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Beilinson's Hodge Conjecture for Smooth Varieties

with an Appendix by Masanori Asakura

by

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Abstract

Let $U/\mathbb{C}$ be a smooth quasi-projective variety of dimension d, $\mathrm{CH}^r(U,m)$ Bloch's higher Chow group, and

$$\mathrm{cl}_{r,m} : \mathrm{CH}^r(U,m) \otimes \mathbb{Q} \to \hom_{\mathrm{MHS}}\big(\mathbb{Q}(0), H^{2r-m}(U,\mathbb{Q}(r))\big)$$

the cycle class map. Beilinson once conjectured $\mathrm{cl}_{r,m}$ to be surjective [Be]; however, Jannsen was the first to find a counterexample in the case $m = 1$ [Ja1]. In this paper we study the image of $\mathrm{cl}_{r,m}$ in more detail (as well as at the "generic point" of U) in terms of kernels of Abel-Jacobi mappings. When $r = m$, we deduce from the Bloch-Kato conjecture (now a theorem) various results, in particular that the cokernel of $\mathrm{cl}_{m,m}$ at the generic point is the same for integral or rational coefficients.

Key Words: Chow group, algebraic cycle, regulator, Beilinson-Hodge conjecture, Abel-Jacobi map, Deligne cohomology.

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1. Introduction

Let $U/\mathbb{C}$ be a smooth and quasi-projective variety of dimension d, $\mathrm{CH}^r(U,m)$ Bloch's higher Chow group, and

$$\mathrm{cl}_{r,m} : \mathrm{CH}^r(U,m;\mathbb{Q}) \to \Gamma\big(H^{2r-m}(U,\mathbb{Q}(r))\big),$$

the Betti cycle class map, where

$$\Gamma\big(H^{2r-m}(U,\mathbb{Q}(r))\big) := \hom_{\mathrm{MHS}}\big(\mathbb{Q}(0), H^{2r-m}(U,\mathbb{Q}(r))\big).$$

If $m = 0$, then by a standard weight argument (see [Ja1, pp.62-63]), the Hodge conjecture implies that $\mathrm{cl}_{r,0}$ is surjective. Beilinson once conjectured that $\mathrm{cl}_{r,m}$ is always surjective [Be]. However unless U is given by base extension from a smooth quasi-projective variety over a number field, it is known that this conjecture is too optimistic [Ja1, Cor. 9.11].

Consider these three statements:

(S1) $\mathrm{cl}_{r,m} : \mathrm{CH}^r(X,m;\mathbb{Q}) \to \Gamma\big(H^{2r-m}(X,\mathbb{Q}(r))\big)$ is surjective for all smooth complex projective varieties X;

(S2) $\mathrm{cl}_{r,m} : \mathrm{CH}^r(U,m;\mathbb{Q}) \to \Gamma\big(H^{2r-m}(U,\mathbb{Q}(r))\big)$ is surjective for all smooth complex quasi-projective varieties U;

(S3) $\lim(\mathrm{cl}_{r,m}) : \mathrm{CH}^r(\mathrm{Spec}(\mathbb{C}(X)),m;\mathbb{Q}) \to \Gamma\big(H^{2r-m}(\mathbb{C}(X),\mathbb{Q}(r))\big)$ is surjective for all smooth complex projective varieties X.

Note that (S1) for $m = 0$ is equivalent with the Hodge conjecture (with rational coefficients as opposed to the original version with integral coefficients that was

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disproven by Atiyah-Hirzebruch by showing the existence of non-algebraic torsion classes), and that it is trivially true for $m > 0$ because then $\Gamma\big(H^{2r-m}(X,\mathbb{Q}(r))\big) = 0$. Also, $\mathrm{CH}^r(\mathrm{Spec}(\mathbb{C}(X)),m;\mathbb{Q}) = 0$ for $r > m$ in (S3) because of dimension reasons.

When $m = 0$, all three statements (for all $r \geq 0$) are equivalent (as one sees using a localization sequence argument, and Deligne's mixed Hodge theory, as in [Ja1, pp.62-63]). However as we shall see in this paper, the statements are independent of each other: (S1) is expected to be true, we conjecture (S3) to be true, and show (S2) to be false in general. There is some evidence ([A-S], [Sa], [A-K], [MSa]) that (S2) holds in the special case $r = m$, and the results in this paper are consistent with this. In [SJK-L] we provided some evidence that (S3) is always true, and in particular, (S3) can be viewed as an appropriate generalization of the Hodge conjecture.

In this paper we address a number of issues, namely:

• Necessary and sufficient conditions for $\mathrm{cl}_{r,m}$, and $\lim(\mathrm{cl}_{r,m})$ to be surjective, in terms of kernels of (reduced) higher Abel-Jacobi maps. This is worked out in Theorem 4.9 and subsequent examples, as well as in Corollary 6.8 below. Naturally this leads to a generalized notion of decomposable classes, which is discussed in Section 8. It is aimed at giving a partial geometric interpretation of a certain subspace mentioned in Corollary 6.8, where it plays a role in capturing the kernel of the Abel-Jacobi map. Also, in Theorem 5.1 we exhibit counterexamples to the surjectivity of $\mathrm{cl}_{r,m}$ in (S2) in all cases where it is not trivially true or where one might reasonably expect this surjectivity (namely $r = m$ or $m = 0$).

• The story can be worked out with integral coefficients in the case $r = m$, and in particular, we are interested in the nature of the map

$$d\log_m : \mathrm{CH}^m(\mathrm{Spec}(\mathbb{C}(X)),m) \to H^m(\mathbb{C}(X),\mathbb{Z}(m)) \cap F^m H^m(\mathbb{C}(X),\mathbb{C}).$$

We prove in Section 7 that the torsion subgroup $H^m(\mathbb{C}(X),\mathbb{Z}(m))_{\mathrm{tor}}$ of $H^m(\mathbb{C}(X),\mathbb{Z}(m))$ is trivial[1] (hence this intersection makes sense!). The combination of Theorem 1.1 and Conjecture 1.2 below would imply that $d\log_m$ is surjective. We also relate $d\log_m$ to the map

$$\frac{\mathrm{CH}^m(\mathrm{Spec}(\mathbb{C}(X)),m)}{l} \to H^m_{\mathrm{et}}(\mathrm{Spec}(\mathbb{C}(X)),\mu_l^{\otimes_{\mathbb{Z}} m}),$$

for l a non-zero integer, which is now known to be an isomorphism. (This is the former Bloch-Kato conjecture[2], for the field $\mathbb{C}(X)$.) Thus the conjectured

[1] Already known to some experts.

[2] This is now a theorem ([We]).

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surjectivity of $d\log_m$ can be thought of as a Hodge theoretic version of the Bloch-Kato conjecture. Note that $\lim(\mathrm{cl}_{m,m})$ equals

$$d\log_m \otimes \mathbb{Q} : \mathrm{CH}^m(\mathrm{Spec}(\mathbb{C}(X)),m;\mathbb{Q}) \to \Gamma\big(H^m(\mathbb{C}(X),\mathbb{Q}(m))\big).$$

As mentioned above, the classical Hodge conjecture, as originally formulated by Hodge with integral coefficients, is false. But we wish to remind the reader that it is false with integral coefficients even *modulo torsion* (see [Lew, p.67]), albeit expected (by optimists) to be true with rational coefficients. The following statement, again proven in Section 7, therefore seems rather interesting.

Theorem 1.1 $\mathrm{coker}(d\log_m) \simeq \mathrm{coker}(\lim(\mathrm{cl}_{m,m}))$. *In particular,* $d\log_m$ *is surjective* $\Leftrightarrow$ $\lim(\mathrm{cl}_{m,m})$ *is surjective.*

What this theorem tells us is that the Hodge theoretic analog of the Bloch-Kato conjecture is the surjectivity of $\lim(\mathrm{cl}_{m,m})$. Quite generally we expect that the following is true.

Conjecture 1.2 *For all* $r,m \geq 0$*, statement (S3) holds.*

By our earlier remarks this conjecture includes the Hodge conjecture and relates it to the (now proved) Bloch-Kato conjecture.

In our attempt to come up with explicit examples, we describe a method in Section 9 to produce interesting (to us) families of open surfaces where (S2) for $r = m = 2$ holds trivially for a (very) general member of the family, and yet a study of special members could open up a new line of enquiry on test cases for this conjecture.

In the second appendix, M. Asakura gives an elementary proof of the following statement: *Let* U *be a smooth complex variety. Then* $d\log : \mathrm{CH}^2(U,2) \to \Gamma H^2(U,\mathbb{Z}(2))$ *is surjective* $\Leftrightarrow$ $d\log\otimes\mathbb{Q} : \mathrm{CH}^2(U,2;\mathbb{Q}) \to \Gamma H^2(U,\mathbb{Q}(2))$ *is surjective.* Assuming it turns out that $d\log$ is surjective, one could think of this as analogous to the Lefschetz (1,1) theorem (which involves integral coefficients).

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2. Notation

(i) Unless otherwise specified, X is a smooth complex projective variety of dimension d, and U is a smooth complex quasi-projective variety.

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Let $\mathbb{A} \subseteq \mathbb{R}$ be a subring.

(ii) $\mathbb{A}(r) := (2\pi \mathrm{i})^r \mathbb{A}$ (Tate twist).

(iii) If H is an $\mathbb{A}$-mixed Hodge structure (MHS), then we write $\Gamma(H) := \hom_{\mathbb{A}-\mathrm{MHS}}(\mathbb{A}(0), H)$ and $J(H) := \mathrm{Ext}^1_{\mathbb{A}-\mathrm{MHS}}(\mathbb{A}(0), H)$.

(iv) For a quasi-projective variety V, $\mathrm{CH}^r(V,m)$ is the higher Chow group defined in [Bl], and $\mathrm{CH}^r(V) := \mathrm{CH}^r(V,0)$.

(v) $\mathrm{CH}^r(V,m;\mathbb{Q}) := \mathrm{CH}^r(V,m) \otimes \mathbb{Q}$.

(vi) We write

$$H^{2r-m}(\mathbb{C}(X),\mathbb{Q}(r)) = \lim_{U \vec{\subset} X} H^{2r-m}(U,\mathbb{Q}(r)),$$

the limit taken over all Zariski open subsets of X.

(vii) We let $H^*_{\mathcal{D}}(U,\mathbb{A}(r))$ denote the (algebraic) Deligne-Beilinson cohomology [EV] of U.

3. Weight filtered spectral sequence

We provide a breezy review of some of the ideas in [K-L, Section 3.1]. We first recall the definition of the higher Chow groups. Let $V/\mathbb{C}$ be a quasi-projective variety. Put $z^r(V) =$ free abelian group generated by subvarieties of codimension r in V, Δ^m the standard m-simplex, and $z^r(V,m) = \{\xi \in z^r(V \times \Delta^m) \mid \xi \text{ meets all faces properly}\}$. We let $\partial = \sum_i (-1)^i \partial_i$ where ∂_i is the restriction to the i-th codimension 1 face.

Definition 3.1 ([Bl]) $\mathrm{CH}^\bullet(V,\bullet) =$ homology of $\{z^\bullet(V,\bullet),\partial\}$. We put $\mathrm{CH}^k(V) := \mathrm{CH}^k(V,0)$.

We also need to recall the cubical version. Let $\square^m := (\mathbb{P}^1 \backslash \{1\})^m$ with coordinates z_i and 2^m codimension one faces obtained by setting $z_i = 0,\infty$, and boundary maps $\partial = \sum (-1)^{i-1}(\partial_i^0 - \partial_i^\infty)$, where ∂_i^0, ∂_i^∞ denote the restriction maps to the faces $z_i = 0$, $z_i = \infty$ respectively. The rest of the definition is completely analogous for $z^r(V,m) \subset z^r(V \times \square^m)$, except that one has to quotient out by a subgroup of degenerate cycles. It is known that both complexes are quasi-isomorphic ([Bl]).

