Math. Struct. in Comp. Science (2017), *vol.* 27, *pp.* 1521–1552. © Cambridge University Press 2016
doi:10.1017/S0960129516000165 First published online 22 June 2016

On naturally continuous non-dcpo domains

VLADIMIR SAZONOV

Department of Computer Science, University of Liverpool, Liverpool, L69 3BX, U.K.
Email: sazonov@liverpool.ac.uk

Received 31 March 2014; revised 4 December 2015

After works of Normann and the author on sequentiality (Normann 2006 *Mathematical Structures in Computer Science* **16** (2) 279–289; Normann and Sazonov 2012 *Annals of Pure and Applied Logic* **163** (5) 575–603; Sazonov 2007 *Logical Methods in Computer Science* **3** (3:7) 1–50), the necessity and possibility of a *non-dcpo* domain theory became evident. In this paper, the category of continuous dcpo domains is generalized to a category of 'naturally' continuous *non-dcpo* domains with 'naturally' continuous maps as arrows. A full subcategory of the latter, assuming a kind of bounded-completeness requirement of domains and presence of $\perp$ in each, proves to be Cartesian closed and equivalent to a subclass of Ershov's general A-spaces (Ershov 1974 *Algebra and Logics* **12** (4) 369–416). This extends a non-dcpo generalization of Scott (algebraic) domains introduced and proved to be equivalent to Ershov's general f-spaces (Ershov 1972 *Algebra and Logic* **11** (4) 367–437) in Sazonov (2007 op. cit.; 2009 *Annals of Pure and Applied Logic* **159** (3) 341–355).

The current approach to natural domains (v-domains) is different from f-spaces and A-spaces in that it has arisen in Sazonov (2007 op. cit.) in a different way from defining fully abstract models for some versions of the language PCF over Integers, whereas the Ershov's approach was not initially related with full abstraction, and non-dcpo version of f-spaces and A-spaces were originally considered in an abstract (mainly topological) style. In this paper devoted to naturally continuous natural domains (v-continuous v-domains), we also work in an abstract (mainly order-theoretic) style but with the hope to relate it in the future with the ideas of PCF over Reals by exploring and adapting the ideas in Escardó (1996 *Theoretical Computer Science* **162** (1) 79–115), Escardó et al. (2004 *Mathematical Structures in Computer Science*, **14** (6), Cambridge University Press 803–814), Marcial-Romero and Escardó (2007 *Theoretical Computer Science* **379** (1-2) 120–141), Sazonov (2007 op. cit.).

1. Introduction

1.1. *Traditional dcpo domains*

The traditional approach to denotational semantics of programming languages assumes a dcpo domain theory (Abramsky and Jung 1994; Amadio and Curien 1998; Gierz et al. 2003) intended to represent both primitive data types, such as integers, and functional (higher order) data types for program procedures. Let us briefly recall some standard concepts. Traditional domain theory assumes that a domain D is a set with a partial order $\sqsubseteq^D$ meaning an informational approximation order such that any increasing chain of elements $x_1 \sqsubseteq x_2 \sqsubseteq \ldots$ or, in general, any directed family of elements $\{x_i\}_{i\in I}$ has a least upper bound (lub) $\bigsqcup_{i\in I} x_i$ in D. That is, the domains are typically assumed to be

https://doi.org/10.1017/S0960129516000165 Published online by Cambridge University Press

directed-complete partial orders (dcpo). In the case of a domain for integers $\mathbb{N}$, the partial order $\sqsubseteq$ has nothing in common with the natural ordering $0 < 1 < 2 < \cdots$. We rather consider an extension $\mathbb{N}_\perp = \mathbb{N} \cup \{\perp\}$ of $\mathbb{N}$ by the so called *undefined element* $\perp$ intuitively meaning an infinite or non-finished computation, whereas any ordinary number $0, 1, \ldots$ can be thought as generated by a finite computation outputting this number. Then, $x \sqsubseteq y$ on the domain $\mathbb{N}_\perp$ means $x = \perp \vee x = y$, and such kind of ordering $S_\perp$ over a set S is called a *flat* one. We also consider *monotonic* functions $f \in [\mathbb{N}_\perp^k \to \mathbb{N}_\perp]$, i.e., such that $\vec{x} \sqsubseteq \vec{y} \Rightarrow f(\vec{x}) \sqsubseteq f(\vec{y})$ because this corresponds to the intuition of computability so that the more informative is the input, the more informative will be the output. Functionals $F \in [[\mathbb{N}_\perp^k \to \mathbb{N}_\perp] \to \mathbb{N}_\perp]$ (as well as higher level functionals) are also required to be monotonic and, additionally, *continuous*, i.e., preserving lub of directed families of functions f_i: $F(\bigsqcup_i f_i) = \bigsqcup_i F(f_i)$. Here, a pointwise ordering $f \sqsubseteq g \rightleftharpoons \forall \vec{x} \in \mathbb{N}_\perp^k (f(\vec{x}) \sqsubseteq g(\vec{x}))$ is assumed (and similarly at higher levels). Note that continuity essentially follows from computability of functionals. All so arising domains of continuous functionals prove to be dcpo's and satisfy other nice domain theoretic properties – are *Scott–Ershov domains* (i.e., *algebraic and bounded-complete*).

There are also some natural abstract concepts of computability of functions and higher type functionals in these domains. One such an important notion is that of *sequentially computable* functions $f(x_1, \ldots, x_k)$ for $f \in [\mathbb{N}_\perp^k \to \mathbb{N}_\perp]$, i.e., those for which there is a deterministic strategy of requesting values $x_i = ?$ of their arguments to get the value of $f(\vec{x})$. The typical example is the ordinary three place **if-then-else-fi** function. The crucial point is that if the sequence of requesting arguments is chosen wrongly and the requested value of x_i is $\perp$, i.e., undefined, then getting this value results in an infinite process thereby not allowing to finish this process for getting the value of $f(\vec{x})$. (For example, **if true then** 5 **else** $\perp$ **fi** $= 5$ despite one argument is $\perp$.) The case of sequential higher level functionals is more complicated and we omit here the details of various approaches to sequentiality such as Abramsky et al. (2000), Hyland and Ong (2000), Nickau (1996), Plotkin (1977), Sazonov (1976a), Sazonov (1976b), Sazonov (2007) and Normann and Sazonov (2012). Not all functions and functionals are sequential (even if we allow non-effectively computable strategies for their computation). For example, there are evidently no such sequential strategies for the ordinary multiplication function $x \cdot y$ on $\mathbb{N}$ extended to $\mathbb{N}_\perp$ monotonically as $0 \cdot \perp = \perp \cdot 0 \rightleftharpoons 0$ and $x \cdot \perp = \perp \cdot x = \perp$ for all $x \neq 0$. The older approaches (Plotkin 1977; Sazonov 1976a,b) consider and characterize in a reasonable way sequential higher level functionals amongst all $\sqsubseteq$-monotonic and continuous ones. This allows staying in the framework of dcpo domain theory. However, the latter class includes also non-sequential functionals.

1.2. *Non-dcpo domains*

However, in the case of *hereditarily sequential* functionals (when all non-sequential functionals are appropriately 'excluded' from the above mentioned domains) (Abramsky et al. 2000; Hyland and Ong 2000; Nickau 1996; Sazonov 2007) or in the case of *hereditarily wittingly consistent* functionals (Sazonov 2007), it is known that these are non-dcpo's (Normann 2006), resp., (Sazonov 2007), contain non-continuous in the traditional

https://doi.org/10.1017/S0960129516000165 Published online by Cambridge University Press

sense functionals and have even more domain theoretic anomalies (Normann and Sazonov 2012) – an expected conflict between old dcpo and new non-dcpo views. Before these works, the domain theoretic nature of hereditarily sequential functionals was rather unclear.

The research initiated in Sazonov (2007) presents an appropriate angle of view, a so called 'natural' non-dcpo domain theory further developed in more general, abstract terms in Sazonov (2009), showing things as quite nice despite the above mentioned anomalies. In fact, we often use 'non-dcpo' as 'in general, non-dcpo.' The context determines whether 'non-dcpo' is understood literally. It proves that hereditarily sequential functionals actually have good properties, being 'natural' non-dcpo generalization of the traditional Scott–Ershov dcpo domains. This is done in terms of non-dcpo partial ordered sets endowed by a partially defined 'natural' lub operator† $\lfloor\cdot\rfloor$ which is a restriction of the ordinary lub operator $\bigsqcup$ (in general also partial). In the context of high-level functionals, $\lfloor\cdot\rfloor$ is just the *pointwise* lub operator. Recall from Sazonov (2007) (Definition 2.3).

Definition 1.1. In any monotonic, order extensional finite type applicative structure a *pointwise lub* $\lfloor\cdot\rfloor_i f_i$ of an arbitrary (not necessarily directed) family of functionals of the same type is the ordinary lub $\bigsqcup_i f_i$, in the case of the basic type, and, for higher types, it is the ordinary lub which is also required to satisfy, inductively, the pointwise identity $(\lfloor\cdot\rfloor_i f_i)x = \lfloor\cdot\rfloor_i(f_i x)$ (with $\lfloor\cdot\rfloor_i(f_i x)$ also pointwise) for all x of appropriate type. (Thus, $\lfloor\cdot\rfloor \sqsubseteq \bigsqcup$.)

A corresponding general theory of such 'natural' non-dcpo domains which are, additionally, *'naturally' algebraic and 'naturally' bounded-complete*, i.e., are 'natural' non-dcpo versions of Scott–Ershov domains is presented in Sazonov (2009). This requirement actually means that there are 'sufficiently many' *naturally finite* elements, i.e., such elements a that for any directed family $\{x_i\}$, $\lfloor\cdot\rfloor_i x_i \sqsupseteq a$ implies $x_i \sqsupseteq a$ for some i. This general theory is not necessarily related to sequentiality. We mentioned the latter just to be more specific and to give an independent or external motivation for considering non-dcpo domains. This is actually a straightforward generalization of the ordinary dcpo domain theory with using $\lfloor\cdot\rfloor$ – a generalization (actually a natural restriction) of $\bigsqcup$.

Note that some other approach to non-dcpo domains have already been considered long time ago by Ershov as an abstract theory of T_0-topological f-spaces (Ershov 1972) and A-spaces (Ershov 1974) where 'complete' versions of these spaces are, respectively, equivalent to *Scott domains* (i.e., *algebraic and bounded-complete*) and *continuous bounded-complete domains* (Abramsky and Jung 1994; Amadio and Curien 1998; Gierz et al. 2003), being dcpo's. 'Non-complete,' i.e., non-dcpo versions of these spaces had only a very abstract motivation (Ershov 1972, 1974) of a more academic character with no realistic/independent non-trivial examples like hereditarily sequential functionals. (Of course, there is an example of non-dcpo f-spaces of finite type computable functionals over $\mathbb{N}_\perp$, but, unlike hereditarily sequential functionals, it can be obtained quite straightforwardly from all continuous functionals by simple omitting non-computable ones in all types.)

† Instead of $\lfloor\cdot\rfloor$, the denotation $\biguplus$ was used in Sazonov (2007, 2009).

https://doi.org/10.1017/S0960129516000165 Published online by Cambridge University Press

In other words, non-dcpo domain theory is related with and can be motivated by some special versions of computability like sequential one. Our approach arose in Sazonov (2007) from quite concrete considerations on (hereditarily) sequential and wittingly consistent finite type functionals and is based on a 'natural,' actually pointwise lub operator $\lfloor\cdot\rfloor$. In Sazonov (2009), it was introduced a general abstract, approach to $\lfloor\cdot\rfloor$, and f-spaces were characterized by a representation theorem in terms of 'natural' non-dcpo domains, in fact, in terms of $\lfloor\cdot\rfloor$. Let us stress, that in the (realistic) case of sequentiality, $\lfloor\cdot\rfloor$ arises just as a pointwise lub and only after some further non-trivial technical considerations we can find out that the resulting domains of finite type functionals are f-spaces. We see no immediate way to find out the topological structure of f-spaces here. In this sense, our approach to non-dcpo domains is different from Ershov (1972, 1974) and serves as a bridge to the old general concept of f-spaces and, as we will show in this paper, to more general A-spaces, thereby also hopefully giving them a new life. Indeed, till the recent time, there were seemingly no reasons, besides the very abstract approach (Ershov 1972, 1974), to consider non-dcpo domains.

There is also another approach to generalizing domain theory via 'Z subset systems,' assuming a general condition on subsets that should have a supremum (Wright et al. 1978). In our case, we see no such condition restricting the class of e.g., directed subsets of all posets in a uniform way which would be appropriate for our purposes. In an appropriate sense, our goal is to get all sequential (or wittingly consistent or something like that) specific finite type functionals, but it does not look like such a general restriction on subsets of all posets. We know that we need to deal with non-dcpo's, but no general condition seems to be appropriate, i.e., in our general theory, each domain considered is a non-dcpo in its own way.

1.3. *This paper*

This paper considers, more generally than in Sazonov (2009), a class of 'natural' non-dcpo domains which are analogies of *continuous bounded-complete dcpo domains* (in particular, Scott's *continuous lattices*) where finite elements $\neq \perp$ may not exist. Our presentation does not require explicitly the concept of sequential computability or the like, as in Sazonov (2007), and is done rather in the line of Sazonov (2009) in some order-theoretical and induced topological considerations extending the ordinary continuous dcpo domain theory to the case of non-dcpo's. One main result says that some 'natural' non-dcpo version of continuous bounded-complete domains constitutes a Cartesian closed category (ccc). Another main result is a representation theorem, characterizing Ershov's A-spaces in terms of 'natural' non-dcpo domains and natural continuity.

A possible future application of such non-dcpo domains is a 'fully abstract' non-dcpo semantics of quite concrete programming languages such as Real PCF of Martin Escardó (Escardó 1996) dealing with the interval domain of Real Numbers $\mathcal{R}$. Note that this domain is continuous, but not a Scott/Ershov (algebraic) domain. Such an application is actually not considered explicitly in this paper but serves as our more concrete motivation for 'natural' non-dcpo domains and a direction of further research,

https://doi.org/10.1017/S0960129516000165 Published online by Cambridge University Press

likewise the previous works on non-dcpo domains (Sazonov 2007, 2009) were mainly motivated by considering the ordinary PCF (Plotkin 1977) and sequentiality over $\mathbb{N}_\perp$.

Although somewhat out of scope of this paper, we can also mention some alternative approach to (possibly) non-dcpo domains arising from versions of PCF in terms of 'rational' ascending chains leading to operational and synthetic domain theory by Escardó and Ho. (See Escardó and Ho (2009), also for the related work, and Normann and Sazonov (2012) for some more comments.)

1.3.1. *Organization of the paper.* We will start in Section 2 with a general definition of 'natural' non-dcpo domains: ν-domains and δ-domains with a 'natural' lub operator $\bigsqcup\!\!\!\!\bigsqcup$. We compare these two versions in the Appendix because the rest of the paper with a minor exception is completely devoted to ν-domains. 'Naturally' continuous functions and 'natural' Scott topology on such non-dcpo domains are considered in Section 3. Section 4 is devoted to 'natural' non-dcpo generalizations of Scott–Ershov dcpo domains and also continuous dcpo domains with various versions of bounded-completeness such as 'natural' continuity of binary lub operator $x \sqcup y$ under any element z. Here, the main result is Theorem 4.1 that corresponding categories with 'naturally' continuous maps are Cartesian closed. Section 5 is devoted to a *semi-formal outline of a future work* on other kind of ccc's (actually, finite type structures) – such as hereditarily sequential functionals over the interval domain of real numbers with using natural continuous non-dcpo domains. These ideas are based on a previous work (Escardó 1996; Escardó et al. 2004; Marcial-Romero and Escardó 2007; Sazonov 2007). Section 6 relates our approach to continuous non-dcpo domains with an old work of Ershov. We define topological A-spaces (Ershov 1974) and their special case of f-spaces (Ershov 1972). The main result here is a characterization in Theorems 6.1 and 6.2 of A-spaces (as well as f-spaces in Theorems 6.3 and 6.4 considered earlier in Sazonov (2009)) in terms of 'naturally' base-continuous non-dcpo domains Finally, Section 7 presents concluding remarks.

2. Defining natural non-dcpo domains (ν-domains and δ-domains)

Here, we mostly repeat some basic material from Sazonov (2009) with the addition for δ-domains. The terms ν-domains and δ-domains are also new.

