case that $\rho = 1$. In [7] the case $\rho \in (-1, 1)$ is dealt with, and [22] extends [2, 3] to the d-dimensional setup of non-degenerated risk processes of Sparre–Andersen type.

For the bivariate BRM we have that the asymptotic behaviour of the simultaneous ruin probability for a finite time horizon when $u \to \infty$ compared with the results of [7] is completely different. The leading term in the asymptotics of the finite time horizon ruin probability is $\mathrm{e}^{-q_{a,\rho}u^2/2}$ with

$$q_{a,\rho} = \frac{1-2a\rho+a^2}{1-\rho^2}\mathbf{1}(a > \rho) + \mathbf{1}(a \le \rho). \tag{5}$$

Note that if $a \in (\rho, 1)$, then $q_{a,\rho} > 1$. In the infinite time horizon, the leading term in the asymptotic of simultaneous ruin probability equals $\mathrm{e}^{-c_{a,\rho}u}$ for some positive $c_{a,\rho}$; see [5, 7]. Consequently, in the bivariate BRM the same relation as (2) holds between the simultaneous ruin probability in the finite and infinite time horizon. Moreover, the pre-factor in the latter case is given in terms of the multi-dimensional Pickands constant, whereas, as we show in the next section, in the former case the pre-factor is given in terms of a new constant (called the bivariate Piterbarg constant). The Piterbarg and Pickands constants are very different from each other; see, e.g., [6]. These facts show in particular that the results of [7] are not useful for the approximation of $\psi(u, au)$ as $u \to \infty$.

In the literature, two-dimensional risk models are mainly concerned with heavy-tailed setups; see, e.g., [10, 11, 14, 25] and the references therein. In view of the tail behaviour of the Gaussian distribution, the two-dimensional BRM corresponds to the light-tail case, which is very different from the heavy-tailed one; see [1] for some explanations and the difficulties in the light-tailed settings.

The rest of the paper is organised as follows. In the next section we give short discussions of our results and the approximation of the conditional ruin time. All the proofs are displayed in Section 3.

2. Results

Let (W_1, W_2) have representation (4) with $\rho \in (-1, 1)$, and set $W_i^*(t) = W_i(t) - c_i t$, $i = 1, 2$. For $a \in (\rho, 1]$ let the positive constant $C_{a,\rho}$ be given by

$$C_{a,\rho} = \int_{\mathbb{R}^2} \mathbb{P}\left\{\text{there exists } t \ge 0 : \begin{matrix} W_1(t) - t > x \\ W_2(t) - at > y \end{matrix}\right\} \mathrm{e}^{\lambda_1 x + \lambda_2 y}\, \mathrm{d}x\, \mathrm{d}y,$$

where

$$\lambda_1 = \frac{1-a\rho}{1-\rho^2}, \qquad \lambda_2 = \frac{a-\rho}{1-\rho^2} \tag{6}$$

are both positive. The finiteness of $C_{a,\rho}$ is not obvious, and in fact is implied by the particular choise of λ_1 and λ_2 defined above. The constant $C_{a,\rho}$ is new, appearing for the first time in our contribution, in the approximation of $\psi(u, au)$. It is a bivariate counterpart of the Piterbarg constant; see, e.g., [6] for its definition. As we shall show below, for $a \in (\rho, 1]$ the approximation that we obtain is

$$\psi(u, au) \sim C_{a,\rho} u^{-2} \varphi_\rho(u + c_1, au + c_2), \qquad u \to \infty, \tag{7}$$

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where φ_ρ stands for the joint probability density function (PDF) of $(W_1(1), W_2(1))$, and $\sim$ means the asymptotic equivalence of two functions when the argument u tends to infinity. In view of [13, Lemma 2] we also have

$$\mathbb{P}\left\{W_1^*(1) > u,\, W_2^*(1) > au\right\} \sim \frac{1}{\lambda_1\lambda_2} u^{-2}\varphi_\rho(u + c_1,\, au + c_2), \qquad u \to \infty,$$

so the claim in (7) says that $\psi(u, au)$ is asymptotically, up to the constant $C_{a,\rho}/(\lambda_1\lambda_2)$, the same as the joint survival probability $\mathbb{P}\left\{W_1^*(1) > u,\, W_2^*(1) > au\right\}$. This is expected since both W_1 and W_2 have maximal variance on [0,1] at 1. In order to derive the exact asymptotics of $\psi(u, au)$ we need to find an interval around the point of maximum of variance (sometimes called the informative interval), say $[T_u, 1]$, such that

$$\psi(u, au) \sim \mathbb{P}\left\{\text{there exists } t \in [T_u, 1] : W_1^*(t) > u,\, W_2^*(t) > au\right\},$$

where $\lim_{u\to\infty} T_u = 1$. From the analysis of the one-dimensional BRM we know that the informative interval is $[1 - \ln u/u^2, 1]$. Utilising a novel idea, we show that the informative interval can be taken to be shorter. This is achieved with the help of Lemma 3.2 in Section 3. The asymptotics of

$$\mathbb{P}\left\{\text{there exists } t \in [1 - \Delta/h^2, 1] : W_1^*(t) > u,\, W_2^*(t) > au\right\}$$

for any $\Delta > 0$ can be derived using the relation with the Brownian bridge and the self-similarity of Brownian motion, see Lemma 3.2.

The case $a \le \rho$ is slightly easier, but requires similar steps to the above to obtain a rigorous proof.

Theorem 2.1 *Let c_1, c_2 be two given constants and let $\rho \in (-1, 1)$.*

(i) If $a \in (\rho, 1]$ then (7) holds.

(ii) If $a \le \rho$ then, as $u \to \infty$,

$$\psi(u, au) \sim 2\sqrt{2\pi(1-\rho^2)}\,\Phi^*(c_1\rho - c_2)\exp\left\{\frac{(c_2 - \rho c_1)^2}{2(1-\rho^2)}\right\} u^{-1}\varphi_\rho(u + c_1,\, \rho u + c_2),$$

where $\Phi^*(c_1\rho - c_2) = 1$ if $a < \rho$ and Φ^* is the distribution function of $\sqrt{1-\rho^2}W_1(1)$ when $a = \rho$.

Remark 2.1.

(i) Piterbarg constants are completely different from Pickands constants, see, e.g., [6, 20, 23]. The multivariate Pickands constants which are different from $C_{a,\rho}$ appear in the study of simultaneous ruin in the infinite time horizon; see [7].

