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# Poisson brackets for guiding-centre and gyrocentre theories

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**Abstract.** Poisson brackets in the phase space of averaging Kruskal coordinates are obtained in a clear and straightforward way. The derivation makes use of the equations of motion of guiding centres and gyrocentres derived from a gyroangle-independent Lagrangian, and from generally valid relations of Hamiltonian mechanics. The usual procedure of matrix inversion to obtain the Poisson tensor from the Lagrange tensor is not required.

## 1. Introduction

In Hamiltonian mechanics, the dynamical evolution of a system is conveniently represented in terms of Poisson brackets. Any dynamical Function $G(\mathbf{x}, \mathbf{p}, t)$, which depends on the canonical variables $\mathbf{x}, \mathbf{p}$ and on time $t$, evolves according to the equation

$$\frac{dG}{dt} = [G, H] + \left.\frac{\partial G}{\partial t}\right|_{\mathbf{x},\mathbf{p}}, \tag{1.1}$$

where $H(\mathbf{x}, \mathbf{p}, t)$ is the Hamiltonian and $[G, H]$ is the usual Poisson bracket in the coordinates $\mathbf{x}, \mathbf{p}$. For many applications and, in particular, for the investigation of guiding-centre and gyrocentre motion, it is convenient to use certain non-canonical coordinates, which are better suited for describing a separation of the fast gyration of particles around magnetic field lines from the rest of the motion. The dynamical law (1.1) is still valid in this case. However, it is necessary to express the Poisson brackets in terms of the new set of non-canonical phase-space variables. By making use of the formulation of guiding-centre and gyrocentre motion based on a gyroangle-independent Lagrangian as given in [1,2], we present here a new and relatively easy method for deriving the Poisson brackets as functions of the averaging coordinates. The usual procedure of matrix inversion to obtain the Poisson tensor from the Lagrange tensor, as applied in guiding-centre theories [3–5] and also adopted in gyrokinetic theories [6, 7], is not required.

## 2. Basic description

The motion of particles of mass $m$ and charge $e$ in an electromagnetic field $\mathbf{E}(\mathbf{x}, t)$ and $\mathbf{B}(\mathbf{x}, t)$, with potentials $\Phi(\mathbf{x}, t)$ and $\mathbf{A}(\mathbf{x}, t)$, respectively, is described by the

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usual canonical Hamiltonian for particles in an electromagnetic field, namely

$$H(\mathbf{x},\mathbf{p},t)=\frac{1}{2m}\left[\mathbf{p}-\frac{e}{c}\mathbf{A}(\mathbf{x},t)\right]^2+e\Phi(\mathbf{x},t), \tag{2.1}$$

with corresponding phase-space Lagrangian

$$L(\mathbf{p},\mathbf{x};\dot{\mathbf{x}};t)=\mathbf{p}\cdot\dot{\mathbf{x}}-H(\mathbf{x},\mathbf{p},t). \tag{2.2}$$

Alternatively, the motion can be described by Newton's equations of motion,

$$\frac{d\mathbf{v}}{dt}=\frac{e}{m}\left(\mathbf{E}+\frac{1}{c}\mathbf{v}\times\mathbf{B}\right), \tag{2.3}$$

which can conveniently be reformulated by representing the particle velocity as a superposition of a velocity $u_\parallel$ parallel to the magnetic field, a fast gyration $u_\perp\mathbf{n}_1$ around the field, and a drift $\mathbf{v}_E=(c/B^2)\mathbf{E}\times\mathbf{B}$ across the field,

$$\mathbf{v}=u_\parallel\mathbf{b}+u_\perp\mathbf{n}_1+\mathbf{v}_E, \tag{2.4}$$

where $\mathbf{n}_1=\mathbf{n}_1(\mathbf{x},\theta,t)$ is a unit vector perpendicular to the magnetic field at point $\mathbf{x}$ and time $t$, characterized by the angle $\theta$. Details about this kind of treatment can be found in many articles and books, e.g. the series of fundamental papers by Littlejohn (among them [3–5, 8]), and the book by Balescu [9]. For the present paper, the description given in [1, 2] is useful. In any case, the aim is to find a way of separating the fast gyration motion around the field lines of a strong magnetic field from the rest of the motion, a goal which can be reached by Kruskal's method of constructing averaging coordinates [10], which we prefer to use because of its conceptual clarity and straightforwardness, instead of the method of Lie transforms commonly used in the current literature.

## 3. Averaging coordinates

The basic procedure consists of constructing a set of phase-space coordinates whose dynamical development is independent of a gyroangle $\phi$. In his paper [10], Kruskal exhaustively discusses many important and fundamental aspects of this transformation of variables. One crucial result is the existence of the asymptotic constant of motion $J$ (basically, the magnetic moment). A modification of Kruskal's method allows one to construct the coordinate transformation so as to express the original phase space coordinates in terms of the new ones [1, 2, 11], which is necessary for our treatment. By expressing the position vector $\mathbf{x}$ and the canonical momentum $\mathbf{p}$ in terms of $J$-based averaging variables $z_1, z_2, z_3, z_4, J$ and $\phi$,

$$\mathbf{x}=\mathfrak{x}(z_1,\ldots,z_4,J,\phi,t),\quad \mathbf{p}=\mathfrak{p}(z_1,\ldots,z_4,J,\phi,t), \tag{3.1}$$

where $\mathfrak{x}$ and $\mathfrak{p}$ are periodic functions of $\phi$ with period 1, one obtains from (2.2) the Lagrangian in a form which can be made independent of the gyroangle $\phi$ by means of an appropriate phase-space gauge transformation.