Let $\langle D, \sqsubseteq \rangle$ be a partial ordered set (poset). A non-empty subset $X \subseteq D$ or a family $\{x_i\}_{i\in I}$ ($I \neq \varnothing$) of elements in D is called *directed* if any two elements in X have an upper bound in X or, respectively, if $\langle I, \leqslant_I \rangle$, the range of i, is a directed poset, and, moreover, x_i is monotonic in i, that is, $i \leqslant_I j \Rightarrow x_i \sqsubseteq x_j$. We will sometimes omit mentioning the range I of i, relying on the context. In general, we do not assume that families and their index sets I and J are directed, unless said explicitly. Different subscript parameters i and j may range, in general, over different index sets I and J. A two-parametric family $\{x_{ij}\}_{i\in I, j\in J}$ is *directed* if it is (monotonic and) directed in each parameter. As usual, $\bigsqcup X$ or $\bigsqcup_i x_i$ denote the ordinary lub of a subset $X \subseteq D$ or a family x_i in a poset D which may exist or not. That is, $\bigsqcup$ is, in general, a partial lub operator. If D has a least element, it is denoted as $\perp_D$ or $\perp$ and called *undefined*. For any poset D, define also the *cones*

https://doi.org/10.1017/S0960129516000165 Published online by Cambridge University Press

$\uparrow x \rightleftharpoons \{y \in D \mid x \sqsubseteq y\}$ and $\downarrow x \rightleftharpoons \{y \in D \mid y \sqsubseteq x\}$. Let

$$X \sqsubseteq y \rightleftharpoons X \subseteq \downarrow y, \quad y \sqsubseteq X \rightleftharpoons X \subseteq \uparrow y, \quad \textit{but} \quad Y \sqsubseteq X \rightleftharpoons \forall y \in Y\, \exists x \in X.\, y \sqsubseteq x. \tag{1}$$

Our goal is defining a general notion of a 'natural domain' (v-domain) which is actually a second-order poset structure $\langle D, \sqsubseteq^D, \lfloor\cdot\rfloor^D\rangle$ with a reasonably restricted version $\lfloor\cdot\rfloor \subseteq \bigsqcup$ of the ordinary lub operator. This is in contrast with the ordinary concept of a directed-complete poset (dcpo) represented as a first-order structure $\langle D, \sqsubseteq^D\rangle$ closed under lubs of arbitrary directed families $\bigsqcup x_i$ and possibly also satisfying some additional requirements of continuity or algebraicity, etc. (Abramsky and Jung 1994; Amadio and Curien 1998; Gierz et al. 2003). In fact, any poset with unrestricted $\lfloor\cdot\rfloor \rightleftharpoons \bigsqcup$ is formally also a v-domain.

Definition 2.1 (v-domains and δ-domains).

a. Any non-empty poset $\langle D, \sqsubseteq^D\rangle$ is also called a *domain*.
b. A *directed-complete partial order* (or a *dcpo domain*) is required to be closed under taking arbitrary directed lub $\bigsqcup x_i$. (In general, we omit the usual requirement that a dcpo should contain a least element $\bot$.)
c. A *natural pre-domain (v-pre-domain)* is a domain D (in general, non-dcpo) with a partially defined lub operator (*partial lub*) $\lfloor\cdot\rfloor : 2^D \dashrightarrow D$ satisfying the condition ($\lfloor\cdot\rfloor$1) below.

Comment. To consider other versions of partial lub operators over the same D, we can use notations like $\lfloor\cdot\rfloor$, $\lfloor\triangleq\rfloor$, etc. Of course, $\bigsqcup$ is used for the ordinary lub operator.

Sets $X \subseteq D$ where $\lfloor\cdot\rfloor X$ exists are called *natural sets (v-sets)*.

$\langle D, \sqsubseteq, \lfloor\cdot\rfloor\rangle$ is called a *natural domain (v-domain)* and $\lfloor\cdot\rfloor$ – a *natural lub (v-lub) operator* if all the following conditions ($\lfloor\cdot\rfloor$1)–($\lfloor\cdot\rfloor$5) hold[†]:

($\lfloor\cdot\rfloor$1) $\lfloor\cdot\rfloor \subseteq \bigsqcup$. That is, for all v-subsets $X \subseteq D$, $\bigsqcup X$ exists and $\lfloor\cdot\rfloor X = \bigsqcup X$.

($\lfloor\cdot\rfloor$2) If $X \subseteq Y \subseteq D$, $\lfloor\cdot\rfloor X$ exists, and $Y \sqsubseteq \lfloor\cdot\rfloor X$, then $\lfloor\cdot\rfloor Y$ exists too (and so is equal to $\lfloor\cdot\rfloor X$).

($\lfloor\cdot\rfloor$3) $\lfloor\cdot\rfloor\{x\}$ exists (and is equal to x) for all $x \in D$.

($\lfloor\cdot\rfloor$4) Let $\{Y_i\}_{i\in I}$ be a family of subsets of D. Then, the equality

$$\lfloor\cdot\rfloor_{i\in I} \lfloor\cdot\rfloor Y_i = \lfloor\cdot\rfloor \bigcup_{i\in I} Y_i$$

holds with all participating v-lubs existing provided that all the required internal lubs $y_i = \lfloor\cdot\rfloor Y_i$ in $\lfloor\cdot\rfloor_{i\in I} \lfloor\cdot\rfloor Y_i$ and any one of the external lubs $\lfloor\cdot\rfloor_{i\in I} \lfloor\cdot\rfloor Y_i$ or $\lfloor\cdot\rfloor \bigcup_{i\in I} Y_i$ exists.

($\lfloor\cdot\rfloor$5) If $X \subseteq Y \subseteq D$, $\lfloor\cdot\rfloor Y$ exists, and X is cofinal with Y (i.e., $X \sqsupseteq Y$, see Equation (1)), then $\lfloor\cdot\rfloor X$ exists too (and $= \lfloor\cdot\rfloor Y$).

[†] In comparison with Sazonov (2009), we slightly correct ($\lfloor\cdot\rfloor$4) and postulate ($\lfloor\cdot\rfloor$5) which was only optional there.

https://doi.org/10.1017/S0960129516000165 Published online by Cambridge University Press

Variations to ($\lfloor\cdot\rfloor 4$). The equality in ($\lfloor\cdot\rfloor 4$) taken *elementwise* looks like

$$\underset{i\in I}{\lfloor\cdot\rfloor}\,\underset{j\in J_i}{\lfloor\cdot\rfloor}\,y_{ij} = \underset{i\in I, j\in J_i}{\lfloor\cdot\rfloor}\,y_{ij}. \tag{2}$$

Of course, the equality itself is a consequence of existence of lubs. It follows also that

($\lfloor\cdot\rfloor 4^0$) For any two-parametric family $\{y_{ij}\}_{i\in I, j\in J}$ with fixed both I and J, the equality

$$\underset{i}{\lfloor\cdot\rfloor}\,\underset{j}{\lfloor\cdot\rfloor}\,y_{ij} = \underset{j}{\lfloor\cdot\rfloor}\,\underset{i}{\lfloor\cdot\rfloor}\,y_{ij}$$

holds if all the internal and either one of the external lubs or the joint lub $\lfloor\cdot\rfloor_{ij}\,y_{ij}$ exist.

($\lfloor\cdot\rfloor 4'$) If $\{y_{ij}\}$ is directed with i and j ranging over the same I ($I = J$), then the equality

$$\underset{ij}{\lfloor\cdot\rfloor}\,y_{ij} = \underset{i}{\lfloor\cdot\rfloor}\,y_{ii}$$

holds with both its v-lubs existing, assuming any one of these two lubs exists. Indeed, y_{ii} is cofinal to y_{ij}, so if LHS exists, then RHS exists by ($\lfloor\cdot\rfloor 5$). Another direction follows by ($\lfloor\cdot\rfloor 2$).

d. A *directly natural domain (δ-domain) D* with a *δ-lub operator* $\lfloor\cdot\rfloor$ and *δ-sets* $X \subseteq D$, where $\lfloor\cdot\rfloor X$ exists is defined as in (c) above by postulating *modified* versions of ($\lfloor\cdot\rfloor 1$)–($\lfloor\cdot\rfloor 5$) with requiring additionally in ($\lfloor\cdot\rfloor 1$) that $\lfloor\cdot\rfloor X$ is defined only on directed X and also that sets Y in ($\lfloor\cdot\rfloor 2$ and X in ($\lfloor\cdot\rfloor 5$) are directed, and $\{Y_i\}_{i\in I}$ in ($\lfloor\cdot\rfloor 4$) is a directed under $\sqsubseteq$ (in the sense of Equation (1)) family of directed subsets of D (so that the family $\{y_i\}_{i\in I}$ and set $\bigcup_{i\in I} Y_i$ are also directed).
In the case of δ-domains, ($\lfloor\cdot\rfloor 4^0$) still holds assuming that the family $y_{i\in I, j\in J}$ is directed. Also, ($\lfloor\cdot\rfloor 4'$) still follows from ($\lfloor\cdot\rfloor 1$), ($\lfloor\cdot\rfloor 2$) and ($\lfloor\cdot\rfloor 5$).

e. In particular, taking $\lfloor\cdot\rfloor = \bigsqcup$ trivially makes any poset (in particular, any dcpo) $\langle D, \sqsubseteq\rangle$ a *standard* v-domain. Taking $\lfloor\cdot\rfloor = \bigsqcup \restriction \{\text{directed sets}\}$ makes it a *standard* δ-domain.

It follows from Lemma A.1 in the Appendix that

Corollary 2.1. The class of all v-lub operators $\lfloor\cdot\rfloor$ over a poset D constitute a complete lattice under $\subseteq$. □

Note 2.1. Although we introduced the concept of δ-domains which might be interesting in itself as a reasonable alternative of the exposition of our approach, this concept will no more be used in the main body of this paper except mentioning δ-domains in the first part of Section 3 and in the Appendix where their detailed relation with v-domains is considered. We just wanted to show that restricting attention only to directed sets in the v-lubs is in principle also possible since such a question sometimes arises. Indeed, in many cases (such as continuity), we are interested just in directed sets. However, in the ordinary domain theory, this does not force us to ignore the existence of lub of non-directed sets. So, we prefer this approach here as well – in the form of v-domains.

Note 2.2. Sometimes we do not assert explicitly that (natural) suprema exist. If such a denotation $\lfloor\cdot\rfloor X$ is a part of an assumption, then, evidently, we implicitly assume such

https://doi.org/10.1017/S0960129516000165 Published online by Cambridge University Press

an existence, or if it is written $x = \bigsqcup X$, then this exactly means that x is the lub of X and so it is assumed to exist. In general, the reader should rely on the context or on the common sense. We believe this should be evident. In most essential situations, we will be quite precise in our formulations.

3. Further properties and concepts for natural domains

Definition 3.1 (Subdomain). Let $\langle D, \sqsubseteq^D, \bigsqcup^D_{\bullet}\rangle$ be a ν-pre-domain and $\varnothing \neq D' \subseteq D$. Let us restrict $\sqsubseteq^D$ and $\bigsqcup^D_{\bullet}$ from D to D' by letting for any $x', y' \in D'$ and $X' \subseteq D'$:

$$x' \sqsubseteq^{D'} y' \iff x' \sqsubseteq^D y', \quad \text{in notation, } \sqsubseteq^{D'} = \sqsubseteq^D \restriction D',$$
$$x' = \bigsqcup^{D'} X' \iff x' = \bigsqcup^D X', \quad \text{in notation, } \bigsqcup^{D'}_{\bullet} = \bigsqcup^D_{\bullet} \restriction D'.$$

Then, this makes D' a *subdomain* of D which is also a ν-pre-domain:

$$x' = \bigsqcup^{D'} X' \Longrightarrow x' = \bigsqcup^{D'} X'$$

by using the evident implication

$$X' \subseteq D' \;\&\; \bigsqcup^D X' \in D' \Longrightarrow \bigsqcup^D X' = \bigsqcup^{D'} X'. \tag{3}$$

It also follows that the second equivalence above can be rewritten as

$$X' \text{ is a } \nu\text{-set in } D' \iff X' \text{ is a } \nu\text{-set in } D \;\&\; \bigsqcup^D X' \in D'. \tag{4}$$

Proposition 3.1. Any subdomain D' of a ν-domain D is also a ν-domain. The same holds also for δ-domains.

Proof. This is just a routine check. □

Note 3.1. The numbered definitions and propositions of the rest of this section are taken from Sazonov (2009). The proofs are mostly omitted.

Definition 3.2. *Direct product* of ν-(pre-) domains $D \times E$ is defined componentwise by letting $\langle x, y\rangle \sqsubseteq^{D\times E} \langle x', y'\rangle$ iff $x \sqsubseteq^D x' \;\&\; y \sqsubseteq^E y'$ and $\bigsqcup^{D\times E}_i \langle x_i, y_i\rangle \rightleftharpoons \left\langle \bigsqcup^D_i x_i, \bigsqcup^E_i y_i \right\rangle$ for any tuples $\langle x, y\rangle, \langle x', y'\rangle$ and $\{\langle x_i, y_i\rangle\}_{i\in I}$ in $D \times E$ whenever both ν-lubs $\bigsqcup_i x_i$ and $\bigsqcup_i y_i$ exist. Similarly, define general finite and infinite products $\prod_{k\in K} D_k$.

Proposition 3.2. *The direct product of ν-(pre-) domains $\prod_{k\in K} D_k$ is a ν-(pre-) domain as well.* □

Note 3.2. The above proposition on direct products does not hold, as it is, for δ-domains. Indeed, let $D = \{\bot, d\}$, $E = \{\bot, e\}$, $I = \{\langle\bot,\bot\rangle, \langle\bot, e\rangle, \langle d, \bot\rangle\}$ and $x_i = \mathrm{fst}(i)$, $y_i = \mathrm{snd}(i)$ (first and second element of the pair i) for each $i \in I$. Then, $\bigsqcup_i \langle x_i, y_i\rangle = \left\langle \bigsqcup_i x_i, \bigsqcup_i y_i \right\rangle = \langle d, e\rangle$, however, the family $\langle x_i, y_i\rangle$ is not directed. Thus, $D \times E$ is not a δ-domain.

An alternative approach (by using denotations $^{\uparrow\nu}$ and $^{\downarrow\delta}$ from Propositions A.1 and A.2 in the Appendix) would be replacing original δ-lubs $\bigsqcup^D$ and $\bigsqcup^E$ by $\bigsqcup^D_{\bullet} = \bigsqcup^{D\uparrow\nu}$ and $\bigsqcup^E_{\bullet} = \bigsqcup^{E\uparrow\nu}$ and considering the restricted version $\bigsqcup^{D\times E} = \bigsqcup^{\downarrow\delta}_{\bullet}$ of $\bigsqcup_{\bullet} = \bigsqcup^{D\times E}_{\bullet}$ on

https://doi.org/10.1017/S0960129516000165 Published online by Cambridge University Press

$D \times E$, but we will not go into further details here and stop considering δ-domains in this paper leaving it for another occasion. Anyway, this would unnecessarily over-complicate our considerations having a different goal.

So far so good for subdomains and direct products for v-domains. However, functional v-domains (see Definition 3.4 and Proposition 3.6) over standard v-domains are v-domains but not necessarily standard v-domains. A v-lub operator $\bigsqcup_{\bullet}$ could also be called *pointwise* because of the way how this concept first arose in Sazonov (2007) for functional domains (see Definition 1.1 above). Nevertheless, this term proves to be inappropriate in our present abstract context due to some terminological (linguistic) issues as discussed in Sazonov (2009). See also the related 'pointwise' considerations for functions below (e.g., Definition 3.3). Despite we will often use directed v-lubs, it also makes sense to consider in general the concept of v-domains (rather than δ-, i.e., directly natural domains) with possibly non-directed v-lubs by analogy with the ordinary domain theory where arbitrary lubs $\bigsqcup$ are considered and used in proofs and formulations not only for directed families. See, e.g., Proposition 4.3 (c).