(ii) In view of [13, Lemma 2] and Theorem 2.1 we have, if $a \le \rho$,

$$\begin{aligned}\psi(u, au) &\sim 2\mathbb{P}\left\{W_1(1) > u + c_1,\, W_2(1) > au + c_2\right\}\\ &\sim 2\Phi^*(c_1\rho - c_2)\mathbb{P}\{W_1(1) > u + c_1\}, \qquad u \to \infty.\end{aligned}$$

(iii) By (7), Theorem 2.1, and Proposition 1.1,

$$C_{a,\rho} \le \frac{1}{\lambda_1\lambda_2\mathbb{P}\left\{W_1(1) > \max(0, c_1),\, W_2(1) > \max(0, c_2)\right\}},$$

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provided that $a \in (\rho, 1)$. Since $C_{a,\rho}$ does not depend on c_1 and c_2, letting c_1, c_2 tend to 0 and using the well-known orthant probability formula, the above implies

$$C_{a,\rho} \le \frac{4\pi}{\lambda_1\lambda_2(\pi + 2\arcsin(\rho))}.$$

Theorem 2.1 enables us to analyse the simultaneous ruin time $\tau_{\rm sim}(u)$ defined by

$$\tau_{\rm sim}(u) = \inf\{t \in [0,1] : W_1(t) - c_1 t > u,\, W_2(t) - c_2 t > au\}.$$

Denote by $\tau^*_{\rm sim}(u)$ the conditional ruin time $\tau_{\rm sim}(u) \mid \tau_{\rm sim}(u) \le 1$. Our result below shows that $u^2(1 - \tau^*_{\rm sim}(u))$ converges as $u \to \infty$ to an exponentially distributed random variable.

Theorem 2.2. *If $a \le 1$, $\rho \in (-1, 1)$, and $x \ge 0$, then with $q_{a,\rho}$ defined in (5) we have*

$$\lim_{u\to\infty} \mathbb{P}\left\{u^2(1 - \tau^*_{\rm sim}(u)) \le x\right\} = 1 - \exp\left(-q_{a,\rho}x/2\right).$$

Note that if $a > \rho$ then $q_{a,\rho} > 1$, and $q_{a,\rho} = 1$ for $a \le \rho$. Also note that, contrary to the findings of Theorem 2.2, the conditional ruin time considered in [7] is approximated by an $N(0,1)$.

3. Proofs

First recall that in our notation B_1, B_2 are two independent standard Brownian motions and (W_1, W_2) has the law given by (4) for some $\rho \in (-1, 1)$. In order to shorten the notation, in the following we set $W_i^*(t) = W_i(t) - c_i t$, $i = 1, 2$, with c_1, c_2 two given constants (not necessarily positive). As in Section 2, φ_ρ stands for the PDF of $(W_1(1), W_2(1))$ below.

3.1. Proof of Proposition 1.1

The proof of the lower bound is immediate. For the proof of the upper bound we follow the same idea as in the proof of [17, Theorem 1.1]. We shall use the standard notation for vectors, which are denoted in bold. Let $\boldsymbol{W}(t) = (W_1(t), W_2(t))$ and $\boldsymbol{c} = (c_1, c_2)$. For $\boldsymbol{u} = (u_1, u_2) \in \mathbb{R}^2 \setminus (-\infty, 0]^2$, define $\boldsymbol{B_u} = \{(x, y) \in \mathbb{R}^2 : (x = u_1 \wedge y \ge u_2) \vee (x \ge u_1 \wedge y = u_2)\}$ (where $\wedge$ is logical 'and', $\vee$ is logical 'or') as the boundary of the set $\{(x, y) \in \mathbb{R}^2 : x \ge u_1, y \ge u_2\}$, and $\tau(\boldsymbol{u}) = \inf\{t \in [0,1] : W_1(t) - c_1 t \ge u_1,\, W_2(t) - c_2 t \ge u_2\}$. Observe that

$$\begin{aligned}&\mathbb{P}\{W_1(1) - c_1 \ge u_1,\, W_2(1) - c_2 \ge u_2\}\\ &= \int_0^1 \mathbb{P}\{\tau(\boldsymbol{u}) \in \mathrm{d}t\} \int_{\boldsymbol{B_u}} \mathbb{P}\{\boldsymbol{W}(t) - \boldsymbol{c}t \in \mathrm{d}\boldsymbol{x} \mid \tau(u) = t\}\mathbb{P}\{\boldsymbol{W}(1-t) - \boldsymbol{c}(1-t) \ge \boldsymbol{u} - \boldsymbol{x}\},\end{aligned}$$

where we used the strong Markov property and the fact that $\boldsymbol{W}(1) - \boldsymbol{W}(t)$ has the same law as $\boldsymbol{W}(1-t)$ for any $t \in [0,1]$. Since for $x_1 \ge u_1$, $x_2 \ge u_2$ and any $t \in [0,1]$ we have

$$\begin{aligned}\mathbb{P}\{\boldsymbol{W}(1-t) - \boldsymbol{c}(1-t) \ge \boldsymbol{u} - \boldsymbol{x}\} &\ge \mathbb{P}\{\boldsymbol{W}(1-t) \ge \tilde{\boldsymbol{c}}(1-t)\}\\ &= \mathbb{P}\left\{\boldsymbol{W}(1) \ge \tilde{\boldsymbol{c}}\sqrt{1-t}\right\}\\ &\ge \mathbb{P}\{\boldsymbol{W}(1) \ge \tilde{\boldsymbol{c}}\},\end{aligned}$$

where $\tilde{\boldsymbol{c}} = (\max(c_1, 0), \max(c_2, 0))$, the proof is complete. □

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3.2. Proof of Theorem 2.1

In the following, let $\delta(u, T) = 1 - Tu^{-2}$ for $T, u > 0$. Before proceeding to the proof of Theorem 2.1 we present two lemmas: Lemma 3.2, which provides a sharp upper bound for

$$m(u, T) = \mathbb{P}\{\text{there exists } t \in [0, \delta(u, T)] : W_1(t) - c_1 t > u,\ W_2(t) - c_2 t > au\},$$

and Lemma 3.2, which yields precise asymptotics of

$$M(u, T) = \mathbb{P}\{\text{there exists } t \in [\delta(u, T), 1] : W_1(t) - c_1 t > u,\ W_2(t) - c_2 t > au\}$$

as $u \to \infty$.