Now write $U = X \backslash Y$, where $X/\mathbb{C}$ is a smooth projective variety of dimension d, $Y = Y_1 \cup \cdots \cup Y_n \subset X$ a NCD with smooth components. For an integer $t \geq 0$, put $Y^{[t]} =$ disjoint union of t-fold intersections of the various components of Y, with corresponding simplicial scheme $Y^{[\bullet]} \to Y \hookrightarrow Y^{[0]} := X$. There is a third quadrant

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double complex

$$\mathcal{Z}_0^{i,j}(r) := z^{r+i}(Y^{[-i]},-j),\ i,\ j \le 0; \qquad \begin{array}{ccc} \mathcal{Z}_0^{i,j+1} & & \\ \uparrow \partial & & \\ \mathcal{Z}_0^{i,j} & \xrightarrow{\text{Gy}} & \mathcal{Z}_0^{i+1,j} \end{array} \tag{3.2}$$

whose differentials are ∂ vertically (∂ as coming from the definition of Bloch's higher Chow groups), and Gy (= Gysin) horizontally. To the corresponding total complex $\mathbf{s}^\bullet\mathcal{Z}(r)$ with $D = \partial \pm \text{Gy}$, are associated the two spectral sequences $E_i^{p,q}$ and ${}'E_i^{p,q}$ arising from the double complex in (3.2), with

$$E_2^{p,q} = H^p_{\text{Gy}}\big(H^q_\partial(\mathcal{Z}_0^{\bullet,\bullet}(r))\big);\quad {}'E_2^{p,q} = H^p_\partial(H^q_{\text{Gy}}\big(\mathcal{Z}_0^{\bullet,\bullet}(r))\big).$$

The second spectral sequence, together with Bloch's quasi-isomorphism

$$\frac{z^\bullet(X,*)}{z^\bullet_Y(X,*)} \xrightarrow{\text{Restriction}} z^\bullet(U,*),$$

shows that ([K-L](sect. 3.1))

$$H^{-m}(\mathbf{s}^\bullet\mathcal{Z}(r)) = {}'E_2^{0,-m} = \text{CH}^r(U,m).$$

The first spectral sequence has $E_1^{i,j} = \text{CH}^{r+i}(Y^{[-i]},-j)$ and

$$E_2^{i,j} = \frac{\ker\big(\text{Gy} : \text{CH}^{r+i}(Y^{[-i]},-j) \to \text{CH}^{r+i+1}(Y^{[-i-1]},-j)\big)}{\text{Gy}\big(\text{CH}^{r+i-1}(Y^{[-i+1]},-j)\big)}.$$

The corresponding filtration on $\mathbf{s}^\bullet\mathcal{Z}(r)$ also induces a "weight" filtration

$$W_{-m}\text{CH}^r(U,m) \subseteq \cdots \subseteq W_0\text{CH}^r(U,m) = \text{CH}^r(U,m),$$

which is characterized by the injection

$$\partial_R^{\ell,m} : Gr_W^\ell \text{CH}^r(U,m) = E_\infty^{-\ell-m,\ell} \hookrightarrow \left\{ \begin{array}{c} \text{A subquotient of} \\ \text{CH}^{r-\ell-m}(Y^{[\ell+m]},-\ell) \end{array} \right\},$$

for $\ell = -m,\ldots,0$, where $\partial_R^{\ell,m}$ is called a residue map in [K-L]. It is easy to check that

$$E_\infty^{0,-m} = W_{-m}\text{CH}^r(U,m)\big) = \text{Image}\big(\text{CH}^r(X,m) \to \text{CH}^r(U,m)\big).$$

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4. The image of the cycle class map

Our main goal in this section is to prove Theorem 4.9, which provides necessary and sufficient conditions for the surjectivity of $\mathrm{cl}_{r,m} : \mathrm{CH}^r(U,m;\mathbb{Q}) \to \Gamma\big(H^{2r-m}(U,\mathbb{Q}(r))\big)$. The obstruction to surjectivity will be explained in terms of kernels of Abel-Jacobi maps for the higher Chow groups. We fix U, $r \geq 0$ and $m \geq 0$. Of particular interest is the top residue $\partial_R(\xi) := \partial_R^{0,m}(\xi) \in E_\infty^{-m,0}$ for $\xi \in \mathrm{CH}^r(U,m)$, where

$$E_\infty^{-m,0} \subseteq E_2^{-m,0} = \frac{\ker\big(\mathrm{Gy} : \mathrm{CH}^{r-m}(Y^{[m]}) \to \mathrm{CH}^{r-m+1}(Y^{[m-1]})\big)}{\mathrm{Gy}\big(\mathrm{CH}^{r-m-1}(Y^{[m+1]})\big)}. \tag{4.1}$$

(In general, $E_2^{-m,0} \neq E_\infty^{-m,0}$.) We use this to study the cycle class map $\mathrm{cl}_{r,m}$ via the commutative diagram

$$\begin{array}{ccc} \mathrm{CH}^r(U,m;\mathbb{Q}) & \twoheadrightarrow & E_\infty^{-m,0} \otimes \mathbb{Q} \\ \Big\downarrow{\scriptstyle \mathrm{cl}_{r,m}} & & \Big\downarrow \\ \Gamma\big(H^{2r-m}(U,\mathbb{Q}(r))\big) & \hookrightarrow & \Gamma\big(Gr_0^W H^{2r-m}(U,\mathbb{Q}(r))\big), \end{array} \tag{4.2}$$

where the injectivity of the map on the bottom row follows from the fact that $\Gamma\big(W_{-1}H^{2r-m}(U,\mathbb{Q}(r))\big) = 0$. (In general we use $\mathbb{Q}$-coefficients because weight filtration in Hodge theory is defined for such coefficients. Exceptions to this situation are discussed in Sections 6 and 7.) Note that (4.2) gives us a map $E_\infty^{-m,0} \otimes \mathbb{Q} \to \Gamma\big(H^{2r-m}(U,\mathbb{Q}(r))\big)$. With regard to the morphism

$$\mathrm{Gy} : \mathrm{CH}^{r-m}(Y^{[m]}) \to \mathrm{CH}^{r-m+1}(Y^{[m-1]}),$$

for $m \geq 1$, put

$$\mathrm{CH}^{r-m}(Y^{[m]})^\circ := \mathrm{Gy}^{-1}\big(\mathrm{CH}^{r-m+1}_{\mathrm{hom}}(Y^{[m-1]})\big),$$

where $\mathrm{CH}^{r-m+1}_{\mathrm{hom}}(Y^{[m-1]}) \subset \mathrm{CH}^{r-m+1}(Y^{[m-1]})$ is the subgroup of null-homologous cycles on $Y^{[m-1]}$. Notice that for $m \geq 1$,

$$E_\infty^{-m,0} = E_{m+1}^{-m,0} \subseteq E_2^{-m,0} \hookrightarrow \frac{\mathrm{CH}^{r-m}(Y^{[m]})^\circ}{\mathrm{Gy}\big(\mathrm{CH}^{r-m-1}(Y^{[m+1]})\big)},$$

and for all $m \geq 0$, we have

$$d_m : E_m^{-m,0} \to E_m^{0,-m+1}, \quad E_\infty^{0,-m+1} = \frac{E_m^{0,-m+1}}{d_m(E_m^{-m,0})}.$$

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For $k = 1,...,m+1$ put

$$\widetilde{E}_k^{-m,0} = \begin{cases} \frac{\mathrm{CH}^{r-m}(Y^{[m]};\mathbb{Q})^\circ}{\mathrm{Gy}\left(\mathrm{CH}^{r-m-1}(Y^{[m+1]};\mathbb{Q})\right)} & \text{if } k=1 \\ E_k^{-m,0}\otimes\mathbb{Q} & \text{for } k=2,...,m+1 \end{cases}$$

so that $\widetilde{E}_k^{-m,0} \subseteq \widetilde{E}_1^{-m,0}$ for $k \geq 1$. (Our main reason for introducing $\widetilde{E}_1^{-m,0}$ is so that the map β in diagram (4.6) below has a chance of being surjective, as implied by the Hodge conjecture.) For $k = 1,\ldots,m$ we let $\widetilde{E}_k^{-m+k,-k+1}$ be

$$\begin{cases} \ker\left(\mathrm{CH}^{r-m+1}_{\mathrm{hom}}(Y^{[m-1]};\mathbb{Q}) \xrightarrow{\mathrm{Gy}} \mathrm{CH}^{r-m+2}(Y^{[m-2]};\mathbb{Q})\right) & \text{if } k=1, \\ E_k^{-m+k,-k+1}\otimes\mathbb{Q} & \text{for } k=2,...,m, \end{cases}$$

hence $\widetilde{E}_k^{-m+k,-k+1} \subseteq E_k^{-m+k,-k+1}\otimes\mathbb{Q}$ and $d_k(\widetilde{E}_k^{-m,0}) \subseteq \widetilde{E}_k^{-m+k,-k+1}$. Then (4.2) becomes

$$\begin{array}{ccc} \mathrm{CH}^r(U,m;\mathbb{Q}) & \twoheadrightarrow & \widetilde{E}_\infty^{-m,0} \\ \Big\downarrow{\scriptstyle \mathrm{cl}_{r,m}} & & \Big\downarrow \\ \Gamma(H^{2r-m}(U,\mathbb{Q}(r))) & \hookrightarrow & \Gamma(Gr_0^W) \end{array} \tag{4.3}$$

where we abbreviate $W_jH^{2r-m}(U,\mathbb{Q}(r))$ to W_j and similarly for Gr_j^W, and where $\widetilde{E}_\infty^{-m,0} \overset{\mathrm{def}}{=} \widetilde{E}_{m+1}^{-m,0} = \bigcap_{k\geq 1}\ker(d_k : \widetilde{E}_k^{-m,0} \to \widetilde{E}_k^{-m+k,-k+1}) \subseteq \widetilde{E}_1^{-m,0}$, which equals $E_\infty^{-m,0}\otimes\mathbb{Q}$ because $E_\infty^{-m,0} = E_{m+1}^{-m,0}$. Note that as in [K-L, 3.1],

$$\Gamma(Gr_0^W) = \frac{\ker\begin{pmatrix}\mathrm{Gy} : H^{r-m,r-m}(Y^{[m]},\mathbb{Q}(r-m)) \to \\ H^{r-m+1,r-m+1}(Y^{[m-1]},\mathbb{Q}(r-m+1))\end{pmatrix}}{\mathrm{Gy}\left(H^{r-m-1,r-m-1}(Y^{[m+1]},\mathbb{Q}(r-m-1))\right)},$$

and, for $0 \leq k \leq m$,

$$Gr_{-k}^W = \frac{\ker\begin{pmatrix}\mathrm{Gy} : H^{2r-2m+k}(Y^{[m-k]},\mathbb{Q}(r-m+k)) \to \\ H^{2r-2m+k+2}(Y^{[m-k-1]},\mathbb{Q}(r-m+k+1))\end{pmatrix}}{\mathrm{Gy}\left(H^{2r-2m+k-2}(Y^{[m-k+1]},\mathbb{Q}(r-m+k-1))\right)}. \tag{4.4}$$

Using the differentials $d_1 := \mathrm{Gy}, d_2, \ldots, d_m$ we claim that there is a commutative diagram of exact sequences for each $k = 1,\ldots,m$

$$\begin{array}{ccccccc} 0 & \to & \widetilde{E}_{k+1}^{-m,0} & \to & \widetilde{E}_k^{-m,0} & \xrightarrow{d_k} & \widetilde{E}_k^{-m+k,-k+1} \\ & & \alpha_{k+1}\downarrow & & \alpha_k\downarrow & & \lambda_k\downarrow \\ 0 & \to & J(W_{-k-1}) & \to & J(W_{-k}) & \overset{h_k}{\twoheadrightarrow} & J(Gr_{-k}^W) \end{array} \tag{4.5}$$

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where h_k is the obvious map, λ_k is the Abel-Jacobi map (defined explicitly in Section 10 below), and where the α_k's are characterized as follows. If we assume α_k is defined, then the definition of α_{k+1} is dictated by imposing commutativity in (4.5). Thus we need only define α_1, and show that $h_k \circ \alpha_k = \lambda_k \circ d_k$. The latter will be proven in Section 10. Note that, as implicit in (4.3), $\mathrm{cl}_{r,m}$ is the composition

$$\mathrm{CH}^r(U,m;\mathbb{Q}) \to \widetilde{E}_\infty^{-m,0} = \widetilde{E}_{m+1}^{-m,0} \to \Gamma\big(H^{2r-m}(U,\mathbb{Q}(r))\big),$$

and that there is a map $\beta = \beta_{r,m} : \widetilde{E}_1^{-m,0} \to \Gamma(Gr_0^W)$, which is an isomorphism when $r = m$, and is surjective for all r and m under the assumption of the Hodge conjecture. We let $\alpha_1 = \kappa \circ \beta$ in the diagram (with κ the obvious connecting homomorphism)

$$\begin{array}{ccccccc}
\widetilde{E}_\infty^{-m,0} & \hookrightarrow & \widetilde{E}_1^{-m,0} & \xrightarrow{d_1} & \widetilde{E}_1^{-m+1,0} & & \\
\downarrow & & \downarrow{\scriptstyle\beta} & \searrow{\scriptstyle\alpha_1} & \downarrow{\scriptstyle\lambda_1} & & \\
\Gamma\big(H^{2r-m}(U,\mathbb{Q}(r))\big) & \hookrightarrow & \Gamma(Gr_0^W) & \xrightarrow{\kappa} & J(W_{-1}) & \xrightarrow{h_1} & J(Gr_{-1}^W).
\end{array} \tag{4.6}$$

Then (4.6) and (4.5) commute (see Theorem 10.1).