Definition 3.3 (Monotonic and v-continuous maps).

a. The poset of all monotonic maps $D \to E$ between any domains ordered pointwise

$$f \sqsubseteq^{(D \to E)} f' \rightleftharpoons fx \sqsubseteq^{E} f'x \text{ for all } x \in D$$

is denoted as $(D \to E)$. We will usually omit the superscripts to $\sqsubseteq$.

b. A monotonic map $f : D \to E$ between v-pre-domains is called *naturally continuous (v-continuous)* if

$$f(\bigsqcup_{i}^{\bullet} x_i) = \bigsqcup_{i}^{\bullet} f(x_i)$$

holds for any directed v-lub $\bigsqcup_{i}^{\bullet} x_i$ (i.e., for a directed family $\{x_i\}_i$), assuming it exists. That is, if $\bigsqcup_{i}^{\bullet} x_i$ exists, then $\bigsqcup_{i}^{\bullet} f(x_i)$ is required to exist and satisfy this equality. The set of all (monotonic and) v-continuous maps $D \to E$ is denoted as $[D \to E]$.

c. Given an arbitrary family $f_i : D \to E$ of monotonic maps between v-pre-domains, define a *v-lub* $f = \bigsqcup_{i}^{\bullet} f_i \in (D \to E)$ pointwise, as

$$fx \rightleftharpoons \bigsqcup_{i}^{\bullet}(f_i x),$$

assuming $\bigsqcup_{i}^{\bullet}(f_i x)$ exists for all $x \in D$; otherwise $\bigsqcup_{i}^{\bullet} f_i$ is undefined. Evidently, if $\bigsqcup_{i}^{\bullet} f_i$ exists, then $\bigsqcup_{i}^{\bullet} f_i = \bigsqcup_{i} f_i$.

Proposition 3.3. Let D, E be v-pre-domains and F a v-domain. A two place monotonic function $f : D \times E \to F$ is v-continuous iff it is v-continuous in each argument. □

Proposition 3.4. v-continuous functions between v-continuous domains are closed under composition.

https://doi.org/10.1017/S0960129516000165 Published online by Cambridge University Press

Proof. For example, consider the composition $f(x) = g(h_1(x), h_2(x))$. For any directed v-lub $x = \bigsqcup_i x_i$, we have by ($\bigsqcup$4) (elementwise and also in the form ($\bigsqcup$4′))

$$h(\bigsqcup_i x_i) = g(h_1(\bigsqcup_i x_i), h_2(\bigsqcup_i x_i)) = \bigsqcup_i \bigsqcup_j g(h_1(x_i), h_2(x_j))$$
$$= \bigsqcup_{ij} g(h_1(x_i), h_2(x_j)) = \bigsqcup_i g(h_1(x_i), h_2(x_i)) = \bigsqcup_i h(x_i). \quad \square$$

Proposition 3.5. For any (not necessarily directed) family of v-continuous maps $f_i : D \to E$ from a v-pre-domain D to a v-domain E, the v-lub $f = \bigsqcup f_i$, if it exists (as a monotonic map), is also a v-continuous map. $\square$

Definition 3.4. Let $F \subseteq (D \to E)$ be any non-empty set of monotonic functions between v-pre-domains with the pointwise partial order $\sqsubseteq^F \rightleftharpoons \sqsubseteq^{(D\to E)} \restriction F$ and the ordinary lub operator $\bigsqcup^F$ on this poset. For any family $f_i \in F$, if the v-lub $\bigsqcup_i f_i$ in $(D \to E)$ exists and is also an element of F, then it is denoted as $\bigsqcup_i^F f_i$; otherwise, $\bigsqcup_i^F f_i$, is considered as not defined. Thus, $\bigsqcup^F = \bigsqcup \restriction F$ and, when defined, $\bigsqcup_i^F f_i = \bigsqcup_i f_i = \bigsqcup_i f_i = \bigsqcup_i^F f_i$ (see the proof of ($\bigsqcup$1) below).

Evidently, $F \subseteq F' \Longrightarrow \bigsqcup_i^{F'} f_i \sqsubseteq \bigsqcup_i^F f_i$ when both lubs exist. In contrast with $\bigsqcup^F$, the v-lub $\bigsqcup_i^F f_i = \bigsqcup_i f_i$ is essentially independent on F, except it is required to be in F. We will omit the superscript F when it is evident from the context.

Proposition 3.6. For D and E v-pre-domains, and any non-empty $F \subseteq (D \to E)$, $\left\langle F, \sqsubseteq^F, \bigsqcup^F \right\rangle$ is a v-pre-domain which is also a v-domain if E is a v-domain. In particular, $(D \to E)$ and $[D \to E]$ are v-domains in this case with $[D \to E]$ closed under (existing, not necessarily directed) v-lubs in $(D \to E)$.

Proof. See (Sazonov 2009, Proposition 2.3). Assuming that E is a v-domain, show that F (i.e. $\left\langle F, \sqsubseteq^F, \bigsqcup^F \right\rangle$) is a v-domain too. We present only the proof of ($\bigsqcup$1) and add checking ($\bigsqcup$5).

($\bigsqcup$1) $\bigsqcup_i^F f_i = f \Leftrightarrow \bigsqcup_i f_i = f \;\&\; f_i, f \in F \Rightarrow \bigsqcup_i f_i = f \;\&\; f_i, f \in F \Rightarrow \bigsqcup_i^F f_i = f$.

($\bigsqcup$5) For a family of monotonic functions $\{f_j \in F\}_{j\in J}$ and $I \subseteq J$, assume that $\bigsqcup_{j\in J} f_j = f \in F$ and $\forall j \in J \exists i \in I . f_j \sqsubseteq f_i$. It follows that for each $x \in D$, $\bigsqcup_{j\in J}(f_j x) = fx$ exists in E and $\forall j \in J \exists i \in I . f_j x \sqsubseteq f_i x$ holds. Therefore, by using ($\bigsqcup$5) for E, $\bigsqcup_{i\in I}(f_i x) = fx$ for each x in the v-domain E, and hence $\bigsqcup_{i\in I} f_i = f$ in $(D \to E)$, as required. $\square$

Moreover, the functors (- → -) and [- → -], together with the direct product functor - × -, make the class of v-domains with monotonic, resp., v-continuous morphisms a ccc in two corresponding ways.

Proposition 3.7. There are the natural (in the sense of category theory) order isomorphisms over v-domains preserving additionally in both directions all the existing v-lubs, not

https://doi.org/10.1017/S0960129516000165 Published online by Cambridge University Press

necessarily directed (and so preserving also the ordinary lubs):

$$(D \times E \to F) \cong (D \to (E \to F)), \tag{5}$$

$$[D \times E \to F] \cong [D \to [E \to F]]. \tag{6}$$

The isomorphism (5) and its inverse are defined as usual by

$$f^* \rightleftharpoons \lambda x.\lambda y.f(x,y) \in (D \to (E \to F)),$$
$$\hat{g} \rightleftharpoons \lambda(x,y).gxy \in (D \times E \to F).$$

for any $f \in (D \times E \to F)$ and $g \in (D \to (E \to F))$. Moreover, each side of the second isomorphism (6) is a subset of the corresponding side of the first Equation (5), with embedding making the square diagram commutative. □

This Proposition is only a most primitive approach to functional natural (ν-) domains. In Sazonov (2007, 2009), we generalized the ordinary dcpo domain theory with $\lfloor\cdot\rfloor = \bigsqcup$ to ν-domains and related this to the models of *hereditarily sequential* and *hereditarily wittingly consistent* finite type functionals $\{\mathbb{Q}_\alpha\}$ and $\{\mathbb{W}_\alpha\}$ *over the natural numbers* with $\lfloor\cdot\rfloor \subseteq \bigsqcup$ understood as the pointwise lub. Actually, the more abstract approach of Sazonov (2009) and the current paper arose from more concrete considerations of Sazonov (2007) on $\{\mathbb{Q}_\alpha\}$ and $\{\mathbb{W}_\alpha\}$. In accordance with Proposition 3.6, we have $\mathbb{Q}_{\alpha\to\beta} \hookrightarrow [\mathbb{Q}_\alpha \to \mathbb{Q}_\beta]$ and do not assert that $\mathbb{Q}_{\alpha\to\beta} \cong [\mathbb{Q}_\alpha \to \mathbb{Q}_\beta]$, that is, $\mathbb{Q}_{\alpha\to\beta}$ consists only of some (namely, sequential) function(al)s from $\mathbb{Q}_\alpha$ to $\mathbb{Q}_\beta$ (and similarly for $\mathbb{W}_{\alpha\to\beta}$). So, we do not follow here Proposition 3.7, and that is the point and the difference to the traditional domain theory. Unfortunately, the construction of $\mathbb{Q}_{\alpha\to\beta}$ is not purely domain theoretical, but rather an intensional one based on sequential computability. Our ν-domain theory only gives a supply of domains and most general facts on them where we can construct by a specific way appropriate (functional or others) domains. What is important, all functionals in $\mathbb{Q}_{\alpha\to\beta}$ are ν-continuous, but, in general, not continuous in the sense of the ordinary $\bigsqcup$ (Normann and Sazonov 2012). In fact, all $\mathbb{Q}_\alpha$ and $\mathbb{W}_\alpha$ are ν-analogues (in the sense of $\lfloor\cdot\rfloor$) of Scott–Ershov domains (Sazonov 2007, 2009), but, in general, not in the sense of $\bigsqcup$ (Normann 2006; Normann and Sazonov 2012). That is why our generalized approach via ν-domains is very much reasonable here, extending the standard dcpo approach with $\lfloor\cdot\rfloor = \bigsqcup$.

Now, our goal is considering ccc's of ν-Scott–Ershov domains and mainly more general ν-continuous domains with some versions of bounded-completeness property which are expected to be more appropriate for the concept of sequentiality *over the real numbers*. In particular, a preliminary idea of extending $\{\mathbb{Q}_\alpha\}$ and $\{\mathbb{W}_\alpha\}$ to the case of the Reals is discussed below in Section 5.

Definition 3.5 (Generalization of the ordinary *Scott topology* on a poset). An upward closed set U in a ν-pre-domain D is called *naturally (ν-) Scott open* if for all directed families x_i having the ν-lub, $\lfloor\cdot\rfloor_i x_i \in U$ implies $x_i \in U$ for some i.

https://doi.org/10.1017/S0960129516000165 Published online by Cambridge University Press

Proposition 3.8.

a. Any v-pre-domain $\left\langle D, \sqsubseteq^D, \lfloor\cdot\rfloor^D \right\rangle$ is a T_0-space under its v-Scott topology whose standardly generated partial ordering coincides with the original ordering $\sqsubseteq^D$ on D.

b. Continuous functions $f \in [D \to E]$ between v-pre-domains defined as preserving the existing v-lubs are also continuous relative to the v-Scott topologies in D and E.

c. Conversely, for any continuous f in the sense of v-Scott topologies in D and E and for any directed family x_i with existing $\lfloor\cdot\rfloor_i x_i$, $f(\lfloor\cdot\rfloor_i x_i) = \bigsqcup_i f(x_i)$.

Comment. So, this is a weakened converse. In the special case of $\lfloor\cdot\rfloor \rightleftharpoons \bigsqcup$ and standard Scott topologies, it is the full converse: $f(\lfloor\cdot\rfloor_i x_i) = \lfloor\cdot\rfloor_i f(x_i)$. We will see in Proposition 4.2 that this stronger version holds also for v-algebraic and v-bounded-complete domains as well as for more general v-continuous domains.

Proof. This is essentially standard and goes as in Sazonov (2009) (Proposition 2.6). We only present here the proof of (c) because of the peculiarity of this clause for non-dcpo domains.

Assume $f : D \to E$ is continuous in the sense of v-Scott topologies in D and E, and $\lfloor\cdot\rfloor_i x_i$ exists in D for a directed family. Let us show that $f(\lfloor\cdot\rfloor_i x_i) = \bigsqcup_i f(x_i)$. The inequality $f(\lfloor\cdot\rfloor_i x_i) \sqsupseteq f(x_i)$ follows by monotonicity of f. Assume y is an upper bound of all $f(x_i)$ in E but $f(\lfloor\cdot\rfloor_i x_i) \not\sqsubseteq y$. Define the Scott open set $V_y \rightleftharpoons \{z \in E \mid z \not\sqsubseteq y\}$. Then, $f^{-1}(V_y)$ is v-Scott open containing $\lfloor\cdot\rfloor_i x_i$ and therefore some x_i, implying $f(x_i) \in V_y$, i.e. $f(x_i) \not\sqsubseteq y$ – a contradiction. This means that $f(\lfloor\cdot\rfloor_i x_i) = \bigsqcup_i f(x_i)$. □

4. 'Natural' non-dcpo generalizations of Scott–Ershov domains and continuous domains

We generalize the standard (i.e., assuming $\lfloor\cdot\rfloor = \bigsqcup$) way-below relation $\ll$ (Amadio and Curien 1998; Gierz et al. 2003) to a *natural way-below (v-way-below)* relation $\lll$.

Definition 4.1 (generalizing). Let D be a v-pre-domain.

a. Define a *v-way-below* relation on D and related cone-like sets

$$x \lll y \rightleftharpoons \forall \text{ directed } \Delta\ (y \sqsubseteq \lfloor\cdot\rfloor\Delta \Rightarrow \exists \delta \in \Delta\,.\, x \sqsubseteq \delta),$$
$$\Uparrow x \rightleftharpoons \{y \in D \mid x \lll y\}, \text{ and } \Downarrow x \rightleftharpoons \{y \in D \mid y \lll x\}.$$

b. Sazonov (2009). A *v-finite (v-compact)* element d in D is such one that $d \lll d$. Define like above

$$\Downarrow^{\text{fin}} x = \{y \in D \mid y \text{ is } v\text{-finite and } y \lll x\} = \{y \in D \mid y \text{ is } v\text{-finite and } y \sqsubseteq x\}.$$

Note that the latter equality holds by Equation (7). We use the standard notation $\ll$ for *way-below* relation if $\lfloor\cdot\rfloor = \bigsqcup$. Then, we evidently have

$$x \ll y \Rightarrow x \lll y \Rightarrow x \sqsubseteq y,$$

https://doi.org/10.1017/S0960129516000165 Published online by Cambridge University Press

where for the last implication we need to assume that the trivial directed lub $y = \lfloor\cdot\rfloor\{y\}$ exists (in the v-pre-domain considered; it does exist in any v-domain),

$$x' \sqsubseteq x \lll y \sqsubseteq y' \Rightarrow x' \lll y', \tag{7}$$

$$x, x' \lll y \Rightarrow x \sqcup x' \lll y, \quad (\text{if } x \sqcup x' \text{ exists}), \tag{8}$$

$$\bot \text{ is } v\text{-finite and so } \bot \lll y \text{ for any } y \quad (\text{if } \bot \in D). \tag{9}$$

In particular, we see that if d is v-finite and $d \sqsubseteq y$, then $d \lll y$. It also follows that each *finite* element ($d \ll d$) is also v-finite ($d \lll d$), but in general not vice versa (Normann and Sazonov 2012).

For the standard case of dcpo's, with v-lub $\lfloor\cdot\rfloor = \bigsqcup$, recall the usual terminology in Abramsky and Jung (1994), Amadio and Curien (1998) and Gierz et al. (2003):

Definition 4.2. A dcpo D (without the usual requirement that a least element $\bot$ exists) is called

- *continuous with a base* $D_0 \subseteq D$ if for all $x \in D$, $x = \bigsqcup(\dotdownarrow x \cap D_0)$ is a directed lub for $\dotdownarrow x = \{y \in D \mid y \ll x\}$ with the usual way-below relation $\ll$. (Recall that any superset of D_0, including D, also serves as a base in this sense.)
- *algebraic* if it is continuous with the base $D_0 = \{\text{all finite elements of } D\}$.
- *bounded-complete* if any two bounded elements $x, y \in D$ have $x \sqcup y$ in D.
- *Scott–Ershov domain* if it is algebraic and bounded-complete.

Definition 4.3. Define the following concepts for (in general, non-dcpo) v-domains D called here for simplicity just domains.

Comment. For example, a v-continuous domain, as defined in (d), is understood as a v-continuous v-domain.

a. D is *v-(ω-)algebraic* if $x = \lfloor\cdot\rfloor \Downarrow^{\text{fin}} x$ and $\Downarrow^{\text{fin}} x$ is directed for all $x \in D$ (and, respectively, there are only countably many v-finite elements in D). In such a domain, v-finite elements constitute the *standard base*.

b. D is *v-bounded-complete* if any two bounded in D v-finite elements have a lub (not necessarily a v-lub, but evidently also a v-finite element).

c. D is a *v-Scott–Ershov domain* if it is v-algebraic and v-bounded-complete (Sazonov 2009).

