Ergod. Th. & Dynam. Sys. (2016), **36**, 2498–2511 © *Cambridge University Press, 2015*
doi:10.1017/etds.2015.15

Measurable rigidity for Kleinian groups

WOOJIN JEON† and KEN'ICHI OHSHIKA‡

† *School of Mathematics, KIAS, Hoegiro 87, Dongdaemun-gu, Seoul, 130-722, Korea*
(e-mail: jwoojin@kias.re.kr)

‡ *Department of Mathematics, Graduate School of Science, Osaka University, Toyonaka, Osaka 560-0043, Japan*
(e-mail: ohshika@math.sci.osaka-u.ac.jp)

(*Received* 25 *June* 2014 *and accepted in revised form* 4 *February* 2015)

Abstract. Let G, H be two Kleinian groups with homeomorphic quotients $\mathbb{H}^3/G$ and $\mathbb{H}^3/H$. We assume that G is of divergence type, and consider the Patterson–Sullivan measures of G and H. The measurable rigidity theorem by Sullivan and Tukia says that a measurable and essentially directly measurable equivariant boundary map $\widehat{k}$ from the limit set Λ_G of G to that of H is either the restriction of a Möbius transformation or totally singular. In this paper, we shall show that such $\widehat{k}$ always exists. In fact, we shall construct $\widehat{k}$ concretely from the Cannon–Thurston maps of G and H.

1. *Introduction*

In the theory of Kleinian groups, it is very important to consider deformations and rigidity. Quasi-conformal deformations of Kleinian groups have been studied for a long time, starting from the work of Ahlfors, Bers, Kra, Maskit and Marden. These deformations are given by considering Beltrami differentials supported on the domains of discontinuity of Kleinian groups. On the other hand, Sullivan showed in [**28**] that there are no non-trivial quasi-conformal deformations supported on limit sets, which is derived from the ergodic property of the actions of Kleinian groups on the product of two Riemann spheres. In particular, when the limit set is the entire Riemann sphere, his theorem implies that there is no non-trivial quasi-conformal deformation. This can be regarded as a generalization of the famous Mostow rigidity saying that when a Kleinian group is a co-finite volume, then every other isomorphic Kleinian group is a conformal conjugate.

Sullivan moreover showed what can be called the measurable rigidity in [**29**] as follows. Consider two isomorphic Kleinian groups G and H assuming that G is of divergence type, and suppose that their limit sets Λ_G and Λ_H have conformal measures μ_G and μ_H of the same dimension. Suppose furthermore that there is a Borel isomorphism f from the Riemann sphere to the Riemann sphere conjugating G to H almost everywhere. The

https://doi.org/10.1017/etds.2015.15 Published online by Cambridge University Press

measurable rigidity says that if $f^*\mu_H$ and μ_G are absolutely continuous to each other, then f almost everywhere coincides with a conformal automorphism conjugating G to H.

Tukia showed a similar rigidity in more general settings in [**32**]. For further generalizations, see [**14**, **33**]. We present his result restricting it to the case with which our theorem is concerned. Let G and H be Kleinian groups as in the previous paragraph, and μ_G, μ_H conformal measures where μ_G does not have an atom. It is not assumed that the dimensions of μ_G and μ_H are the same, but is assumed that at least one of their dimensions is positive. Suppose that there is an essential injective (i.e. injective outside a μ_G-null set), measurable and essentially directly measurable (i.e. the image of any measurable set outside some fixed μ_G-null set is measurable) map $f : \Lambda_G \to \Lambda_H$ which conjugates G to H almost everywhere. Then either there is a set A in Λ_G of full measure with $\mu_H(f(A)) = 0$ or f coincides with the restriction of a conformal automorphism to Λ_G almost everywhere and the dimensions of μ_G and μ_H coincide.

In the case when H is a quasi-conformal deformation of G, it is obvious that there is a Borel map conjugating G to H as in the results above of Sullivan and Tukia. But the existence of such a map for general isomorphic Kleinian groups G and H, in particular if we require it to be essentially directly measurable as in Tukia's setting, is not so evident. There is work in a more general framework of [**34**, Proposition 4.3.9], [**25**, Theorem 3.1] and [**7**, Theorem 0.2], which implies the existence of a Borel map conjugating G to H, but still we do not know whether such a map is essentially directly measurable.

On the other hand, recent developments of Kleinian group theory have made it possible to construct an equivariant map from the limit set of G to that of H concretely using Cannon–Thurston maps. In this paper, we shall show that such a map constructed from Cannon–Thurston maps in fact satisfies the conditions of Tukia's theorem provided that G is of divergence type. To be more precise, we shall prove the following.

THEOREM 1.1. *Let F be a geometrically finite, minimally parabolic, non-elementary, torsion-free Kleinian group. Suppose that we are given two representations ρ_1, $\rho_2 \in \mathcal{D}(F)$ with images G and H in* PSL(2, $\mathbb{C}$) *such that there are homeomorphisms from $\mathbb{H}^3/F$ to $\mathbb{H}^3/G$ and $\mathbb{H}^3/H$ which induce ρ_1 and ρ_2 respectively between the fundamental groups. We further assume that G is of divergence type. Let $\widehat{i} : \Lambda_F \to \Lambda_G$ and $\widehat{j} : \Lambda_F \to \Lambda_H$ be the Cannon–Thurston maps, which are guaranteed to exist by virtue of the work of Mj. Let μ_F, μ_G and μ_H be the Patterson–Sullivan measures of F, G and H with the base point at the origin of the Poincaré ball. Then the following hold.*

(1) *There exists a Borel set $Z \subset \Lambda_G$ with $\mu_G(Z) = 0$ such that $\widehat{j}\,\widehat{i}^{-1}$ is a homeomorphism from $\Lambda_G \backslash Z$ to its image.*
(2) *Either $\widehat{j}\,\widehat{i}^{-1}$ is the restriction of a Möbius transformation or there exists a Borel subset A of $\Lambda_G \backslash Z$ such that $\mu_G(A) = \mu_G(\Lambda_G)$ and $\mu_H(\widehat{j}\,\widehat{i}^{-1}(A)) = 0$.*

The Cannon–Thurston map $\widehat{i}$ of a Kleinian group G is defined to be a continuous equivariant map from the limit set Λ_F of F to the limit set Λ_G of G, where F is a minimally parabolic geometrically finite group isomorphic to G (see §2.2). Although $\widehat{i}^{-1}$ may not be well defined as a map, we can show that $\widehat{i}$ is a Borel measurable isomorphism under the condition that G is of divergence type. Then we can define $\widehat{k}$ to be $\widehat{j}\,\widehat{i}^{-1}$ and can show that it is both Borel measurable and Borel essentially directly measurable.

https://doi.org/10.1017/etds.2015.15 Published online by Cambridge University Press

2. *Preliminaries*

2.1. *Hyperbolic 3-manifolds.* We use the symbol $\mathbb{H}^3$ to denote the hyperbolic 3-space. We need to use both the Poincaré ball model and the upper half space model of $\mathbb{H}^3$ depending on the situation. The ball model is the open unit ball $\{x \in \mathbb{R}^3 \mid |x| < 1\}$ with the metric $2|dx|/\sqrt{1-|x|^2}$, where $|x|$ is the usual Euclidean norm of x, and the upper half space model is $\{x = (x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_3 > 0\}$ with the metric $|dx|/x_3$. The ideal boundary of the upper half space can be identified with $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, and is also denoted by S^2_∞. A discrete subgroup G of $\mathrm{PSL}(2, \mathbb{C})$ is called a *Kleinian group*. If G has no elliptic elements, the quotient $N := \mathbb{H}^3/G$ becomes a hyperbolic 3-manifold. Throughout this paper, every Kleinian group is assumed to be finitely generated and without elliptic elements.