With regard to the diagram (4.5) above, it is obvious that

$$\ker(\alpha_{k+1}) = \ker\big(d_k\big|_{\ker(\alpha_k)}\big) \subseteq \ker(\alpha_k).$$

From the isomorphisms $\beta(\ker(\alpha_k)) \simeq \ker(\alpha_k)/\ker(\beta)\cap\ker(\alpha_k)$ and $\beta(\ker(\alpha_{k+1})) \simeq \ker\big(d_k\big|_{\ker(\alpha_k)}\big)/\ker(\beta) \cap \ker\big(d_k\big|_{\ker(\alpha_k)}\big)$, we arrive at the identification

$$\frac{\beta(\ker(\alpha_k))}{\beta(\ker(\alpha_{k+1}))} \simeq \frac{d_k(\ker(\alpha_k))}{d_k\big(\ker(\beta)\cap\ker(\alpha_k)\big)}.$$

We have inclusions

$$\mathrm{im}(\mathrm{cl}_{r,m}) = \beta(\ker(\alpha_{m+1})) \subseteq \cdots \subseteq \beta(\ker(\alpha_1)) \subseteq \Gamma\big(H^{2r-m}(U,\mathbb{Q}(r))\big)$$

where on the left we have equality as $W_{-m-1} = 0$ and $\widetilde{E}_{m+1}^{-m,0} = \widetilde{E}_\infty^{-m,0}$, and the right-most inclusion is an equality if $\Gamma\big(H^{2r-m}(U,\mathbb{Q}(r))\big) \subseteq \mathrm{im}(\beta)$ (e.g., if β is surjective). We mention in passing the following result:

Proposition 4.7 *Suppose that* $\lambda_k\big|_{\mathrm{im}(d_k)}$ *is injective for* $k = 1,\ldots,m$. *If* β *is surjective, then* $\mathrm{cl}_{r,m} : \mathrm{CH}^r(U,m;\mathbb{Q}) \to \Gamma\big(H^{2r-m}(U,\mathbb{Q}(r))\big)$ *is surjective.*

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Proof: From (4.5) we get $\ker(\alpha_{m+1}) = \cdots = \ker(\alpha_1)$, and we apply the inclusions above. □

A slight tweaking of Proposition 4.7 together with diagrams (4.5) and (4.6) leads to:

Proposition 4.8 *[M. Saito [MSa]. Also see [K-L].] Assume the Hodge conjecture, and that the Bloch-Beilinson conjecture holds, viz., for all smooth projective $V/\overline{\mathbb{Q}}$, the Abel-Jacobi map $AJ : \mathrm{CH}^r_{\mathrm{hom}}(V/\overline{\mathbb{Q}}, j; \mathbb{Q}) \to J\big(H^{2r-j-1}(V, \mathbb{Q}(r))\big)$, is injective for all r and j. If U is obtained from a smooth quasi-projective variety over $\overline{\mathbb{Q}}$ by base change to $\mathbb{C}$, then for all r and m, $\mathrm{cl}_{r,m} : \mathrm{CH}^r(U, m; \mathbb{Q}) \to \Gamma\big(H^{2r-m}(U, \mathbb{Q}(r))\big)$ is surjective.*

Proof: The proof, which is similar to the one given in [K-L, Prop. 3.7], is omitted. □

Next, $\ker(\beta) \subseteq \ker(\alpha_1)$ by (4.6), hence $\ker(\beta) \cap \widetilde{E}_k^{-m,0} \subseteq \ker(\alpha_1) \cap \widetilde{E}_k^{-m,0} = \ker(\alpha_k) \subseteq \widetilde{E}_k^{-m,0}$ for $k = 1, \ldots, m+1$, with the last equality because of (4.5). Hence $\ker(\beta) \cap \ker(\alpha_k) = \ker(\beta) \cap \widetilde{E}_k^{-m,0}$. If h_k is an isomorphism then this gives

$$\frac{d_k(\ker(\alpha_k))}{d_k\big(\ker(\beta) \cap \ker(\alpha_k)\big)} = \frac{\ker\big(\lambda_k|_{d_k(\widetilde{E}_k^{-m,0})}\big)}{d_k\big(\ker(\beta) \cap \widetilde{E}_k^{-m,0}\big)}.$$

This is the case when $k = m$ again because $W_{-m-1} = 0$. Putting all these ideas together, we obtain the following.

Theorem 4.9 (i) *if $\mathrm{cl}_{r,m}$ is surjective then*

$$\frac{d_k(\ker(\alpha_k))}{d_k\big(\ker(\beta) \cap \ker(\alpha_k)\big)} = 0 \text{ for all } k = 1, \ldots, m;$$

the converse is true if β is surjective; more precisely: the converse is true if $\Gamma\big(H^{2r-m}(U, \mathbb{Q}(r))\big) \subseteq \mathrm{im}(\beta)$ (e.g., if β is surjective);

(ii) *$\mathrm{cl}_{r,m}$ is surjective implies that*

$$\frac{\ker\big(\lambda_m|_{d_m(\widetilde{E}_m^{-m,0})}\big)}{d_m\big(\ker(\beta) \cap \widetilde{E}_m^{-m,0}\big)} = 0$$

(iii) *if $\Gamma\big(H^{2r-m}(U, \mathbb{Q}(r))\big) \subseteq \mathrm{im}(\beta)$ and*

$$\frac{d_k(\ker(\alpha_k))}{d_k\big(\ker(\beta) \cap \ker(\alpha_k)\big)} = 0$$

for all $k = 1, \ldots, m-1$, then there is a short exact sequence

$$0 \to \mathrm{im}(\mathrm{cl}_{r,m}) \to \Gamma\big(H^{2r-m}(U, \mathbb{Q}(r))\big) \to \frac{\ker\big(\lambda_m|_{d_m(\widetilde{E}_m^{-m,0})}\big)}{d_m\big(\ker(\beta) \cap \widetilde{E}_m^{-m,0}\big)} \to 0.$$

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Note that λ_1 is automatically injective when $r = m$ by the theory of the Picard variety, so that $d_1(\ker(\alpha_1)) = 0$ in this case. Since β is an isomorphism here, we deduce

Corollary 4.10 *Let us assume that $r = m$. Then $\mathrm{cl}_{m,m}$ is surjective if and only if $d_k(\ker(\alpha_k)) = 0$ for all $k = 2,\ldots,m$. In particular, $\mathrm{cl}_{1,1}$ is always surjective. (See also Example 6.3.)*

Example 4.11. *Assume $r \geq m = 1$. Note that $d_1 : \widetilde{E}_1^{-1,0} \to \widetilde{E}_1^{0,0} = \mathrm{CH}^r_{\mathrm{hom}}(X;\mathbb{Q})$. Assuming β is surjective, from Theorem 4.9(iii) we deduce the short exact sequence*

$$0 \to \mathrm{im}(\mathrm{cl}_{r,1}) \to \Gamma\big(H^{2r-1}(U,\mathbb{Q}(r))\big) \to \frac{\ker\big(\lambda_1\big|_{\mathrm{im}(d_1)}\big)}{d_1(\ker(\beta))} \to 0.$$

Recalling $U = X\backslash Y$ of dimension d, we have that

$$\lambda_1 : \mathrm{im}(d_1) \to J\big(H^{2r-1}(X,\mathbb{Q}(r))/H_Y^{2r-1}(X,\mathbb{Q}(r))\big),$$

where the denominator term $H_Y^{2r-1}(X,\mathbb{Q}(r))$ in the jacobian is identified with its image in $H^{2r-1}(X,\mathbb{Q}(r))$, which apparently coincides with $Gy\big(H^{2r-3}(Y^{[1]},\mathbb{Q}(r-1))\big)$ by a standard mixed Hodge theory argument (Deligne). Taking limits, we arrive at the short exact sequence

$$0 \to \mathrm{im}(\lim(\mathrm{cl}_{r,1})) \to \Gamma\big(H^{2r-1}(\mathbb{C}(X),\mathbb{Q}(r))\big) \to$$

$$\frac{\ker\big(\mathrm{CH}^r_{\mathrm{hom}}(X;\mathbb{Q}) \xrightarrow{AJ} J\big(H^{2r-1}(X,\mathbb{Q}(r))/N^1H^{2r-1}(X,\mathbb{Q}(r))\big)\big)}{\varinjlim_U d_1(\ker(\beta))} \to 0,$$

where $N^pH^i(X,\mathbb{Q})$ is the p-th coniveau filtration. However, if for example $r = d$, then using the fact that a zero-cycle on a projective variety is homologous to zero if and only its degree is 0, we see β is surjective in this case, and that

$$\frac{\mathrm{CH}^d_{\mathrm{hom}}(X;\mathbb{Q})}{\varinjlim_U d_1(\ker(\beta))} = 0,$$

owing to the fact that any finite set of points on X lies on a smooth divisor in X. Therefore we arrive at the following result.

Corollary 4.12 ([SJK-L])

$$\lim(\mathrm{cl}_{d,1}) : \mathrm{CH}^d(\mathrm{Spec}(\mathbb{C}(X)),1;\mathbb{Q}) \to \Gamma\big(H^{2d-1}(\mathbb{C}(X),\mathbb{Q}(d))\big)$$

is surjective.

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Remark 4.13. *Corollary 4.12 really says that* $\Gamma\big(H^{2d-1}(\mathbb{C}(X),\mathbb{Q}(d))\big)$ *is trivial for* $d>1$. *But, quite generally,* $H^{2d-1}(\mathbb{C}(X),\mathbb{Q}(d))=0$ *for* $d>1$ *by the affine Lefschetz theorem (using* $2d-1>d \Leftrightarrow d>1$*), even though we did not use this fact in the above corollary.*

Example 4.14. *The case* $r=m=2$. *We observe that* $\ker(\lambda_1)=0$, *and that* β *is an isomorphism. Thus from Theorem 4.9(iii) and (4.5), we arrive at the short exact sequence*

$$0\to \mathrm{im}(\mathrm{cl}_{2,2})\to \Gamma\big(H^2(U,\mathbb{Q}(2))\big)\to \ker\big(\lambda_2\big|_{\mathrm{im}(d_2)}\big)\to 0.$$

We have

$$d_2(\widetilde{E}_2^{-2,0})\subseteq \widetilde{E}_2^{0,-1}=\frac{\mathrm{CH}^2(X,1;\mathbb{Q})}{Gy\big(\mathrm{CH}^1(Y^{[1]},1;\mathbb{Q})\big)}\xrightarrow{\lambda_2} J\left(\frac{H^2(X,\mathbb{Q}(2))}{H^2_Y(X,\mathbb{Q}(2))}\right).$$

There is an exact sequence

$$0\to d_2(\widetilde{E}_2^{-2,0})\to \frac{\mathrm{CH}^2(X,1;\mathbb{Q})}{Gy\big(\mathrm{CH}^1(Y^{[1]},1;\mathbb{Q})\big)}\to \mathrm{CH}^2(U,1;\mathbb{Q}),$$

hence

$$d_2(\widetilde{E}_2^{-2,0})=\text{Image of } \mathrm{CH}^1(Y,1;\mathbb{Q})\to \frac{\mathrm{CH}^2(X,1;\mathbb{Q})}{Gy\big(\mathrm{CH}^1(Y^{[1]},1;\mathbb{Q})\big)}.$$

Note that

$$\mathrm{CH}^1(Y^{[1]},1)=\big(\mathbb{C}^\times\big)^{\oplus n},$$

and recall we have $Y^{[1]}=\coprod_{i=1}^n Y_i$. *Thus* $\mathrm{cl}_{2,2}$ *is surjective if and only if* λ_2 *is injective on the subgroup of cycles in* $\mathrm{CH}^2(X,1;\mathbb{Q})$ *supported on* Y, *modulo the image of the space of decomposables in* $\mathrm{CH}^2(X,1;\mathbb{Q})$, *supported on* Y. *Now let* $\mathrm{CH}^2_{\mathrm{dec}}(X,1;\mathbb{Q}):=$

$$\mathrm{im}\big(\mathrm{CH}^1(X,0;\mathbb{Q})\otimes \mathrm{CH}^1(X,1;\mathbb{Q})\to \mathrm{CH}^2(X,1;\mathbb{Q})\big),$$

under the product on the higher Chow groups, and

$$\mathrm{CH}^2_{\mathrm{ind}}(X,1;\mathbb{Q}):=\frac{\mathrm{CH}^2(X,1;\mathbb{Q})}{\mathrm{CH}^2_{\mathrm{dec}}(X,1;\mathbb{Q})}.$$

Then

$$\lim(\mathrm{cl}_{2,2}):\mathrm{CH}^2\big(\mathrm{Spec}(\mathbb{C}(X)),2;\mathbb{Q}\big)\to \Gamma\big(H^2(\mathbb{C}(X),\mathbb{Q}(2))\big),$$

is surjective if and only if

$$AJ:\mathrm{CH}^2_{\mathrm{ind}}(X,1;\mathbb{Q})\to J\left(\frac{H^2(X,\mathbb{Q}(2))}{N^1H^2(X,\mathbb{Q}(2))}\right),$$

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is injective[3]. *In summary,*

Corollary 4.15

$$\frac{\Gamma\big(H^2(\mathbb{C}(X),\mathbb{Q}(2))\big)}{\lim(\mathrm{cl}_{2,2})\big(\mathrm{CH}^2\big(\mathrm{Spec}(\mathbb{C}(X)),2;\mathbb{Q}\big)\big)} \simeq$$

$$\ker AJ : \mathrm{CH}^2_{\mathrm{ind}}(X,1;\mathbb{Q}) \to J\left(\frac{H^2(X,\mathbb{Q}(2))}{N^1H^2(X,\mathbb{Q}(2))}\right).$$

(See also Corollary 6.5.)