Comment. If D satisfying (c) is a standard dcpo (with $\lfloor\cdot\rfloor = \bigsqcup$), then it is actually the ordinary Scott–Ershov domain and so bounded-complete in the standard sense (not only for finite elements).

d. D is *v-continuous* if $x = \lfloor\cdot\rfloor \Downarrow x$ and $\Downarrow x$ is directed for all $x \in D$.

e. We say that in D the ordinary (partial) binary lub operator $x \sqcup y$ *is v-continuous under any element* if for each $z \in D$ this operator restricted to the cone $\downarrow z$ is totally defined and v-continuous.

Comment. I.e., $(\lfloor\cdot\rfloor x_i) \sqcup y = \lfloor\cdot\rfloor(x_i \sqcup y)$ holds for any directed family $\{x_i\} \sqsubseteq z$ and $y \sqsubseteq z$ with $\lfloor\cdot\rfloor x_i$ existing. In fact, for D v-continuous, this means that $\sqcup$ is total and v-continuous on the v-continuous subdomain $\downarrow z$ for each $z \in D$ (see Definition 3.1 and Corollary 4.2 (b)).

f. A subset $D_0 \subseteq D$ is called *bounded-complete in D* if for any $a, b \in D_0$ bounded in D, the ordinary lub $a \sqcup b$ exists and $\in D_0$.

Comment. Note that v-bounded-completeness of D in (b) means that $D_0 = \{v\text{-finite elements of } D\}$ is bounded-complete in D.

https://doi.org/10.1017/S0960129516000165 Published online by Cambridge University Press

g. D is *v-base-continuous with a base* D_0 if

- D_0 is bounded-complete in D, and
- $\Downarrow x \cap D_0 \neq \varnothing$ and $x = \lfloor\cdot\rfloor(\Downarrow x \cap D_0)$ for all $x \in D$ (with $\lfloor\cdot\rfloor(\Downarrow x \cap D_0)$ directed by using Equation (8)).

  Comment. Like in the comment in (c), if D satisfying (g) is a standard dcpo (with $\lfloor\cdot\rfloor = \bigsqcup$), then it is the ordinary *continuous and bounded-complete domain* (bounded-complete not only for the base elements). Thereby (g) generalizes these standard domains (with the trivial base $D_0 = D$) to non-dcpo v-domains. In general, for standard continuous dcpo domains with a base, the base is not required to be bounded-complete in D. However, this is required for A-spaces; cf. Section 6.

See Corollary 4.2 for some further features of and relationships between these concepts.

Lemma 4.1. In a v-continuous (v-) domain D, the v-lub of an arbitrary family $y_i \in D$, $i \in I$, can be represented as

$$\lfloor\cdot\rfloor_{i\in I} y_i = \lfloor\cdot\rfloor \bigcup_{i\in I} \Downarrow y_i, \tag{10}$$

where both v-lubs either exist or not simultaneously.

The same holds also in a v-base-continuous domain D where we assume that all $y_i \in D_0$ and $\Downarrow y_i$ is replaced by $\Downarrow y_i \cap D_0$.

This is an extension of Sazonov (2009) (Lemma 3.1) to v-continuous domains.

Proof. Let $y_i^0 \lll y_i$ denote an arbitrarily chosen v-way-below approximation of y_i, and let j range over the set $J \subseteq D$ of v-way-below approximations of elements of D. Define $y_{ij} \rightleftharpoons j$ if $j \lll y_i$, and $\rightleftharpoons y_i^0$ otherwise, so that $\{y_{ij} \mid j \in J\} = \Downarrow y_i$. Then, $\lfloor\cdot\rfloor_i y_i = \lfloor\cdot\rfloor_i \lfloor\cdot\rfloor \Downarrow y_i = \lfloor\cdot\rfloor_i \lfloor\cdot\rfloor_j y_{ij} = \lfloor\cdot\rfloor_{ij} y_{ij} = \lfloor\cdot\rfloor \bigcup_i \Downarrow y_i$ by v-continuity and ($\lfloor\cdot\rfloor$4).

The case of v-base-continuous domains is similar. □

Concerning further properties of the v-way-below relation $\lll$ in Lemmas 4.2, 4.3 and 4.4, we mainly follow the exposition of Amadio and Curien (1998) and Abramsky and Jung (1994) for its ordinary version $\ll$.

Lemma 4.2. If D is v-continuous, then

$$x \lll y \iff \text{for all directed } \Delta\ (y = \lfloor\cdot\rfloor \Delta \Rightarrow \exists \delta \in \Delta\,.\, x \sqsubseteq \delta).$$

□

Lemma 4.3. Let D be a v-domain, $x \in D$, $\Delta \subseteq \Delta' \subseteq \Downarrow x$, Δ is directed and $x = \lfloor\cdot\rfloor \Delta$, then Δ' is directed too and $x = \lfloor\cdot\rfloor \Delta'$. In particular, $\Downarrow x$ is directed and $x = \lfloor\cdot\rfloor \Downarrow x$. □

Note the use of ($\lfloor\cdot\rfloor$2) in this Lemma for showing $x = \lfloor\cdot\rfloor \Delta'$.

Corollary 4.1. In Definition 4.3 (a,d,g), we can replace, respectively, $\Downarrow^{\text{fin}} x$, $\Downarrow x$, and $\Downarrow x \cap D_0$ by 'some its directed subset Δ.' □

Lemma 4.4 (on interpolation). In any v-(base-) continuous domain D, for any natural number n, if all $x_1, \ldots, x_n \lll x$, then there exists $z \in D$ ($\in D_0$) such that $x_1, \ldots, x_n \lll z \lll x$.

https://doi.org/10.1017/S0960129516000165 Published online by Cambridge University Press

Proof. It evidently suffices to show that in any v-continuous domain, $\Downarrow\Downarrow x$ is directed and the identity $x = \lfloor\cdot\rfloor\Downarrow\Downarrow x$ holds.

Directedness of $\Delta \rightleftharpoons \{a \in D \mid \exists a' \in D\ a \lll a' \lll x\} = \Downarrow\Downarrow x$ is proved as in the proof in the op. cit. of corresponding version of Lemma 4.4.

Prove the equality $x = \lfloor\cdot\rfloor\Delta$. Let $x_i \lll x$, $i \in I$, be a family of all elements $\lll x$. Then, $\bigcup_i \Downarrow x_i = \Delta$ and hence $x = \lfloor\cdot\rfloor\Downarrow x = \lfloor\cdot\rfloor_i x_i = \lfloor\cdot\rfloor\bigcup_i \Downarrow x_i = \lfloor\cdot\rfloor\Delta$, where all the v-lubs exist and the equalities hold by v-continuity of D and Lemma 4.1.

In particular, we have $\Downarrow\Downarrow x = \Downarrow x$ for all x. □

Recall that in a subdomain D' of a v-domain D (cf. Definition 3.1), $\sqsubseteq^{D'}$ and v-lub $\lfloor\cdot\rfloor^{D'}$ on D' are, respectively, $\sqsubseteq$ and $\lfloor\cdot\rfloor$ on D restricted to D'. Then, the v-way-below relation $\lll^{D'}$ on D' is taken with respect to $\sqsubseteq^{D'}$ and $\lfloor\cdot\rfloor^{D'}$.

Corollary 4.2.

a. If D is v-base-continuous with a base D_0, then
   - for all $x \in D$, approximating sets $\Downarrow x \cap D_0$ are directed.
   - D is v-continuous (i.e. $x = \lfloor\cdot\rfloor(\Downarrow x)$ with directed $\Downarrow x$ for all $x \in D$).
   - all v-finite elements in D must belong to the base D_0.

b. Assume that D is v-continuous and $D' \subseteq D$ is any downward closed subdomain in the sense that $z \in D' \Longrightarrow \downarrow z \subseteq D'$ for all z. Then, D' is v-continuous and $\lll^{D'} = \lll \restriction D'$. In particular, any $\downarrow z$ or $\Downarrow z$ with $z \in D$ is a v-continuous subdomain.

c. Assume that D is v-base-continuous with a base D_0, and $D' \subseteq D$ is any subdomain which is D_0-downward closed in the sense that $z \in D' \Longrightarrow \downarrow z \cap D_0 \subseteq D'$ for all z.
   - Then, D' is v-continuous and $\lll^{D'} = \lll \restriction D'$.
   - If, additionally, D' is closed under applying the binary lub operator $a \sqcup b$ for any $a, b \in D'_0 = D' \cap D_0$ which are upper bounded in D, then D' is v-base-continuous with the base D'_0.
   - In particular, if $D_0 \subseteq D' \subseteq D$ or $\downarrow z \cap D_0 \subseteq D' \subseteq \downarrow z$ or $\Downarrow z \cap D_0 \subseteq D' \subseteq \Downarrow z$ for some $z \in D$, then D' is a v-continuous subdomain and even v-base-continuous with the base, respectively, D_0, or $\downarrow z \cap D_0$ or $\Downarrow z \cap D_0$.

d. Any v-Scott–Ershov domain D is v-base-continuous with the base $D_0 = \{v\text{-finite elements in } D\}$.

   Comment. Here, also $\Downarrow x \cap D_0 = \downarrow x \cap D_0 = \Downarrow^{\mathrm{fin}} x$ holds for all $x \in D$. Hence, we have the identity $x = \lfloor\cdot\rfloor\Downarrow^{\mathrm{fin}} x$ in D.

   Moreover, the binary lub operator $\sqcup$ restricted to the base is totally defined and v-continuous under any element $z \in D_0$, i.e., in the subdomain $\downarrow z \cap D_0$.

   Comment. However, we do not assert that v-continuity of $\sqcup$ in the full subdomain $\downarrow z$ holds. The general class of v-base-continuous domains D with a base D_0, where the binary lub operator $\sqcup$ is total and v-continuous on the base under any element $z \in D_0$ (i.e., on the subdomain $\downarrow z \cap D_0$) and understood like in the comment to Definition 4.3 (e) will play a role in Theorem 4.1 and corresponding Note 6.1 for A-spaces.

Proof.

a.• $\Downarrow x \cap D_0$ is directed by Equation (8) and by bounded-completeness of D_0 in D.

https://doi.org/10.1017/S0960129516000165 Published online by Cambridge University Press

- Use Lemma 4.3 for $\Delta = \Downarrow x \cap D_0$ and $\Delta' = \Downarrow x$.
- Let d be ν-finite. Since d is a directed lub $= \lfloor\cdot\rfloor(\Downarrow d \cap D_0)$, we have $d \in \Downarrow d \cap D_0$.

b. By Proposition 3.1, D' is a ν-domain. As for any subdomain, $x \ll y \Rightarrow x \lll^{D'} y$ for all $x, y \in D'$. For the converse implication, assume that $x, y \in D'$, $x \lll^{D'} y$, $\Delta \subseteq D$ is directed, and $y = \lfloor\cdot\rfloor \Delta$. Then, $\Delta \subseteq D'$ since D' is downward closed, and so $y = \lfloor\cdot\rfloor^{D'} \Delta$ holds implying that $x \sqsubseteq \delta$ for some $\delta \in \Delta$ by definition of $\lll^{D'}$. Hence, since D is ν-continuous, Lemma 4.2 entails $x \ll y$ in D, and we can conclude $\lll^{D'} = \lll \restriction D'$.

Therefore, $x = \lfloor\cdot\rfloor(\Downarrow x) = \lfloor\cdot\rfloor(\Downarrow^{D'} x)$ holds for any $x \in D'$ with $\Downarrow^{D'} x \rightleftharpoons \{y \in \downarrow z \mid y \lll^{D'} x\} = \Downarrow x$ directed. Thus, D' is ν-continuous.

c.
- Proof goes as in (b) except showing the implication $x \lll^{D'} y \Rightarrow x \ll y$ is changed as follows. We assume again that $x, y \in D'$ and $x \lll^{D'} y$, $\Delta \subseteq D$ is directed and $y = \lfloor\cdot\rfloor \Delta$. But we cannot guarantee that $\Delta \subseteq D'$. So let $\Delta_\delta = \downarrow \delta \cap D_0$ for any $\delta \in \Delta$ and $\Delta' = \bigcup_{\delta \in \Delta} \Delta_\delta \subseteq D_0$. Then, ($\lfloor\cdot\rfloor 4$) implies

$$y = \lfloor\cdot\rfloor_{\delta \in \Delta} \delta = \lfloor\cdot\rfloor_{\delta \in \Delta} \lfloor\cdot\rfloor \Delta_\delta = \lfloor\cdot\rfloor \bigcup_{\delta \in \Delta} \Delta_\delta = \lfloor\cdot\rfloor \Delta'.$$

  Now, we have $\Delta' \subseteq D'$ since D' is D_0-downward closed and so $y = \lfloor\cdot\rfloor^{D'} \Delta'$ holds implying that $x \sqsubseteq \delta'$ for some $\delta' \in \Delta'$ by definition of $\lll^{D'}$. However, $\delta' \in \Delta_\delta$ implying that $x \sqsubseteq \delta' \sqsubseteq \delta$ for some $\delta \in \Delta$. Now, continue as in (b) with referring to Lemma 4.2, etc.
- From the given additional assumption on D', show that D' is ν-base-continuous with the base $D'_0 = D' \cap D_0$.

  First, for any $a, b \in D'_0$ upper bounded in D, since $a \sqcup b \in D'$ and D_0 is bounded-complete in D, we have $a \sqcup b \in D' \cap D_0$ and hence $a \sqcup^{D'} b = a \sqcup b \in D'_0$, and so D'_0 is bounded-complete in D'.

  Second, for any $x \in D'$, we have

$$x = \lfloor\cdot\rfloor(\Downarrow x \cap D_0) = \lfloor\cdot\rfloor(\Downarrow x \cap D' \cap D' \cap D_0) = \lfloor\cdot\rfloor(\Downarrow^{D'_0} x \cap D' \cap D_0) = \lfloor\cdot\rfloor^{D'}(\Downarrow^{D'_0} x \cap D'_0).$$

  Step-by-step: because D is ν–base-continuous, D' is D_0-downward closed, $\lll^{D'} = \lll \restriction D'$, giving $\Downarrow^{D'} x = \Downarrow x \cap D'$, and by definitions of $\lfloor\cdot\rfloor^{D'}$ and D'_0. Thus, we are done.
- This clause is an immediate consequence of the above.

d. The first part follows straightforwardly from definitions.

ν-continuity of $x \sqcup y$ in $\downarrow z \cap D_0$ follows from monotonicity of $x \sqcup y$ and by a trivial reason that any directed ν-set in D_0 has a lub (in D or in $\downarrow z \cap D_0$) iff it has a largest element (which is this lub).

Comment. Note that the ν-continuity of $\sqcup$ degenerates here because of this special case of ν-continuous domains (ν-Scott–Ershov domains).

□

The interpolation Lemma 4.4 implies

Proposition 4.1. In a ν-continuous domain D, $\Uparrow x$ is the interior of the cone $\uparrow x$ in the ν-Scott topology in D, and these sets $\Uparrow x$ constitute a base of this topology. □

https://doi.org/10.1017/S0960129516000165 Published online by Cambridge University Press

In any ν-(base-) continuous domain, we evidently have

$$x \sqsubseteq y \iff \Downarrow x \subseteq \Downarrow y \quad (\iff \Downarrow x \cup D_0 \subseteq \Downarrow y \cup D_0). \tag{11}$$

Let us extend Proposition 3.1 (a) in Sazonov (2009) to ν-continuous domains.

Proposition 4.2. For D and E ν-continuous domains and a monotonic map $f : D \to E$, the following three conditions are equivalent:

a. f is ν-continuous (in the sense of preserving directed ν-lubs).
b. $f(x) = \bigsqcup f(\Downarrow x)$ for all $x \in D$. (Or, $f(x) = \bigsqcup f(\Downarrow x \cap D_0)$ for D ν-base-continuous.)
c. (ϵ-δ-continuity) For all $x \in D$ and $b \lll fx$, there exists $a \lll x$ such that $b \lll fa$.
(a and b can range, respectively, over D_0 and E_0 for D and E ν-base-continuous.)

By Proposition 4.1, ϵ-δ-condition means the ν-Scott topological continuity of f.