Lemma 3.1. *For any $T > 0$, $a \in (-\infty, 1]$, and all sufficiently large u,*

$$m(u, T) \le \mathrm{e}^{-T/8} \frac{\mathbb{P}\{W_1(1) \ge u + c_1,\ W_2(1) \ge au + c_2\}}{\mathbb{P}\{W_1(1) > \max(c_1, 0),\ W_2(1) > \max(c_2, 0)\}}.$$

Proof. For notational simplicity we suppress the argument u, writing only $\delta(T)$ instead of $\delta(u, T)$ in the following. By the self-similarity of Brownian motion, combined with Proposition 1.1, for any $u > 0$,

$$\begin{aligned} m(u, T) &= \mathbb{P}\big\{\text{there exists } t \in [0, 1] : W_1(t) - c_1\delta^{1/2}(T)t > \delta^{-1/2}(T)u, \\ &\qquad\qquad W_2(t) - c_2\delta^{1/2}(T)t > \delta^{-1/2}(T)au\big\} \\ &\le \frac{\mathbb{P}\{W_1(\delta(T)) \ge u + c_1\delta(T),\ W_2(\delta(T)) \ge au + c_2\delta(T)\}}{\mathbb{P}\{W_1(1) > \max(c_1, 0),\ W_2(1) > \max(c_2, 0)\}}. \end{aligned}$$

Since, for sufficiently large u (setting $\nu = \delta^{-1/2}(T)$, $\overline{\nu} = \delta^{-1/2}(T/2)$, and recalling that both ν and $\overline{\nu}$ depend on u),

$$\nu u + c_1/\nu \ge \overline{\nu}(u + c_1), \qquad \nu au + c_2/\nu \ge \overline{\nu}(au + c_2),$$

then we have

$$\begin{aligned} &\mathbb{P}\{W_1(1) \ge \nu u + c_1/\nu,\ W_2(1) \ge \nu au + c_2/\nu\} \\ &\le \mathbb{P}\{W_1(1) \ge \overline{\nu}(u + c_1),\ W_2(1) \ge \overline{\nu}(au + c_2)\} \\ &= \overline{\nu}^2 \int_{u+c_1}^{\infty} \int_{au+c_2}^{\infty} \varphi_\rho(\overline{\nu}x, \overline{\nu}y)\,\mathrm{d}x\,\mathrm{d}y \end{aligned}$$

for sufficiently large u. Taking into account that

$$\begin{aligned} \varphi_\rho(\overline{\nu}x, \overline{\nu}y) &= \frac{1}{2\pi(1-\rho^2)} \exp\Big(-\frac{\overline{\nu}^2}{2(1-\rho^2)}(x^2 - 2\rho xy + y^2)\Big) \\ &\le \frac{1}{2\pi(1-\rho^2)} \exp\Big(-\frac{1 + Tu^{-2}/2}{2(1-\rho^2)}((\rho x - y)^2 + (1-\rho^2)x^2)\Big) \\ &\le \varphi_\rho(x, y) \exp\Big(-\frac{Tu^{-2}}{4}x^2\Big), \end{aligned}$$

we obtain, for sufficiently large u,

$$\begin{aligned} \overline{\nu}^2 \int_{u+c_1}^{\infty} \int_{au+c_2}^{\infty} \varphi_\rho(\overline{\nu}x, \overline{\nu}y)\,\mathrm{d}x\,\mathrm{d}y &\le \overline{\nu}^2 \int_{u+c_1}^{\infty} \int_{au+c_2}^{\infty} \varphi_\rho(x, y) \exp\Big(-\frac{Tu^{-2}}{4}x^2\Big)\,\mathrm{d}x\,\mathrm{d}y \\ &\le \exp\Big(-\frac{T}{8}\Big)\mathbb{P}\{W_1(1) \ge u + c_1,\ W_2(1) \ge au + c_2\}. \end{aligned}$$

This completes the proof. □

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Lemma 3.2.

(i) *For any* $a \in (\rho, 1]$ *and any* $T > 0$ *we have*

$$M(u, T) \sim u^{-2}\varphi_\rho(u + c_1, au + c_2)I(T), \qquad u \to \infty,$$

where

$$I(T) = \int_{\mathbb{R}^2} \mathbb{P}\left\{\text{there exists } t \in [0, T] : \begin{matrix} W_1(t) - t > x \\ W_2(t) - at > y \end{matrix}\right\} \mathrm{e}^{\lambda_1 x + \lambda_2 y}\, \mathrm{d}x\, \mathrm{d}y \in (0, \infty).$$

(ii) *For any* $a \le \rho$ *and* $T > 0$ *with* $\rho \in (-1, 1)$ *we have*

$$M(u, T) \sim u^{-1}\varphi_\rho(u + c_1, \rho u + c_2)I(T), \qquad u \to \infty,$$

where

$$I(T) = \int_{\mathbb{R}^2} \mathbb{P}\left\{\sup_{t\in[0,T]} (W_1(t) - t) > x\right\}\left[\mathbf{1}(a < \rho) + \mathbf{1}(y<0, a = \rho)\right] \times \exp\left\{x - \frac{y^2 - 2y(c_2 - c_1\rho)}{2(1 - \rho^2)}\right\} \mathrm{d}x\, \mathrm{d}y.$$

Proof. For any x, y and $u_x = u + c_1 - x/u$, $u_y = au + c_2 - y/u$ we have (recall that the PDF of $(W_1(1), W_2(1))$ is denoted by φ_ρ)

$$\varphi_\rho(u_x, u_y) =: \varphi_\rho(u + c_1, au + c_2)\psi_u(x, y) \sim \varphi_\rho(u + c_1, au + c_2)\mathrm{e}^{\lambda_1 x + \lambda_2 y}, \qquad u \to \infty,$$

where the λ_i are defined in (6) and $a =: b$ means that b is defined by a. Set

$$u_{x,y} = u_y - \rho u_x = (a - \rho)u - (y - \rho x)/u + c_2 - \rho c_1,$$

and let B_1, B_2 be two independent standard Brownian motions. The above convergence holds for almost all $x, y \in \mathbb{R}$. For any $u > 0$, set $\bar{t}_u = 1 - t/u^2$.