For a Kleinian group G, its *limit set* Λ_G is the set of accumulation points on S^2_∞ of an orbit $G \cdot o$ with $o \in \mathbb{H}^3$. The complement $\Omega_G := S^2_\infty \setminus \Lambda_G$ is the domain of discontinuity for the action of G on $\hat{\mathbb{C}}$. The *convex hull* of G is defined to be the smallest convex set in $\mathbb{H}^3$ whose closure in $\mathbb{H}^3 \cup S^2_\infty$ contains Λ_G. The *convex core* of G is the quotient by G of its convex hull. The Kleinian group G is called *geometrically finite* (respectively *convex cocompact*) if the ϵ-neighbourhood of its convex core for some $\epsilon > 0$ has finite volume (respectively if its convex core is compact). Under our assumption that G is finitely generated, by Ahlfors' finiteness theorem, this is equivalent to the condition that the convex core itself has finite volume. The Kleinian group G is called *minimally parabolic* if every parabolic element is contained in a rank-two free abelian subgroup of G.

The thin part $N_\epsilon \subset N$ is the set of points in N where the injectivity radius is less than ϵ. It is known that there exists some $\epsilon > 0$, called the Margulis constant, such that each component of N_ϵ is either a *cusp neighbourhood* or a solid torus neighbourhood of a closed geodesic, the latter of which is called a *Margulis tube*. Let N_0 be the *non-cuspidal* part of N, which is the complement of the cusp neighbourhoods in N for some Margulis constant ϵ. Then we can find a *relative compact core* (see [**11**, **23**]) $C \subset N_0$, which is a compact 3-manifold with boundary such that the inclusion of $(C, C \cap \partial N_0)$ into $(N_0, \partial N_0)$ is a homotopy equivalence of pairs. The ends of N_0 are in one-to-one correspondence with the components of $\mathrm{Fr}\, C$ in N_0. When an end e corresponds to a component S_e of $\mathrm{Fr}\, C$, we say that *e faces S_e*.

An end of N_0 is called *geometrically finite* if it has a neighbourhood which is disjoint from the convex core of G. Clearly, G is geometrically finite if and only if every end of $\mathbb{H}^3/G$ is geometrically finite. A geometrically finite end e corresponds to a component Σ_e of Ω_G/G attached to e in the Kleinian manifold $N_G := (\mathbb{H}^3 \cup \Omega_G)/G$. The point of the Teichmüller space of Σ_e determined by the conformal structure coming from Ω_G is called the *conformal structure at infinity* of e.

2.2. *Deformation spaces.* For a group F, we denote by $\mathcal{D}(F)$ the deformation space of F, that is, the space of faithful discrete representations of F into $\mathrm{PSL}(2, \mathbb{C})$ modulo conjugacy, endowed with the topology induced from the representation space. To denote a point in $\mathcal{D}(F)$, we use a representation $\phi : F \to \mathrm{PSL}(2, \mathbb{C})$ in the class or the Kleinian group $\phi(F)$ or, when necessary, the pair (F, ϕ). By Thurston's uniformization theorem, for every finitely generated Kleinian group G there is a minimally parabolic geometrically

https://doi.org/10.1017/etds.2015.15 Published online by Cambridge University Press

finite Kleinian group with isomorphism $\phi : F \to G$ such that (G, ϕ) lies in $\mathcal{D}(F)$ and there is a homeomorphism from a compact core of $\mathbb{H}^3/F$ to that of $\mathbb{H}^3/G$ inducing an isomorphism conjugate to ϕ between the fundamental groups.

2.3. *Ending laminations.* For a closed hyperbolic surface S, a *geodesic lamination* λ on S is a closed set consisting of disjoint simple geodesics called *leaves*. A leaf of λ is said to be isolated if no point of the leaf is an accumulation point of points on the other leaves. A minimal component of λ is a sublamination μ of λ such that the closure of each leaf of μ coincides with μ. A geodesic lamination is decomposed into finitely many minimal components and finitely many isolated leaves which are not closed leaves.

Each component V of $S\backslash\lambda$ is called a *complementary region* of λ in S. A lift of a complementary region to $\mathbb{H}^2$ is an ideal polygon with countably many ideal vertices. If a complementary region itself is simply connected, it is an ideal polygon with finitely many vertices, which coincides with its lift in $\mathbb{H}^2$. We say that a geodesic lamination is *filling* when each of its complementary regions is simply connected.

Now we shall define the ending lamination λ_e for a geometrically infinite end e of $N = \mathbb{H}^3/G$ as follows. Let N be a hyperbolic 3-manifold and e an end of N_0. As explained before, e faces a component S_e of the frontier of the relative compact core C. Suppose that e is geometrically infinite. By the tameness theorem proved by Bonahon [**5**], Agol [**1**] and Calegari and Gabai [**8**], we see that we can take C so that the complement of C containing e, which we denote by U_e, is homeomorphic to $S_e \times (0, \infty)$. Since e is geometrically infinite, every neighbourhood of e intersects a closed geodesic and, moreover, as was shown in Bonahon [**5**] and Canary [**9**], there is a sequence of simple closed curves γ_n in S_e which are homotopic in $U_e \cup S_e$ to closed geodesics tending to e. Fix a hyperbolic metric on S_e and regard γ_n as a closed geodesic on S_e. We consider the Hausdorff limit of γ_n. It can be shown that this limit geodesic lamination has only one minimal component λ_e, which is filling. We call λ_e the *ending lamination* of e following Thurston [**31**], which can be proved to be unique by Bonahon's intersection lemma appearing in [**5**]. The ending lamination theorem proved by Brock *et al* in [**6**, **18**] says that the end invariant consisting of conformal structures at infinity and of ending laminations uniquely determines the Kleinian group G up to conjugacy.

For our later use, we need to consider the union of ending laminations together with the parabolic locus, which we call the total ending lamination and is defined as a geodesic lamination on the boundary of a compact core of a minimally parabolic geometrically finite Kleinian group. Let us define it more formally. Consider a Kleinian group (G, ϕ) contained in $\mathcal{D}(F)$ for a minimally parabolic geometrically finite Kleinian group F, and suppose that there is a homeomorphism $\Phi : \mathbb{H}^3/F \to \mathbb{H}^3/G$ inducing an isomorphism conjugate to ϕ between the fundamental groups. Let C be the convex core of $\mathbb{H}^3/F$. We can isotope Φ so that $\Phi(C) \cap (\mathbb{H}^3/G)_0$ is a relative compact core of $(\mathbb{H}^3/G)_0$. Then each component U of $(\mathbb{H}^3/G)_0\backslash\Phi(C)$ is homeomorphic to $S \times (0, \infty)$, where S is the frontier component of $\Phi(C)$ touching U, and in the case when the end contained in U is geometrically infinite, an ending lamination is defined on S. We fix a hyperbolic metric on the union of non-torus components of ∂C, which we denote by Σ. Then we consider the union λ of all ending laminations and core curves of the annuli among $\Phi(C) \cap \mathrm{Fr}(\mathbb{H}^3/G)_0$ and pull them back to

https://doi.org/10.1017/etds.2015.15 Published online by Cambridge University Press

Σ by Φ^{-1}. We call the geodesic lamination on Σ isotopic to λ the *total ending lamination* for (G, ϕ).