Proof.
(a)⇒(b) and (b)⇒(c) are trivial.
(c)⇒(a): Assume f satisfies (c), i.e., the ϵ-δ-continuity property and $x = \bigsqcup_i x_i$ is a directed ν-lub in D. Let us show that $fx = \bigsqcup_i fx_i$. The inclusion $\bigcup_i \Downarrow fx_i \subseteq \Downarrow fx$ holds by monotonicity of f. The converse inclusion $\Downarrow fx \subseteq \bigcup_i \Downarrow fx_i$ holds as well. Indeed, assume $b \lll fx$. Then, (ϵ-δ) implies $b \lll fa$ for some $a \lll x = \bigsqcup_i x_i$. It follows that $a \sqsubseteq x_i$ for some i and hence $b \lll fa \sqsubseteq fx_i$.Thus,

$$\Downarrow fx = \bigcup_i \Downarrow fx_i.$$

It remains to apply ν-continuity of E, the above equality and Equation (10) for $y_i = fx_i$:

$$fx = \bigsqcup \Downarrow fx = \bigsqcup \bigcup_i \Downarrow fx_i = \bigsqcup_i fx_i,$$

where all participating $\bigsqcup$ evidently exist. This shows (a). □

Definition 4.4. Let D and E be ν-continuous domains, where E has a least element $\bot_E$. For $x \in D$ and $y \in E$, define a *step function* $[[x \Rightarrow y]] : D \to E$ by the formula

$$[[x \Rightarrow y]](z) = y \quad \text{if} \quad x \lll z, \quad \text{and} \quad = \bot_E, \quad \text{otherwise.}$$

Proposition 4.3. (Non-dcpo adaptation and variations of Gierz et al. (2003, II-2.31) and Ershov (1974, Theorem 2)) *For D and E ν-continuous domains where E has a least element $\bot_E$, the following clauses hold:*

a. *Any step function $[[x \Rightarrow y]]$ with $x \in D, y \in E$, is ν-continuous.*
b. *If $f \in [D \to E]$ and $y \lll f(x)$, then $[[x \Rightarrow y]] \lll f$ in $[D \to E]$.*
c. • *For all $f \in [D \to E]$, $f = \bigsqcup\{[[x \Rightarrow y]] \mid y \lll f(x)\}$.*
   • *If the partial binary lub $\sqcup$ on E is totally defined and ν-continuous under any $z \in E$, then $[D \to E]$ is a ν-continuous domain with the binary $\sqcup$ totally defined and ν-continuous under any $f \in [D \to E]$.*
d. *If both D and E are ν-base-continuous with the bases D_0 and E_0, then*

https://doi.org/10.1017/S0960129516000165 Published online by Cambridge University Press

- $[D \to E]$ *is* ν*-base-continuous with a base* $[D \to E]_0$ *consisting of finite lubs*

$$[[a_1 \Rightarrow b_1]] \sqcup \cdots \sqcup [[a_n \Rightarrow b_n]], a_i \in D_0, b_i \in E_0,$$

actually existing whenever $[[a_1 \Rightarrow b_1]], \ldots, [[a_n \Rightarrow b_n]]$ *are bounded in* $[D \to E]$.

- *If, additionally, the partial binary* $\sqcup$ *on* E *restricted to the base* E_0 *is totally defined and* ν*-continuous under any base element, then so is the binary* $\sqcup$ *on* $[D \to E]$ *restricted to* $[D \to E]_0$.

Proof.

a.We only note that $\lfloor\cdot\rfloor\{\bot, y\} = y$, and so ($\lfloor\cdot\rfloor 2$) and ($\lfloor\cdot\rfloor 3$) are used in the proof.

b.This follows by using the evident equivalence for any ν-continuous $f : D \to E$:

$$[[x \Rightarrow y]] \sqsubseteq f \iff \forall z \ggg x \,.\, y \sqsubseteq f(z). \tag{12}$$

Assume $f \in [D \to E]$ and $y \lll f(x)$. Then, $[[x \Rightarrow y]] \sqsubseteq f$ by Equation (12). Let also $f \sqsubseteq \lfloor\cdot\rfloor_i f_i$ for a directed family of ν-continuous functions f_i. Since $\sqsubseteq$ and $\lfloor\cdot\rfloor_i f_i$ on $[D \to E]$ are determined pointwise, we have $f(x) \sqsubseteq \lfloor\cdot\rfloor_i f_i(x)$ and so $y \sqsubseteq f_i(x)$ for some i. Then, for all $z \sqsupseteq x$, $y \sqsubseteq f_i(z)$. This implies $[[x \Rightarrow y]] \sqsubseteq f_i$ by (12), as required.

c.• It suffices to show that $f = \lfloor\cdot\rfloor\{[[x \Rightarrow y]] \mid y \lll f(x)\}$ holds pointwise with any fixed $z \in D$ and varying $x \in D$ and $y \in E$:

$$\begin{aligned} f(z) = \underset{x \lll z}{\lfloor\cdot\rfloor} f(x) &= \underset{x \lll z}{\lfloor\cdot\rfloor} \underset{y \lll f(x)}{\lfloor\cdot\rfloor} y = \underset{x \lll z, y \lll f(x)}{\lfloor\cdot\rfloor} y \\ &= \underset{x \lll z, y \lll f(x)}{\lfloor\cdot\rfloor} [[x \Rightarrow y]](z) = \underset{y \lll f(x)}{\lfloor\cdot\rfloor} [[x \Rightarrow y]](z). \end{aligned}$$

The first equality is just the continuity of f. In the following equality, the internal $\lfloor\cdot\rfloor$ is actually $\lfloor\cdot\rfloor \Downarrow f(x) = f(x)$. Then, we have just an elementwise version of ($\lfloor\cdot\rfloor 4$). The next equality holds by the definition of step function. At last, ($\lfloor\cdot\rfloor 2$) can be applied since $[[x \Rightarrow y]](z) \sqsubseteq f(z)$ evidently holds if $y \lll f(x)$ (and even if $y \sqsubseteq f(x)$).

- For any $f \in [D \to E]$, it follows by ($\lfloor\cdot\rfloor 2$) that $f = \lfloor\cdot\rfloor \Downarrow f$ with $\Downarrow f \neq \varnothing$ by (b). Moreover, $\Downarrow f$ is directed. By Equation (8), it suffices to show that $[D \to E]$ is bounded-complete. If $h_1, h_2 \sqsubseteq g$ holds in $[D \to E]$, then $h_1(x), h_2(x) \sqsubseteq g(x)$ for all x and so $h(x) \rightleftharpoons h_1(x) \sqcup h_2(x)$ exists for all x and is a ν-continuous function on x as a composition of h_1 and h_2 with ν-continuous $u \sqcup v$ on any $\downarrow g(x) \subseteq E$. Hence, $h = h_1 \sqcup h_2$, and this shows the bounded-completeness, and so that $\Downarrow f$ is directed. Thus, $[D \to E]$ is a ν-continuous domain.

Now, show that the binary $\sqcup$ on $[D \to E]$ is ν-continuous on any cone $\downarrow g \subseteq [D \to E]$. (See also Corollary 4.2 (b).) That is, $(\lfloor\cdot\rfloor f_i) \sqcup f = \lfloor\cdot\rfloor(f_i \sqcup f)$ holds for any directed family f_i, with existing $\lfloor\cdot\rfloor f_i$, and any f under g. Like above, the proof goes pointwise for any $x \in D$. This allows using the continuity of $\sqcup$ in E:

$$\begin{aligned} ((\underset{i}{\lfloor\cdot\rfloor} f_i) \sqcup f)(x) = (\underset{i}{\lfloor\cdot\rfloor} f_i)(x) \sqcup f(x) &= (\underset{i}{\lfloor\cdot\rfloor} f_i(x)) \sqcup f(x) \\ = \underset{i}{\lfloor\cdot\rfloor}(f_i(x) \sqcup f(x)) &= (\underset{i}{\lfloor\cdot\rfloor}(f_i \sqcup f))(x). \end{aligned}$$

https://doi.org/10.1017/S0960129516000165 Published online by Cambridge University Press

d. *For the first part*, we essentially follow the ideas of the proof of Ershov (1974) (Theorem 2) for A-spaces in our framework of ν-domains. Note that here we do not have and so cannot use the continuity of $\sqcup$ on E.

1. $f = \lfloor\cdot\rfloor\{[[a \Rightarrow b]] \mid b \lll f(a), a \in D_0, b \in E_0\}$ for any $f \in [D \to E]$.

   The proof is like in (c) with using $f(a) = \lfloor\cdot\rfloor(\Downarrow f(a) \cap E_0)$ instead of $f(x) = \lfloor\cdot\rfloor \Downarrow f(x)$.

2. For $\vec{a} \in D_0^n$ and $\vec{b} \in E_0^n$, $n \geqslant 0$, define an evidently monotonic *tabular* function $[[\vec{a} \Rightarrow \vec{b}]] : D \to E$, in fact, $D \to E_0$, by the formula

$$[[\vec{a} \Rightarrow \vec{b}]](z) \rightleftharpoons \bigsqcup_{a_k \lll z} b_k \quad (= \bot_E \text{ if no } a_k \lll z \text{ or if } n = 0).$$

   This definition is *correct (consistent) iff* for any $K \subseteq \{1, \ldots, n\}$ if $\{a_k \mid k \in K\}$ is $\lll$-bounded in D, then $\{b_k \mid k \in K\}$ is $\sqsubseteq$-bounded in E and so the above $\bigsqcup$ does exist.

3. Any consistently defined tabular function $[[\vec{a} \Rightarrow \vec{b}]]$ is ν-continuous.

   Indeed, let $z_i \in D$ be a directed family and $z = \lfloor\cdot\rfloor_i z_i$ exist in D. Since there are only finitely many of a_k, for all $a_k \lll z$ we have, by using interpolation, that $a_k \lll z' \lll z$ for some fixed z', and so $z' \lll z_{i_0}$ for some i_0 and all these $a_k \lll z_{i_0} \sqsubseteq z$. It follows that

$$[[\vec{a} \Rightarrow \vec{b}]](z) \rightleftharpoons \bigsqcup_{a_k \lll z} b_k = \bigsqcup_{a_k \lll z_{i_0}} b_k = \lfloor\cdot\rfloor_i \bigsqcup_{a_k \lll z_i} b_k \rightleftharpoons \lfloor\cdot\rfloor_i [[\vec{a} \Rightarrow \vec{b}]](z_i).$$

   Here, all $\bigsqcup$ are defined by bounded-completeness of E_0 in E and $\lfloor\cdot\rfloor_i(\ldots)$ is defined by ($\lfloor\cdot\rfloor 2$) and ($\lfloor\cdot\rfloor 3$) because it can actually be represented as $\lfloor\cdot\rfloor B$ for a finite set B with $b = \bigsqcup_{a_k \lll z_{i_0}} b_k$ its largest element.

4. Note that the definition of $[[\vec{a} \Rightarrow \vec{b}]]$ is consistent also *iff* the individual step functions $[[a_k \Rightarrow b_k]]$, $1 \leqslant k \leqslant n$, are bounded by some ν-continuous function in $[D \to E]$.

5. $[[\vec{a} \Rightarrow \vec{b}]] = [[a_1 \Rightarrow b_1]] \sqcup \cdots \sqcup [[a_n \Rightarrow b_n]]$ holds in the case of consistency.

   Indeed, $[[a_k \Rightarrow b_k]] \sqsubseteq [[\vec{a} \Rightarrow \vec{b}]]$ evidently holds for all $1 \leqslant k \leqslant n$, and the implication

$$[[a_1 \Rightarrow b_1]] \sqsubseteq f, \ldots, [[a_n \Rightarrow b_n]] \sqsubseteq f \Longrightarrow [[\vec{a} \Rightarrow \vec{b}]] \sqsubseteq f$$

   for all ν-continuous f follows from Equation (12): $[[a_k \Rightarrow b_k]] \sqsubseteq f \Leftrightarrow \forall z \ggg a_k \,.\, b_k \sqsubseteq f(z)$.

6. If $\{[[\vec{a} \Rightarrow \vec{b}]], [[\vec{c} \Rightarrow \vec{d}]]\}$ is bounded in $[D \to E]$, then $[[\vec{a}\vec{c} \Rightarrow \vec{b}\vec{d}]] = [[\vec{a} \Rightarrow \vec{b}]] \bigsqcup [[\vec{c} \Rightarrow \vec{d}]]$. Thus, the set of tabular functions $[D \to E]_0$ is bounded-complete in $[D \to E]$.

7. (Generalization of (b)) If $f \in [D \to E]$ and $\vec{b} \lll f(\vec{a})$ componentwise (i.e., $b_k \lll f(a_k)$ in E) with $\vec{a} \in D_0^n$ and $\vec{b} \in E_0^n$, then $[[\vec{a} \Rightarrow \vec{b}]]$ is consistent and $\lll f$ in $[D \to E]$.

   Indeed, we conclude that all $[[a_k \Rightarrow b_k]] \lll f$ by (b). It remains to apply (d5) and Equation (8).

https://doi.org/10.1017/S0960129516000165 Published online by Cambridge University Press

8. For any $f \in [D \to E]$, $f = \bigsqcup(\Downarrow f \cap [D \to E]_0)$ with $\Downarrow f \cap [D \to E]_0 (\neq \varnothing)$ directed. Use (d1), (d7) for $n = 1$, ($\bigsqcup 2$), $\bot \in [D \to E]_0$ (to show non-emptiness), (d5,d6) and Equation (8).
9. This and (d6) mean that $[D \to E]$ is ν-base continuous with the base $[D \to E]_0$.
10. Finally, *the second part* of (d) is proved as the second part of (c), but assuming that all f, f_i and $\bigsqcup_i f_i$ are in $[D \to E]_0$ and $\sqsubseteq g \in [D \to E]_0$ (so that $f(x), f_i(x)$ and $(\bigsqcup_i f_i)(x)$ are in E_0 and $\sqsubseteq g(x) \in E_0$; see also Corollary 4.2 (c)). □

Without going into further details, we can conclude:

Theorem 4.1. The following categories are Cartesian closed with respect to the functors [- → -] and [- × -]:

- The category of ν-continuous domains with $\bot$ and a partial binary lub operator $\sqcup$ which is total and ν-continuous under any element and with ν-continuous maps as the arrows.
- The category of ν-base-continuous domains with $\bot$.
- The category of ν-base-continuous domains with $\bot$ and a partial binary lub operator $\sqcup$ which is total and ν-continuous over the base under any given base element.
- The category of ν-Scott–Ershov (i.e., ν-algebraic and ν-bounded-complete) domains with $\bot$ (Sazonov 2009).

However, recall that in practice, we rather expect another kind of Cartesian closed categories, or, in other terms, finite type domains of some kind of functionals, with the function domain $[D \to E]$ of *all* ν-continuous functions (assumed above) to be replaced by a suitable *subdomain* $[D \to E]' \subseteq [D \to E]$ consisting of sequential function(al)s, or of any other kind. Evidently, at this place, we should come to a non-dcpo domain $[D \to E]'$ (see also Definition 3.4 and Proposition 3.6), and *it is this reason for all this story on non-dcpo domain theory*. Thereby, we also come both to non-dcpo anomalies like those of hereditarily sequential finite type functionals over natural numbers (Normann 2006; Normann and Sazonov 2012) and, despite these anomalies, to non-dcpo 'naturality' of these functionals and corresponding general non-dcpo domain theory (Sazonov 2007, 2009) including the current work potentially oriented rather towards functionals over the *interval domain* of 'partial' real numbers (see, e.g., Escardó (1996) and Section 5 below). The 'naturality' means that this non-dcpo domain theory goes quite smoothly, rather analogous to the usual dcpo domain theory, so that *dcpo requirement looks now unnecessarily restrictive*. We know from the op. cit. that these expectations on Cartesian closed categories more general than in the above theorem are indeed satisfied for ν-Scott–Ershov domains (in the form of hereditarily sequential or wittingly consistent finite type functionals over the naturals), and such expectations are also highly plausible for the general case of ν-(base-) continuous domains considered here.

An additional and complicated research like in Sazonov (2007) for the case of real numbers instead of natural ones is required to exploit this possibility not covered by the above theorem on ccc's. Such an elaborated research goes beyond the scope of current general theory. It is rather about some concrete and rather intricate examples of ccc's of

https://doi.org/10.1017/S0960129516000165 Published online by Cambridge University Press

v-continuous domains for which there is no space in this paper and whose details are still to be developed. Nevertheless, we present here some preliminary considerations.