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## 4. Gyroangle-independent Lagrangian

In [1, 2], the following Lagrangian based on averaging Kruskal coordinates was derived:

$$L(z_1,\ldots,z_4,J;\dot z_1,\ldots,\dot z_4,\dot J,\dot\phi;t)=\sum_{i=1}^{4}\dot z_i\left\langle \mathfrak{p}\cdot\frac{\partial\mathfrak{x}}{\partial z_i}\right\rangle+J\dot\phi$$
$$+\dot J\left\langle \mathfrak{p}\cdot\frac{\partial\mathfrak{x}}{\partial J}\right\rangle-\langle H\rangle+\left\langle \mathfrak{p}\cdot\frac{\partial\mathfrak{x}}{\partial t}\right\rangle. \tag{4.1}$$

This Lagrangian does not depend on $\phi$. There are no approximations in the reduced (i.e. $\phi$-independent) Lagrangian $L$. In particular, there is no approximation concerning $\phi$. Any results obtained from this Lagrangian are therefore exact.

In contrast to the treatment in [1,2], we keep here the terms $J\dot\phi$ and $\dot J\langle\mathfrak{p}\cdot(\partial\mathfrak{x}/\partial J)\rangle$ because we now also want to describe the fast gyromotion in order to be able to obtain all Poisson brackets in the space $(\mathbf{R},U,J,\phi;t)$ later. The Lagrangian of (4.1) agrees with Larsson's results [11] obtained in a somewhat different way within the framework of guiding-centre theory.

## 5. The equations of motion

As in [1, 2], we use a convenient vector notation to express the Lagrangian $L$ and the corresponding Euler–Lagrange equations, introducing a vector $\mathbf{R}$ with three Cartesian components,

$$\mathbf{R}=(R_1,R_2,R_3),\quad R_i:=z_i,\quad i=1,2,3. \tag{5.1}$$

The fourth coordinate in phase space is designated by $U$,

$$U:=z_4. \tag{5.2}$$

For the gradient with respect to $\mathbf{R}$ we write either $\partial/\partial\mathbf{R}$ or $\nabla_{\mathbf{R}}$, as appears convenient. The Lagrangian $L$ (4.1) can then be written as

$$L(\mathbf{R},U,J;\dot{\mathbf{R}},\dot U,\dot J,\dot\phi;t)=\frac{e}{c}[\dot{\mathbf{R}}\cdot\hat{\mathbf{A}}(\mathbf{R},U,J;t)+\dot U\hat A_U+\dot J\hat A_J+\dot\phi\hat A_\phi]$$
$$-e\hat\Phi(\mathbf{R},U,J;t), \tag{5.3}$$

where

$$\hat{\mathbf{A}}(\mathbf{R},U,J;t):=\frac{c}{e}\left\langle\frac{\partial\mathfrak{x}}{\partial\mathbf{R}}\cdot\mathfrak{p}\right\rangle, \tag{5.4}$$

$$\hat A_U(\mathbf{R},U,J;t):=\frac{c}{e}\left\langle\frac{\partial\mathfrak{x}}{\partial U}\cdot\mathfrak{p}\right\rangle, \tag{5.5}$$

$$\hat A_J(\mathbf{R},U,J;t):=\frac{c}{e}\left\langle\frac{\partial\mathfrak{x}}{\partial J}\cdot\mathfrak{p}\right\rangle, \tag{5.6}$$

$$\hat A_\phi(\mathbf{R},U,J;t):=\frac{c}{e}\left\langle\frac{\partial\mathfrak{x}}{\partial\phi}\cdot\mathfrak{p}\right\rangle=\frac{c}{e}J, \tag{5.7}$$

$$e\hat\Phi(\mathbf{R},U,J;t):=\langle H\rangle-\left\langle\mathfrak{p}\cdot\frac{\partial\mathfrak{x}}{\partial t}\right\rangle. \tag{5.8}$$

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The Euler–Lagrange equations obtained by varying $\phi$, $J$, $U$ and $\mathbf{R}$ are, respectively,

$$\dot{J}(\mathbf{R}, U, J; t) = 0, \tag{5.9}$$

$$-\frac{1}{c}\frac{d}{dt}\hat{A}_J + \frac{1}{c}\frac{\partial \hat{\mathbf{A}}}{\partial J}\cdot\dot{\mathbf{R}} + \frac{1}{c}\frac{\partial \hat{A}_U}{\partial J}\dot{U} + \frac{1}{e}\dot{\phi} - \frac{\partial \hat{\Phi}}{\partial J} = 0, \tag{5.10}$$

$$-\frac{1}{c}\frac{d}{dt}\hat{\mathbf{A}} + \frac{1}{c}\frac{\partial \hat{\mathbf{A}}}{\partial \mathbf{R}}\cdot\dot{\mathbf{R}} + \frac{1}{c}\frac{\partial \hat{A}_U}{\partial \mathbf{R}}\dot{U} - \frac{\partial \hat{\Phi}}{\partial \mathbf{R}} = 0, \tag{5.11}$$

and

$$-\frac{1}{c}\frac{d}{dt}\hat{A}_U + \frac{1}{c}\frac{\partial \hat{\mathbf{A}}}{\partial U}\cdot\dot{\mathbf{R}} + \frac{1}{c}\frac{\partial \hat{A}_U}{\partial U}\dot{U} - \frac{\partial \hat{\Phi}}{\partial U} = 0. \tag{5.12}$$