2.4. *Cannon–Thurston maps.* Let (G, ρ) be a finitely generated Kleinian group in $\mathcal{D}(F)$ for a minimally parabolic geometrically finite Kleinian group F such that there is a homeomorphism from $\mathbb{H}^3/F \to \mathbb{H}^3/G$ inducing an isomorphism conjugate to ρ. Fixing a point $o \in \mathbb{H}^3$, we can construct a ρ-equivariant map i between the orbit $F \cdot o$ and the orbit $G \cdot o$ by setting $i(o) = o$. If i extends to a continuous equivariant map between the limit sets $\widehat{i} : \Lambda_F \to \Lambda_G$, the latter map is called the Cannon–Thurston map. The existence of Cannon–Thurston maps was proved by Mj for Kleinian surface groups in [**19**], and the general case was dealt with in [**20**].

Mj also gave a criterion for limit points of F to be non-injective under $\widehat{i}$ as follows. Consider the Kleinian manifold $(\mathbb{H}^3 \cup \Omega_F)/F$. Then, by the nearest point map, we can identify ∂C with Ω_F/F. Let λ be the total ending lamination for (G, ϕ) regarded as lying on Ω_F/F. We consider the lift $\widetilde{\lambda}$ of λ to the universal cover Ω_F. We define an equivalence relation R on the points on Λ_F to be the smallest one having the property that xRy if x and y are the two end points of a leaf of $\widetilde{\lambda}$. What Mj showed is that $\widehat{i}(x) = \widehat{i}(y)$ if and only if xRy.

2.5. *Conformal density.* For x in $\mathbb{H}^3$ and $\eta \in S^2_\infty$, we take a geodesic ray $\alpha(t)$ from $\alpha(0) = x \in \mathbb{H}^3$ toward $\eta \in S^2_\infty$. Then we define the *Busemann function* $b_{x,\eta}(\cdot)$ at η with $b_{x,\eta}(x) = 0$ by

$$b_{x,\eta}(y) = \lim_{t\to\infty} (d_{\mathbb{H}^3}(y, \alpha(t)) - d_{\mathbb{H}^3}(x, \alpha(t))).$$

For x, y in $\mathbb{H}^3$ and $s \geq 0$, we consider the Poincaré series for a Kleinian group G, defined by

$$g_s(x, y) = \sum_{g\in G} e^{-sd_{\mathbb{H}^3}(x, g\cdot y)}.$$

The critical exponent δ_G of G is defined to be

$$\delta_G = \limsup_{r\to\infty} \frac{1}{r} \log(\#\{g \in G \mid d^3_{\mathbb{H}}(o, g \cdot o) \leq r\}).$$

We say that G is of *divergence type* if $g_s(x, y)$ diverges at $s = \delta_G$. Otherwise G is said to be of *convergence type*. These definitions are independent of the choice of $x, y \in \mathbb{H}^3$.

For $x \in \mathbb{H}^3$ and $s \geq 0$, we consider a measure defined by

$$\mu^s_x = \frac{1}{g_s(y, y)} \sum_{g\in G} e^{-sd_{\mathbb{H}^3}(x, g\cdot y)} \delta_{g\cdot y},$$

where $\delta_{g\cdot y}$ denotes the Dirac measure of weight one at $g \cdot y$. Then we can find a sequence $\{s_i\}$ with $s_i \to \delta_G + 0$ such that $\mu^{s_i}_x$ weakly converges to a finite measure μ_x on the compact space $\mathbb{H}^3 \cup S^2_\infty$. When G is of divergence type, μ_x has its support on the limit set Λ_G. The family of measures $\{\mu_x\}$ has two properties:

$$\frac{d\mu_x}{d\mu_y}(\eta) = e^{-\delta_G b_{x,\eta}(y)}, \quad g^*\mu_x = \mu_{g^{-1}\cdot x},$$

https://doi.org/10.1017/etds.2015.15 Published online by Cambridge University Press

where the pull-back measure $g^*\mu_x$ is defined by setting $(g^*\mu_x)(A) := \mu_x(g \cdot A)$ for any μ_x-measurable set $A \subset \Lambda_G$. A family of measures parametrized by $x \in \mathbb{H}^3$ having these properties is called a G-invariant *conformal density* of dimension δ_G.

Even when G is of convergence type, it is still possible to construct a conformal density $\{\mu_x\}$ of dimension δ_G by increasing the Dirac mass on each orbit point suitably, as was shown in [**22**, **27**]. We call an element of a G-invariant conformal density a *conformal measure*. In particular, the conformal measure μ_x constructed above is called the *Patterson–Sullivan measure* of G based at x whether or not G is of divergence type.

The dimension δ_G of a conformal density $\{\mu_x\}$ is always positive if G is *non-elementary*, i.e. if Λ_G has more than two points (see [**3**, Theorem 7], and also [**27**, Corollary 2] and [**10**, Proposition 3.8]). A trivial example of a two-dimensional conformal density on S^2_∞ is the area density $\{\text{area}_x\}$ obtained by identifying the visual sphere based at the point $x \in \mathbb{H}^3$ with S^2_∞. If we use the Poincaré ball model of $\mathbb{H}^3$ setting o to be the origin of the ball, then area_o is equal to the Lebesgue measure on S^2_∞ up to a constant multiple. As an application of the tameness theorem combined with the work of Culler and Shalen in [**10**, Proposition 3.9], it is known that if a Kleinian group G satisfies $\Lambda_G = S^2_\infty$ then any G-invariant conformal measure μ_x is equal to the area density up to homothety. In fact, we have $K\mu_x = \text{area}_x$ with a constant $K > 0$ for any $x \in \mathbb{H}^3$.

A point in Λ_G is called *a conical limit point* if it is an accumulation point of an orbit $G \cdot o$ in $\mathbb{H}^3$ contained in a fixed radial neighbourhood of a geodesic ray ending at that point. We denote the set of conical limit points by $\Lambda_{G,c}$. We use the symbol Δ to denote the diagonal set in general. For a subset X of S^2_∞, we use the symbol $X^{(2)}$ to denote $(X \times X)\backslash\Delta$.

THEOREM 2.1. (Hopf, Tsuji, Sullivan, Thurston, Canary and Yue) *Let G be a finitely generated torsion-free Kleinian group, and μ_x its Patterson–Sullivan measure. Then the following are equivalent.*

(1) *The conical limit set $\Lambda_{G,c}$ has full μ_x-measure.*
(2) *The conical limit set $\Lambda_{G,c}$ has positive μ_x-measure.*
(3) *The diagonal action of G on $(S^2_\infty)^{(2)}$ is $\mu_x \times \mu_x$-ergodic.*
(4) *G is of divergence type.*
(5) *G is either geometrically finite or $\Lambda_G = S^2_\infty$.*

Proof. The fourth and the fifth conditions are equivalent, as was shown in §9 of the paper [**9**] by Canary. By Sullivan [**27**, p. 420], we see that the first three conditions are equivalent and imply the fourth, and that the fourth implies the first three if $\delta_G > 1$. Suppose that G is of divergence type; then we have the following. If G is not geometrically finite, then $\delta_G = 2$ because $\Lambda_G = S^2_\infty$ by the fifth condition and $K\mu_o = \text{area}_o$ for some constant $K > 0$. Therefore, as explained above, this implies the first three. If H is geometrically finite, then Λ_G is the union of $\Lambda_{G,c}$ and parabolic fixed points, which are countable, and hence $\Lambda_{G,c}$ has the full μ_x-measure. Thus, the fourth implies the first three in both of the two cases. The equivalence of the first four conditions also follows from a more general result by Yue in [**33**, p. 77]. □

https://doi.org/10.1017/etds.2015.15 Published online by Cambridge University Press

2.6. *Measure theoretic notions.* Let X be a complete separable metric space with the Borel σ-algebra generated by the open sets. A measure space (Y, ν) is called *standard* or *Lebesgue* if it is measurably isomorphic to a Borel subset of X. Two standard measure spaces are Borel isomorphic if and only if they have the same cardinality.