(i) Recalling that $W_i^*(t) = W_i(t) - c_i t$, we have

$$\begin{aligned} M(u, T) &= u^{-2} \int_{\mathbb{R}^2} \mathbb{P}\big\{\text{there exists } t \in [\delta(u, T), 1] : \\ &\quad W_1^*(t) > u,\, W_2^*(t) > au \,\big|\, W_1(1) = u_x,\, W_2(1) = u_y\big\}\varphi_\rho(u_x, u_y)\, \mathrm{d}x\, \mathrm{d}y \\ &= A(u) \int_{\mathbb{R}^2} \psi_u(x, y)\, \mathrm{d}x\, \mathrm{d}y \\ &\qquad \mathbb{P}\left\{\text{there exists } \{t \in [0, T] : \begin{matrix} B_1(\bar{t}_u) - c_1\bar{t}_u > u \\ \rho B_1(\bar{t}_u) + \rho^* B_2(\bar{t}_u) - c_2\bar{t}_u > au \end{matrix} \middle| \begin{matrix} B_1(1) = u_x \\ \rho^* B_2(1) = u_{x,y} \end{matrix}\right\} \\ &=: A(u) \int_{\mathbb{R}^2} h_u(T, x, y)\psi_u(x, y)\, \mathrm{d}x\, \mathrm{d}y, \end{aligned}$$

where $\rho^* = \sqrt{1 - \rho^2}$ and $A(u) = u^{-2}\varphi_\rho(u + c_1, au + c_2)$. For notational simplicity we also define

$$\begin{aligned} B_{u,1}(t) &= B_1(\bar{t}_u) \mid (B_1(1) = u_x) - \bar{t}_u u_x, && t \ge 0, \\ B_{u,2}(t) &= B_2(\bar{t}_u) \mid (\rho^* B_2(1) = u_{x,y}) - \bar{t}_u u_{x,y}/\rho^*, && t \ge 0. \end{aligned}$$

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Since $\mathrm{Cov}(B_1(1), B_1(t) - tB_1(1)) = 0$ for $t \in [0, 1]$, $B_1(1)$ is independent of $B_1(t) - tB_1(1)$ for all $t \in [0, 1]$. Hence we have the equality in distribution (abbreviated as $\stackrel{\mathrm{d}}{=}$)

$$\begin{aligned} B_1(\bar{t}_u) \,|\, (B_1(1) = u_x) &= B_1(\bar{t}_u) - \bar{t}_u B_1(1) + \bar{t}_u B_1(1) \,|\, (B_1(1) = u_x) \\ &\stackrel{\mathrm{d}}{=} \bar{t}_u u_x + B_1(\bar{t}_u) - \bar{t}_u B_1(1). \end{aligned}$$

Consequently, for any $s, t \in [0, T]$,

$$\lim_{u\to\infty} u^2 \mathrm{Var}(B_{u,1}(t) - B_{u,1}(t)) = \lim_{u\to\infty} u^2 \mathrm{Var}(B_1(t_u) - B_1(s_u)) = |t - s|\,. \tag{8}$$

Since, further,

$$\lim_{u\to\infty} u\bar{t}_u(u_x - c_1 - u) = -t - x$$

for all $t \in [0, T]$, $x \in \mathbb{R}$, applying [24, Proposition 9.7] we obtain the following weak convergence as $u \to \infty$ on the space $C([0, T])$ of continuous functions on $[0, T]$ equipped with the supremum norm:

$$u\big[B_{u,1}(t) + \bar{t}_u u_x - c_1\bar{t}_u - u\big] \to B_1(t) - t - x, \qquad t \in [0, T].$$

The above implies the weak convergence, as $u \to \infty$,

$$\begin{aligned} &u\rho^* B_{u,2}(t) + u\big[\bar{t}_u u_{x,y} - (c_2 - \rho c_1)\bar{t}_u - u(a - \rho)\big] \\ &\quad \to \rho^* B_2(t) - (a - \rho)t - (y - \rho x), \qquad t \in [0, T], \end{aligned}$$

for any $x, y \in \mathbb{R}$. Next, define, for $x, y \in \mathbb{R}$,

$$h(T, x, y) = \mathbb{P}\left\{\sup_{t\in[0,T]} \min\big(B_1(t) - t - x, B_{12}(t) - y\big) > 0\right\},$$

where $B_{12}(t) = \rho[B_1(t) - t] + \rho^*[B_2(t) - t(a - \rho)/\rho^*]$. Note that by the independence of B_1 and B_2 we have that $(B_1(t) - t, B_{12}(t))$, $t \geq 0$, has the same law as

$$(W_1(t) - t, W_2(t) - at), \qquad t \geq 0,$$

implying that $h(T, x, y) > 0$.
Observe that, for any $(x, y) \in \mathbb{R}^2$ and $\delta > 0$,

$$\begin{aligned} &h(T, x - \delta, y - \delta) - h(T, x, y) \\ &\quad = h(T, x - \delta, y - \delta) - h(T, x - \delta, y) + h(T, x - \delta, y) - h(T, x, y) \\ &\quad \leq \mathbb{P}\left\{\text{there exists } t \in [0, T] : \begin{matrix} W_1(t) - t > x - \delta \\ W_2(t) - at \in (y - \delta, y] \end{matrix}\right\} \\ &\qquad + \mathbb{P}\left\{\text{there exists } t \in [0, T] : \begin{matrix} W_1(t) - t \in (x - \delta, x] \\ W_2(t) - at > y \end{matrix}\right\} \\ &\quad \leq \mathbb{P}\left\{\sup_{t\in[0,T]} (W_2(t) - at) \in (y - \delta, y]\right\} + \mathbb{P}\left\{\sup_{t\in[0,T]} (W_1(t) - t) \in (x - \delta, x]\right\} \to 0 \end{aligned}$$

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as $\delta \to 0$, by (1). Consequently, the function

$$f(z) = \mathbb{P}\left\{ \sup_{t\in[0,T]} \min\Big(W_1(t) - t - x,\, W_2(t) - at - y\Big) > z \right\}$$

is left continuous at $z = 0$ for all $(x, y) \in \mathbb{R}^2$, since $f(z) = h(T, x + z, y + z)$. Moreover, using that $f(z)$ is a survival function, it is also right continuous at $z = 0$. Hence, $f(z)$ is continuous at $z = 0$ for any $(x, y) \in \mathbb{R}^2$.