Since $\dot{J} = 0$, the total time derivative is given by

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \dot{\mathbf{R}}\cdot\frac{\partial}{\partial \mathbf{R}} + \dot{U}\frac{\partial}{\partial U} + \dot{\phi}\frac{\partial}{\partial \phi}. \tag{5.13}$$

With the fields $\hat{\mathbf{B}}$, $\hat{\mathbf{E}}$ and $\mathbf{b}_U$ defined by

$$\hat{\mathbf{B}} := \nabla_{\mathbf{R}} \times \hat{\mathbf{A}}, \tag{5.14}$$

$$\hat{\mathbf{E}} := -\frac{1}{c}\frac{\partial \hat{\mathbf{A}}}{\partial t} - \nabla_{\mathbf{R}}\hat{\Phi}, \tag{5.15}$$

$$\mathbf{b}_U := \frac{e}{mc}\left(\frac{\partial \hat{\mathbf{A}}}{\partial U} - \frac{\partial \hat{A}_U}{\partial \mathbf{R}}\right), \tag{5.16}$$

(5.11) and (5.12) become, respectively,

$$\frac{1}{c}\dot{\mathbf{R}}\times\hat{\mathbf{B}} + \hat{\mathbf{E}} - \frac{m}{e}\dot{U}\mathbf{b}_U = 0 \tag{5.17}$$

and

$$\frac{m}{e}\dot{\mathbf{R}}\cdot\mathbf{b}_U - \frac{\partial \hat{\Phi}}{\partial U} - \frac{1}{c}\frac{\partial \hat{A}_U}{\partial t} = 0. \tag{5.18}$$

After crossing (5.17) with $\mathbf{b}_U$ and inserting $\dot{\mathbf{R}}\cdot\mathbf{b}_U$ from (5.18), one can obtain the solution for $\dot{\mathbf{R}}$. With the definition

$$\mathscr{B}(\mathbf{R}, U, J; t) := \hat{\mathbf{B}}\cdot\mathbf{b}_U = \hat{\mathbf{B}}\cdot\frac{e}{mc}\left(\frac{\partial \hat{\mathbf{A}}}{\partial U} - \frac{\partial \hat{A}_U}{\partial \mathbf{R}}\right) \tag{5.19}$$

one obtains the velocity $\dot{\mathbf{R}}$,

$$\dot{\mathbf{R}} = \mathbf{V}_{\mathbf{R}}(\mathbf{R}, U, J; t), \tag{5.20}$$

with

$$\mathbf{V}_{\mathbf{R}}(\mathbf{R}, U, J; t) = \frac{e}{m\mathscr{B}}\left(\frac{\partial \hat{\Phi}}{\partial U} + \frac{1}{c}\frac{\partial \hat{A}_U}{\partial t}\right)\hat{\mathbf{B}} + \frac{c}{\mathscr{B}}\hat{\mathbf{E}}\times\mathbf{b}_U. \tag{5.21}$$

The scalar product of (5.17) with $\hat{\mathbf{B}}$ yields

$$\dot{U} = V_U(\mathbf{R}, U, J; t), \tag{5.22}$$

where

$$V_U(\mathbf{R}, U, J; t) = \frac{e}{m\mathscr{B}}\hat{\mathbf{B}}\cdot\hat{\mathbf{E}}, \tag{5.23}$$

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which, using (5.21), can be written in the alternative form

$$V_U = \frac{\mathbf{V}_{\mathbf{R}} \cdot \hat{\mathbf{E}}}{\partial\hat{\Phi}/\partial U + (1/c)\partial\hat{A}_U/\partial t}. \tag{5.24}$$

By replacing $\hat{\mathbf{E}}$ according to (5.15), (5.24) yields the energy-related equation,

$$\frac{d(e\hat{\Phi})}{dt} = -\mathbf{V}_{\mathbf{R}} \cdot \frac{e}{c}\frac{\partial\hat{\mathbf{A}}}{\partial t} - \frac{e}{c}V_U\frac{\partial\hat{A}_U}{\partial t} + \left.\frac{\partial(e\hat{\Phi})}{\partial t}\right|_{\mathbf{R},U,J}. \tag{5.25}$$

It is seen that $\hat{\Phi}$ is conserved when $\hat{\mathbf{A}}$, $\hat{A}_U$ and $\hat{\Phi}$ do not depend explicitly on time and $e\,\hat{\Phi} = \langle H\rangle$. Finally, (5.10) can be written as

$$\begin{aligned}\dot{\phi} = V_\phi(\mathbf{R},U,J;t) &:= e\frac{\partial\hat{\Phi}}{\partial J} + \frac{e}{c}\frac{\partial\hat{A}_J}{\partial t} + \frac{e}{c}\mathbf{V}_{\mathbf{R}} \cdot \left(\frac{\partial\hat{A}_J}{\partial\mathbf{R}} - \frac{\partial\hat{\mathbf{A}}}{\partial J}\right) + \frac{e}{c}V_U\left(\frac{\partial\hat{A}_J}{\partial U} - \frac{\partial\hat{A}_U}{\partial J}\right)\\ &= e\frac{\partial\hat{\Phi}}{\partial J} + \frac{e}{c}\frac{\partial\hat{A}_J}{\partial t} + \frac{1}{c}\frac{e}{m\mathscr{B}}\left[\hat{\mathbf{B}} \cdot \left(\frac{\partial\hat{A}_J}{\partial\mathbf{R}} - \frac{\partial\hat{\mathbf{A}}}{\partial J}\right)\right] e\left(\frac{\partial\hat{\Phi}}{\partial U} + \frac{1}{c}\frac{\partial\hat{A}_U}{\partial t}\right)\\ &\quad + \frac{1}{c}\frac{e}{m\mathscr{B}}\left[\frac{mc}{e}\mathbf{b}_U \times \left(\frac{\partial\hat{A}_J}{\partial\mathbf{R}} - \frac{\partial\hat{\mathbf{A}}}{\partial J}\right) + \left(\frac{\partial\hat{A}_J}{\partial U} - \frac{\partial\hat{A}_U}{\partial J}\right)\hat{\mathbf{B}}\right] \cdot e\hat{\mathbf{E}}.\end{aligned} \tag{5.26}$$