For a discrete group G with the counting measure, a *measurable action* of G on a standard measure space (Y, ν) is defined to be a measurable map $\Theta : G \times Y \to Y$ from the product measure space satisfying the following properties.

(1) $\Theta(\mathrm{id}, y) = y$ for all $y \in Y$.
(2) $\Theta(g_1, \Theta(g_2, y)) = \Theta(g_1 g_2, y)$.

Since G is a group, $\Theta(g, \cdot)$ has the measurable inverse $\Theta(g^{-1}, \cdot)$ for any $g \in G$. We denote $\Theta(g, y)$ simply by $g \cdot y$. The measure ν is said to be *G-invariant* if $\nu(A) = \nu(g \cdot A)$ for any measurable set A in Y and $g \in G$. For an element $g \in G$, we define the push-forward $g_*\nu$ and the pull-back $g^*\nu$ by setting $g_*\nu(A) = \nu(g^{-1} \cdot A)$ and $g^*\nu(A) = \nu(g \cdot A)$. We say that ν is *quasi-invariant* under G if $g_*\nu$ and ν are equivalent for all $g \in G$, i.e. if $\nu(A) = 0 \Leftrightarrow g_*\nu(A) = 0$.

A measurable *G-space* is a standard measure space (Y, ν) with a measurable G-action such that ν is quasi-invariant under G. We say that a set $Z \subset Y$ is *null* in (Y, ν) if Z is a subset of a measurable set Z' with $\nu(Z') = 0$. A set is said to be *conull* when it is the complement of a null set. A measurable G-space is called *ergodic* if any G-invariant measurable set is either null or conull. This definition is equivalent to the following weaker condition (see [**12**, Ch. 2], for example): a measurable G-space is ergodic if and only if any essentially invariant measurable set is either null or conull. Another way to express this condition is: a G-space is ergodic if and only if any essentially G-invariant measurable function from the G-space is constant almost everywhere.

For a Kleinian group G with a conformal measure μ_x on S^2_∞, both (S^2_∞, μ_x) and $((S^2_\infty)^{(2)}, \mu_x \times \mu_x)$ are G-spaces. If G is of divergence type, both of them are ergodic G-spaces. If G is geometrically infinite and of divergence type, as mentioned in §2.5, we know that a conformal density is equal to the area density up to a constant multiple. If G is geometrically finite, then any two conformal densities of dimension equal to the critical exponent are equal up to a constant multiple (see [**30**]). Thus, the conformal density of dimension equal to the critical exponent is always unique (up to constant multiples) if G is of divergence type. On the other hand, if G is of convergence type, conformal densities are neither unique nor ergodic in general. For this, see [**2**, **4**].

3. *Non-injective points*

Let F be a minimally parabolic geometrically finite Kleinian group and (G, ϕ) a Kleinian group in $\mathcal{D}(F)$ such that there is a homeomorphism from $\mathbb{H}^3/F$ to $\mathbb{H}^3/G$ inducing an isomorphism conjugate to ϕ between the fundamental groups. We let $\widehat{i} : \Lambda_F \to \Lambda_G$ be the Cannon–Thurston map, which is guaranteed to exist by the work of Mj. We shall need the following proposition to show that $\widehat{i}$ is a Borel measurable isomorphism.

PROPOSITION 3.1. *There are only countably many points $\eta \in \Lambda_F$ such that the cardinality of $\widehat{i}^{-1}(\widehat{i}(\eta))$ is greater than two.*

https://doi.org/10.1017/etds.2015.15 Published online by Cambridge University Press

Let N_F be $(\mathbb{H}^3 \cup \Omega_F)/F$, which is either a compact 3-manifold or a 3-manifold with only finitely many ends, all of which are toral. Let $\Phi_1 : \operatorname{Int} N_F \to \mathbb{H}^3/G$ be a homeomorphism, which is assumed to exist. Let S be a component of Ω_F/F. Let λ_G be the total ending lamination for (G, ϕ) as explained in §2.3, and consider its restriction $\lambda_G \cap S$ to S, which we call the total ending lamination on S.

Now let Ω_0 be a component of Ω_F which covers S by the canonical projection from Ω_F to Ω_F/F, and $p_0 : \Omega_F \to S$ the restriction of the projection. Each leaf of the total ending lamination on S lifts to open arcs which are geodesics with respect to a hyperbolic metric on Ω_0 inducing one on S, which we call *end invariant leaves* for S. More specifically, we call open arcs which are lifts of leaves of ending laminations *ending lamination leaves* and those which are lifts of parabolic curves *parabolic leaves*. A point in Λ_G is said to be an *end invariant point* for S when it is an end point at infinity of a lift of an end invariant leaf of S.

To prove the proposition, we shall first show the following lemma.

LEMMA 3.2. *Suppose that there are three distinct points $x, y, z \in \Lambda_F$ with $\widehat{i}(x) = \widehat{i}(y) = \widehat{i}(z)$. Then there is a component S of Ω_F/F and the total ending lamination $\lambda := \lambda_S(G)$ of G on S as follows. Either there is a lift U in Ω_F of a complementary region of λ whose vertices at infinity contain all of x, y, z or there exist two adjacent lifts U_1, U_2 in Ω_F of complementary regions of λ such that $\overline{U}_1 \cap \overline{U}_2$ contains a parabolic leaf and all of x, y, z are contained in the vertices of $U_1 \cup U_2$.*

Proof. By Mj's work explained in §2.4, the relation between two points x, y in Λ_F defined by $\widehat{i}(x) = \widehat{i}(y)$ is the transitive closure of the relation that two points are the end points of the same end invariant leaf. This implies that if we only take into account one component S of Ω_F/F, three points x, y, z which are equivalent must be situated as in the statement. Therefore, to show our lemma, it is enough to prove that for two distinct components S_1 and S_2 of Ω_F/F, an end invariant point for S_1 cannot be that of S_2 at the same time.

Now, seeking a contradiction, suppose that there is a point $z \in \Lambda_F$ which is an end invariant point for both S_1 and S_2. Let Ω_1 and Ω_2 be the components of Ω_F covering S_1 and S_2, respectively. Then there exist geodesic rays r_1 and r_2 ending at z which are lifts of parabolic curves or leaves of ending laminations to Ω_1 and Ω_2, respectively. Let l_1 and l_2 be their projections on S_1 and S_2. Since the total ending lamination is decomposed into minimal components, the closures of l_1 and l_2 are minimal components of the total ending lamination, which we denote by λ_1 and λ_2. Now the argument of [**16**, Lemma C.1] (see also [**17**]) implies that λ_1 and λ_2 are homotopic in N_F. Here two laminations λ_1 and λ_2 are said to be homotopic if there is a sequence of incompressible annuli A_i properly embedded in N_F such that $A_i \cap \partial N_F$ converges to $\lambda_1 \cup \lambda_2$ in the Hausdorff topology. Since λ_1 and λ_2 are parabolic curves or ending laminations of G lying on distinct components of Ω_F/F, this is a contradiction. □

Proof of Proposition 3.1. Suppose that $z \in \Lambda_F$ is a point such that $\widehat{i}^{-1}(\widehat{i}(z))$ has more than two points. Then, by Lemma 3.2, there is a component S of Ω_F/F, and z is an ideal vertex of a lift of a complementary region of the total ending lamination for G on S, which we denote by λ. Note that there are only finitely many complementary regions of λ, and

https://doi.org/10.1017/etds.2015.15 Published online by Cambridge University Press

each of them has only finitely many ideal vertices. Therefore, there are only countably many ideal vertices for the lifts of complementary regions. This implies that there are only countably many points z with $\#(\widehat{i}^{-1}(\widehat{i}(z))) > 2$.