Since B_1, B_2 are independent, it follows by the continuous mapping theorem (the functional below is continuous on the space $C([0, T])$) that

$$\begin{aligned}
&h_u(T, x, y) = \\
&\mathbb{P}\left\{\text{there exists } t \in [0, T] : \begin{array}{c} B_{u,1}(t) + \bar{t}_u u_x - c_1\bar{t}_u > u \\ \rho[B_{u,1}(t) + \bar{t}_u u_x] + \rho^*[B_{u,2}(t) + \bar{t}_u u_{x,y}/\rho^*] - c_2\bar{t}_u > au \end{array}\right\} \\
&= \mathbb{P}\Big\{ \sup_{t\in[0,T]} \min\Big(u(B_{u,1}(t) + \bar{t}_u(u_x - c_1) - u),\, u(\rho[B_{u,1}(t) + \bar{t}_u u_x] \\
&\quad + \rho^*[B_{u,2}(t) + \bar{t}_u u_{x,y}/\rho^*] - c_2\bar{t}_u - au)\Big) > 0\Big\} \\
&\to h(T, x, y), \qquad u \to \infty.
\end{aligned}$$

Hence, using the dominated convergence theorem, we have

$$\begin{aligned}
M(u, T) &= A(u) \int_{\mathbb{R}^2} h_u(T, x, y)\psi_u(x, y)\,\mathrm{d}x\,\mathrm{d}y \\
&\sim A(u) \int_{\mathbb{R}^2} h(T, x, y)\mathrm{e}^{\lambda_1 x + \lambda_2 y}\,\mathrm{d}x\,\mathrm{d}y, \qquad u \to \infty.
\end{aligned}$$

The application of the dominated convergence theorem can be justified as follows. First note that for $\boldsymbol{c} = (c_1, c_2)^\top$, $\boldsymbol{z} = (x, y)^\top$, and Σ the covariance matrix of $(W_1(1), W_2(1))$,

$$\begin{aligned}
\psi_u(x, y) &= \frac{\varphi_\rho(u_x, u_y)}{\varphi_\rho(u + c_1, au + c_2)} \\
&= \exp\left\{\lambda_1 x + \lambda_2 y + \boldsymbol{c}^\top \Sigma^{-1}\boldsymbol{z}/\boldsymbol{u} - \boldsymbol{z}^\top \Sigma^{-1}\boldsymbol{z}/(2u^2)\right\}. \qquad (9)
\end{aligned}$$

Moreover, for $x, y \geq 0$, some positive constants C, $\bar{C}$, C_1, and $\varepsilon > 0$ sufficiently small, in view of (8), the Piterbarg inequality (see, e.g., [23, Theorem 8.1]) implies that

$$\begin{aligned}
&h_u(T, x, y) = \\
&\mathbb{P}\Big\{ \sup_{t\in[0,T]} \min\Big(u(B_{u,1}(t) + \bar{t}_u(u_x - c_1) - u),\, u(\rho[B_{u,1}(t) + \bar{t}_u u_x] \\
&\quad + \rho^*[B_{u,2}(t) + \bar{t}_u u_{x,y}/\rho^*] - c_2\bar{t}_u - au)\Big) > 0\Big\} \\
&\leq \mathbb{P}\Big\{ \sup_{t\in[0,T]} \Big(u(B_{u,1}(t) + \bar{t}_u(u_x - c_1) - u) + u(\rho[B_{u,1}(t) + \bar{t}_u u_x] \\
&\quad + \rho^*[B_{u,2}(t) + \bar{t}_u u_{x,y}/\rho^*] - c_2\bar{t}_u - au)\Big) > 0\Big\}
\end{aligned}$$

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$$\le \mathbb{P}\left\{ \sup_{t\in[0,T]} \left(uB_{u,1}(t) + u\rho B_{u,1}(t) + u\rho^* B_{u,2}(t) - (t(a+1) + x + y - \epsilon) \right) > 0 \right\}$$
$$\le \mathbb{P}\left\{ \sup_{t\in[0,T]} \left(uB_{u,1}(t) + u\rho B_{u,1}(t) + u\rho^* B_{u,2}(t) \right) > x + y - C_1 \right\}$$
$$\le \bar{C}\mathrm{e}^{-C(x+y)^2} \le \bar{C}\mathrm{e}^{-C(x^2+y^2)};$$

for $x \ge 0$, $y \le 0$,

$$h_u(T, x, y) \le \mathbb{P}\left\{ \sup_{t\in[0,T]} u(B_{u,1}(t) + \bar{t}_u(u_x - c_1) - u) > 0 \right\} \le \bar{C}\mathrm{e}^{-Cx^2};$$

and for $x \le 0$, $y \ge 0$,

$$h_u(T, x, y) \le \mathbb{P}\left\{ \sup_{t\in[0,T]} u(\rho[B_{u,1}(t) + \bar{t}_u u_x] + \rho^*[B_{u,2}(t) + \bar{t}_u u_{x,y}/\rho^*] - c_2\bar{t}_u - au) > 0 \right\}$$
$$\le \bar{C}\mathrm{e}^{-Cy^2}.$$

Hence, using (9), we have

$$\int_{x\ge 0, y\ge 0} h_u(T, x, y)\psi_u(x, y)\,\mathrm{d}x\,\mathrm{d}y \le \bar{C}\int_0^\infty \mathrm{e}^{(\lambda_1+\varepsilon)x}\mathrm{e}^{-Cx^2}\,\mathrm{d}x \int_0^\infty \mathrm{e}^{(\lambda_2+\varepsilon)y}\mathrm{e}^{-Cy^2}\,\mathrm{d}y < \infty$$
$$\int_{x\ge 0, y\le 0} h_u(T, x, y)\psi_u(x, y)\,\mathrm{d}x\,\mathrm{d}y \le \bar{C}\int_{-\infty}^0 \mathrm{e}^{(\lambda_2-\varepsilon)y}\,\mathrm{d}y \int_0^\infty \mathrm{e}^{(\lambda_1+\varepsilon)x}\mathrm{e}^{-Cx^2}\,\mathrm{d}x < \infty$$
$$\int_{x\le 0, y\ge 0} h_u(T, x, y)\psi_u(x, y)\,\mathrm{d}x\,\mathrm{d}y \le \bar{C}\int_{-\infty}^0 \mathrm{e}^{(\lambda_1-\varepsilon)x}\,\mathrm{d}x \int_0^\infty \mathrm{e}^{(\lambda_2+\varepsilon)y}\mathrm{e}^{-Cy^2}\,\mathrm{d}y < \infty$$
$$\int_{x\le 0, y\le 0} h_u(T, x, y)\psi_u(x, y)\,\mathrm{d}x\,\mathrm{d}y \le \int_{-\infty}^0 \mathrm{e}^{(\lambda_1-\varepsilon)x}\,\mathrm{d}x \int_{-\infty}^0 \mathrm{e}^{(\lambda_2-\varepsilon)y}\,\mathrm{d}y < \infty,$$

which confirms the validity of the dominated convergence theorem.