In particular, $\lim(\mathrm{cl}_{2,2})$ *above is surjective if* $\mathrm{CH}^2_{\mathrm{ind}}(X,1;\mathbb{Q}) = 0$. *We recall Bloch's conjecture which says in the case that* X *is a surface,* $p_g(X) = 0 \Leftrightarrow$ *the Albanese map* $\kappa : \mathrm{CH}^2_{\deg 0}(X) \to Alb(X)$ *is an isomorphism. Equivalently, this amounts to saying that* $p_g(X) = 0 \Leftrightarrow$ *the motive of* X *degenerates. The degeneration of the motive of* X *implies that* $\mathrm{CH}^2_{\mathrm{ind}}(X,1;\mathbb{Q}) = 0$ *([CS]). So according to Bloch's conjecture, if* X *is a surface with* $p_g(X) = 0$, *then* $\lim(\mathrm{cl}_{2,2})$ *is surjective.*

5. Amending the Beilinson-Hodge conjecture

For any smooth quasi-projective variety $U/\mathbb{C}$ of dimension d we consider the following three regions for the pair (r,m) (see Figure 5.1):

I: $r > m > 0$ and $r \leq d$;

II: $r > m$ and $r > d$;

III: $r < m$.

The corresponding cycle class map

$$\mathrm{cl}_{r,m} : \mathrm{CH}^r(U,m;\mathbb{Q}) \to \Gamma\big(H^{2r-m}(U,\mathbb{Q}(r))\big)$$

is surjective in regions II and III since there the right-hand side is trivial (see Corollary 6.6 on page 85 in [Ja1]). We shall show below that at every point in region I surjectivity fails in general. Thus the only open cases are the diagonal $(r = m)$, where surjectivity is the Beilinson-Hodge conjecture as formulated in [A-S], and the r-axis, where surjectivity corresponds to the Hodge conjecture extended to such U.

Theorem 5.1 *For a fixed* d, *assume* (r,m) *lies in region I. Then there exists a smooth quasi-projective variety* $U/\mathbb{C}$ *of dimension* d *such that* $\mathrm{cl}_{r,m}$ *fails to be surjective.*

[3]This is also pointed out in [MSa] (Remark 4.4).

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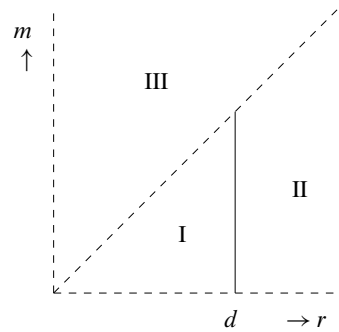


Figure 5.1: The three regions

Proof: Let $U/\mathbb{C}$ be any smooth quasi-projective variety of dimension d. Note that the MHS $H^1(\mathbb{C}^\times,\mathbb{Q}(1))$ is pure Tate of type (0,0). The Künneth formula gives an inclusion of MHS:

$$H^{2r-m-1}(U,\mathbb{Q}(r-1)) \otimes H^1(\mathbb{C}^\times,\mathbb{Q}(1)) \subset H^{2r-m}(U\times\mathbb{C}^\times,\mathbb{Q}(r)),$$

and hence the residue map

$$H^{2r-m}(U\times\mathbb{C}^\times,\mathbb{Q}(r)) \to H^{2r-m-1}(U,\mathbb{Q}(r-1)),$$

is a split surjection of MHS. Accordingly

$$\Gamma H^{2r-m}(U\times\mathbb{C}^\times,\mathbb{Q}(r)) \to \Gamma H^{2r-m-1}(U,\mathbb{Q}(r-1))$$

is surjective. There is a commutative diagram

$$\begin{array}{ccc} \mathrm{CH}^r(U\times\mathbb{C}^\times,m;\mathbb{Q}) & \xrightarrow{\partial} & \mathrm{CH}^{r-1}(U,m-1;\mathbb{Q}) \\ \Big\downarrow \mathrm{cl}_{r,m} & & \Big\downarrow \mathrm{cl}_{r-1,m-1} \\ \Gamma H^{2r-m}(U\times\mathbb{C}^\times,\mathbb{Q}(r)) & \overset{\mathrm{Res}}{\twoheadrightarrow} & \Gamma H^{2r-m-1}(U,\mathbb{Q}(r-1)). \end{array}$$

Start with any (r,m) in region I, and consider this schema:

$$\begin{array}{ll} (r,m) & d \geq r > m \\ \downarrow & \\ (r-1,m-1) & d-1 \geq r-1 > m-1 \\ \downarrow & \\ \vdots & \\ \downarrow & \\ (r-m+1,1) & d-m+1 \geq r-m+1 > 1. \end{array}$$

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Let U_0 be of dimension $d-m+1$ such that $\mathrm{cl}_{r-m+1,1}$ is not onto for U_0 ([Ja1, p. 156]). Put $U = U_0 \times \left(\mathbb{C}^\times\right)^{\times(m-1)}$. Then $\mathrm{cl}_{r,m}$ is not onto for U. □

Remark 5.2. (i) Let $U/\mathbb{C}$ be a smooth quasi-projective variety, and let $z \in H^0(\mathbb{C}^\times, \mathcal{O}^\times_{\mathbb{C}^\times}) = \mathrm{CH}^1(\mathbb{C}^\times,1)$, $z \mapsto \{z\} \in \mathrm{CH}^1(\mathbb{C}^\times,1)$ be given. Then the obvious map induced by cup product with $\{z\}$ induces a map

$$\mathrm{CH}^{r-1}(U,m-1) \to \mathrm{CH}^r(U \times \mathbb{C}^\times, m),$$

and accordingly a split short exact sequence

$$0 \to \mathrm{CH}^r(U,m) \to \mathrm{CH}^r(U \times \mathbb{C}^\times, m) \to \mathrm{CH}^{r-1}(U,m-1) \to 0.$$

This gives us a commutative diagram:

$$\begin{array}{ccc} \mathrm{CH}^r(U \times \mathbb{C}^\times, m) & \simeq & \mathrm{CH}^r(U,m) \oplus \mathrm{CH}^{r-1}(U,m-1) \\ \downarrow & & \downarrow \\ H^{2r-m}(U \times \mathbb{C}^\times, \mathbb{Z}(r)) & \simeq & H^{2r-m}(U,\mathbb{Z}(r)) \oplus H^{2r-m-1}(U,\mathbb{Z}(r-1)), \end{array}$$

which adds some clarity to the above proof.

(ii) A key ingredient in [Ja1, p. 156] is the failure of injectivity of the classical Abel-Jacobi map for smooth complex projective varieties, outside of the case of the Picard variety. It is no surprise then that with regard to the above proof, the vanishing condition in Theorem 4.9(i) fails in the case $k = 1$, that being the part involving the aforementioned Abel-Jacobi map.

6. Integral coefficients I

This section serves as a necessary forerunner to Section 7. Along the way we prove some results that are either new, or appear to be known only among experts. Let $X/\mathbb{C}$ be a smooth projective variety and $Y \subset X$ a proper subvariety. There is a short exact sequence

$$0 \to \frac{H^{2r-m}(X,\mathbb{Z}(r))}{H^{2r-m}_Y(X,\mathbb{Z}(r))} \to H^{2r-m}(X \backslash Y, \mathbb{Z}(r)) \to H^{2r-m+1}_Y(X,\mathbb{Z}(r))^0 \to 0,$$

where we write $\frac{H^{2r-m}(X,\mathbb{Z}(r))}{H^{2r-m}_Y(X,\mathbb{Z}(r))}$ for $\frac{H^{2r-m}(X,\mathbb{Z}(r))}{\mathrm{im}(H^{2r-m}_Y(X,\mathbb{Z}(r)))}$ in order to simplify notation, and let

$$H^{2r-m+1}_Y(X,\mathbb{Z}(r))^0 := \ker\left(H^{2r-m+1}_Y(X,\mathbb{Z}(r)) \to H^{2r-m+1}(X,\mathbb{Z}(r))\right).$$

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Let us assume for the moment that

$$\frac{H^{2r-m}(X,\mathbb{Z}(r))}{H^{2r-m}_Y(X,\mathbb{Z}(r))}$$

is torsion-free. Except for the obvious case $2r-m=0$ this also holds in the following two cases:

(i) $2r-m=1$. Here

$$\frac{H^1(X,\mathbb{Z}(r))}{H^1_Y(X,\mathbb{Z}(r))} = H^1(X,\mathbb{Z}(r)),$$

is torsion-free, as can be seen from the long exact sequence of cohomology of X associated to the short exact sequence

$$0 \to \mathbb{Z}(1) \to \mathbb{C} \to \mathbb{C}^\times \to 0. \tag{6.1}$$

(ii) $2r-m=2$. Let Y be a divisor such that the image $H^2_Y(X,\mathbb{Z}(2)) \to H^2(X,\mathbb{Z}(2))$ is precisely the algebraic part $H^2_{\text{alg}}(X,\mathbb{Z}(2))$. Then by the Lefschetz (1,1) theorem, $H^2(X,\mathbb{Z}(2))\big/H^2_{\text{alg}}(X,\mathbb{Z}(2))$ is torsion-free, and $H^2(X,\mathbb{Z}(2))/H^2_Y(X,\mathbb{Z}(2))$ is isomorphic to this group.

Then by purity of negative weight and torsion-freeness,

$$\Gamma\left(\frac{H^{2r-m}(X,\mathbb{Z}(r))}{H^{2r-m}_Y(X,\mathbb{Z}(r))}\right) = 0 \text{ for } m>0.$$

Corresponding to this is a commutative diagram (use the fact that cycle class maps are compatible with localization sequences for the left-hand square, and the commutativity of the right-hand square can be deduced from an extension class interpretation of the Abel-Jacobi map (see [KLM]))

$$\begin{array}{ccccc}
\frac{\mathrm{CH}^r(X\setminus Y,m)}{\mathrm{CH}^r(X,m)} & \hookrightarrow & \mathrm{CH}^r_Y(X,m-1)^0 & \xrightarrow{\alpha} & \mathrm{CH}^r_{\text{hom}}(X,m-1) \\
\Big\downarrow \underline{\mathrm{cl}_{r,m}} & & \Big\downarrow \underline{\beta} & & \Big\downarrow \underline{AJ} \\
\Gamma H^{2r-m}(X\setminus Y,\mathbb{Z}(r)) & \hookrightarrow & \Gamma H^{2r-m+1}_Y(X,\mathbb{Z}(r))^0 & \to & J\left(\frac{H^{2r-m}(X,\mathbb{Z}(r))}{H^{2r-m}_Y(X,\mathbb{Z}(r))}\right),
\end{array} \tag{6.2}$$

where

$$\mathrm{CH}^r_{\text{hom}}(X,m-1) = \ker\big(\mathrm{CH}^r(X,m-1) \to H^{2r-m+1}(X,\mathbb{Z}(r))\big),$$

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and $\mathrm{CH}^r_Y(X,m-1)^0 :=$

$$\ker\big(\mathrm{CH}^r_Y(X,m-1) \to H^{2r-m+1}(X,\mathbb{Z}(r))\big) = \alpha^{-1}\mathrm{CH}^r_{\mathrm{hom}}(X,m-1).$$

Example 6.3. *Suppose that $r = m = 1$. Then $\underline{\beta}$ and $\underline{AJ}$ are isomorphisms, hence the same holds for $\underline{\mathrm{cl}}_{1,1}$ by* (6.2)*. If we make the identifications $\mathrm{CH}^1(X,1) \simeq \mathbb{C}^\times$, $\mathrm{CH}^1(X\backslash Y,1) = \mathcal{O}^\times_{X\backslash Y}(X\backslash Y)$, then we arrive at the short exact sequence*

$$0 \to \mathbb{C}^\times \to \mathcal{O}^\times_{X\backslash Y}(X\backslash Y) \xrightarrow{d\log} \Gamma H^1(X\backslash Y,\mathbb{Z}(1)) \to 0,$$

where $\mathrm{cl}_{1,1} = d\log$ is well-known (see [KLM]). This is also a consequence of the identification

$$\mathcal{O}^\times_{X\backslash Y}(X\backslash Y) \simeq H^1_{\mathcal{D}}(X\backslash Y,\mathbb{Z}(1)).$$

The surjectivity of $d\log$ in this case is also proven in [A-K].