4. *Measurable rigidity*

In this section, we shall prove our main theorem.

THEOREM 1.1. *Let F be a geometrically finite, minimally parabolic, non-elementary, torsion-free Kleinian group. Suppose that we are given two representations $\rho_1, \rho_2 \in \mathcal{D}(F)$ with images G and H in* PSL(2, $\mathbb{C}$) *such that there are homeomorphisms from $\mathbb{H}^3/F$ to $\mathbb{H}^3/G$ and $\mathbb{H}^3/H$ which induce ρ_1 and ρ_2 respectively between the fundamental groups. We further assume that G is of divergence type. Let $\widehat{i} : \Lambda_F \to \Lambda_G$ and $\widehat{j} : \Lambda_F \to \Lambda_H$ be the Cannon–Thurston maps, which are guaranteed to exist by virtue of the work of Mj. Let μ_F, μ_G and μ_H be the Patterson–Sullivan measures of F, G and H with the base point at the origin of the Poincaré ball. Then the following hold.*

(1) *There exists a Borel set $Z \subset \Lambda_G$ with $\mu_G(Z) = 0$ such that $\widehat{j}\,\widehat{i}^{-1}$ is a homeomorphism from $\Lambda_G \backslash Z$ to its image.*
(2) *Either $\widehat{j}\,\widehat{i}^{-1}$ is the restriction of a Möbius transformation or there exists a Borel subset A of $\Lambda_G \backslash Z$ such that $\mu_G(A) = \mu_G(\Lambda_H)$ and $\mu_H(\widehat{j}\,\widehat{i}^{-1}(A)) = 0$.*

The following Lemma 4.1 follows from a more general result [**13**, Proposition 7.5.2]. See also [**15**, Theorem A].

LEMMA 4.1. *If the cardinality of $\widehat{i}^{-1}(\xi)$ is greater than one, then $\xi \in \Lambda_G$ is not a conical limit point.*

The following lemma appeared as [**21**, Theorem 3.5.8].

LEMMA 4.2. *For a Kleinian group G of divergence type, the Patterson–Sullivan measure μ_G has no atom. Therefore, every countable set in Λ_G has μ_G-measure zero.*

We shall construct a Borel null set Z in Λ_G such that $\widehat{j}\,\widehat{i}^{-1}$ is well defined and injective in $\Lambda_G \backslash Z$. Set

$$\mathrm{End}_F(G) := \{(\xi_1, \xi_2) \in \Lambda_F^{(2)} \mid \widehat{i}(\xi_1) = \widehat{i}(\xi_2)\}$$

and

$$\mathrm{End}_F(H) := \{(\eta_1, \eta_2) \in \Lambda_F^{(2)} \mid \widehat{j}(\eta_1) = \widehat{j}(\eta_2)\}.$$

LEMMA 4.3. $\mathrm{End}_F(G)$ *and* $\mathrm{End}_F(H)$ *are closed F-invariant Borel null sets in the F-space* $(\Lambda_F^{(2)}, \mu_F \times \mu_F)$.

Proof. We have only to show this for $\mathrm{End}_F(G)$, for the argument for $\mathrm{End}_F(H)$ is the same. Since $\widehat{i}$ is continuous, $\mathrm{End}_F(G)$ is closed in $\Lambda_F^{(2)}$. We also see that $\mathrm{End}_F(G)$ is F-invariant because $\widehat{i}$ is an equivariant map. Since $(\Lambda_F^{(2)}, \mu_F \times \mu_F)$ is an ergodic F-space, we only need to show that $\mathrm{End}_F(G)$ cannot have full $\mu_F \times \mu_F$-measure. Since $\mathrm{End}_F(G)$ is closed in $\Lambda_F^{(2)}$ and $\Lambda_F^{(2)}$ is open in $\Lambda_F \times \Lambda_F$, it is sufficient to show that $\mathrm{End}_F(G)$ is not the entire $\Lambda_F^{(2)}$.

https://doi.org/10.1017/etds.2015.15 Published online by Cambridge University Press

Choose a loxodromic element $\gamma \in F$ such that $\rho_1(\gamma)$ is also loxodromic. Such an element exists because there are only finitely many conjugacy classes of accidental parabolic elements. Let $\xi_1, \xi_2 \in \Lambda_F$ be the fixed points of γ. Then $\widehat{i}(\xi_1)$ and $\widehat{i}(\xi_2)$ are conical limit points, since they are loxodromic fixed points. It follows that (ξ_1, ξ_2) does not belong to $\mathrm{End}_F(G)$ by Lemma 4.1. Therefore, $\mathrm{End}_F(G)$ cannot be the entire $\Lambda_F^{(2)}$. □

Now we consider the projection maps proj_1 and proj_2 from $S^2_\infty \times S^2_\infty$ to S^2_∞ sending (ξ_1, ξ_2) to ξ_1 and ξ_2, respectively. We set

$$\mathrm{end}_F(G) := \mathrm{proj}_1(\mathrm{End}_F(G)) = \mathrm{proj}_2(\mathrm{End}_F(G)).$$

LEMMA 4.4. *$\mathrm{end}_F(G)$ is a Borel null set in (Λ_F, μ_F). Moreover, $\widehat{i}$ is a homeomorphism from $\Lambda_F \setminus \mathrm{end}_F(G)$ to $\Lambda_G \setminus \widehat{i}(\mathrm{end}_F(G))$.*

Proof. We shall first prove that $\mathrm{end}_F(G)$ is a Borel set. We consider a countable covering $\{A_i \times B_i\}_{i \in I}$ of $\Lambda_F^{(2)}$ by products of disjoint closed subsets A_i, B_i of Λ_F. For each $A \times B \in \{A_i \times B_i\}$, we consider the sets $\mathrm{end}_{A, A\times B} := \mathrm{proj}_1(\mathrm{End}_F(G) \cap A \times B)$ and $\mathrm{end}_{B, A\times B} := \mathrm{proj}_2(\mathrm{End}_F(G) \cap A \times B)$. In this setting, we claim that $\mathrm{end}_{A, A\times B}$ is a closed subset of A. To show this, we take a sequence $\{a_j\}$ in $\mathrm{end}_{A, A\times B}$ converging to $a \in A$. Then there exists a sequence $\{b_j\} \subset \mathrm{end}_{B, A\times B}$ such that $(a_j, b_j) \in \mathrm{End}_F(G)$ for each j. Since $\mathrm{End}_F(G)$ is closed and $A \times B$ is compact, $\{(a_j, b_j)\}$ converges to $(a, b) \in \mathrm{End}_F(G)$ for some $b \in B$ after passing to a subsequence. This shows that $a \in \mathrm{end}_{A, A\times B}$ and we have thus shown that $\mathrm{end}_{A, A\times B}$ is closed. Now, since $\mathrm{end}_F(G)$ is expressed as a countable union $\bigcup_{i \in I} \mathrm{end}_{A_i, A_i \times B_i}$ of closed subsets, we see that $\mathrm{end}_F(G)$ is a Borel set.