Thus, one advantage of using the vector representation is that it is easy to solve for $\dot{\mathbf{R}}$, $\dot{U}$ and $\dot{\phi}$ as functions of $\mathbf{R},U,J,t$, while the (equivalent) solution of the equations for $\dot{z}_j$ $(z_1,\ldots,z_4,J,\phi;t)$, $j=1,\ldots,4$, obtained from Hamilton's principle with the Lagrangian $L$ (4.1), is more involved.

## 6. Time derivatives

We now wish to derive the relation between the time derivatives of functions in the phase spaces $(\mathbf{R},U,J,\phi;t)$ and $(\mathfrak{x},\mathfrak{p},t)$.

### *6.1. Partial time derivatives*

Let $G(\mathfrak{x},\mathfrak{p},t) = G(\mathfrak{x}(\mathbf{R},U,J,\phi;t),\mathfrak{p}(\mathbf{R},U,J,\phi;t),t)$ be an arbitrary function of the phase-space coordinates and of time. One then has, with the summation convention with respect to the index $i$,

$$\begin{aligned}\left.\frac{\partial G}{\partial t}\right|_{\mathbf{R},U,J,\phi} &= \frac{\partial\mathfrak{x}}{\partial t}\cdot\frac{\partial G}{\partial\mathfrak{x}} + \frac{\partial\mathfrak{p}}{\partial t}\cdot\frac{\partial G}{\partial\mathfrak{p}} + \left.\frac{\partial G}{\partial t}\right|_{\mathfrak{x},\mathfrak{p}}\\ &= \frac{\partial\mathfrak{x}}{\partial t}\cdot[G,\mathfrak{p}] - \frac{\partial\mathfrak{p}}{\partial t}\cdot[G,\mathfrak{x}] + \left.\frac{\partial G}{\partial t}\right|_{\mathfrak{x},\mathfrak{p}}\\ &= \frac{\partial\mathfrak{x}}{\partial t}\cdot\left\{[G,R_i]\frac{\partial\mathfrak{p}}{\partial R_i} + [G,U]\frac{\partial\mathfrak{p}}{\partial U} + [G,J]\frac{\partial\mathfrak{p}}{\partial J} + [G,\phi]\frac{\partial\mathfrak{p}}{\partial\phi}\right\}\\ &\quad - \frac{\partial\mathfrak{p}}{\partial t}\cdot\left\{[G,R_i]\frac{\partial\mathfrak{x}}{\partial R_i} + [G,U]\frac{\partial\mathfrak{x}}{\partial U} + [G,J]\frac{\partial\mathfrak{x}}{\partial J} + [G,\phi]\frac{\partial\mathfrak{x}}{\partial\phi}\right\} + \left.\frac{\partial G}{\partial t}\right|_{\mathfrak{x},\mathfrak{p}}\\ &= \sum_{q_\nu = R_i,U,J,\phi}[G,q_\nu]\left(\frac{\partial\mathfrak{x}}{\partial t}\cdot\frac{\partial\mathfrak{p}}{\partial q_\nu} - \frac{\partial\mathfrak{p}}{\partial t}\cdot\frac{\partial\mathfrak{x}}{\partial q_\nu}\right) + \left.\frac{\partial G}{\partial t}\right|_{\mathfrak{x},\mathfrak{p}},\end{aligned} \tag{6.1}$$

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where $[G, q_\nu]$, $q_\nu = R_i, U, J, \phi$, are the usual Poisson brackets between $G$ and the $q_\nu$. With (A 11) and (A 12) from Appendix A taken into account, this can be written as

$$\begin{aligned}\left.\frac{\partial G}{\partial t}\right|_{\mathbf{R},U,J,\phi} &= \sum_{q_\nu = R_i,U,J} [G, q_\nu] \left\langle \frac{\partial \mathfrak{x}}{\partial t} \cdot \frac{\partial \mathfrak{p}}{\partial q_\nu} - \frac{\partial \mathfrak{p}}{\partial t} \cdot \frac{\partial \mathfrak{x}}{\partial q_\nu} \right\rangle \\ &\quad + \sum_{q_\nu = R_i,U,J,\phi} [G, q_\nu] \frac{\partial \widetilde{H}}{\partial q_\nu} + \left.\frac{\partial G}{\partial t}\right|_{\mathfrak{x},\mathfrak{p}} \\ &= \sum_{q_\nu = R_i,U,J} [G, q_\nu] \left\{ \frac{\partial}{\partial q_\nu} \left\langle \frac{\partial \mathfrak{x}}{\partial t} \cdot \mathfrak{p} \right\rangle - \frac{\partial}{\partial t} \left\langle \frac{\partial \mathfrak{x}}{\partial q_\nu} \cdot \mathfrak{p} \right\rangle \right\} \\ &\quad + \sum_{q_\nu = R_i,U,J,\phi} [G, q_\nu] \frac{\partial \widetilde{H}}{\partial q_\nu} + \left.\frac{\partial G}{\partial t}\right|_{\mathfrak{x},\mathfrak{p}} \\ &= \left[ G, \left\langle \frac{\partial \mathfrak{x}}{\partial t} \cdot \mathfrak{p} \right\rangle \right] - \sum_{q_\nu = R_i,U,J} [G, q_\nu] \frac{\partial}{\partial t} \left\langle \frac{\partial \mathfrak{x}}{\partial q_\nu} \cdot \mathfrak{p} \right\rangle + [G, \widetilde{H}] + \left.\frac{\partial G}{\partial t}\right|_{\mathfrak{x},\mathfrak{p}},\end{aligned}$$