Example 6.4. *Suppose that $r = m = 2$ and $Y = Y_1 \cup \cdots \cup Y_n \subset X$ is a divisor. We observe that there is a short exact sequence*

$$\frac{H^2_Y(X,\mathbb{C})}{F^2H^2_Y(X,\mathbb{C}) + H^2_Y(X,\mathbb{Z}(2))} \hookrightarrow H^3_{\mathcal{D},Y}(X,\mathbb{Z}(2)) \twoheadrightarrow \Gamma H^3_Y(X,\mathbb{Z}(2)),$$

where the first term may be identified with

$$\oplus_{j=1}^n H^0(Y_j,\mathbb{C}/\mathbb{Z}(1)) \simeq (\mathbb{C}^\times)^{\oplus n} =: \mathrm{CH}^2_{Y,\mathrm{dec}}(X,1).$$

There is a canonical isomorphism $\mathrm{CH}^2_Y(X,1) \xrightarrow{\sim} H^3_{\mathcal{D},Y}(X,\mathbb{Z}(2))$ by [Ja2, Lemma 3.1]. (In loc. cit. this is formulated in terms of K-theory, but because of the particular indices, it gives exactly this result.) We therefore obtain a short exact sequence (using the fact that $\underline{\beta}$ factors through Deligne cohomology)

$$0 \to \mathrm{CH}^2_{Y,\mathrm{dec}}(X,1) \to \mathrm{CH}^2_Y(X,1)^0 \xrightarrow{\beta} \Gamma H^3_Y(X,\mathbb{Z}(2))^0 \to 0,$$

so that the snake lemma applied to (6.2) *yields an exact sequence*

$$0 \to \ker(\underline{\mathrm{cl}}_{2,2}) \to \mathrm{CH}^2_{Y,\mathrm{dec}}(X,1) \to \ker\big(\underline{AJ}\big|_{\mathrm{Im}(\alpha)}\big) \to \mathrm{cok}(\underline{\mathrm{cl}}_{2,2}) \to 0.$$

When taking limits over Y in the above example α becomes surjective, and $\lim(\underline{\mathrm{cl}}_{2,2}) = d\log_2$ in Section 1. So using the description of $\ker(\underline{\beta})$ for each Y above we obtain the following result.

Corollary 6.5

$$\frac{\Gamma H^2(\mathbb{C}(X),\mathbb{Z}(2))}{\mathrm{im}(d\log_2)} \simeq \ker\left[\frac{\mathrm{CH}^2_{\mathrm{hom}}(X,1)}{\mathrm{CH}^2_{\mathrm{dec}}(X,1)} \xrightarrow{\underline{AJ}} J\left(\frac{H^2(X,\mathbb{Z}(2))}{H^2_{\mathrm{alg}}(X,\mathbb{Z}(2))}\right)\right].$$

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Let us now consider the general situation of (r,m) with $m > 0$, so that we have the diagram (6.2) tensored with $\mathbb{Q}$. Then $\underline{\beta} \otimes \mathbb{Q}$ need not be surjective; moreover a detailed description of this map when Y is a NCD leads to the same kind of analysis as in Section 4. Recall that by a weight argument, $\Gamma H^{2r-m}(X \backslash Y, \mathbb{Q}(r)) = 0$ for $r \leq m-1$. With this in mind let us assume that $r \geq m$. Then as $Y \subset X$ ranges over all pure codimension one subvarieties, the image of α in (6.2) is $\mathrm{CH}^r_{\mathrm{hom}}(X,m-1)$ because of dimensions. Referring to $(6.2)_{\mathbb{Q}}$, let us put

$$N^1\mathrm{CH}^r(X,m-1;\mathbb{Q}) := \varinjlim_{Y} \alpha(\ker \underline{\beta} \otimes \mathbb{Q}), \tag{6.6}$$

where $Y \subset X$ ranges over all pure codimension one[4] algebraic subsets of X. Note that from Example 6.4, $N^1\mathrm{CH}^2(X,1;\mathbb{Q}) = \mathrm{CH}^2_{\mathrm{dec}}(X,1;\mathbb{Q})$. Now fix a $j : Y \hookrightarrow X$ of pure codimension one, with desingularization $\lambda : \widetilde{Y} \xrightarrow{\approx} Y$, and composite morphism $\sigma = j \circ \lambda : \widetilde{Y} \to X$. By a weight argument (Deligne) together with the purity of $H^{2r-m}(X,\mathbb{Q}(r))$, both Gysin images (σ_*, j_*) in the commutative diagram below are the same.

$$\begin{array}{ccc} H^{2r-m-2}(\widetilde{Y},\mathbb{Q}(r-1)) & & \\ & \overset{\sigma_*}{\searrow} & \\ \Big\downarrow{\scriptstyle \lambda_*} & & H^{2r-m}(X,\mathbb{Q}(r)) \\ & \overset{j_*}{\nearrow} & \\ H^{2r-m}_Y(X,\mathbb{Q}(r)). & & \end{array}$$

Assuming the Hodge conjecture, we can find w in $\mathrm{CH}^{d-1}(X \times \widetilde{Y};\mathbb{Q})$ with $\sigma_* \circ [w]_* = \mathrm{Id}_{\mathrm{im}(\sigma_*)}$, where

$$[w]_* : H^{2r-m}(X,\mathbb{Q}(r)) \to H^{2r-m-2}(\widetilde{Y},\mathbb{Q}(r-1))$$

is induced by w (see [Lew, Prop. 7.4]). Note that

$$w_*\mathrm{CH}^r_{\mathrm{hom}}(X,m-1;\mathbb{Q}) \subset \mathrm{CH}^{r-1}_{\mathrm{hom}}(\widetilde{Y},m-1;\mathbb{Q}) \to \ker \underline{\beta}_{\mathbb{Q}},$$

where $\underline{\beta}_{\mathbb{Q}} := \underline{\beta} \otimes \mathbb{Q}$. Similarly, recall we write $(6.2)_{\mathbb{Q}}$ for (6.2) tensored with $\mathbb{Q}$, and let

$$AJ_{\mathbb{Q}} : \mathrm{CH}^r_{\mathrm{hom}}(X,m-1;\mathbb{Q}) \to J\left(H^{2r-m}(X,\mathbb{Q}(r))\right)$$

[4]Working with subvarieties of higher pure codimension, this gives rise to a descending filtration $\{N^p\mathrm{CH}^i(X,j;\mathbb{Q})\}_{p \geq 0}$ which is finer that the coniveau filtration on $\mathrm{CH}^i(X,j;\mathbb{Q})$.

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be the (full) Abel-Jacobi map. Referring to $(6.2)_{\mathbb{Q}}$, we deduce from the Hodge conjecture that

$$\ker\big(\underline{AJ}_{\mathbb{Q}}\big|_{\mathrm{im}(\alpha)}\big) = \ker\big(AJ_{\mathbb{Q}}\big|_{\mathrm{im}(\alpha)}\big) + \alpha(\ker\underline{\beta}_{\mathbb{Q}}).$$

Indeed, if $\xi \in \ker\big(\underline{AJ}_{\mathbb{Q}}\big|_{\mathrm{im}(\alpha)}\big)$, then $AJ_{\mathbb{Q}}(\xi) = AJ_{\mathbb{Q}}(\sigma_* \circ w_*(\xi))$ by functoriality of the Abel-Jacobi map. Thus $\xi = \big(\xi - \sigma_* \circ w_*(\xi)\big) + \sigma_* \circ w_*(\xi) \in \ker\big(AJ_{\mathbb{Q}}\big|_{\mathrm{im}(\alpha)}\big) + \alpha(\ker\underline{\beta}_{\mathbb{Q}})$ by $(6.2)_{\mathbb{Q}}$. In particular,

Proposition 6.7 *Under the assumption of the Hodge conjecture, for a fixed Y as above there are short exact sequences*

$$\frac{\ker AJ_{\mathbb{Q}}\big|_{\mathrm{im}(\alpha)} + \alpha(\ker\underline{\beta}_{\mathbb{Q}})}{\alpha(\ker\underline{\beta}_{\mathbb{Q}})} \hookrightarrow \frac{\Gamma H^{2r-m}(X\backslash Y,\mathbb{Q}(r))}{\mathrm{im}(\mathrm{cl}_{r,m})} \twoheadrightarrow \mathrm{cok}(\underline{\beta}_{\mathbb{Q}})^0,$$

where

$$\mathrm{cok}(\underline{\beta}_{\mathbb{Q}})^0 := \ker\big(\mathrm{cok}(\underline{\beta}_{\mathbb{Q}}) \to \mathrm{cok}(\underline{AJ}_{\mathbb{Q}})\big),$$

$$\underline{AJ}_{\mathbb{Q}} : \mathrm{im}(\alpha) \to J\left(\frac{H^{2r-m}(X,\mathbb{Q}(r))}{H^{2r-m}_Y(X,\mathbb{Q}(r))}\right).$$

Taking the direct limit over Y we obtain a short exact sequence

$$\frac{\ker AJ_{\mathbb{Q}} + N^1\mathrm{CH}^r(X,m-1;\mathbb{Q})}{N^1\mathrm{CH}^r(X,m-1;\mathbb{Q})} \hookrightarrow \frac{\Gamma H^{2r-m}(\mathbb{C}(X),\mathbb{Q}(r))}{\mathrm{im}(\mathrm{limcl}_{r,m})} \twoheadrightarrow \mathrm{limcok}(\underline{\beta}_{\mathbb{Q}})^0,$$

where

$$\mathrm{limcok}(\underline{\beta}_{\mathbb{Q}})^0 = \ker\big(\mathrm{limcok}(\underline{\beta}_{\mathbb{Q}}) \to \mathrm{cok}(\underline{AJ}_{\mathbb{Q}})\big),$$

$$\underline{AJ}_{\mathbb{Q}} : \mathrm{CH}^r_{\mathrm{hom}}(X,m-1;\mathbb{Q}) \to J\left(\frac{H^{2r-m}(X,\mathbb{Q}(r))}{N^1H^{2r-m}(X,\mathbb{Q}(r))}\right).$$

Corollary 6.8 *Assume the Hodge conjecture and let $r \geq m$. Then*

$$\frac{\Gamma H^{2r-m}(\mathbb{C}(X),\mathbb{Q}(r))}{\mathrm{im}(\mathrm{limcl}_{r,m})} = 0 \text{ implies } \ker AJ_{\mathbb{Q}} \subset N^1\mathrm{CH}^r(X,m-1;\mathbb{Q}).$$

Remark 6.9. *(i) Note that Corollary 6.8 for $m = 1$ is essentially a conjectural type question of Jannsen [Ja3, p.227].*

(ii) Let us (again) take the direct limit over Y of diagram $(6.2)_{\mathbb{Q}}$. By applying the snake lemma to the limit diagram, we deduce (unconditionally) that for $r \geq m$,

$$\frac{\Gamma H^{2r-m}(\mathbb{C}(X),\mathbb{Q}(r))}{\mathrm{im}(\mathrm{limcl}_{r,m})} = 0$$

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implies that

$$\underline{AJ}_{\mathbb{Q}} : \frac{\mathrm{CH}^r_{\mathrm{hom}}(X,m-1;\mathbb{Q})}{N^1\mathrm{CH}^r(X,m-1;\mathbb{Q})} \to J\left(\frac{H^{2r-m}(X,\mathbb{Q}(r))}{N^1H^{2r-m}(X,\mathbb{Q}(r))}\right),$$

is injective, which in turn implies by a generalization of Beilinson's rigidity theorem given in [MS] that

$$\frac{\mathrm{CH}^r(X,m-1;\mathbb{Q})}{N^1\mathrm{CH}^r(X,m-1;\mathbb{Q})}$$

is countable for $m \geq 2$. *Note that in the case* $r = m = 2$, *we have*

$$\mathrm{CH}^2_{\mathrm{ind}}(X,1;\mathbb{Q}) = \frac{\mathrm{CH}^2(X,1;\mathbb{Q})}{N^1\mathrm{CH}^2(X,1;\mathbb{Q})},$$

where $\mathrm{CH}^2_{\mathrm{ind}}(X,1;\mathbb{Q})$ *was defined in Example 4.14, and the statement of countability of* $\mathrm{CH}^2_{\mathrm{ind}}(X,1;\mathbb{Q})$ *is a conjecture of Voisin.*