Secondly, we shall prove that $\mathrm{end}_F(G)$ is a null set by extending Soma's argument in [**24**]. Since (Λ_F, μ_F) is an ergodic F-space and $\mathrm{end}_F(G)$ is an invariant set, $\mathrm{end}_F(G)$ is either null or conull with respect to μ_F. Suppose that $\mathrm{end}_F(G)$ is conull. Under this assumption, in the following we shall construct an invariant real valued function Ψ which is defined on a conull set in $\Lambda_F^{(2)}$ and is Borel measurable. Since $(\Lambda_F^{(2)}, \mu_F \times \mu_F)$ is an ergodic F-space, Ψ has to be constant almost everywhere and we shall reach a contradiction.

For simplicity, we denote $\mathrm{end}_{A, A\times B}$ and $\mathrm{end}_{B, A\times B}$ by A and B, respectively. Each $a \in \mathsf{A}$ corresponds to some $b \in \mathsf{B}$ with $(a, b) \in \mathrm{End}_F(G)$, which may not be unique. Let Z_2 be the set of the points $a \in \Lambda_F$ such that $\widehat{i}^{-1}(\widehat{i}(a))$ has more than two points. Then, by Proposition 3.1, Z_2 is a countable set and hence has measure zero by Lemma 4.2.

Now we claim that $\widehat{i}|_{\mathsf{A}\setminus Z_2}$ is a homeomorphism to its image. Since we know that $\widehat{i}|_{\mathsf{A}\setminus Z_2}$ is a continuous injective map, we only need to show that it is also a closed map to its image. Set $\mathsf{A}' := \mathsf{A}\setminus Z_2$ and let C' be a closed subset of A'. Then $C' = C \cap \mathsf{A}'$ for some compact set C in Λ_F. It is clear that $\widehat{i}(C') \subset \widehat{i}(C) \cap \widehat{i}(\mathsf{A}\setminus Z_2)$. On the other hand, since $\widehat{i}^{-1}(\widehat{i}(Z_2)) \cap \mathsf{A} = Z_2 \cap \mathsf{A}$, we have $\widehat{i}(C') = \widehat{i}(C) \cap \widehat{i}(\mathsf{A}\setminus Z_2)$. Since $\widehat{i}(C)$ is compact, this shows that $\widehat{i}(C')$ is a closed set in $\widehat{i}(\mathsf{A}\setminus Z_2)$ and hence $\widehat{i}|_{\mathsf{A}\setminus Z_2}$ is a homeomorphism to its image.

In the same way, $\widehat{i}|_{\mathsf{B}'}$ for $\mathsf{B}' := \mathsf{B}\setminus Z_2$ is a homeomorphism to its image. Therefore, we have a homeomorphism $\psi := \widehat{i}|_{\mathsf{B}'}^{-1}\, \widehat{i}|_{\mathsf{A}'}$ from A' to B'. When $A \times B = A_j \times B_j$ for some index $j \in I$, we denote this homeomorphism by ψ_j.

https://doi.org/10.1017/etds.2015.15 Published online by Cambridge University Press

Now we define a Borel measurable function $\Psi : (\mathrm{end}_F(H)\backslash Z_2)^{(2)} \to \mathbb{R}$ as follows. For a given point $(a, b) \in (\mathrm{end}_F(H)\backslash Z_2)^{(2)}$, we choose two sets $A_j \times B_j$, $A_k \times B_k$ in the covering so that $a \in \mathsf{A}_j$ and $b \in \mathsf{A}_k$. Then we define $\Psi(a, b)$ to be the absolute cross ratio

$$|\mathrm{cr}(a, b, \psi_k(b), \psi_j(a))| = \frac{|a - \psi_k(b)||\psi_j(a) - b|}{|a - b||\psi_k(b) - \psi_j(a)|}.$$

By the definition of ψ_j and ψ_k, it is clear that $\Psi(a, b)$ is determined only by a and b, independently of the choice of j, k. We shall drop the subscript of $\psi_j(a)$ when we do not need to specify in which A_j lies a. Since each ψ_j is a homeomorphism, Ψ is Borel measurable. Moreover, Ψ is F-invariant because each ψ_j is equivariant and the absolute cross ratio is preserved by Möbius transformations.

We shall show that Ψ is not constant almost everywhere with respect to $\mu_F \times \mu_F$. We choose an element $A \times B$ in the covering $\{A_i \times B_i\}_{i\in I}$ of $\Lambda_F^{(2)}$ such that A' has a positive measure with respect to μ_F. This is possible because we are assuming that $\mathrm{end}_F(G)$ is conull. We claim that there exists a sequence $\{B_{\epsilon_n}\}$ of ϵ_n-open balls with respect to the spherical metric on S^2_∞ such that the following holds. For each n, $B_{\epsilon_n} \cap \mathsf{A}'$ has a positive μ_F-measure and $B_{\epsilon_{n+1}} \subset B_{\epsilon_n}$, and the sequence $\{\epsilon_n\}$ converges to 0 as $n\to\infty$. The existence of such balls can be seen by considering a finite covering of the compact set A by ϵ_n-balls.

Next we shall show that as $\epsilon_n \to 0$, the infimum of $\Psi|_{(B_{\epsilon_n}\cap \mathsf{A}')^{(2)}}$ goes to infinity. Suppose not. Then there exist two sequences $\{a_n\}, \{a'_n\}$ with $(a_n, a'_n) \in (B_{\epsilon_n} \cap \mathsf{A}')^{(2)}$ such that $\Psi(a_n, a'_n) < K$ for some constant K. Now, since $\psi(\mathsf{A}')$ is contained in B, there is a positive constant L such that $|a_n - \psi(a'_n)| \geq L$ and $|a'_n - \psi(a_n)| \geq L$. On the other hand, since $|a_n - a'_n| \leq \epsilon_n \to 0$, it can be easily checked that $\Psi(a_n, a'_n) \to \infty$ as $n\to\infty$, which is a contradiction. Thus, we have shown that $\inf \Psi|_{(B_{\epsilon_n}\cap \mathsf{A}')^{(2)}}$ goes to ∞ as $n \to \infty$.

Suppose now, seeking a contradiction, that Ψ coincides with a constant K' almost everywhere with respect to $\mu_F \times \mu_F$. Then we choose n large enough so that the infimum of $\Psi|_{(B_{\epsilon_n}\cap \mathsf{A}')^{(2)}}$ is greater than K'. This is a contradiction, because $B^{(2)}_{\epsilon_n}$ has a positive $\mu_F \times \mu_F$ measure and thus there must be a point $(a, b) \in B^{(2)}_{\epsilon_n}$ such that $\Psi(a, b) = K'$. Thus, we have shown that Ψ cannot be constant almost everywhere.