$$\widetilde{H} := H - \langle H \rangle. \tag{6.2}$$

*6.2. Total time derivatives*

The total time derivative of a function

$$G(\mathfrak{x}, \mathfrak{p}, t) = G(\mathfrak{x}(\mathbf{R}, U, J, \phi; t), \mathfrak{p}(\mathbf{R}, U, J, \phi; t), t)$$

is given by the generally valid relation of Hamiltonian mechanics,

$$\frac{dG}{dt} = [G, H] + \left.\frac{\partial G}{\partial t}\right|_{\mathfrak{x},\mathfrak{p}}. \tag{6.3}$$

After inserting the partial derivative of $G$ with respect to time at constant $\mathfrak{x}, \mathfrak{p}$ obtained from (6.2) and replacing $H$ by $\langle H \rangle + \widetilde{H} = e\hat{\Phi} + \langle \mathfrak{p} \cdot (\partial \mathfrak{x}/\partial t) \rangle + \widetilde{H}$, one obtains

$$\begin{aligned}&\frac{d}{dt} G(\mathbf{R}, U, J, \phi; t) \\ &\quad = [G, e\hat{\Phi}] + \sum_{q_\nu = R_1,R_2,R_3,U,J} [G, q_\nu] \frac{\partial}{\partial t} \left\langle \frac{\partial \mathfrak{x}}{\partial q_\nu} \cdot \mathfrak{p} \right\rangle + \left.\frac{\partial G}{\partial t}\right|_{R_i,U,J,\phi} \\ &\quad = [G, e\hat{\Phi}] + [G, \mathbf{R}] \cdot \frac{e}{c} \frac{\partial \hat{\mathbf{A}}}{\partial t} + [G, U] \frac{e}{c} \frac{\partial \hat{A}_U}{\partial t} + [G, J] \frac{e}{c} \frac{\partial \hat{A}_J}{\partial t} + \left.\frac{\partial G}{\partial t}\right|_{R_i,U,J,\phi},\end{aligned} \tag{6.4}$$

which can also be written as

$$\begin{aligned}\frac{d}{dt} G(\mathbf{R}, U, J, \phi; t) &= -[G, \mathbf{R}] \cdot e\hat{\mathbf{E}} + [G, U] e \left( \frac{\partial \hat{\Phi}}{\partial U} + \frac{1}{c} \frac{\partial \hat{A}_U}{\partial t} \right) \\ &\quad + [G, J] e \left( \frac{\partial \hat{\Phi}}{\partial J} + \frac{1}{c} \frac{\partial \hat{A}_J}{\partial t} \right) + \left.\frac{\partial G}{\partial t}\right|_{R_i,U,J,\phi}.\end{aligned} \tag{6.5}$$

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## 7. Poisson brackets

### *7.1. Fundamental Poisson brackets*

Equation (6.5) yields the time derivatives of $\mathbf{R}, U, J$ and $\phi$ in terms of Poisson brackets

$$\frac{dR_i}{dt} = -[R_i, \mathbf{R}] \cdot e\hat{\mathbf{E}} + [R_i, U]e\left(\frac{\partial\hat{\Phi}}{\partial U} + \frac{1}{c}\frac{\partial\hat{A}_U}{\partial t}\right) + [R_i, J]e\left(\frac{\partial\hat{\Phi}}{\partial J} + \frac{1}{c}\frac{\partial\hat{A}_J}{\partial t}\right), \tag{7.1}$$

$$\frac{dU}{dt} = -[U, \mathbf{R}] \cdot e\hat{\mathbf{E}} + [U, J]e\left(\frac{\partial\hat{\Phi}}{\partial J} + \frac{1}{c}\frac{\partial\hat{A}_J}{\partial t}\right), \tag{7.2}$$

$$\frac{dJ}{dt} = -[J, \mathbf{R}] \cdot e\hat{\mathbf{E}} + [J, U]e\left(\frac{\partial\hat{\Phi}}{\partial U} + \frac{1}{c}\frac{\partial\hat{A}_U}{\partial t}\right) \tag{7.3}$$

and

$$\frac{d\phi}{dt} = -[\phi, \mathbf{R}] \cdot e\hat{\mathbf{E}} + [\phi, U]e\left(\frac{\partial\hat{\Phi}}{\partial U} + \frac{1}{c}\frac{\partial\hat{A}_U}{\partial t}\right) + [\phi, J]e\left(\frac{\partial\hat{\Phi}}{\partial J} + \frac{1}{c}\frac{\partial\hat{A}_J}{\partial t}\right). \tag{7.4}$$