7. Integral coefficients II

As always, $X/\mathbb{C}$ is a smooth projective variety. This section concerns the following integrally defined map:

$$\begin{aligned} d\log_m : \mathrm{CH}^m(\mathrm{Spec}(\mathbb{C}(X)),m) &\to \Gamma\big(H^m(\mathbb{C}(X),\mathbb{Z}(m))\big), \\ \{f_1,\ldots,f_m\} &\mapsto \bigwedge_1^m d\log(f_j) \end{aligned} \tag{7.1}$$

mentioned in Section 1. Of course, $d\log_m \otimes \mathbb{Q} = \lim(\mathrm{cl}_{m,m})$. We shall prove in this section that $H^m(\mathbb{C}(X),\mathbb{Z}(m))$ is torsion-free, so that by a weight argument

$$\Gamma\big(H^m(\mathbb{C}(X),\mathbb{Z}(m))\big) = H^m(\mathbb{C}(X),\mathbb{Z}(m)) \cap F^mH^m(\mathbb{C}(X),\mathbb{C}).$$

Clearly, if $d\log_m$ is surjective then so is $\lim(\mathrm{cl}_{m,m})$, but we shall show that the converse also holds. In fact, we expect the following to be true:

Conjecture 7.2 *The map in (7.1) is surjective.*

For the moment, let us restrict to the case $r = m = 1$. It is then easy (and also follows from Example 6.3) that

$$d\log : \mathrm{CH}^1(\mathbb{C}(X),1) = \mathbb{C}(X)^\times \to \Gamma\big(H^1(\mathbb{C}(X),\mathbb{Z}(1))\big),$$

is surjective, with divisible kernel: $\ker(d\log) = \mathbb{C}^\times$. Thus for any integer $l \neq 0$,

$$\frac{\mathrm{CH}^1(\mathbb{C}(X),1)}{l\cdot\mathrm{CH}^1(\mathbb{C}(X),1)} \simeq \frac{\Gamma\big(H^1(\mathbb{C}(X),\mathbb{Z}(1))\big)}{l\cdot\Gamma\big(H^1(\mathbb{C}(X),\mathbb{Z}(1))\big)}.$$

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If U is a Zariski open part of X, then there is an exact sequence

$$0 \to \frac{H^1(U,\mathbb{Z}(1))}{\Gamma\big(H^1(U,\mathbb{Z}(1))\big)} \to \frac{H^1(U,\mathbb{C})}{F^1H^1(U,\mathbb{C})} \to \underline{H}^2_{\mathcal{D}}(U,\mathbb{Z}(1)), \tag{7.3}$$

where

$$\underline{H}^2_{\mathcal{D}}(U,\mathbb{Z}(1)) := \mathrm{Image}\big(H^2_{\mathcal{D}}(X,\mathbb{Z}(1)) \to H^2_{\mathcal{D}}(U,\mathbb{Z}(1))\big) = \mathrm{CH}^1(U). \tag{7.4}$$

Indeed, from mixed Hodge theory (see [De](Cor. 3.2.13(ii)), the restriction map induces an isomorphism

$$H^1(X,\mathcal{O}_X) \simeq \frac{H^1(U,\mathbb{C})}{F^1H^1(U,\mathbb{C})}.$$

Thus we have an exact sequence:

$$H^2_{\mathcal{D},X\setminus U}(X,\mathbb{Z}(1)) \to H^2_{\mathcal{D}}(X,\mathbb{Z}(1)) \to \underline{H}^2_{\mathcal{D}}(U,\mathbb{Z}(1)) \to 0;$$

moreover by a weak purity argument,

$$H^2_{\mathcal{D},X\setminus U}(X,\mathbb{Z}(1)) \simeq H^2_{X\setminus U}(X,\mathbb{Z}(1)).$$

Via the Deligne cycle class maps, we have identifications, $\mathrm{CH}^1(X) = H^2_{\mathcal{D}}(X,\mathbb{Z}(1))$, $\mathrm{CH}^1_{X\setminus U}(X) = H^2_{X\setminus U}(X,\mathbb{Z}(1))$, and hence by a localization sequence argument, the aforementioned identification $\underline{H}^2_{\mathcal{D}}(U,\mathbb{Z}(1)) = \mathrm{CH}^1(U)$. Next, by shrinking U, and using that $\mathrm{CH}^1(\mathrm{Spec}(\mathbb{C}(X)) = 0$, we obtain from (7.3) and (7.4) a short exact sequence

$$0 \to \Gamma H^1(\mathbb{C}(X),\mathbb{Z}(1)) \to H^1(\mathbb{C}(X),\mathbb{Z}(1)) \to H^1(X,\mathcal{O}_X) \to 0,$$

where the last term is uniquely divisible. Hence, for $l \neq 0$, we find an isomorphism

$$\frac{\Gamma H^1(\mathbb{C}(X),\mathbb{Z}(1))}{l\cdot\Gamma H^1(\mathbb{C}(X),\mathbb{Z}(1))} \simeq \frac{H^1(\mathbb{C}(X),\mathbb{Z}(1))}{l\cdot H^1(\mathbb{C}(X),\mathbb{Z}(1))}.$$

Next, we observe that $H^2(\mathbb{C}(X),\mathbb{Z}(1))$ is torsion-free, since by the Lefschetz (1,1) theorem the torsion in $H^2(X,\mathbb{Z}(1))$ is algebraic. Namely, let us go back to the setting in the case $2r - m = 2$ immediately following (6.1). Then from the localization sequence associated to the pair (X,Y), it suffices to show that $H^3_Y(X,\mathbb{Z})$ is torsion-free. But after excising the singular set of Y, this amounts to the fact that $H^1(\tilde{Y},\mathbb{Z})$ is torsion-free for any desingularization $\tilde{Y} \to Y$.

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From our statement about $H^2(\mathbb{C}(X),\mathbb{Z}(1))$ we deduce that

$$\frac{H^1(\mathbb{C}(X),\mathbb{Z}(1))}{l\cdot H^1(\mathbb{C}(X),\mathbb{Z}(1))} \simeq H^1\left(\mathbb{C}(X),\frac{\mathbb{Z}(1))}{l\cdot\mathbb{Z}(1)}\right).$$

Finally, we deduce the well-known fact

$$\frac{\mathrm{CH}^1(\mathbb{C}(X),1)}{l\cdot \mathrm{CH}^1(\mathbb{C}(X),1)} \simeq H^1\left(\mathbb{C}(X),\frac{\mathbb{Z}(1))}{l\cdot\mathbb{Z}(1)}\right).$$

To see this in another context, let us work in the étale topology on a variety $V/\mathbb{C}$, and consider the sheaf μ_l on V, where for $U \to V$ étale, $\mu_l(U) = \{\xi \in \Gamma(U,\mathcal{O}_U) \mid \xi^l = 1\}$. Now let $V = \mathrm{Spec}(\mathbb{C}(X))$. Then by Hilbert 90,

$$H^1_{\mathrm{et}}(\mathbb{C}(X),\mu_l) \simeq \mathbb{C}(X)^\times/[\mathbb{C}(X)^\times]^l,$$

where $[\mathbb{C}(X)^\times]^l = \{x^l \mid x \in \mathbb{C}(X)^\times\}$. There are exact sequences

$$0 \to \mathbb{C}^\times \to \mathbb{C}(X)^\times \xrightarrow{d\log} \Omega^1_{\mathbb{C}(X)/\mathbb{C}},$$

$$0 \to \mathbb{C}^\times \to [\mathbb{C}(X)^\times]^l \xrightarrow{d\log} \Omega^1_{\mathbb{C}(X)/\mathbb{C}},$$

where $\Omega^1_{\mathbb{C}(X)/\mathbb{C}}$ are the Kähler differentials, which induces the short exact sequences

$$0 \to \mathbb{C}^\times \to \mathbb{C}(X)^\times \xrightarrow{d\log} \Gamma\big(H^1(\mathbb{C}(X),\mathbb{Z}(1))\big) \to 0$$

and

$$0 \to \mathbb{C}^\times \to [\mathbb{C}(X)^\times]^l \xrightarrow{d\log} l\cdot\Gamma\big(H^1(\mathbb{C}(X),\mathbb{Z}(1))\big) \to 0.$$

Thus going from the étale to Betti cohomology with finite coefficients can be traced via the isomorphisms:

$$\begin{aligned} H^1_{\mathrm{et}}(\mathbb{C}(X),\mu_l) &\simeq \mathbb{C}(X)^\times/[\mathbb{C}(X)^\times]^l \qquad (7.5)\\ &\simeq \frac{\Gamma\big(H^1(\mathbb{C}(X),\mathbb{Z}(1))\big)}{l\cdot\Gamma\big(H^1(\mathbb{C}(X),\mathbb{Z}(1))\big)} \simeq \frac{H^1(\mathbb{C}(X),\mathbb{Z}(1))}{l\cdot H^1(\mathbb{C}(X),\mathbb{Z}(1))}\\ &\simeq H^1\left(\mathbb{C}(X),\frac{\mathbb{Z}(1)}{l\cdot\mathbb{Z}(1)}\right). \end{aligned}$$

Note that $d\log : \mathrm{CH}^1(\mathrm{Spec}(\mathbb{C}(X),1) = \mathbb{C}(X)^\times \to H^1(\mathbb{C}(X),\mathbb{Z}(1))$, and $\mathrm{CH}^1(\mathrm{Spec}(\mathbb{C}(X),1) = \mathbb{C}(X)^\times \to H^1_{\mathrm{et}}(\mathbb{C}(X),\mu_l)$ from (7.5) are the respective cycle class maps to Betti cohomology (analytic topology) and étale cohomology. In

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summary, we have a commutative diagram corresponding to a morphism of sites from the étale to the analytic topologies.

$$\begin{array}{ccc} \mathbb{C}(X)^\times & \xrightarrow{d\log} & H^1(\mathbb{C}(X),\mathbb{Z}(1)) \\ \downarrow & & \downarrow \\ H^1_{\text{et}}(\mathbb{C}(X),\mu_l) & \xrightarrow{\sim} & H^1\big(\mathbb{C}(X),\tfrac{\mathbb{Z}(1)}{l\cdot\mathbb{Z}(1)}\big) \end{array}$$

where the isomorphism in the bottom row is from (7.5). Taking cup products, we have a similar diagram

$$\begin{array}{ccc} \big(\mathbb{C}(X)^\times\big)^{\otimes m} & \xrightarrow{(d\log)^{\otimes m}} & H^1(\mathbb{C}(X),\mathbb{Z}(1))^{\otimes m} \\ \downarrow & & \downarrow \\ H^1_{\text{et}}(\mathbb{C}(X),\mu_l)^{\otimes m} & \xrightarrow{\sim} & H^1\big(\mathbb{C}(X),\tfrac{\mathbb{Z}(1)}{l\cdot\mathbb{Z}(1)}\big)^{\otimes m} \\ \cup\downarrow & & \downarrow\cup \\ H^m_{\text{et}}(\mathbb{C}(X),\mu_l^{\otimes m}) & \xrightarrow{\sim} & H^m\big(\mathbb{C}(X),\tfrac{\mathbb{Z}(m)}{l\cdot\mathbb{Z}(m)}\big) \end{array}$$

where the isomorphism

$$H^m_{\text{et}}(\mathbb{C}(X),\mu_l^{\otimes_{\mathbb{Z}} m}) \xrightarrow{\sim} H^m\left(\mathbb{C}(X),\frac{\mathbb{Z}(m)}{l\cdot\mathbb{Z}(m)}\right)$$

here (and in the previous diagram for $m = 1$), which really arises from the Leray spectral sequence associated to a morphism of sites, can be deduced from [Mi, Thm 3.12 on p.117]. The Bloch-Kato/Milnor conjectures (now theorems [We]) tell us that for l a non-zero integer, the induced map

$$\frac{\mathrm{CH}^m(\mathrm{Spec}(\mathbb{C}(X)),m)}{l\cdot\mathrm{CH}^m(\mathrm{Spec}(\mathbb{C}(X)),m)} \to H^m_{\text{et}}(\mathrm{Spec}(\mathbb{C}(X)),\mu_l^{\otimes_{\mathbb{Z}} m})$$

is an isomorphism. [5] In our situation, this translates to saying

[5] This generalizes the Merkurjev-Suslin theorem, where $m = 2$.