Finally, we see that $\widehat{i}|_{\Lambda_F \setminus \mathrm{end}_F(G)}$ is also a homeomorphism to its image $\Lambda_H \backslash \widehat{i}(\mathrm{end}_F(G))$ by noting that $\widehat{i}^{-1}(\widehat{i}(\mathrm{end}_F(G))) = \mathrm{end}_F(G)$ and that $\widehat{i}|_{\Lambda_F \setminus \mathrm{end}_F(G)}$ is a closed map. □

We can define $\mathrm{end}_F(H)$ in the same way so that $\widehat{j}|_{\Lambda_F \setminus \mathrm{end}_F(H)}$ becomes a homeomorphism to its image. Now we define a subset Z of Λ_G as follows:

$$Z := \widehat{i}(\mathrm{end}_F(G) \cup \mathrm{end}_F(H)).$$

LEMMA 4.5. *Z is a null set in (Λ_G, μ_G).*

Proof. By Lemma 4.1, $\widehat{i}(\mathrm{end}_F(G))$ consists of non-conical limit points. Since G is of divergence type, it has measure zero by Theorem 2.1. To see that $\widehat{i}(\mathrm{end}_F(H))$ is also null, we first note that $\widehat{j}\,\widehat{i}^{-1}|_{\Lambda_G \setminus \widehat{i}(\mathrm{end}_F(G))}$ is a well-defined equivariant continuous map.

https://doi.org/10.1017/etds.2015.15 Published online by Cambridge University Press

This follows from the fact that $\widehat{i}$ gives a homeomorphism between $\Lambda_F \setminus \mathrm{end}_F(G)$ and $\Lambda_G \setminus \widehat{i}(\mathrm{end}_F(G))$ as in Lemma 4.4 and $\widehat{j}$ is continuous. If we set

$$\mathrm{End}_G(H) := \{(\xi_1, \xi_2) \in (\Lambda_G \setminus \widehat{i}(\mathrm{end}_F(G)))^{(2)} | \widehat{j}\,\widehat{i}^{-1}(\xi_1) = \widehat{j}\,\widehat{i}^{-1}(\xi_2)\},$$

then $\mathrm{End}_G(H)$ is a G-invariant closed subset in $(\Lambda_G \setminus \widehat{i}(\mathrm{end}_F(G)))^{(2)}$. We set $\mathrm{end}_G(H) := \mathrm{proj}_1(\mathrm{End}_G(H))$, and can use the same argument as in the proof of Lemma 4.4 to prove that it is a Borel null set.

In fact, we can prove this as follows. We consider a countable covering $\{C_j \times D_j\}_{j \in J}$ of $(\mathrm{end}_G(H))^{(2)}$, where C_j and D_j are disjoint closed subsets of Λ_G for each $j \in J$. Set $\mathsf{C}_j := \mathrm{end}_{C_j, C_j \times D_j} = \mathrm{proj}_1(\mathrm{end}_G(H) \cap C_j \times D_j)$, which is closed in $\Lambda_G \setminus \widehat{i}(\mathrm{end}_F(G))$. We note that $\mathrm{end}_G(H) = \bigcup_{j \in J} \mathsf{C}_j$ to see that it is a Borel set in $\Lambda_G \setminus \widehat{i}(\mathrm{end}_F(G))$. Then we apply Proposition 3.1 to $\widehat{j}$ to see that $\{x \in \Lambda_F \mid \#(\widehat{j}^{-1}(\widehat{i}(x))) > 2\}$ is countable and hence its image Z_2' by $\widehat{i}$ is also countable. We set $\mathsf{C}_j' := \mathsf{C}_j \setminus Z_2'$ and $\mathsf{D}_j' := \mathsf{D}_j \setminus Z_2'$ as before. Then we also have a homeomorphism $\psi_j' := (\widehat{j}\,\widehat{i}^{-1}|_{\mathsf{C}_j'})^{-1}(\widehat{j}\,\widehat{i}^{-1}|_{\mathsf{D}_j'})$ between C_j' and D_j'. Using the ψ_j', we can define a G-invariant Borel measurable function Ψ' from $(\mathrm{end}_G(H) \setminus Z_2')^{(2)}$ in the same way as before. Since $\widehat{i}(\mathrm{end}_F(G))$ is a G-invariant null set, $\mathrm{end}_G(H)$ is H-invariant and, since we are assuming that G is of divergence type, $\mathrm{end}_G(H)$ is either null or conull. If it is conull, we conclude that Ψ' has to be constant almost everywhere. This leads to a contradiction as before. Since $\widehat{i}(\mathrm{end}_F(H))$ is contained in $\mathrm{end}_G(H) \cup \widehat{i}(\mathrm{end}_F(G))$, it is also a null set. □

LEMMA 4.6. *$Z = \widehat{i}(\mathrm{end}_F(G) \cup \mathrm{end}_F(H))$ is a Borel set.*

Proof. It is well known that the image of a Borel set in a standard measure space by a countable-to-one continuous map is also Borel. See [**26**, Theorem 4.12.4], for example. We already know that $\mathrm{end}_F(G)$ is Borel. Since $\mathrm{end}_F(H)$ is also a Borel set, which can be shown by the argument as in the first half of Lemma 4.4, we see that Z is a Borel set. □

Thus, we conclude that Z is a Borel null set. We have proved (1) of Theorem 1.1, and (2) follows from the argument by Sullivan and Tukia in [**29**, **32**]. Here we shall present a simplified version of the proof of [**32**, Theorem 3C].

First we need to restrict the underlying σ-algebra of the measure spaces (S^2_∞, μ_F) and $((S^2_\infty)^{(2)}, \mu_F)$ to the Borel σ-algebra. Let us denote $\widehat{j}\,\widehat{i}^{-1}$ by $\widehat{k}$. Then $\widehat{k}|_{\Lambda_G \setminus Z}$ is Borel measurable and also Borel *directly measurable* in the sense that it sends every Borel subset of $\Lambda_G \setminus Z$ to a Borel set. We define the pull-back measure $\widehat{k}^*\mu_H$ on Λ_G by setting $(\widehat{k}^*\mu_H)(B) := \mu_H(\widehat{k}(B \setminus Z))$ for any Borel set B in Λ_G. Next we apply the Lebesgue decomposition theorem to the measure $\widehat{k}^*\mu_H$ on (Λ_G, μ_G). Then we get a conull Borel subset A of $\Lambda_G \setminus Z$ such that $\widehat{k}^*\mu_H$ is absolutely continuous with respect to μ_G on A. If $(\widehat{k}^*\mu_H)(A) = 0$, then the second case in (2) of Theorem 1.1 holds and we are done. Therefore, we assume that $(\widehat{k}^*\mu_H)(A) > 0$ from now on. Then we have the Radon–Nikodym derivative $D = d(\widehat{k}^*\mu_H)/d\mu_G$ on A, which cannot be a μ_G-almost everywhere zero function.