(ii) Next, when $a \le \rho$ we shall apply a different transformation, namely

$$u_x = u + c_1 - x/u, \qquad u_y = \rho u + c_2 - y.$$

With this notation we have

$$u_{x,y} = u_y - \rho u_x = \rho x/u - y + c_2 - \rho c_1$$

and

$$\varphi_\rho(u_x, u_y) =: \psi_u(x, y)\exp\left\{x - \frac{y^2}{2(1-\rho^2)}\right\}$$
$$\sim \varphi_\rho(u + c_1, \rho u + c_2)\exp\left\{x - \frac{y^2 - 2y(c_2 - c_1\rho)}{2(1-\rho^2)}\right\}, \qquad u \to \infty.$$

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For any $x, y \in \mathbb{R}$ we thus have

$$\begin{aligned}\lim_{u\to\infty} h_u(T,x,y) &= \mathbb{P}\left\{\sup_{t\in[0,T]} (W_1(t)-t) > x\right\}\left[\mathbf{1}(a<\rho)+\mathbf{1}(y<0,a=\rho)\right]\\ &\quad+\mathbb{P}\left\{\sup_{t\in[0,T]} \min\left(W_1(t)-t-x, W_2(t)-at\right) > 0\right\}\mathbf{1}(y=0,a=\rho)\\ &=: h(T,x,y).\end{aligned}$$

Setting $A(u) = u^{-1}\varphi_\rho(u+c_1, \rho u + c_2)$ with similar arguments as above, we have

$$\begin{aligned}&M(u,T)\\ &= u^{-1}\int_{\mathbb{R}^2} \mathbb{P}\big\{\text{there exists } t\in[\delta(u,T),1]:\\ &\quad W_1^*(t) > u, W_2^*(t) > au \,\big|\, W_1(1)=u_x, W_2(1)=u_y\big\}\varphi_\rho(u_x,u_y)\,\mathrm{d}x\,\mathrm{d}y\\ &= u^{-1}\int_{\mathbb{R}^2} \psi_u(x,y)\exp\left\{x-\frac{y^2}{2(1-\rho^2)}\right\}\times\\ &\mathbb{P}\left\{\text{there exists } t\in[0,T]: \begin{array}{l} B_1(\bar{t}_u)-c_1\bar{t}_u > u\\ \rho B_1(\bar{t}_u)+\rho^* B_2(\bar{t}_u)-c_2\bar{t}_u > au\end{array}\middle|\begin{array}{l} B_1(1)=u_x\\ \rho^* B_2(1)=u_{x,y}\end{array}\right\}\mathrm{d}x\mathrm{d}y\\ &\sim A(u)\int_{\mathbb{R}^2} h(T,x,y)\exp\left\{x-\frac{y^2-2y(c_2-c_1\rho)}{2(1-\rho^2)}\right\}\mathrm{d}x\,\mathrm{d}y, \qquad u\to\infty.\end{aligned}$$

The application of the dominated convergence theorem is simpler in this case and is therefore omitted. □

Proof of Theorem 2.1. Recall first that we define $\delta(u,T) = 1 - Tu^{-2}$. In view of Lemma 3.2 combined with Proposition 1.1, we immediately obtain that

$$\lim_{T\to\infty}\lim_{u\to\infty} \frac{\mathbb{P}\left\{\text{there exists } t\in[0,\delta(u,T)]: W_1^*(t)>u, W_2^*(t)>au\right\}}{\psi(u,au)} = 0.$$

Hence, recalling that $M(u,T) = \mathbb{P}\left\{\text{there exists } t\in[\delta(u,T),1]: W_1^*(t)>u, W_2^*(t)>au\right\}$, and using that

$$M(u,T) \le \psi(u,au) \le \mathbb{P}\left\{\text{there exists } t\in[0,\delta(u,T)]: W_1^*(t)>u, W_2^*(t)>au\right\} + M(u,T),$$

we obtain

$$\lim_{T\to\infty}\lim_{u\to\infty}\frac{M(u,T)}{\psi(u,au)} = 1.$$

Consequently, in view of Lemma 3.2, it suffices to prove that

$$\lim_{T\to\infty} I(T) \in (0,\infty),$$

where $I(T)$ is defined in Lemma 3.2. We derive this considering separately $a\in(\rho,1]$ and $a\le\rho$.

(i) If $a\in(\rho,1]$, then

$$\begin{aligned}\lim_{T\to\infty} I(T) &= \lim_{T\to\infty}\int_{\mathbb{R}^2} h(T,x,y)\mathrm{e}^{\lambda_1 x+\lambda_2 y}\,\mathrm{d}x\,\mathrm{d}y\\ &= \int_{\mathbb{R}^2}\widetilde{h}(T,x,y)\mathrm{e}^{\lambda_1 x+\lambda_2 y}\,\mathrm{d}x\,\mathrm{d}y,\end{aligned}$$

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where the function h is as in the proof of Lemma 3.2, λ_1, λ_2 are positive constants defined in (6), and

$$\widetilde{h}(x, y) = \lim_{T\to\infty} h(T, x, y).$$

We have the following upper bound:

$$\begin{aligned}
&\int_{\mathbb{R}^2} \widetilde{h}(x, y)\mathrm{e}^{\lambda_1 x+\lambda_2 y}\,\mathrm{d}x\,\mathrm{d}y\\
&\le \sum_{i=0}^{\infty} \int_{\mathbb{R}^2} h([i, i+1], x, y)\mathrm{e}^{\lambda_1 x+\lambda_2 y}\,\mathrm{d}x\,\mathrm{d}y\\
&= \sum_{i=0}^{\infty} \int_{\mathbb{R}^2} \mathbb{P}\left\{\text{there exists } t\in[i, (i+1)] : \begin{matrix} W_1(t)-t > x \\ W_2(t)-at > y\end{matrix}\right\} \mathrm{e}^{\lambda_1 x+\lambda_2 x_2}\,\mathrm{d}x\,\mathrm{d}y\\
&\le \sum_{i=0}^{\infty} \int_{\mathbb{R}^2} \mathbb{P}\left\{\sup_{t\in[i,(i+1)]} (W_1(t)-t) > x, \sup_{t\in[i,(i+1)]} (W_2(t)-at) > y\right\} \mathrm{e}^{\lambda_1 x+\lambda_2 x_2}\,\mathrm{d}x\,\mathrm{d}y\\
&= \frac{1}{\lambda_1\lambda_2} \sum_{i=0}^{\infty} \mathbb{E}\left\{\mathrm{e}^{\lambda_1 M_i+\lambda_2 M_i^*}\right\}.
\end{aligned}$$