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Theorem 7.6 *For $m \geq 0$, the map*

$$\frac{\mathrm{CH}^m(\mathrm{Spec}(\mathbb{C}(X)),m)}{l \cdot CH^m(\mathrm{Spec}(\mathbb{C}(X)),m)} \to H^m\left(\mathbb{C}(X), \frac{\mathbb{Z}(m)}{l \cdot \mathbb{Z}(m)}\right)$$

is an isomorphism for any integer $l \neq 0$.

We can now prove the following result. Note that part (i) for $i = 1$ is immediate from the short exact sequence (6.1), and for $i = 2$ follows from the Lefschetz (1,1) theorem.

Theorem 7.7 (i) *$H^i(\mathbb{C}(X),\mathbb{Z})$ is torsion-free for all i. In particular, the torsion subgroup of $H^i(X,\mathbb{Z})$ is supported in codimension* 1.

(ii) $\ker(d\log_m)$ *in (7.1) is divisible.*

(iii) *The groups*

$$\frac{H^m(\mathbb{C}(X),\mathbb{Z}(m))}{\Gamma\big(H^m(\mathbb{C}(X),\mathbb{Z}(m))\big)} \quad \textit{and} \quad \frac{\Gamma\big(H^m(\mathbb{C}(X),\mathbb{Z}(m)\big)}{\mathrm{im}(d\log_m)}$$

are uniquely divisible.

Proof: First observe that the map in Theorem 7.6 is the composition

$$\frac{\mathrm{CH}^m(\mathrm{Spec}(\mathbb{C}(X)),m)}{l} \to \frac{H^m(\mathbb{C}(X),\mathbb{Z}(m))}{l} \to H^m\left(\mathbb{C}(X), \frac{\mathbb{Z}(m)}{l \cdot \mathbb{Z}(m)}\right).$$

Notice that the short exact sequence $0 \to \mathbb{Z} \xrightarrow{\times l} \mathbb{Z} \to \mathbb{Z}/l\mathbb{Z} \to 0$ induces the short exact sequence

$$\frac{H^m(\mathbb{C}(X),\mathbb{Z}(m))}{l} \hookrightarrow H^m\left(\mathbb{C}(X), \frac{\mathbb{Z}(m)}{l \cdot \mathbb{Z}(m)}\right) \twoheadrightarrow H^{m+1}(\mathbb{C}(X),\mathbb{Z}(m))_{l-\mathrm{tor}}.$$

By Theorem 7.6, it follows that $H^{m+1}(\mathbb{C}(X),\mathbb{Z}(m))_{l-\mathrm{tor}} = 0$, thus proving part (i). Next observe that

$$\Gamma\big(H^m(\mathbb{C}(X),\mathbb{Z}(m))\big) = F^m H^m(\mathbb{C}(X),\mathbb{C}) \cap H^m(\mathbb{C}(X),\mathbb{Z}(m)),$$

and hence

$$\Gamma\big(H^m(\mathbb{C}(X),\mathbb{Z}(m))\big) \cap l \cdot H^m(\mathbb{C}(X),\mathbb{Z}(m)) = l \cdot \Gamma\big(H^m(\mathbb{C}(X),\mathbb{Z}(m))\big).$$

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Using also Theorem 7.6, we have the commutative diagram

$$\begin{array}{ccc}
\dfrac{\mathrm{CH}^m(\mathrm{Spec}(\mathbb{C}(X)),m)}{l\cdot\mathrm{CH}^m(\mathrm{Spec}(\mathbb{C}(X)),m)} & \xrightarrow{\simeq} & H^m\left(\mathbb{C}(X),\dfrac{\mathbb{Z}(m)}{l\cdot\mathbb{Z}(m)}\right) \\
\downarrow & & \uparrow \\
\dfrac{\mathrm{im}(d\log_m)}{l\cdot\mathrm{im}(d\log_m)} & & \simeq \\
\downarrow & & \uparrow \\
\dfrac{\Gamma\left(H^m(\mathbb{C}(X),\mathbb{Z}(m))\right)}{l\cdot\Gamma\left(H^m(\mathbb{C}(X),\mathbb{Z}(m))\right)} & \hookrightarrow & \dfrac{H^m(\mathbb{C}(X),\mathbb{Z}(m))}{l\cdot H^m(\mathbb{C}(X),\mathbb{Z}(m))},
\end{array}$$

where all maps must be isomorphisms. Part (ii) follows by applying the snake lemma to multiplication by $l \neq 0$ on the short exact sequence

$$0 \to \ker(d\log_m) \to \mathrm{CH}^m(\mathrm{Spec}(\mathbb{C}(X)),m) \to \mathrm{im}(d\log_m) \to 0$$

as $\mathrm{im}(d\log_m)$ is torsion-free. Using the obvious abbreviations, part (iii) follows similarly from $0 \to \Gamma \to H^m \to H^m/\Gamma \to 0$ and $0 \to \mathrm{im} \to \Gamma \to \Gamma/\mathrm{im} \to 0$. □

Corollary 7.8

$$\frac{\Gamma\left(H^m(\mathbb{C}(X),\mathbb{Z}(m))\right)}{\mathrm{im}(d\log_m)} = \frac{\Gamma\left(H^m(\mathbb{C}(X),\mathbb{Q}(m))\right)}{\mathrm{im}(d\log_m\otimes\mathbb{Q})}.$$

Corollary 7.9

$$\ker\left[\underline{AJ} : \frac{\mathrm{CH}^2_{\mathrm{hom}}(X,1)}{\mathrm{CH}^2_{\mathrm{dec}}(X,1)} \to J\left(\frac{H^2(X,\mathbb{Z}(2))}{H^2_{\mathrm{alg}}(X,\mathbb{Z}(2))}\right)\right]$$

is uniquely divisible.

Proof: Apply Theorem 7.7(iii) with $m = 2$ to Corollary 6.5. □

Note that one can define $N^1\mathrm{CH}^r(X,m-1)$ with integral coefficients analogous to (6.6). We assume $r \geq m$. Since $H^{2r-m}(\mathbb{C}(X),\mathbb{Z}(r))$ is torsion-free by Theorem 7.7(i), diagram (6.2) becomes valid after passing to the generic point of X. After applying the snake lemma, we arrive at the fact that

$$\ker\left(\underline{AJ} : \frac{\mathrm{CH}^r_{\mathrm{hom}}(X,m-1)}{N^1\mathrm{CH}^r(X,m-1)} \to J\left(\frac{H^{2r-m}(X,\mathbb{Z}(r))}{N^1H^{2r-m}(X,\mathbb{Z}(r))}\right)\right)$$

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injects into

$$\frac{\Gamma\big(H^{2r-m}(\mathbb{C}(X),\mathbb{Z}(r))\big)}{\mathrm{im}(\lim(\mathrm{cl}_{r,m}))}.$$

In the case $r = m$ we deduce

Corollary 7.10

$$\ker\left(\underline{AJ} : \frac{\mathrm{CH}^m_{\mathrm{hom}}(X,m-1)}{N^1\mathrm{CH}^m(X,m-1)} \to J\left(\frac{H^m(X,\mathbb{Z}(m))}{N^1H^m(X,\mathbb{Z}(m))}\right)\right)$$

is torsion-free. Hence we have an injection of torsion subgroups

$$\underline{AJ} : \left\{\frac{\mathrm{CH}^m_{\mathrm{hom}}(X,m-1)}{N^1\mathrm{CH}^m(X,m-1)}\right\}_{\mathrm{tor}} \hookrightarrow \left\{J\left(\frac{H^m(X,\mathbb{Z}(m))}{N^1H^m(X,\mathbb{Z}(m))}\right)\right\}_{\mathrm{tor}}.$$

Remark 7.11. *Conjecture 7.2 would imply that the maps $\underline{AJ}$ in Corollaries 7.9 and 7.10 are injective.*

8. Decomposables

A primary goal in this section is to have a better understanding of the space $N^1\mathrm{CH}^r(X,m-1;\mathbb{Q})$ defined in (6.6), since it plays a role in capturing the kernel of the Abel-Jacobi map, as evident in Corollary 6.8. Towards this goal, for $r \geq m \geq 1$ we introduce subspaces and establish inclusions (see Proposition 8.6):

$$\mathrm{CH}^r_{\mathrm{dec}}(X,m-1;\mathbb{Q}) \subseteq \Xi^{r,m}_X \subseteq N^1\mathrm{CH}^r(X,m-1;\mathbb{Q}).$$

The second inclusion is somewhat formal from the definitions, and is an equality if $r = m$. We do not know if the first inclusion is an equality, but this holds when $r = m = 2$. One should think of $\mathrm{CH}^r_{\mathrm{dec}}(X,m-1;\mathbb{Q})$ as what the notion of decomposable classes in $\mathrm{CH}^r(X,m-1;\mathbb{Q})$ should mean naively.

A further goal of this section is the following. Fix $r,m \geq 1$. If $r = m$, then according to Theorem 4.9, surjectivity of $\mathrm{cl}_{m,m}$ implies that λ_m is injective on $d_m(\widetilde{E}_m^{-m,0}) \subset \widetilde{E}_m^{0,-m+1}$. Thus it makes sense to calculate $\mathrm{im}(d_m)$ in general.

Let $j : Y \hookrightarrow X$ be the inclusion of a NCD Y (with smooth components). Then

$$\frac{\widetilde{E}_m^{0,-m+1}}{\mathrm{im}(d_m)} = E_\infty^{0,-m+1} \otimes \mathbb{Q} = \frac{\mathrm{CH}^r(X,m-1;\mathbb{Q})}{j_*\mathrm{CH}^{r-1}(Y,m-1;\mathbb{Q})} \subset \mathrm{CH}^r(U,m-1;\mathbb{Q}).$$

Note that $\mathrm{CH}^{r-1}(Y,m-1;\mathbb{Q})$ can be calculated from the simplicial complex $Y^{[\bullet]} \to Y$. So there are residue maps

$$\partial_R^Y : \mathrm{CH}^{r-1}(Y,m-1;\mathbb{Q}) \to \frac{\mathrm{CH}^{r-m}(Y^{[m]};\mathbb{Q})}{\mathrm{Gy}\big(\mathrm{CH}^{r-m-1}(Y^{[m+1]};\mathbb{Q})\big)},$$

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$$\partial_R^{X\setminus Y} : \mathrm{CH}^r(X\setminus Y,m;\mathbb{Q}) \to \frac{\mathrm{CH}^{r-m}(Y^{[m]};\mathbb{Q})}{\mathrm{Gy}\big(\mathrm{CH}^{r-m-1}(Y^{[m+1]};\mathbb{Q})\big)}.$$

Proposition 8.1 *For $m \geq 1$,*

$$\mathrm{im}(d_m) = \frac{j_*\mathrm{CH}^{r-1}(Y,m-1;\mathbb{Q})}{j_*(\ker\partial_R^Y)}.$$

Corollary 8.2 *If $m \geq 1$ and $\mathrm{cl}_{m,m}$ is surjective, then the Abel-Jacobi map λ_m in* (4.5) *induces an injection*

$$\frac{j_*\mathrm{CH}^{m-1}(Y,m-1;\mathbb{Q})}{j_*(\ker\partial_R^Y)} \hookrightarrow J(Gr_{-m}^W).$$

Proof of Proposition 8.1: Consider the exact sequence

$$\mathrm{CH}^r(X,m) \to \mathrm{CH}^r(X\setminus Y,m) \to \mathrm{CH}^{r-1}(Y,m-1) \to \mathrm{CH}^r(X,m-1).$$

In the weight filtered spectral sequence obtained from (3.2) involved in computing $\mathrm{CH}^r(X\setminus Y,m)$ we can restrict our attention to those $\mathcal{Z}_0^{i,j}(r) = z^{r+i}(Y^{[-i]},-j)$ with $i \leq -1$, which converges to $\mathrm{CH}^{r-1}(Y,m-1)$. It also exists for $\mathrm{CH}^r(X,m-1)$ (using the column where $i = 0$). We use indices to distinguish between the various spectral sequences.