We define a G-invariant ergodic measure ν_G on $\Lambda_G^{(2)}$ as follows:

$$d\nu_G(\xi_1, \xi_2) = \frac{d\mu_G(\xi_1)\, d\mu_G(\xi_2)}{|\xi_1 - \xi_2|^{\delta_G}}.$$

https://doi.org/10.1017/etds.2015.15 Published online by Cambridge University Press

We can define an H-invariant measure ν_H on $\Lambda_H^{(2)}$ in the same way, although ν_H may not be ergodic. Let ν be a measure on $A^{(2)}$ defined by setting $\nu(B') = \nu_H((\widehat{k} \times \widehat{k})(B' \cap A^{(2)})$ for any Borel set B' in $\Lambda_G^{(2)}$. Then ν can be written as follows:

$$d\nu(\xi_1, \xi_2) = \frac{d(\widehat{k}^*\mu_H)(\xi_1)\, d(\widehat{k}^*\mu_H)(\xi_2)}{|\widehat{k}(\xi_1) - \widehat{k}(\xi_2)|^{\delta_H}}$$

for $(\xi_1, \xi_2) \in A^{(2)}$. Since $\widehat{k}$ is equivariant, ν is also G-invariant. Thus, $d\nu/d\nu_G$ is a G-invariant measurable function defined ν_G-almost everywhere. Due to the ergodicity of ν_G, it has to be a constant ν_G-almost everywhere. We can write $d\nu/d\nu_G$ as follows:

$$\frac{d\nu}{d\nu_G}(\xi_1, \xi_2) = D(\xi_1)D(\xi_2)\frac{|\xi_1 - \xi_2|^{\delta_G}}{|\widehat{k}(\xi_1) - \widehat{k}(\xi_2)|^{\delta_H}}.$$

Thus, $d\nu/d\nu_G$ is a non-zero constant function ν_G-almost everywhere and this implies that for almost every quadruple $(\xi_1, \xi_2, \xi_3, \xi_4)$ in Λ_G^4,

$$\mathrm{cr}(\xi_1, \xi_2, \xi_3, \xi_4)^{\delta_G} = \mathrm{cr}(\widehat{k}(\xi_1), \widehat{k}(\xi_2), \widehat{k}(\xi_3), \widehat{k}(\xi_4))^{\delta_H}.$$

Now, since G is non-elementary, we have $\delta_G > 0$ by [**3**, Theorem 7]. Therefore, [**32**, Theorem 2B] implies that $\delta_G = \delta_H$. Thus, the above equality shows the invariance of cross ratio by $\widehat{k}$ and hence that $\widehat{k}$ is the restriction of a Möbius transformation.

References

[**1**] I. Agol. Tameness of hyperbolic 3-manifolds. *Preprint*, 2004, arXiv:math.GT/0405568.
[**2**] J. Anderson, K. Falk and P. Tukia. Conformal measures associated to ends of hyperbolic n-manifolds. *Q. J. Math.* **58** (2007), 1–15.
[**3**] A. Beardon. The exponent of convergence of Poincaré series. *Proc. Lond. Math. Soc.* (3) **18** (1968), 461–483.
[**4**] C. J. Bishop and P. W. Jones. The law of iterated logarithm for Kleinian groups. *Contemp. Math.* **211** (1997), 17–50.
[**5**] F. Bonahon. Bout des variétés hyperboliques de dimension 3. *Ann. of Math.* (2) **124** (1986), 71–158.
[**6**] J. Brock, D. Canary and Y. Minsky. The classification of Kleinian surface groups II: the ending lamination conjecture. *Ann. of Math.* (2) **176** (2012), 1–149.
[**7**] M. Burger and S. Mozes. CAT(–1)-spaces, divergence groups and their commensurators. *J. Amer. Math. Soc.* **9**(1) (1996), 57–93.
[**8**] D. Calegari and D. Gabai. Shrinkwrapping and the taming of hyperbolic 3-manifolds. *J. Amer. Math. Soc.* **19**(2) (2006), 385–446.
[**9**] D. Canary. Ends of hyperbolic 3-manifolds. *J. Amer. Math. Soc.* **6**(1) (1993), 1–35.
[**10**] M. Culler and P. Shalen. Paradoxical decompositions, 2-generator Kleinian groups, and volumes of hyperbolic 3-manifolds. *J. Amer. Math. Soc.* **5**(2) (1992), 231–288.
[**11**] M. Feighn and D. McCullough. Finiteness conditions for 3-manifolds with boundary. *Amer. J. Math.* **109** (1987), 1155–1169.
[**12**] R. Feres and A. Katok. Ergodic theory and dynamics of G-spaces. *Handbook of Dynamical Systems.* Vol 1A. Eds. B. Hasselblatt and A. Katok. North-Holland, Amsterdam, 2002, pp. 665–763.
[**13**] V. Gerasimov. Floyd maps for relatively hyperbolic groups. *Geom. Funct. Anal.* **22** (2012), 1–39.
[**14**] S. Hersonsky and F. Paulin. On the rigidity of discrete isometry groups of negatively curved spaces. *Comment. Math. Helv.* **72** (1997), 349–388.
[**15**] W. Jeon, I. Kapovich, C. Leininger and K. Ohshika. Conical limit points and the Cannon–Thurston map. *Preprint*, 2014, arXiv:1401.2638.
[**16**] C. Lecuire. Plissage des variétés hyperboliques de dimension 3. *Invent. Math.* **164** (2006), 85–141.
[**17**] C. Lecuire. Structure hyperboliques convexes sur les variétés de dimension 3. *Thèse*, ENS Lyon, 2004.

https://doi.org/10.1017/etds.2015.15 Published online by Cambridge University Press

[**18**] Y. N. Minsky. The classification of Kleinian surface groups. I. Models and bounds. *Ann. of Math.* (2) **171** (2010), 1–107.
[**19**] M. Mj. Cannon–Thurston map for surface groups. *Ann. of Math.* (2) **179** (2014), 1–80.
[**20**] M. Mj. Cannon–Thurston maps for Kleinian groups. *Preprint*, 2010, arXiv:1002.0996.
[**21**] P. J. Nicholls. *The Ergodic Theory of Discrete Groups (London Mathematical Society Lecture Note Series, 143)*. Cambridge University Press, Cambridge, 1989.
[**22**] S. J. Patterson. The limit set of a Fuchsian group. *Acta Math.* **136** (1976), 241–273.
[**23**] P. Scott. Compact submanifolds of a 3-manifold. *J. Lond. Math. Soc.* (2) **7** (1973), 246–250.
[**24**] T. Soma. Equivariant, almost homeomorphic maps between S^1 and S^2. *Proc. Amer. Math. Soc.* **123**(9) (1995), 2915–2920.
[**25**] R. Spatzier and R. Zimmer. Fundamental groups of negatively curved manifolds and actions of semisimple groups. *Topology* **30** (1991), 591–601.
[**26**] S. M. Srivastava. *A Course on Borel Sets (Graduate Text in Mathematics, 180)*. Springer, New York, 1998.
[**27**] D. Sullivan. The density at infinity of a discrete group of hyperbolic motions. *Publ. Math. Inst. Hautes Études Sci.* **50** (1979), 171–202.
[**28**] D. Sullivan. On the ergodic theory at infinity of an arbitrary discrete group of hyperbolic motions. *Riemann Surfaces and Related Topics: Proceeding of the 1978 Stony Brook Conference (Annals of Mathematics Studies, 97)*. Princeton University Press, Princeton, NJ, 1981, pp. 465–496.
[**29**] D. Sullivan. Discrete conformal groups and measurable dynamics. *Bull. Amer. Math. Soc. (N.S.)* **6** (1982), 57–73.
[**30**] D. Sullivan. Entropy, Hausdorff measures old and new, and limit sets of geometrically finite Kleinian groups. *Acta Math.* **153** (1984), 259–277.
[**31**] W. P. Thurston. *The Geometry and Topology of 3-Manifolds (Princeton University Lecture Notes)*. 1980, available at http://library.msri.org/nonmsri/gt3m/.
[**32**] P. Tukia. A rigidity theorem for Mobius groups. *Invent. Math.* **97** (1989), 405–431.
[**33**] C. Yue. Mostow rigidity of rank 1 discrete groups with ergodic Bowen–Margulis measure. *Invent. Math.* **125** (1996), 75–102.
[**34**] R. Zimmer. *Ergodic Theory and Semi-simple Groups*. Birkhäuser, Boston, MA, 1984.

https://doi.org/10.1017/etds.2015.15 Published online by Cambridge University Press