5. Some insights on hereditarily sequential and wittingly consistent functionals over the Reals

Following Escardó (1996), define $\mathcal{I}_\perp$ as a set of closed subintervals of the unit interval $\perp_{\mathcal{I}} = [0,1] \subseteq \mathbb{R}$ in the real line partially ordered by $x \sqsubseteq y \iff x \supseteq y$, and let $\mathcal{I} = \mathcal{I}_\perp \setminus \{\perp_{\mathcal{I}}\}$. Then, by Definition 4.2,

Proposition 5.1. $\mathcal{I}$ is a continuous bounded-complete dcpo with a base $\mathcal{I}_0 \subseteq \mathcal{I}$ consisting of non-zero length $\mathcal{I}$-intervals with rational end points. Similarly, $(\mathcal{I}_0)_\perp = \mathcal{I}_0 \cup \{\perp\}$ is a base for the continuous bounded-complete dcpo $\mathcal{I}_\perp$ (denoted also as D_{I}). The flat domain $\mathbb{N}_\perp = \mathbb{N} \cup \{\perp\}$ is also a continuous bounded-complete dcpo domain with the base $\mathbb{N}_\perp$ (denoted also as D_{N}; in fact, a Scott–Ershov domain). □

Define inductively finite type standard continuous bounded-complete dcpo domains $D_{\sigma\to\tau} = [D_\sigma \to D_\tau]$ consisting of *all* continuous maps between such domains D_σ and D_τ and also a set of *basic values* $\mathcal{V} = \mathbb{N} \cup \mathcal{I}_0$ and its extension $\mathcal{V}_\perp = \mathcal{V} \cup \{\perp\}$ which serve as a set of *partial answers to queries* used in computations and, respectively, as a set of *partial intermediate output results* of a computation.

We extend the idea of Sazonov (2007) for the simpler case of $\mathcal{V} = \mathbb{N}$; the type I for the Reals $D_{\mathsf{I}} = \mathcal{I}_\perp$ was not considered there.

> *Typed weak sequential computational strategies* $m : \alpha$ for various types α can be defined as an abstraction of Turing Machines with a kind of Oracle answering queries about the strategies (having the form '$A \sqsupseteq ?$' with $A : \mathsf{N}$ or $: \mathsf{I}$ an applicative term involving strategies and also typed variables).

Each such a strategy computes its denotational semantic value $[[m]] \in D_\alpha$ as a finite type *sequential* functional. Thereby, the *partial* values from $\mathcal{V} = \mathbb{N} \cup \mathcal{I}_0$ are used in the computations. Partial – mainly because of the continuous nature of $\mathcal{I}$.

We call these strategies 'weak' because over the Reals (unlike the naturals), they will not be able to compute even the addition operation (or the average operation in our case of $\mathcal{I}_\perp$) as it is argued in a different context by Escardó et al. (2004). However, in the case of $\mathcal{V} = \mathbb{N}$, i.e., when the real interval numbers are not involved, this is the full-fledged sequentiality with no weakness.

Non-dcpo domain theory is *not* involved at this point. However, we are interested in domains of *hereditarily (weakly) sequential functionals over the Reals* $\mathbb{Q}^{wr}_\alpha$ with [[-]] redefined over these domains (instead of dcpo's D_α) so that $\bigcup_\alpha \mathbb{Q}^{wr}_\alpha =$ the image of [[-]] for all possible strategies. Actually, these will be non-dcpo v-continuous domains such that $\mathbb{Q}^{wr}_{\mathsf{N/I}} = D_{\mathsf{N/I}}$, $\mathbb{Q}^{wr}_\alpha \not\subseteq D_\alpha$ (not a subset) and $\mathbb{Q}^{wr}_{\sigma\to\tau} \subsetneq [\mathbb{Q}^{wr}_\sigma \to \mathbb{Q}^{wr}_\tau]$ (a proper subset consisting of only sequential functionals) in general, and their precise definition should be technically involved (requiring a lot of space and a separate publication), like in the special case of v-Scott–Ershov domains $\mathbb{Q}_\alpha$ of sequential finite type functionals over $\mathbb{N}_\perp$ in Sazonov (2007).

https://doi.org/10.1017/S0960129516000165 Published online by Cambridge University Press

To overcome the 'weakly' above, we can consider, like in Sazonov (1976c, Ph.D. Thesis, Ch. II, Section 4) and Sazonov (2007), *non-deterministic* strategies, both over $\mathbb{N}_\perp$ and $\mathcal{I}_\perp$.

> Such strategies are called *wittingly consistent* if
>
> as soon as two different computational histories involving also non-deterministic steps lead to two *contradictory* (not upper bounded) output values in $\mathcal{V}$ by the same strategy m, then some query '$A \sqsupseteq$?' was answered contradictorily by the Oracle during these two computations.

This way, we have left the (deterministic) sequentiality framework. That is a pity, of course, but according to Escardó (1996) and Escardó et al. (2004), whose approach to Real PCF is covered by the above definition, this leads to a satisfactory computability power over the Reals $D_{\mathsf{I}} = \mathcal{I}_\perp$. This again should give rise to non-dcpo v-continuous domains $\mathbb{W}^r_\alpha$ of *hereditarily wittingly consistent functionals over the Reals*.

Nevertheless, we can still remain in the sequentiality framework by shifting from the interval Reals $D_{\mathsf{I}} = \mathcal{I}_\perp$ to $D_{\mathsf{I}} = \mathcal{P}^H\mathcal{I}_\perp$, and similarly for D_{N}. Here, $\mathcal{P}^H D$ denotes the *Hoare powerdomain* of any dcpo D (Abramsky and Jung 1994) consisting of all non-empty Scott-closed subsets of D ordered by inclusion ($\sqsubseteq_{\mathcal{P}^H D} = \subseteq$). If D (with $\perp_D$) is a continuous bounded-complete domain with a base D_0, then so is $\mathcal{P}^H D$ (with $\perp_{\mathcal{P}^H D} = \{\perp_D\}$, and so even a continuous lattice) with a base consisting of finitely generated Scott-closed sets $\downarrow F$ for finite $F \subseteq D_0$. Then, our intervals $x \in \mathcal{I}_\perp$ and, in particular, the real numbers $r = [r, r] \in \mathcal{I}_\perp$ in $[0, 1]$ can be embedded and so represented in the powerdomain $\mathcal{P}^H\mathcal{I}_\perp$ as

$$\eta(x) = \{y \in \mathcal{I}_\perp \mid x \subseteq y\} = \{y \in \mathcal{I}_\perp \mid x \sqsupseteq_{\mathcal{I}} y\} \quad \text{and} \quad \eta(r) = \{y \in \mathcal{I}_\perp \mid r \in y\}.$$

> Now, we can define *many-valued sequential (deterministic) computational strategies* the same way as was assumed above for the weak version, except requiring that the set of values $\mathcal{V}$ is *redefined* as the set of Scott-closed sets $\downarrow V \in \mathcal{P}^H\mathbb{N}_\perp$ or $\in \mathcal{P}^H\mathcal{I}_\perp$ for finite $\varnothing \neq V \subseteq \mathbb{N}$ or $\subseteq \mathcal{I}_0$.

Then, we can define (like for $\mathbb{Q}^{wr}_\alpha$) the denotational semantics $[\![m]\!]$ of strategies and v-continuous non-dcpo domains $\mathbb{Q}^{mr}_\alpha$ of *hereditarily sequential functionals over the Reals in* this new *multiple valued framework*, where $\mathbb{Q}^{mr}_{\mathsf{N}} = \mathcal{P}^H\mathbb{N}_\perp$ and $\mathbb{Q}^{mr}_{\mathsf{I}} = \mathcal{P}^H\mathcal{I}_\perp$. The point is that the expressive power of new sequential strategies and functionals becomes satisfactory for the real numbers $r \in [0, 1]$ represented as $\eta(r)$. We refer to Marcial-Romero and Escardó (2007) for more detail (not in terms of our strategies) and hope that our approach via strategies and non-dcpo v-domains will make things more clear and uniform. Note that the term 'strategy' is used in the op. cit. as well, but with a different sense and purpose. Also, unlike ours, their approach is more (PCF) language dependent whereas our key notion is an abstract computational strategy.

Overall, the hope is to show that non-dcpo naturally continuous domains should work well also in the cases of fully abstract models over the Reals (of hereditarily sequential functionals or the like) in contrast to traditional dcpo domains. *Carrying out this programme will be the subject of further work.*

https://doi.org/10.1017/S0960129516000165 Published online by Cambridge University Press

6. A-spaces vs. v-base-continuous domains

Consider any T_0-space D – a topological space in which any two different points x, y are separated one from another by some open set U: $x \in U$ & $y \notin U$ or $y \in U$ & $x \notin U$. Define a partial ordering $\sqsubseteq$ on D and other related concepts generated by the T_0-topology:

$$x \sqsubseteq y \rightleftharpoons \forall \text{ open } U(x \in U \Rightarrow y \in U).$$
$$\check{x} \rightleftharpoons \mathbf{I}nt \uparrow x - \text{the greatest open set contained in } \uparrow x = \{y \in D \mid x \sqsubseteq y\},$$
$$x \prec y \rightleftharpoons y \in \check{x}.$$

Definition 6.1 (Ershov (1972, 1974)). A T_0-space D is called an *A-space* if it has a *base* subspace $D_0 \subseteq D$, i.e., such a subset that satisfies the following two conditions.

A1. D_0 is *bounded-complete in* D, i.e., any two elements $x_0, y_0 \in D_0$ upper bounded in D have a lub $z_0 = x_0 \sqcup y_0$ in D and $z_0 \in D_0$.
A2. For any open $U \subseteq D$ and $x \in U$, there is an $x_0 \in D_0 \cap U$ such that $x_0 \prec x$.

A *full-base A-space* D is such one with the base $D_0 = D$.

An element a of a T_0-space D is called an *f-element* if the set $\uparrow a$ is open. Open sets of this form are called *f-sets*. The set of f-elements of D is denoted as $F(D)$. D is called an f-*space* if

F1. Intersection of any two f-sets, if non-empty, is also an f-set: $\uparrow a \cap \uparrow b = \uparrow c$. (Then, $c = a \sqcup b$ in D.)
F2. f-sets constitute a base of the given topology on D (so that open sets in D are exactly arbitrary unions of f-sets).

An A_0-*space* is an A-space D which has a least element $\perp_D$ (evidently an f-element). An f_0-*space* is defined similarly.

Evidently, any f-space is also an A-space with the standard base $F(D)$ consisting of all its f-elements. By (A2), all f-elements of an A-space must belong to any its base. Note that an A-space may have various base subspaces.

The topologically *continuous maps* $f : D \to E$ between A-spaces are also characterized by the following ϵ-δ-condition:

$$\forall x \in D \forall b \in D_0(b \prec fx \Longrightarrow \exists a \in D_0(a \prec x \;\&\; b \prec fa)). \tag{13}$$

In fact, $\Longleftrightarrow$ holds. For any A-space D, $x \in D$ and $Z \subseteq D$ let

$$\hat{x} = \{x_0 \in D_0 \mid x_0 \sqsubseteq x\}, \quad \widehat{x} = \{x_0 \in D_0 \mid x_0 \prec x\} \quad \text{and} \quad \widehat{Z} \rightleftharpoons \bigcup_{z \in Z} \widehat{z}$$

be the set of *base approximations* of x and, respectively, the set of *base strong approximations* of x and Z. (In Ershov (1974), definitions are different: $\widehat{x} = \{y \in D \mid y \prec x\}$, and similarly for $\hat{x}$.) Then, we have

$$\widehat{x} \subseteq \hat{x}, \quad \widehat{x} \text{ is directed (and } \neq \varnothing), \quad x = \bigsqcup \widehat{x}, \quad \text{and} \quad x \sqsubseteq y \Longleftrightarrow \widehat{x} \subseteq \widehat{y}. \tag{14}$$

https://doi.org/10.1017/S0960129516000165 Published online by Cambridge University Press

Moreover, any $x \in D$ is also a topological *limit point* of the set of its D_0-strong approximations. In particular, Equation (13) can be rewritten as

$$\widehat{f(x)} = \bigcup\{\widehat{f(a)} \mid a \in D_0 \ \&\ a \prec x\}$$

so that any continuous f is completely determined by its behaviour on the base subspace

$$f(x) = f(\bigsqcup \hat{x}) = \bigsqcup f(\hat{x}).$$

Now, Definitions 3.5, 4.3 (g) and Proposition 4.2 imply

Theorem 6.1. Each v-base-continuous domain becomes an A-space under the v-Scott topology.

Let us show the inverse of Theorem 6.1, i.e., how any A-space D with a base subspace D_0 can be represented as a v-base-continuous domain. If $Z \subseteq \hat{x}$ is any directed set of D_0-elements cofinal with $\hat{x}$, i.e., $\forall a \in \hat{x} \exists z \in Z (a \sqsubseteq z)$, then evidently $\bigsqcup Z = \bigsqcup \hat{x} = x$. For a directed $Z \subseteq \hat{x}$, it is possible in principle that $\bigsqcup Z = x$ but Z is not cofinal with $\hat{x}$. In this case, we call $\bigsqcup Z = x$ a *non-v-lub*. Otherwise (the case of cofinality), the lub is called a *v-lub* and denoted as $\bigsqcup^{\bullet} Z$.

More general,

Definition 6.2. An arbitrary *directed* set Z in an A-space D is said to have a *standard (A-standard) v-lub* x in D, $\bigsqcup^{\bullet} Z = x$, if any one of the following three equivalent conditions holds:

a. $\widehat{Z} \rightleftharpoons \bigcup_{z \in Z} \hat{z}$ (which is also directed for directed Z) is a cofinal subset of $\hat{x}$,
b. $\widehat{Z} = \hat{x}$,
c. x is both an upper bound and a (topological) limit point of Z.

Let us show all the stated equivalences.

Proof. We assume below that x_0, z_0 etc. range over the base D_0. First, show that $\widehat{Z} \rightleftharpoons \bigcup_{z \in Z} \hat{z}$ is directed. Indeed, if $u_0 \prec u \in Z$, $v_0 \prec v \in Z$ and $u, v \sqsubseteq w \in Z$, then $u \in \breve{u}_0$, $v \in \breve{v}_0$, $w \in \breve{u}_0 \cap \breve{v}_0$, $u_0 \sqcup v_0 \sqsubseteq \breve{u}_0 \cap \breve{v}_0$ (i.e., for each element of the open set $\breve{u}_0 \cap \breve{v}_0$), and so $u_0 \sqcup v_0 \prec w$.

(a) $\Rightarrow$ (b): It suffices to show that (a) implies $\hat{x} \subseteq \widehat{Z}$. Indeed, $x_0 \prec x \Rightarrow x_0 \sqsubseteq z_0 \prec z \in Z$ for some z_0 and z by cofinality, and so $\Rightarrow x_0 \prec z \in Z$.

(b) $\Rightarrow$ (c): $\widehat{Z} \subseteq \hat{x}$ implies that x is an upper bound of Z. Indeed, $z \in Z \Rightarrow \hat{z} \subseteq \hat{x} \Rightarrow z \sqsubseteq x$. $\widehat{Z} \supseteq \hat{x}$ implies that x is a limit point of Z. Indeed, $x_0 \prec x \Rightarrow x_0 \prec z \in Z$ for some z.

(c) $\Rightarrow$ (a): Since x is an upper bound of Z, $z_0 \prec z \in Z$ implies $z_0 \prec x$, so $\widehat{Z} \subseteq \hat{x}$. If x is also a limit point of Z, then $x_0 \prec x \Rightarrow x_0 \prec z \in Z$ for some z, and so $\Rightarrow x_0 \prec z_0 \prec z \in Z$ for some z_0 by (A2). This gives $x_0 \sqsubseteq z_0 \in \widehat{Z}$, and the cofinality condition holds. □

Thus, if a standard v-lub of a directed Z exists, then $\bigsqcup^{\bullet} Z = \bigsqcup Z$ and we have for any *directed* set Z or *directed* family z_i:

$$\bigsqcup^{\bullet} Z = x \iff \widehat{Z} = \hat{x}, \quad \text{and} \quad \bigsqcup_i^{\bullet} z_i = x \iff \bigcup_i \widehat{z_i} = \hat{x}. \tag{15}$$

https://doi.org/10.1017/S0960129516000165 Published online by Cambridge University Press

Hence, a directed v-lub corresponds to set theoretic union and in this sense it is indeed quite natural – 'pointwise.' For any $x \in D$ and (not necessarily directed) $Z \subseteq D$, we have by (A2)

$$\widehat{\widehat{x}} = \widehat{x}, \quad \widehat{\widehat{Z}} = \widehat{Z}, \quad x = \bigsqcup\nolimits^{\bullet} \widehat{x}. \tag{16}$$

Definition 6.3 (Arbitrary, not necessarily directed sets). A set $Z \subseteq D$ in an A-space D with a base subspace D_0 is said to have a *standard v-lub* x, $\bigsqcup^{\bullet} Z = x$, if any of the following four equivalent conditions holds:

a. The set of base elements

$$\bar{Z} \rightleftharpoons \overline{\widehat{Z}}, \quad \text{where} \quad \overline{X} \rightleftharpoons \{\bigsqcup F \in D_0 \mid F \subseteq X, F \neq \varnothing \text{ is finite and } \bigsqcup F \text{ exists}\} \tag{17}$$

is a cofinal subset of $\widehat{x}$. (In this case, all these $\bigsqcup F$ actually exist, $\bar{Z}$ is directed, and $x = \bigsqcup Z$.)

b. $\hat{\bar{Z}} = \widehat{x}$.

c. $\bar{\widehat{Z}} = \widehat{x}$ (or, equivalently, $\widehat{\overline{Z}} = \widehat{x}$).

d. x is both an upper bound and a (topological) limit point of $\bar{Z}$.