Comparison of the coefficients of (7.1)–(7.4) with those of (5.9) and (5.21)–(5.26) yields the Poisson brackets of the phase-space coordinates among themselves,

$$\begin{gathered}
[R_i, \mathbf{R}] = \frac{c}{e\mathscr{B}}\mathbf{e}_i \times \mathbf{b}_U, \\
[\mathbf{R}, U] = \frac{1}{m\mathscr{B}}\hat{\mathbf{B}}, \\
[\mathbf{R}, J] = 0, \\
[U, J] = 0, \\
[\phi, \mathbf{R}] = -\frac{1}{c}\frac{e}{m\mathscr{B}}\left\{\frac{mc}{e}\mathbf{b}_U \times \left(\frac{\partial\hat{A}_J}{\partial\mathbf{R}} - \frac{\partial\hat{\mathbf{A}}}{\partial J}\right) + \left(\frac{\partial\hat{A}_J}{\partial U} - \frac{\partial\hat{A}_U}{\partial J}\right)\hat{\mathbf{B}}\right\}, \\
[\phi, U] = \frac{1}{c}\frac{e}{m\mathscr{B}}\hat{\mathbf{B}} \cdot \left(\frac{\partial\hat{A}_J}{\partial\mathbf{R}} - \frac{\partial\hat{\mathbf{A}}}{\partial J}\right), \\
[\phi, J] = 1.
\end{gathered} \tag{7.5}$$

Here $\mathbf{e}_i, i = 1, 2, 3$, is a Cartesian basis. Note that, in contrast to the usual procedure [3–7], it was not necessary to invert the Lagrange matrix $(\omega_{\nu\mu})$ of Appendix A in order to obtain the Poisson brackets. Although it is, of course, not necessary here, one can show (by matrix multiplication in a tedious but straightforward way) that the well-known, generally valid and easily derivable relation

$$\delta_{\lambda\mu} = \sum_{\nu=1}^{6}[q_\lambda, q_\nu]\left[\frac{\partial\mathfrak{x}}{\partial q_\mu} \cdot \frac{\partial\mathfrak{p}}{\partial q_\nu} - \frac{\partial\mathfrak{p}}{\partial q_\mu} \cdot \frac{\partial\mathfrak{x}}{\partial q_\nu}\right], \tag{7.6}$$

where $\delta_{\lambda\mu}$ is the Kronecker symbol, is satisfied with the results of (7.5) for the Poisson brackets and the expressions (A 1)–(A 3) and (A 7) in Appendix A for the Lagrange brackets. This confirms the validity of (7.5).

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*7.2. Poisson bracket of arbitrary functions*

Having obtained the fundamental Poisson brackets, it is easy to express the Poisson bracket of any two arbitrary functions $G(\mathbf{R}, U, J, \phi; t)$, $K(\mathbf{R}, U, J, \phi; t)$:

$$
\begin{aligned}
[G,K] &= \sum_{q_\nu, q_\mu = R_1, R_2, R_3, U, J, \phi} \frac{\partial G}{\partial q_\mu}[q_\mu, q_\nu]\frac{\partial K}{\partial q_\nu} \\
&= \sum_{i,j=1}^{3} \frac{\partial G}{\partial R_i}\frac{1}{m\mathscr{B}}(\mathbf{e}_j \times \mathbf{e}_i)\cdot\frac{mc}{e}\mathbf{b}_U\frac{\partial K}{\partial R_j} \\
&\quad + \sum_{i=1}^{3}\frac{1}{m\mathscr{B}}\hat{B}_i\left(\frac{\partial G}{\partial R_i}\frac{\partial K}{\partial U} - \frac{\partial G}{\partial U}\frac{\partial K}{\partial R_i}\right) + \left(\frac{\partial G}{\partial \phi}\frac{\partial K}{\partial J} - \frac{\partial G}{\partial J}\frac{\partial K}{\partial \phi}\right) \\
&\quad + \sum_{i=1}^{3}[R_i,\phi]\left(\frac{\partial G}{\partial R_i}\frac{\partial K}{\partial \phi} - \frac{\partial G}{\partial \phi}\frac{\partial K}{\partial R_i}\right) + [U,\phi]\left(\frac{\partial G}{\partial U}\frac{\partial K}{\partial \phi} - \frac{\partial G}{\partial \phi}\frac{\partial K}{\partial U}\right) \\
&= -\frac{c}{e\mathscr{B}}(\nabla_{\mathbf{R}} G \times \nabla_{\mathbf{R}} K)\cdot\mathbf{b}_U + \frac{\hat{\mathbf{B}}}{m\mathscr{B}}\cdot\left(\frac{\partial K}{\partial U}\nabla_{\mathbf{R}} G - \frac{\partial G}{\partial U}\nabla_{\mathbf{R}} K\right) \\
&\quad + \left(\frac{\partial G}{\partial \phi}\frac{\partial K}{\partial J} - \frac{\partial G}{\partial J}\frac{\partial K}{\partial \phi}\right) + [\mathbf{R},\phi]\cdot\left(\frac{\partial K}{\partial \phi}\nabla_{\mathbf{R}} G - \frac{\partial G}{\partial \phi}\nabla_{\mathbf{R}} K\right) \\
&\quad + [U,\phi]\left(\frac{\partial G}{\partial U}\frac{\partial K}{\partial \phi} - \frac{\partial G}{\partial \phi}\frac{\partial K}{\partial U}\right). \qquad (7.7)
\end{aligned}
$$

## 8. Summary

A method for easily deriving the Poisson brackets in the phase space of averaging Kruskal coordinates appropriate for the description of guiding-centre and gyrocentre motion has been given. The method is based on the Lagrangian formulation of guiding-centre and gyrocentre motion using a gyroangle-independent Lagrangian as given in [1, 2]. The usual cumbersome procedure of inverting the Lagrange matrix to obtain the Poisson matrix is not necessary.