Using further the independence of increments of the Brownian motion, the following equality in distribution holds:

$$\begin{aligned}
(M_i, M_i^*) &= \left(\sup_{t\in[i,i+1]} (W_1(t)-t), \sup_{t\in[i,i+1]} (W_2(t)-at)\right)\\
&\stackrel{\mathrm{d}}{=} \left(\sup_{t\in[0,1]} (W_1(t)-t), \sup_{t\in[0,1]} (W_2(t)-at)\right) + \Big(V_1(i)-i, V_2(i)-ai\Big)\\
&=: (Q_1, Q_2) + \Big(V_1(i)-i, V_2(i)-ai\Big),
\end{aligned}$$

with (V_1, V_2) an independent copy of (W_1, W_2). By the definition of λ_1 and λ_2 we have $\lambda_1+\lambda_2\rho = 1$. Consequently, since for $\tilde{V}_2$ an independent copy of V_1

$$\lambda_1 V_1(i)+\lambda_2 V_2(i) \stackrel{\mathrm{d}}{=} (\lambda_1+\lambda_2\rho)V_1(i)+\lambda_2\rho^*\tilde{V}_2(i) = V_1(i)+\lambda_2\rho^*\tilde{V}_2(i),$$

we obtain

$$\begin{aligned}
\ln \mathbb{E}\left\{\mathrm{e}^{\lambda_1 M_i+\lambda_2 M_i^*}\right\} - \ln \mathbb{E}\left\{\mathrm{e}^{\lambda_1 Q_1+\lambda_2 Q_2}\right\} &= \frac{i}{2} + \frac{(a-\rho)^2}{2(1-\rho^2)} i - \frac{1-a\rho}{1-\rho^2} i - \frac{a-\rho}{1-\rho^2} ai\\
&= -\frac{i}{2(1-\rho^2)}[2+2a^2-(1-\rho^2)-(a-\rho)^2]\\
&= -i\frac{\kappa}{2},
\end{aligned}$$

where $\kappa = \frac{1-2a\rho+a^2}{1-\rho^2} > 1$. Consequently, by the monotone convergence theorem,

$$\lim_{T\to\infty} \int_{\mathbb{R}^2} h(T, x, y)\mathrm{e}^{\lambda_1 x+\lambda_2 y}\,\mathrm{d}x\,\mathrm{d}y = \int_{\mathbb{R}^2} \widetilde{h}(x, y)\mathrm{e}^{\lambda_1 x+\lambda_2 y}\,\mathrm{d}x\,\mathrm{d}y \in (0, \infty),$$

implying the claim.

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(ii) Next, suppose that $a \le \rho$. Again by Lemma 3.2, the proof that $\lim_{T\to\infty} I(T) \in (0, \infty)$ follows if we show that

$$\int_{\mathbb{R}} \mathrm{e}^x \, \mathbb{P}\left\{\sup_{t\ge 0} (W_1(t) - t) > x\right\} \mathrm{d}x < \infty.$$

In view of the fact that $\mathbb{P}\left\{\sup_{t\ge 0} (W_1(t) - t) > x\right\} = \mathrm{e}^{-2x}$ for $x \ge 0$, we have that

$$\int_{\mathbb{R}} \mathrm{e}^x \, \mathbb{P}\left\{\sup_{t\ge 0} (W_1(t) - t) > x\right\} \mathrm{d}x = 2,$$

implying that, for $a < \rho$,

$$\begin{aligned}\int_{\mathbb{R}^2} \widetilde{h}(x, y) \exp\left\{x - \frac{y^2 - 2y(c_2 - c_1\rho)}{2(1-\rho^2)}\right\} \mathrm{d}x\, \mathrm{d}y &= 2\int_{\mathbb{R}} \exp\left\{-\frac{y^2 - 2y(c_2 - c_1\rho)}{2(1-\rho^2)}\right\} \mathrm{d}y \\ &= 2\sqrt{2\pi(1-\rho^2)} \exp\left\{\frac{(c_2 - \rho c_1)^2}{2(1-\rho^2)}\right\},\end{aligned}$$

and for $a = \rho$,

$$\begin{aligned}&\int_{\mathbb{R}\times(-\infty,0)} \widetilde{h}(x, y) \exp\left\{x - \frac{y^2 - 2y(c_2 - c_1\rho)}{2(1-\rho^2)}\right\} \mathrm{d}x\, \mathrm{d}y \\ &\quad = 2\int_0^\infty \exp\left\{-\frac{y^2 - 2y(c_1\rho - c_2)}{2(1-\rho^2)}\right\} \mathrm{d}y \\ &\quad = 2\exp\left\{\frac{(c_2 - \rho c_1)^2}{2(1-\rho^2)}\right\} \int_0^\infty \exp\left\{-\frac{(y - (c_1\rho - c_2))^2}{2(1-\rho^2)}\right\} \mathrm{d}y \\ &\quad = 2\exp\left\{\frac{(c_2 - \rho c_1)^2}{2(1-\rho^2)}\right\} \sqrt{2\pi(1-\rho^2)}\Phi\left((c_1\rho - c_2)/\sqrt{1-\rho^2}\right),\end{aligned}$$

where Φ is the standard normal distribution, hence the proof follows easily. □

3.3. Proof of Theorem 2.2

We focus only on the case $c_1, c_2 \ge 0$, since other scenarios follow by similar arguments. Recalling that $W_i^*(t) = W_i(t) - c_i t$, and using that, for any $u > 0$ and $x > 0$,

$$\begin{aligned}&\mathbb{P}\left\{u^2(1 - \tau_{\mathrm{sim}}(u)) > x \mid \tau_{\mathrm{sim}}(u) \le 1\right\} \\ &\qquad = \frac{\mathbb{P}\left\{\text{there exists } t \in \left[0, 1 - \frac{x}{u^2}\right] : W_1^*(t) > u,\, W_2^*(t) > au\right\}}{\mathbb{P}\left\{\text{there exists } t \in [0, 1] : W_1^*(t) > u,\, W_2^*(t) > au\right\}},\end{aligned}$$