Correctness proof. Let us show all the required equivalences and other implicit statements.

(a) $\Rightarrow$ (b): $\hat{\bar{Z}} \subseteq \widehat{x}$ holds by $\bar{Z} \subseteq \widehat{x}$: $x_0 \sqsubseteq \bigsqcup F \in \bar{Z} \Rightarrow x_0 \prec x$. $\hat{\bar{Z}} \supseteq \widehat{x}$ holds by cofinality.

(b) $\Rightarrow$ (c): By (A2) and (b), $x_0 \prec x \Rightarrow x_0 \prec x_1 \prec x \Rightarrow x_0 \prec x_1 \sqsubseteq \bigsqcup F \Rightarrow x_0 \prec \bigsqcup F$.

(c) $\Rightarrow$ (d): By (c), $z \in \bar{Z} \Rightarrow \widehat{z} \subseteq \widehat{x} \Rightarrow z \sqsubseteq x$ and $x_0 \prec x \Rightarrow x_0 \prec z \in \bar{Z}$ for some z.

(d) $\Rightarrow$ (a): Given $\bar{Z} \sqsubseteq x$, we also have $\widehat{Z} \sqsubseteq x$ since $\widehat{Z} \subseteq \bar{Z}$ by definition of $\bar{Z}$. Then, Equations (14) and (16) give $\widehat{Z} = \widehat{\widehat{Z}} \subseteq \widehat{x}$. Hence, by Equation (8), all $\bigsqcup F$ above exist and are $\prec x$, and so $\bar{Z}$ is directed and $\bar{Z} \subseteq \widehat{x}$. Finally, limit point property implies cofinality: $x_0 \prec x \Rightarrow x_0 \prec z \in \bar{Z} \Rightarrow x_0 \sqsubseteq z \in \bar{Z}$.

It remains to show that (any of) the above clauses (a–d) (i.e., formally, $x = \bigsqcup^{\bullet} Z$) imply $x = \bigsqcup Z$. Indeed, let us assume (a–d). Then, $\bar{\widehat{Z}} = \hat{\bar{Z}} = \widehat{x}$, being also directed (as well as $\bar{Z}$). Hence, these identical sets and, by (a) and Equation (16), the sets $\bar{Z} \supseteq \widehat{Z}$, and so the original set Z are all bounded by x. At last, show that x is the ordinary lub of all mentioned 'Z-sets,'

$$x = \bigsqcup Z = \bigsqcup \widehat{Z} = \bigsqcup \bar{Z} = \bigsqcup \hat{\bar{Z}} = \bigsqcup \bar{\widehat{Z}}.$$

So, $x = \bigsqcup \bar{Z} = \bigsqcup \bar{\widehat{Z}}$ holds by (c) and Equation (15) for directed sets, and by $\widehat{\widehat{Z}} = \bar{\widehat{Z}}$ following from Equation (16). The case of $\bigsqcup \hat{\bar{Z}}$ follows trivially. For the first two lubs $x = \bigsqcup Z = \bigsqcup \widehat{Z}$, it suffices to consider $\widehat{Z}$. By definition of $\bar{Z}$ with the known lub x, $\widehat{Z}$ and $\bar{Z}$ have the same upper bounds, resulting in $x = \bigsqcup \widehat{Z}$. □

As we noticed earlier, if Z is directed, then so is $\widehat{Z}$, and hence $\widehat{Z} = \bar{Z} = \widehat{\widehat{Z}} = \bar{\widehat{Z}}$, by using also Equation (16). Therefore, $\widehat{Z} = \widehat{x} \iff \bar{\widehat{Z}} = \widehat{x}$ for directed Z and so both Definitions 6.2 (b) and 6.3 (c) agree.

https://doi.org/10.1017/S0960129516000165 Published online by Cambridge University Press

Theorem 6.2 ('Natural' representation of A-spaces). The above definition of standard $\lfloor\cdot\rfloor$ makes any A-space D a v-base-continuous domain with the same partial order. Thereby, the original open sets of an A-space are exactly those v-Scott open.

Proof. First, show that D is a v-domain satisfying ($\lfloor\cdot\rfloor$1)–($\lfloor\cdot\rfloor$5)

$\lfloor\cdot\rfloor$1. Already considered.

$\lfloor\cdot\rfloor$2. Assume $\bar{\bar{X}} = \hat{x}$, $X \subseteq Y$ and Y is bounded by x. The latter implies $\hat{Y} \subseteq \hat{x}, \bar{Y} \subseteq \hat{x}$ and $\bar{\bar{Y}} \subseteq \hat{x}$. Since $\hat{Z}$, $\bar{Z}$ and so $\bar{\bar{Z}}$ depend monotonically in $Z \subseteq D$ under set inclusion, we have $\bar{\bar{X}} \subseteq \bar{\bar{Y}} \subseteq \hat{x}$ and so $\bar{\bar{X}} = \bar{\bar{Y}} = \hat{x}$.

$\lfloor\cdot\rfloor$3. This is evidently satisfied by $\overbrace{\{x\}} = \hat{x}$.

$\lfloor\cdot\rfloor$4. First, note that for any (non-necessarily directed) family z_i for which $z = \lfloor\cdot\rfloor_{i\in I}\, z_i$ exists we have by definition $\hat{z} = \overbrace{\lfloor\cdot\rfloor_{i\in I}\, z_i} = \overbrace{\overbrace{\{\hat{z}_i\}_{i\in I}}}$, so for $a, a_i, \ldots$ ranging over base elements,

$$a \in \hat{z} \iff a \prec a_{i_1} \sqcup \cdots \sqcup a_{i_m}\ (m > 0) \text{ for some } a_{i_r} \in \hat{z}_{i_r}.$$

Now, for ($\lfloor\cdot\rfloor$4) in terms of families of elements (2), assume (even a weaker condition) that either all the participating (internal and external) v-lubs in $\lfloor\cdot\rfloor_{i\in I} \lfloor\cdot\rfloor_{j\in J_i}\, y_{ij}$ or the lub $\lfloor\cdot\rfloor_{i\in I, j\in J_i}\, y_{ij}$ exist. Then, in each of these cases, the two-parametric family y_{ij} is bounded, and, by Definition 6.3(c), we have, respectively, two equalities for sets of base elements:

$$\overbrace{\lfloor\cdot\rfloor_i \lfloor\cdot\rfloor_j\, y_{ij}} = \overbrace{\overbrace{\{\lfloor\cdot\rfloor_j\, y_{ij}\}_i}} = \overbrace{\overbrace{\{\overbrace{\{\hat{y}_{ij}\}_j}\}_i}} \tag{18}$$

$$\overbrace{\lfloor\cdot\rfloor_{ij}\, y_{ij}} = \overbrace{\overbrace{\{\hat{y}_{ij}\}_{ij}}}. \tag{19}$$

Under any of the above assumptions, the rightmost expressions in Equations (18) and (19) can be shown to be equal sets of base elements. This is evidently proves ($\lfloor\cdot\rfloor$4).

(18) $\subseteq$ (19) : Assume that a belongs to (the last set in Equation) (18), i.e., $a \prec a_{i_1} \sqcup \cdots \sqcup a_{i_m}$ (with the lub existing) for some $a_{i_r} \in \overbrace{\overbrace{\{\hat{y}_{i_r j}\}_j}}$. The latter membership can be analogously rewritten as $a_{i_r} \prec a_{i_r j_1} \sqcup \cdots \sqcup a_{i_r j_n}$ (it suffices $\sqsubseteq$) for some $a_{i_r j_s} \in \hat{y}_{i_r j_s}$. That is, our assumption on a implies $a \prec (a_{i_1 j_1} \sqcup \cdots \sqcup a_{i_1 j_n}) \sqcup \cdots \sqcup (a_{i_m j_1} \sqcup \cdots \sqcup a_{i_m j_n})$ for some $a_{i_r j_s} \in \hat{y}_{i_r j_s}$ (with the lub existing since the family y_{ij} is bounded), that is a belongs to the set in (19), as required.

(19) $\subseteq$ (18) : If $a \prec a_{i_1 j_1} \sqcup \cdots \sqcup a_{i_m j_n}$ for some $a_{i_r j_s} \in \hat{y}_{i_r j_s}$, then we can group the members of this lub as above by the first subscript so that $a \prec (a_{i_1 j_1} \sqcup \cdots \sqcup a_{i_1 j_n}) \sqcup \cdots \sqcup (a_{i_m j_1} \sqcup \cdots \sqcup a_{i_m j_n})$. Let us name the expressions in parentheses as a_{i_r} what gives rise to $a \prec a_{i_1} \sqcup \cdots \sqcup a_{i_m}$. Then, $a_{i_r} \in \overbrace{\{\hat{y}_{i_r j}\}_j}$. Finally, by continuity of $\sqcup$ (A3), there

https://doi.org/10.1017/S0960129516000165 Published online by Cambridge University Press

should exist $a'_{i_1} \prec a_{i_1}, \ldots, a'_{i_m} \prec a_{i_m}$ such that $a \prec a'_{i_1} \sqcup \cdots \sqcup a'_{i_m}$ and $a'_{i_r} \in \overbrace{\{\widehat{y}_{i_r j}\}_j}$. This leads to the required inclusion (19) $\subseteq$ (18).

$\lfloor\cdot\rfloor$ 5. If X is a cofinal subset of Y, then we have straightforwardly $\widehat{X} = \widehat{Y}$, and hence $\bar{X} = \bar{Y}$ and $\widehat{\bar{X}} = \widehat{\bar{Y}}$ so that $\widehat{y} = \widehat{\bar{Y}} \Rightarrow \widehat{y} = \widehat{\bar{X}}$.

Thus, D is a ν-domain with the same $\sqsubseteq$. The rest of proof consists of the following steps:

- $x \lll y \iff x \prec y$ ($\iff y \in \check{x}$).
  ($\Rightarrow$): This follows immediately by the definition of $\lll$ and by $y = \lfloor\cdot\rfloor \widehat{y}$ according to Equation (16) with $\widehat{y}$ directed.
  ($\Leftarrow$): Assume $x \prec y \sqsubseteq z$ and $z = \lfloor\cdot\rfloor \Delta$, that is $\widehat{z} = \widehat{\Delta}$ (15), for any directed Δ. Then, $x \prec z$ and so for some $\delta \in \Delta$, we have $x \prec \delta$, and $x \sqsubseteq \delta$, resulting in $x \lll y$.
- D is ν-base-continuous domain with the base D_0. This follows from (A1), (A2), coincidence of $\lll$ with $\prec$, the equalities $x = \lfloor\cdot\rfloor \check{x}$ and $\check{x} = \Downarrow x \cap D_0$.
- The statement on topologies follows from $\lll \; = \; \prec$ and Proposition 4.1.

□

Note 6.1. Theorem 4.1 on Cartesian closed categories of some classes of ν-base-continuous domains suggests the corresponding classes of A-spaces (or A_0-spaces): with partially defined binary operator $\sqcup$ continuous on the base D_0 (under each base element), in particular, with $D_0 = D$, and also f-spaces discussed below.

Similar results to Theorems 6.1 and 6.2 on correspondence between f-spaces and ν-Scott–Ershov (i.e., ν-algebraic ν-bounded-complete) non-dcpo domains were obtained in Sazonov (2009):

Theorem 6.3. Each ν-algebraic ν-bounded-complete domain becomes an f-space under the topology consisting of ν-Scott open sets and with f-elements exactly those ν-finite.

Theorem 6.4 (On 'Natural' Representation of f-spaces). The above definition of standard $\lfloor\cdot\rfloor$ makes any f-space D a ν-algebraic ν-bounded-complete domain with the same partial order, f-elements exactly those ν-finite, and open sets exactly those ν-Scott open.

We see that *purely topological* f- and A-spaces look technically as simpler concepts than their versions of ν-Scott–Ershov domains and, respectively, ν-base-continuous domains. However, when considering finite type hereditarily-sequential functionals in Sazonov (2007), and, as we expect, in the approach of Section 5, the topology does not appear as a primary concept here since it is rather *derived* as ν-Scott topology via ν-domains with $\lfloor\cdot\rfloor$ just the *pointwise* lub; see Definition 1.1. Actually, the way from this definition and the definition of sequential functionals in the op. cit. to the good properties of ν-domains and the derived (ν-Scott) topology is quite long and complicated, and no shorter and direct way is expectable at present. Hence – the *order theoretic approach* of this paper.

7. Conclusion

The reason of the general non-dcpo approach to domain theory and in particular of writing this paper was that non-dcpo ν-domains arise quite naturally when considering finite type

https://doi.org/10.1017/S0960129516000165 Published online by Cambridge University Press

models like hereditarily sequential functionals $\{\mathbb{Q}_\alpha\}$ over one base type $\mathbb{Q}_{\mathsf{N}} = \mathbb{N}_\perp$ of the natural numbers where the ν-lub operator $\lfloor\cdot\rfloor$ is defined (see Definition 1.1 above) just as a pointwise lub – a restricted version of the ordinary lub operator $\bigsqcup$ (Sazonov 2007, 2009). This makes such a typed model having very natural domain theoretic properties, in contrast to the case when the ordinary $\bigsqcup$ is considered. Thus, we expect that in particular realistic situations such as typed functional models $\lfloor\cdot\rfloor$ will be defined quite simply and naturally. On the other hand, a general axiomatic approach requires axioms ($\lfloor\cdot\rfloor$1)–($\lfloor\cdot\rfloor$5) looking somewhat more complicated than just saying 'pointwise lub.' Anyway, we see that they are quite manageable and the theory is rather smooth, just a natural generalization of the ordinary dcpo domain theory.

Another point initiated here, which is crucial for further developing this approach, is considering a wider class of ν–continuous domains (with $\sqcup$ total and ν–continuous under any element) rather than ν-algebraic (and ν-bounded-complete) domains in the op. cit. The goal could be a general domain theoretic as we demonstrated in this paper, in particular, by considering ccc of such domains in Theorem 4.1. But more specifically, we intend to deal with finite type functionals over the real numbers as outlined in Section 5 by involving into consideration the continuous interval domain $\mathcal{I}_\perp$ of the real numbers (in $[0,1]$) or its Hoare powerdomain $\mathcal{P}^H\mathcal{I}_\perp$ and relating ν-base-continuous non-dcpo ν–domains with programming languages like Real PCF (Escardó 1996; Marcial-Romero and Escardó 2007). This is the subject of further work analogous to considering sequential computational strategies and hereditarily sequential functionals over $\mathbb{N}_\perp$ (Sazonov 2007). This way quite abstract considerations of this paper will interplay with a PCF-like programming language and with a kind of computability in the line of how and why domain theory arose originally.

The third point is that non-dcpo domains have already been introduced long time ago as general f-spaces and A-spaces (Ershov 1972, 1974). However, these non-dcpo spaces were considered mainly from a general theoretical perspective, mainly topological. Our approach in this paper is also just a general order theoretic one, but in Sazonov (2007, 2009), it has already been demonstrated how particular examples of non-dcpo f-spaces can arise naturally, from some concrete considerations on (hereditarily) sequential and wittingly consistent functionals over $\mathbb{N}_\perp$. The current more general approach, if unified with a more concrete consideration on Real PCF or the like, could hopefully lead us to realistic, more computationally motivated examples and considerations of ν-continuous domains and A-spaces.

The author is grateful to the editor of this issue Dieter Spreen for inviting to present a paper, and also to the anonymous referees for numerous useful comments helping to considerably improve the exposition.