## Appendix A. Gyroangle-independent Lagrange brackets

We prove here that the Lagrange brackets of the coordinates $R_i, U, J, \phi$ among themselves are all independent of $\phi$.

It was shown in [1, 2] using the properties of the averaging coordinates, that the following relations are valid:

$$
\frac{\partial \mathfrak{p}}{\partial R_i}\cdot\frac{\partial \mathfrak{x}}{\partial \phi} - \frac{\partial \mathfrak{p}}{\partial \phi}\cdot\frac{\partial \mathfrak{x}}{\partial R_i} = 0, \quad i = 1, \ldots, 3, \qquad (A\,1)
$$

$$
\frac{\partial \mathfrak{p}}{\partial U}\cdot\frac{\partial \mathfrak{x}}{\partial \phi} - \frac{\partial \mathfrak{p}}{\partial \phi}\cdot\frac{\partial \mathfrak{x}}{\partial U} = 0, \qquad (A\,2)
$$

$$
\frac{\partial \mathfrak{p}}{\partial J}\cdot\frac{\partial \mathfrak{x}}{\partial \phi} - \frac{\partial \mathfrak{p}}{\partial \phi}\cdot\frac{\partial \mathfrak{x}}{\partial J} = 1, \qquad (A\,3)
$$

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and

$$\frac{\partial \mathfrak{p}}{\partial t}\cdot\frac{\partial \mathfrak{x}}{\partial \phi}-\frac{\partial \mathfrak{p}}{\partial \phi}\cdot\frac{\partial \mathfrak{x}}{\partial t}+\frac{\partial H}{\partial \phi}=0. \tag{A 4}$$

Consider now the Lagrange bracket

$$\omega_{\nu\mu}:=\frac{\partial \mathfrak{x}}{\partial q_\nu}\cdot\frac{\partial \mathfrak{p}}{\partial q_\mu}-\frac{\partial \mathfrak{x}}{\partial q_\mu}\cdot\frac{\partial \mathfrak{p}}{\partial q_\nu},\quad q_\nu,q_\mu=R_1,R_2,R_3,U,J, \tag{A 5}$$

and build its partial derivative with respect to the gyroangle $\phi$,

$$\begin{aligned}\frac{\partial \omega_{\nu\mu}}{\partial \phi}&=\frac{\partial^2 \mathfrak{x}}{\partial q_\nu\partial\phi}\cdot\frac{\partial \mathfrak{p}}{\partial q_\mu}+\frac{\partial \mathfrak{x}}{\partial q_\nu}\cdot\frac{\partial^2 \mathfrak{p}}{\partial q_\mu\partial\phi}-\frac{\partial^2 \mathfrak{x}}{\partial q_\mu\partial\phi}\cdot\frac{\partial \mathfrak{p}}{\partial q_\nu}-\frac{\partial \mathfrak{x}}{\partial q_\mu}\cdot\frac{\partial^2 \mathfrak{p}}{\partial q_\nu\partial\phi}\\
&=\frac{\partial}{\partial q_\nu}\left[\frac{\partial \mathfrak{x}}{\partial\phi}\cdot\frac{\partial \mathfrak{p}}{\partial q_\mu}\right]-\frac{\partial \mathfrak{x}}{\partial\phi}\cdot\frac{\partial^2 \mathfrak{p}}{\partial q_\nu\partial q_\mu}+\frac{\partial}{\partial q_\mu}\left[\frac{\partial \mathfrak{x}}{\partial q_\nu}\cdot\frac{\partial \mathfrak{p}}{\partial\phi}\right]-\frac{\partial^2 \mathfrak{x}}{\partial q_\nu\partial q_\mu}\cdot\frac{\partial \mathfrak{p}}{\partial\phi}\\
&\quad-\frac{\partial}{\partial q_\mu}\left[\frac{\partial \mathfrak{x}}{\partial\phi}\cdot\frac{\partial \mathfrak{p}}{\partial q_\nu}\right]+\frac{\partial \mathfrak{x}}{\partial\phi}\cdot\frac{\partial^2 \mathfrak{p}}{\partial q_\nu\partial q_\mu}-\frac{\partial}{\partial q_\nu}\left[\frac{\partial \mathfrak{x}}{\partial q_\mu}\cdot\frac{\partial \mathfrak{p}}{\partial\phi}\right]+\frac{\partial^2 \mathfrak{x}}{\partial q_\nu\partial q_\mu}\cdot\frac{\partial \mathfrak{p}}{\partial\phi}\\
&=\frac{\partial}{\partial q_\nu}\left[\frac{\partial \mathfrak{x}}{\partial\phi}\cdot\frac{\partial \mathfrak{p}}{\partial q_\mu}-\frac{\partial \mathfrak{x}}{\partial q_\mu}\cdot\frac{\partial \mathfrak{p}}{\partial\phi}\right]-\frac{\partial}{\partial q_\mu}\left[\frac{\partial \mathfrak{x}}{\partial\phi}\cdot\frac{\partial \mathfrak{p}}{\partial q_\nu}-\frac{\partial \mathfrak{x}}{\partial q_\nu}\cdot\frac{\partial \mathfrak{p}}{\partial\phi}\right]\\
&=0\end{aligned} \tag{A 6}$$