in conjunction with Theorem 2.1, we are left with finding the asymptotics of

$$\mathbb{P}\left\{\text{there exists } t \in \left[0, 1 - \frac{x}{u^2}\right] : W_1^*(t) > u,\, W_2^*(t) > au\right\}$$

as $u \to \infty$. By the self-similarity of W_1, W_2 we have, denoting $\bar{x}_u = 1 - x/u^2$,

$$\begin{aligned}\Pi(u, x) &:= \mathbb{P}\left\{\text{there exists } t \in [0, \bar{x}_u] : W_1^*(t) > u,\, W_2^*(t) > au\right\} \\ &= \mathbb{P}\left\{\text{there exists } t \in [0, 1] : \sqrt{\bar{x}_u} W_1(t) - c_1 t\bar{x}_u > u,\, \sqrt{\bar{x}_u} W_2(t) - c_2 t\bar{x}_u > au\right\}.\end{aligned}$$

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Consequently, for all u, x positive,

$$\Pi(u,x) \geq \psi\left(\frac{u}{\sqrt{\bar{x}_u}}, \frac{au}{\sqrt{\bar{x}_u}}\right) \tag{10}$$

and

$$\Pi(u,x) \leq \psi\left(\frac{u}{\sqrt{\bar{x}_u}} - \frac{c_1 x}{u^2}, \frac{au}{\sqrt{\bar{x}_u}} - \frac{c_2 x}{u^2}\right). \tag{11}$$

Hence, the proof follows by a direct application of Theorem 2.1 to (10) and (11). □

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References

[1] Asmussen, S., Hashorva, E., Laub, P. J. and Taimre, T. (2017). Tail asymptotics of light-tailed Weibull-like sums. *Prob. Math. Statist.* **37**, 235–256.

[2] Avram, F., Palmowski, Z. and Pistorius, M. R. (2008). A two-dimensional ruin problem on the positive quadrant. *Insurance Math. Econom.* **42**, 227–234.

[3] Avram, F., Palmowski, Z. and Pistorius, M. R. (2008). Exit problem of a two-dimensional risk process from the quadrant: Exact and asymptotic results. *Ann. Appl. Prob.* **18**, 2421–2449.

[4] Bai, L., Dębicki, K. and Liu, P. (2018). Extremes of vector-valued Gaussian processes with trend. *J. Math. Anal. Appl.* **465**, 47–74.

[5] Borovkov, K. and Palmowski, Z. (2019). The exact asymptotics for hitting probability of a remote orthant by a multivariate Lévy process: The Cramér case. In *2017 MATRIX Annals* (MATRIX Book Series **2**), eds D. Wood, J. de Gier, C. Praeger, and T. Tao. Springer, Cham.

[6] Dębicki, K. and Hashorva, E. (2017). On extremal index of max-stable stationary processes. *Prob. Math. Statist.* **37**, 299–317.

[7] Dębicki, K., Hashorva, E., Ji, L. and Rolski, T. (2018). Extremal behavior of hitting a cone by correlated Brownian motion with drift. *Stoch. Process. Appl.* **128**, 4171–4206.

[8] Dębicki, K. and Mandjes, M. (2015). *Queues and Lévy Fluctuation Theory*. Springer, New York.

[9] Delsing, G. A., Mandjes, M., Spreij, P. J. and Winands, E. M. (2018). Asymptotics and approximations of ruin probabilities for multivariate risk processes in a Markovian environment. Preprint, arXiv:1812.09069.

[10] Dombry, C. and Rabehasaina, L. (2017). High order expansions for renewal functions and applications to ruin theory. *Ann. Appl. Prob.* **27**, 2342–2382.

[11] Foss, S., Korshunov, D., Palmowski, Z. and Rolski, T. (2017). Two-dimensional ruin probability for subexponential claim size. *Prob. Math. Statist.* **37**, 319–335.

[12] Furrer, H, Michna, Z. and Weron, A. (1997). Stable Lévy motion approximation in collective risk theory. *Insurance Math. Econom.* **20**, 97–114.

[13] Hashorva, E. (2019). Approximation of some multivariate risk measures for Gaussian risks. *J. Multivar. Anal.* **169**, 330–340.

[14] Hu, Z. and Jiang, B. (2013). On joint ruin probabilities of a two-dimensional risk model with constant interest rate. *J. Appl. Prob.* **50**, 309–322.

[15] Iglehart, D. (1969). Diffusion approximations in collective risk theory. *J. Appl. Prob.* **6**, 285–292.

[16] Ji, L. and Robert, S. (2018). Ruin problem of a two-dimensional fractional Brownian motion risk process. *Stoch. Models* **34**, 73–97.

[17] Korshunov, D. and Wang, L. (2019). Tail asymptotics for Shepp-statistics of Brownian motion in R^d. *Extremes* **23**, 35–54.

[18] Lieshout, P. and Mandjes, M. (2007). Tandem Brownian queues. *Math. Meth. Operat. Res.* **66**, 275–298.

https://doi.org/10.1017/jpr.2020.14 Published online by Cambridge University Press

[19] LIFSHITS, M. (2012). *Lectures on Gaussian Processes* (Springer Briefs in Mathematics). Springer, Heidelberg.
[20] MICHNA, Z. (2017). Remarks on Pickands' theorem. *Prob. Math. Statist.* **37**, 373–393.
[21] MICHNA, Z. (2018). Ruin probabilities for two collaborating insurance companies. Preprint, arXiv:1804.06598. To appear in *Prob. Math. Statist.*
[22] PAN, Y. AND BOROVKOV, K. A. (2019). The exact asymptotics of the large deviation probabilities in the multivariate boundary crossing problem. *Adv. Appl. Prob.* **51**, 835–864.
[23] PITERBARG, V. I. (1996). *Asymptotic Methods in the Theory of Gaussian Processes and Fields* (Translations of Mathematical Monographs **148**). American Mathematical Society, Providence, RI.
[24] PITERBARG, V. I. (2015). *Twenty Lectures About Gaussian Processes*. Atlantic Financial Press, London.
[25] SAMORODNITSKY, G. AND SUN, J. (2016). Multivariate subexponential distributions and their applications. *Extremes* **19**, 171–196.

https://doi.org/10.1017/jpr.2020.14 Published online by Cambridge University Press