Appendix A. ν-Domains vs. δ-Domains

Our goal here is to establish a relationship between ν-domains and δ-domains. We will eventually confine ourselves to ν-domains. However, the point of considering δ-domains could be the fact that, as in the ordinary domain theory, the main stress will be on lubs of directed families and so δ-domains could seem more appropriate concept. Here, we

https://doi.org/10.1017/S0960129516000165 Published online by Cambridge University Press

will show that a technically simpler concept of v-domains is in a sense sufficient and 'equivalent' to that of δ-domains. In principle, we could continue demonstrating and exploring such an equivalence through the rest of the paper, but will refrain from doing that and will mention δ-domains until the first part of Section 3 only.

Proposition A.1. In a v-domain $\langle D, \sqsubseteq, \lfloor\cdot\rfloor\rangle$, let $\lfloor\cdot\rfloor^{\downarrow\delta} \subseteq \lfloor\cdot\rfloor$ be $\lfloor\cdot\rfloor$ restricted to directed subsets where it is defined. This makes $\langle D, \sqsubseteq, \lfloor\cdot\rfloor^{\downarrow\delta}\rangle$ a δ-domain.

Proof. It consists in checking straightforwardly versions of ($\lfloor\cdot\rfloor$1)–($\lfloor\cdot\rfloor$5) for δ-domains (see Definition 2.1 (d)) in the case of $\lfloor\cdot\rfloor^{\downarrow\delta}$ and is left for the reader. □

Thus, by forgetting non-directed sets v-domains can also be treated as δ-domains. Conversely,

Proposition A.2. Any δ-domain $\langle D, \sqsubseteq, \lfloor\cdot\rfloor\rangle$ (in fact, any v-pre-domain) is expandable to a v-domain $\langle D, \sqsubseteq, \lfloor\cdot\rfloor^{\uparrow v}\rangle$ with $\lfloor\cdot\rfloor \subseteq \lfloor\cdot\rfloor^{\uparrow v}$, by taking a '$v$-closure' so that

$$\lfloor\cdot\rfloor^{\uparrow v} \text{ is the least } v\text{-lub operator on } D \text{ extending } \lfloor\cdot\rfloor.$$

Similarly, any v-pre-domain where all v-sets are directed can be expanded to a δ-domain by an appropriate extending (closing) $\lfloor\cdot\rfloor$ denoted as $\lfloor\cdot\rfloor^{\uparrow\delta}$.

Proof. For showing the existence of required (the least...) $\lfloor\cdot\rfloor^{\uparrow v}$, it mainly suffices to note that the above clauses ($\lfloor\cdot\rfloor$1)–($\lfloor\cdot\rfloor$5) in the definition (c) (or, resp., (d)) can equivalently be reformulated as properties (N1)–(N5) below of the class of v-sets in $\langle D, \sqsubseteq, \lfloor\cdot\rfloor\rangle$ where (N2)–(N5) are closure properties for this class:

- N1. For all v-sets $X \subseteq D$, $\bigsqcup X$ exists (and, resp., X is directed).
- N2. If $X \subseteq Y \subseteq D$, X is a v-set, (and, resp., Y is directed) and $Y \sqsubseteq \bigsqcup X$, then Y is a v-set too.
- N3. Any singleton set $\{x\}$ is a v-set.
- N4. Let $\{Y_i\}_{i\in I}$ be a (resp., directed) family of (resp., directed) v-subsets of D. Then, the lubs $y_i = \bigsqcup Y_i$ constitute a v-set *iff* the union $\bigcup_{i\in I} Y_i$ is a v-set. (In this case, $\bigsqcup_{i\in I} y_i = \bigsqcup \bigcup_{i\in I} Y_i$.)
- N5. If $X \subseteq Y \subseteq D$, Y is a v-set, and X is (resp., directed and) cofinal with Y, then X is a v-set too.

Any $X \subseteq D$ such that $\bigsqcup X$ exists is called a $\bigsqcup$-*set* . Let $D_{\bigsqcup}$ be a class of all $\bigsqcup$-subsets of D.

Lemma A.1. Evidently, $D_{\bigsqcup}$ is a class of sets in a domain (poset) D satisfying (N1)–(N5). In fact, it is the largest possible such a class of sets in D. The least class of sets in D satisfying (N1)–(N5) consists of sets $X \subseteq D$ each having a largest element $x \in X$ so that $\bigsqcup X = x$. Moreover, the classes of sets in a domain D satisfying (N1)–(N5) are closed under intersection and so constitute a complete lattice under the class inclusion relation $\subseteq$. □

Then (N2)–(N5) evidently define the corresponding monotonic operator $\mathcal{N} : 2^{D_{\bigsqcup}} \to 2^{D_{\bigsqcup}}$ relative to $\subseteq$. Assuming that $\mathcal{X}, \mathcal{Y}, \ldots$ range over subsets of $D_{\bigsqcup}$, define the *closure* of

https://doi.org/10.1017/S0960129516000165 Published online by Cambridge University Press

$\mathcal{Y} \subseteq D_{\bigsqcup}$ by $\mathcal{N}$ as

$$Cl_{\mathcal{N}}(\mathcal{Y}) = \bigcap\{\mathcal{X} \subseteq D_{\bigsqcup} \mid \mathcal{Y} \cup \mathcal{N}(\mathcal{X}) \subseteq \mathcal{X}\}.$$

Equivalently, by theory of inductive definitions (Aczel 1977), $Cl_{\mathcal{N}}(\mathcal{Y})$ is the least fixed point of the operator

$$\mathcal{N}_{\mathcal{Y}}(\mathcal{X}) = \mathcal{Y} \cup \mathcal{N}(\mathcal{X}) \tag{20}$$

so that for all $\mathcal{Y}$, we have

$$\begin{aligned} &Cl_{\mathcal{N}}(\mathcal{Y}) = \mathcal{Y} \cup \mathcal{N}(Cl_{\mathcal{N}}(\mathcal{Y})), \quad \mathcal{Y} \subseteq Cl_{\mathcal{N}}(\mathcal{Y}), \\ &\mathcal{N}(Cl_{\mathcal{N}}(\mathcal{Y})) \subseteq Cl_{\mathcal{N}}(\mathcal{Y}), \quad \text{and} \quad Cl_{\mathcal{N}}(Cl_{\mathcal{N}}(\mathcal{Y})) = Cl_{\mathcal{N}}(\mathcal{Y}). \end{aligned} \tag{21}$$

In fact, all, except the last equality (actually, for $\subseteq$) are evident. We can re-define equivalently the least fixed point $Cl_{\mathcal{N}}(\mathcal{Y})$ by transfinite recursion as in Aczel (1977) and use transfinite induction. Alternatively (by suggestion of the referee), we can do that even without using transfinite recursion. To this end, note that for any set $\mathcal{Z}$, $Cl_{\mathcal{N}}(\mathcal{Z})$ is the least fixed point of the operator $\mathcal{N}_{\mathcal{Z}}$, and so for $\mathcal{Z} = Cl_{\mathcal{N}}(\mathcal{Y})$, $Cl_{\mathcal{N}}(Cl_{\mathcal{N}}(\mathcal{Y}))$ is the least fixed point of the operator $\mathcal{N}_{Cl_{\mathcal{N}}(\mathcal{Y})}$. Thus, in order to establish $Cl_{\mathcal{N}}(Cl_{\mathcal{N}}(\mathcal{Y})) \subseteq Cl_{\mathcal{N}}(\mathcal{Y})$, it suffices to show $\mathcal{N}_{Cl_{\mathcal{N}}(\mathcal{Y})}(Cl_{\mathcal{N}}(\mathcal{Y})) \subseteq Cl_{\mathcal{N}}(\mathcal{Y})$, that is, by (20), $Cl_{\mathcal{N}}(\mathcal{Y}) \cup \mathcal{N}(Cl_{\mathcal{N}}(\mathcal{Y})) \subseteq Cl_{\mathcal{N}}(\mathcal{Y})$. But the latter holds by the first part of Equation (21).

At last, define $\lfloor\cdot\rfloor^{\uparrow\nu} = \lfloor\bigsqcup\rfloor \uparrow Cl_{\mathcal{N}}(\{X \subseteq D \mid \lfloor\cdot\rfloor X \text{ exists}\})$. □

Proposition A.3. Transformations

$$\uparrow \nu : \{\delta\text{-lubs } \lfloor\vartriangle\rfloor\} \to \{\nu\text{-lubs } \lfloor\cdot\rfloor\} \quad \text{and} \quad \downarrow \delta : \{\nu\text{-lubs } \lfloor\cdot\rfloor\} \to \{\delta\text{-lubs } \lfloor\vartriangle\rfloor\}$$

over the same poset D constitute a *monotone Galois connection*, i.e., satisfy

$$\lfloor\vartriangle\rfloor^{\uparrow\nu} \subseteq \lfloor\cdot\rfloor \iff \lfloor\vartriangle\rfloor \subseteq \lfloor\cdot\rfloor^{\downarrow\delta}$$

or, equivalently, Equations (22) and (24) hold, as well as all the following conditions:

$$\lfloor\cdot\rfloor^{\downarrow\delta\uparrow\nu} \subseteq \lfloor\cdot\rfloor \tag{22}$$

$$\lfloor\cdot\rfloor^{\downarrow\delta\uparrow\nu\downarrow\delta} = \lfloor\cdot\rfloor^{\downarrow\delta}$$

$$\lfloor\cdot\rfloor^{\downarrow\delta\uparrow\nu\downarrow\delta\uparrow\nu} = \lfloor\cdot\rfloor^{\downarrow\delta\uparrow\nu} \tag{23}$$

$$\lfloor\vartriangle\rfloor^{\uparrow\nu\downarrow\delta} \supseteq \lfloor\vartriangle\rfloor \tag{24}$$

$$\lfloor\vartriangle\rfloor^{\uparrow\nu\downarrow\delta\uparrow\nu} = \lfloor\vartriangle\rfloor^{\uparrow\nu}$$

$$\lfloor\vartriangle\rfloor^{\uparrow\nu\downarrow\delta\uparrow\nu\downarrow\delta} = \lfloor\vartriangle\rfloor^{\uparrow\nu\downarrow\delta} \tag{25}$$

Hence, compositions $\uparrow \nu \downarrow \delta$ and $\downarrow \delta \uparrow \nu$ are, respectively, *closure* (24), (25) and *kernel* (22, 23) operators over δ-lubs and, resp., ν-lubs. It also proves that

- $\uparrow \nu$ transforms the class of all δ-lubs $\lfloor\vartriangle\rfloor$ *onto* the class of all *kernel* ν-lubs $\lfloor\cdot\rfloor$ (i.e., such that $\lfloor\cdot\rfloor^{\downarrow\delta\uparrow\nu} = \lfloor\cdot\rfloor$).
  Thus, $\lfloor\vartriangle\rfloor^{\uparrow\nu}$ ranges over all *kernel* ν-lubs.
- $\downarrow \delta$ transforms the class of all ν-lubs $\lfloor\cdot\rfloor$ *onto* the class of all *closed* δ-lubs $\lfloor\vartriangle\rfloor$ (i.e., such that $\lfloor\vartriangle\rfloor^{\uparrow\nu\downarrow\delta} = \lfloor\vartriangle\rfloor$).
  Thus, $\lfloor\cdot\rfloor^{\downarrow\delta}$ ranges over all *closed* δ-lubs.

https://doi.org/10.1017/S0960129516000165 Published online by Cambridge University Press

- Actually, these two transformations establish isomorphism under $\subseteq$ between *closed* δ-lubs $\lfloor \triangleq \rfloor$ and *kernel* ν-lubs $\lfloor \cdot \rfloor$.

Proof. This is standard and straightforward for Galois connections. □

The last item means that closed δ-lubs and kernel ν-lubs are uniquely determined one by another via $\uparrow \nu$ and $\downarrow \delta$ and, in a reasonable sense, can be identified. But we will not develop further this identification through the whole approach of this paper and will mostly concentrate on ν-domains rather than δ-domains. We only make a last comment on this topic that closed δ-lubs and kernel ν-lubs are quite natural concepts since

- A closed δ-lub $\lfloor \triangleq \rfloor$ is, by definition, a δ-lub whose class of δ-sets is non-extendable by new directed ν-sets by $\uparrow \nu$.
- A kernel ν-lub $\lfloor \cdot \rfloor$ is, by definition, a ν-lub whose class of ν-sets is 'non-restrictable irrevocably' by $\downarrow \delta$, i.e., after restricting all non-directed ν-sets are recoverable by $\uparrow \nu$.

References

Abramsky, S., Jagadeesan, R. and Malacaria, P. (2000). Full abstraction for PCF. *Information and Computation* **163** (2) 409–470.

Abramsky, S. and Jung, A. (1994). Domain theory. In: Abramsky, S., Gabbay, D. and Maibaum T.S.E. (eds.) *Handbook of Logic in Computer Science*, volume III, Oxford University Press, 1–168.

Aczel, P. (1977). An introduction to inductive definitions. In: *Handbook of Mathematical Logic*. Studies in Logic and the Foundations of Mathematics, volume 90, Elsevier Science Publishers B.V., North Holland, 739–782.

Amadio, R.M. and Curien, P.-L. (1998). *Domains and Lambda Calculus*, Cambridge Tracts in Theoretical Computer Science, Cambridge University Press.

Ershov, Yu. L. (1972). Computable functionals of finite types. *Algebra and Logic* **11** (4) 367–437. (The journal is translated in English.)

Ershov, Yu.L. (1974). A theory of A-spaces. *Algebra and Logics* **12** (4) 369–416. (The journal is translated in English.)

Escardó, M.H. (1996). PCF extended with real numbers. *Theoretical Computer Science* **162** (1) 79–115.

Escardó, M.H., Hofmann, M. and Streicher, Th. (2004). On the non-sequential nature of the interval-domain model of real-number computation. In: *Mathematical Structures in Computer Science*, **14** (6), Cambridge University Press, 803–814.

Escardó, M.H. and Ho, W.K. (2009). Operational domain theory and topology of sequential programming languages. *Information and Computation* **207** (3) 411–437.

Gierz, G., Hofmann, K.H., Keimel, K., Lawson, J.D., Mislove, M. and Scott, D.S. (2003). *Continuous Lattices and Domains*, Cambridge University Press.

Hyland, J. M. E. and Ong, C.-H.L. (2000). On full abstraction for PCF: I, II, and III. *Information and Computation* **163** (2) 285–408.

Marcial-Romero, J.R. and Escardó, M.H. (2007). Semantics of a sequential language for exact real-number computation. *Theoretical Computer Science* **379** (1–2) 120–141.

Nickau, H. (1996). *Hereditarily-Sequential Functionals: A Game-Theoretic Approach to Sequentiality*, Ph.D. Thesis, Siegen University, Siegen.

https://doi.org/10.1017/S0960129516000165 Published online by Cambridge University Press

Normann, D. (2006). On sequential functionals of type 3. *Mathematical Structures in Computer Science* **16** (2) 279–289.

Normann, D. and Sazonov, V.Yu. (2012). The extensional ordering of the sequential functionals. *Annals of Pure and Applied Logic* **163** (5) 575–603.

Plotkin, G. (1977). LCF considered as a programming language. *Theoretical Computer Science* **5** (3) 223–255.

Sazonov, V.Yu. (1976a). Functionals computable in series and in parallel. *Sibirskii Matematicheskii Zhurnal* **17** (3) 648–672. The journal is translated in English as *Siberian Mathematical Journal.*

Sazonov, V.Yu. (1976b). Expressibility of functionals in D.Scott's LCF language. *Algebra and Logic* **15** (3) 308–330. (The journal is translated in English.).

Sazonov, V.Y. (1976c). *On Semantics of the Applicative Algorithmic Languages*, Ph.D. Thesis, Institute of Mathematics, Novosibirsk (in Russian).

Sazonov, V.Y. (2007). Inductive definition and domain theoretic properties of fully abstract models for PCF and PCF^{+}. *Logical Methods in Computer Science* **3** (3:7) 1–50. `http://www.lmcs-online.org`.

Sazonov, V. (2009). Natural non-dcpo domains and f-spaces. *Annals of Pure and Applied Logic* **159** (3) 341–355.

Wright, J.B., Wagner, E.G. and Thatcher, J.W. (1978). A uniform approach to inductive posets and inductive closure. *Theoretical Computer Science* **7** (1) 57–77.

https://doi.org/10.1017/S0960129516000165 Published online by Cambridge University Press