because of (A 1)–(A 3). Therefore, not only the $\omega_{\nu\phi}$ but also all other $\omega_{\nu\mu}$ are $\phi$-independent:

$$\begin{aligned}\omega_{\nu\mu}&=\frac{\partial \mathfrak{x}}{\partial q_\nu}\cdot\frac{\partial \mathfrak{p}}{\partial q_\mu}-\frac{\partial \mathfrak{x}}{\partial q_\mu}\cdot\frac{\partial \mathfrak{p}}{\partial q_\nu}=\left\langle\frac{\partial \mathfrak{x}}{\partial q_\nu}\cdot\frac{\partial \mathfrak{p}}{\partial q_\mu}-\frac{\partial \mathfrak{x}}{\partial q_\mu}\cdot\frac{\partial \mathfrak{p}}{\partial q_\nu}\right\rangle\\
&=\frac{\partial}{\partial q_\mu}\left\langle\frac{\partial \mathfrak{x}}{\partial q_\nu}\cdot\mathfrak{p}\right\rangle-\frac{\partial}{\partial q_\nu}\left\langle\frac{\partial \mathfrak{x}}{\partial q_\mu}\cdot\mathfrak{p}\right\rangle\\
&=\frac{e}{c}\left(\frac{\partial \hat{A}_\nu}{\partial q_\mu}-\frac{\partial \hat{A}_\mu}{\partial q_\nu}\right),\quad q_\nu,q_\mu=R_1,R_2,R_3,U,J,\end{aligned} \tag{A 7}$$

with the $\hat{A}_\nu,\hat{A}_\mu$ from Sec. 5.

For $q_\nu=R_i,U,J$, terms involving derivatives with respect to time yield, after some simple transformations, the following relations

$$\begin{aligned}\frac{\partial}{\partial\phi}\left[\frac{\partial \mathfrak{p}}{\partial q_\nu}\cdot\frac{\partial \mathfrak{x}}{\partial t}-\frac{\partial \mathfrak{p}}{\partial t}\cdot\frac{\partial \mathfrak{x}}{\partial q_\nu}\right]&=\frac{\partial}{\partial q_\nu}\left[\frac{\partial \mathfrak{p}}{\partial\phi}\cdot\frac{\partial \mathfrak{x}}{\partial t}-\frac{\partial \mathfrak{p}}{\partial t}\cdot\frac{\partial \mathfrak{x}}{\partial\phi}\right]+\frac{\partial}{\partial t}\left[\frac{\partial \mathfrak{p}}{\partial q_\nu}\cdot\frac{\partial \mathfrak{x}}{\partial\phi}-\frac{\partial \mathfrak{p}}{\partial\phi}\cdot\frac{\partial \mathfrak{x}}{\partial q_\nu}\right]\\
&=\frac{\partial}{\partial q_\nu}\frac{\partial H}{\partial\phi},\end{aligned} \tag{A 8}$$

where (A 1)–(A 4) have been used in the last step. From (A 8), one obtains

$$\frac{\partial H}{\partial q_\nu}+\frac{\partial \mathfrak{p}}{\partial t}\cdot\frac{\partial \mathfrak{x}}{\partial q_\nu}-\frac{\partial \mathfrak{p}}{\partial q_\nu}\cdot\frac{\partial \mathfrak{x}}{\partial t}=f_\nu(R_i,U,J;t),\quad q_\nu=R_i,U,J \tag{A 9}$$

and

$$\frac{\partial\langle H\rangle}{\partial q_\nu}+\left\langle\frac{\partial \mathfrak{p}}{\partial t}\cdot\frac{\partial \mathfrak{x}}{\partial q_\nu}-\frac{\partial \mathfrak{p}}{\partial q_\nu}\cdot\frac{\partial \mathfrak{x}}{\partial t}\right\rangle=f_\nu(R_i,U,J;t),\quad q_\nu=R_i,U,J, \tag{A 10}$$

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where $f_\nu(R_i, U, J; t)$ are functions independent of $\phi$. From these equations and (A 4), one obtains, respectively,

$$\frac{\partial \mathfrak{x}}{\partial t}\cdot\frac{\partial \mathfrak{p}}{\partial q_\nu}-\frac{\partial \mathfrak{x}}{\partial q_\nu}\cdot\frac{\partial \mathfrak{p}}{\partial t}=\left\langle\frac{\partial \mathfrak{x}}{\partial t}\cdot\frac{\partial \mathfrak{p}}{\partial q_\nu}-\frac{\partial \mathfrak{x}}{\partial q_\nu}\cdot\frac{\partial \mathfrak{p}}{\partial t}\right\rangle+\frac{\partial \widetilde{H}}{\partial q_\nu},\quad q_\nu=R_i,U,J \tag{A 11}$$

and

$$\frac{\partial \mathfrak{x}}{\partial t}\cdot\frac{\partial \mathfrak{p}}{\partial \phi}-\frac{\partial \mathfrak{x}}{\partial \phi}\cdot\frac{\partial \mathfrak{p}}{\partial t}=\frac{\partial \widetilde{H}}{\partial \phi}, \tag{A 12}$$

$$\widetilde{H}:=H-\langle H\rangle. \tag{A 13}$$

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