Note that $E_{1,X\setminus Y}^{0,-m+1} = E_{\infty,X}^{0,-m+1} = \mathrm{CH}^r(X,m-1)$ and that for $\ell \geq 1$,

$$E_{\ell,X\setminus Y}^{-\ell,\ell-m} = E_{\ell,Y}^{-\ell,\ell-m} \twoheadrightarrow E_{\infty,Y}^{-\ell,\ell-m} \tag{8.3}$$

since $d_\ell^Y = 0$ on $E_{\ell,Y}^{-\ell,\ell-m}$ as its target is trivial. Non-canonically, we have

$$\mathrm{CH}^{r-1}(Y,m-1;\mathbb{Q}) \simeq \widetilde{E}_{\infty,Y}^{-m,0} \oplus \ker(\partial_R^Y)$$

and

$$\mathrm{CH}^r(X\setminus Y,m;\mathbb{Q}) \simeq \widetilde{E}_{\infty,X\setminus Y}^{-m,0} \oplus \ker(\partial_R^{X\setminus Y}).$$

Thus

$$j_*\mathrm{CH}^{r-1}(Y,m-1;\mathbb{Q}) \simeq \frac{\mathrm{CH}^{r-1}(Y,m-1;\mathbb{Q})}{\mathrm{CH}^r(X\setminus Y,m;\mathbb{Q})} \simeq \frac{\widetilde{E}_{\infty,Y}^{-m,0}}{\widetilde{E}_{\infty,X\setminus Y}^{-m,0}} \oplus \frac{\ker(\partial_R^Y)}{\ker(\partial_R^{X\setminus Y})},$$

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with the last term isomorphic with $j_*(\ker(\partial_R^Y))$. One has a commutative diagram

$$\begin{array}{ccccccc} & & & & \widetilde{E}^{-m,0}_{m,X\setminus Y} & & \\ & & & & \downarrow & \searrow^{d_m} & \\ 0 & \to & \widetilde{E}^{-m,0}_{\infty,X\setminus Y} & \to & \widetilde{E}^{-m,0}_{\infty,Y} & \to & \widetilde{E}^{0,-m+1}_{m,X\setminus Y} \end{array}$$

with surjective vertical map by (8.3). One sees the bottom row is exact by comparing the spectral sequences $E^{p,q}_{m,X\setminus Y}$ and $E^{p,q}_{m,Y}$ at $(p,q) = (-m,0)$. Hence $j_*\mathrm{CH}^{r-1}(Y,m-1;\mathbb{Q}) \simeq \mathrm{im}(d_m) \oplus j_*\ker(\partial_R^Y)$. □

We now restrict to the case $r \geq m$ because $\mathrm{im}(d_m) = 0$ for $r < m$ as $\mathrm{CH}^{r-m}(Y^{[m]}) = 0$ (see (4.1)). Let $\xi \in \mathrm{CH}^r(X,m-1)$. Then $|\xi| \subset X \times \Delta^{m-1}$ is of codimension r, and hence $W := \overline{\mathrm{Pr}_1(|\xi|)}$ is of codimension $(r-m)+1 \geq 1$ in X. By Hironaka, there is a proper modification diagram

$$\begin{array}{ccccc} \sigma^{-1}(W) & =: & \overline{Y} & \overset{\overline{j}}{\hookrightarrow} & \overline{X} \\ & & \downarrow & & \wr\wr \downarrow \sigma \\ & & W & \overset{\overline{j}_1}{\hookrightarrow} & X \end{array} \tag{8.4}$$

where $\overline{Y}$ is a NCD. Let $U = X\setminus W$. Before proceeding further, we first introduce some notation. The higher Chow groups form a homology theory but the index r is given by the codimension of cycles. In order to avoid complicated notation when $W \subset X$ is not of pure dimension, we define

$$z^r_W(X,m) = \ker\big(z^r(X,m) \to z^r(U,m)\big),$$

and let $\mathrm{CH}^r_W(X,m)$ denote the middle homology of

$$z^r_W(X,m+1) \to z^r_W(X,m) \to z^r_W(X,m-1).$$

Next, we have a corresponding diagram with $\mathrm{CH}^r_{\overline{Y}}(\overline{X},m-1;\mathbb{Q}) = \mathrm{CH}^{r-1}(\overline{Y},m-1;\mathbb{Q})$.

$$\begin{array}{ccccc} \overline{j}_*\mathrm{CH}^r_{\overline{Y}}(\overline{X},m-1;\mathbb{Q}) & \hookrightarrow & \mathrm{CH}^r(\overline{X},m-1;\mathbb{Q}) & \to & \mathrm{CH}^r(U,m-1;\mathbb{Q}) \\ \sigma_{\overline{Y},*}\downarrow & & \downarrow \sigma_* & & || \\ \overline{j}_{1,*}\mathrm{CH}^r_W(X,m-1;\mathbb{Q}) & \hookrightarrow & \mathrm{CH}^r(X,m-1;\mathbb{Q}) & \to & \mathrm{CH}^r(U,m-1;\mathbb{Q}) \end{array} \tag{8.5}$$

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where ξ is in $\overline{j}_{1,*}\mathrm{CH}^r_W(X,m-1;\mathbb{Q})$. Obviously $\sigma_*\circ\sigma^* = \text{Identity}$ implies that σ_* is onto. A diagram chase shows that $\sigma_{\overline{Y},*}$ is a surjection as well. As $\overline{X}$ is obtained from X by a sequence of blow-ups with non-singular centres, it is clear from the well-known motivic decomposition of $\overline{X}$ with respect to X, that the (higher) Chow group of $\overline{X}$ involves that of X and of smooth irreducible divisors. Hence it follows that $\ker(\sigma_*)\subset\overline{j}_*(\ker\partial_R^{\overline{Y}})$. We deduce the isomorphisms

$$\frac{\mathrm{CH}^r(\overline{X},m-1;\mathbb{Q})}{\overline{j}_*\mathrm{CH}^r_{\overline{Y}}(\overline{X},m-1;\mathbb{Q})}\simeq\frac{\mathrm{CH}^r(X,m-1;\mathbb{Q})}{\overline{j}_{1,*}\mathrm{CH}^r_W(X,m-1;\mathbb{Q})},$$

$$\frac{\overline{j}_*\mathrm{CH}^r_{\overline{Y}}(\overline{X},m-1;\mathbb{Q})}{\overline{j}_*(\ker\partial_R^{\overline{Y}})}\simeq\frac{\overline{j}_{1,*}\mathrm{CH}^r_W(X,m-1;\mathbb{Q})}{\sigma_*\circ\overline{j}_*(\ker\partial_R^{\overline{Y}})}.$$

Let

$$\Xi^{r,m}_X:=\sum_{(\overline{X},\overline{Y})}\sigma_*\circ\overline{j}_*(\ker\partial_R^{\overline{Y}}),$$

where $(\overline{X},\overline{Y})$ runs over all proper modifications (8.4) for all $W\subset X$ of codimension at least 1. By construction we have

$$\mathrm{CH}^r(X,m-1;\mathbb{Q})=\sum_{(\overline{X},\overline{Y})}\sigma_*\circ\overline{j}_*\big(\mathrm{CH}^r_{\overline{Y}}(\overline{X},m-1;\mathbb{Q})\big).$$

Now recall the subspace $N^1\mathrm{CH}^r(X,m-1;\mathbb{Q})\subset\mathrm{CH}^r_{\mathrm{hom}}(X,m-1;\mathbb{Q})$ introduced in equation (6.6). It follows from the definitions that

$$\Xi^{r,m}_X\subseteq N^1\mathrm{CH}^r(X,m-1;\mathbb{Q}),$$

since, by (4.6), $\partial_R^{\overline{Y}}(\xi)=0$ in $\widetilde{E}_1^{-m,0}$ implies $\beta(\partial_R^{\overline{Y}}(\xi))=0$ in $\Gamma\big(Gr_0^W\big)$. The above inclusion is an equality if $r=m$, since in this case β is an isomorphism. Let $\mathrm{CH}^r_{\mathrm{dec}}(X,m-1;\mathbb{Q})\subseteq\mathrm{CH}^r(X,m-1;\mathbb{Q})$ be the subspace generated by images of the form

$$\mathrm{CH}^a(X,b;\mathbb{Q})\otimes\mathrm{CH}^s(X,t;\mathbb{Q})\to\mathrm{CH}^r(X,m-1;\mathbb{Q})$$

under the product where

$$a+s=r,\quad b+t=m-1,\quad (a,b)\neq(0,0),\quad (b,t)\neq(0,0),\quad (s,t)\neq(0,0).$$

Proposition 8.6 *For $r\geq m\geq 1$ we have $\mathrm{CH}^r_{\mathrm{dec}}(X,m-1;\mathbb{Q})\subseteq\Xi^{r,m}_X$.*

Remark 8.7. *It is not clear if equality holds in the above proposition.*

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Proof of Proposition 8.6: Take $\xi_1 \otimes \xi_2$ in $\mathrm{CH}^a(X,b;\mathbb{Q}) \otimes \mathrm{CH}^s(X,t;\mathbb{Q})$, where $b, t \leq m-1$. Note that $\mathrm{CH}^i(X,j)$ is supported in codimension $i-j$, namely

$$\mathrm{CH}^i(X,j) = \sum_{\mathrm{cd}_X Y \geq i-j} \mathrm{im}\big(\mathrm{CH}^i_Y(X,j) \to \mathrm{CH}^i(X,j)\big).$$

Also $a \leq b$ and $s \leq t \Rightarrow r = a+s \leq b+t = m-1$, which is not the case as we are assuming $r \geq m$. Therefore we can assume say $s > t$. Next, if $t = m-1$, then $b = 0$, and thus $a > 0 = b$ implies ξ_1 is supported on a divisor in X. This scenario can be handled in the same way as in the case where we assume that $s > t$ and $t < m-1$. Namely, since $s > t$, we can assume that ξ_2 is supported on some $Y \subset X$ of codimension $1 \leq s-t$, and by the surjectivity of $\sigma_{\overline{Y},*}$ in diagram (8.5), we can assume without loss of generality that $Y \subset X$ is a NCD. Thus for some $\xi_2^Y \in \mathrm{CH}^{s-1}(Y,t;\mathbb{Q})$, $\xi_2^Y \mapsto \xi_2$. Since X is smooth, we have the product ([Bl]):

$$\mathrm{CH}^a(X,b) \times \mathrm{CH}^{s-1}(Y,t;\mathbb{Q}) \to \mathrm{CH}^{r-1}(Y,m-1;\mathbb{Q}),$$

which defines $\xi_1 \cap \xi_2^Y \in \mathrm{CH}^{r-1}(Y,m-1;\mathbb{Q})$. But since ξ_1 can be assumed in general position with respect to Y, and together with $t < m-1$, we have $\partial_R^Y\big(\xi_1 \cap \xi_2^Y\big) = 0$. □

9. Noether-Lefschetz for a family of surfaces

In Theorem 9.1 we give an explicit method of construction of a family of smooth open surfaces $\{U_t'\}_{t\in B}$ such that:

- one can partially "calculate" $\Gamma H^2(U_t',\mathbb{Q}(2))$ in terms of residues of algebraic 2-forms;
- for very general $t \in B$, $\Gamma H^2(U_t',\mathbb{Q}(2))$ is trivial;
- $\Gamma Gr_0^W H^2(U_t',\mathbb{Q}(2))$ can be highly non-trivial.

Let $X/\mathbb{C}$ be a smooth projective surface, $Y = \bigcup_{j=1}^n Y_i \subset X$ a NCD (with smooth Y_i's) with open complement $U := X \backslash Y$. The work of Deligne ([De], Cor. 3.2.13 and 3.2.14) implies that $F^2H^2(U,\mathbb{C}) = H^0(X,\Omega_X^2\langle Y\rangle)$, where $\Omega_X^2\langle Y\rangle$ is the sheaf of rational 2-forms on X, regular on U with logarithmic poles along Y. In particular, if we denote by $H(U)$, the space of regular algebraic 2-forms on U with $\mathbb{Q}(2)$-periods, then

$$\Gamma H^2(U,\mathbb{Q}(2)) := H^2(U,\mathbb{Q}(2)) \cap F^2H^2(U,\mathbb{C}) \subset H(U),$$

where we allow for the possibility that the left hand side here is non-zero. Let $Z(Y)$ be the singular set of Y. For each Y_j we choose distinct points $\{P_j, Q_j\} \subset Y_j \backslash \{Y_j \cap$

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$Z(Y)\}$. Now let's modify X by blowing it up along $\{P_1, Q_1, ..., P_n, Q_n\}$ and call this X'. So in particular the strict transform Y' of Y is a copy of Y itself. Now put $U' := X' \backslash Y'$. We observe from (4.4) that $Gr_0^W H^2(U, \mathbb{Q}(2)) \simeq Gr_0^W H^2(U', \mathbb{Q}(2))$. Thus

$$\Gamma H^2(U, \mathbb{Q}(2)) \neq 0 \Rightarrow \Gamma Gr_0^W H^2(U', \mathbb{Q}(2)) \neq 0.$$

On U' we have for each $j = 1, ..., n$ an interesting real 2-cycle γ_j obtained as follows: take the complement of a small disc in the blowup of P_j, the complement of a small disc in the blowup of Q_j, and a small tube in $X \backslash Y$ along a path in Y_j from P_j to Q_j that so that the end circles meet the circles of the two previous parts. Then by a standard residue argument, integrating an element $\omega \in H(U')$ against γ_j gives us essentially $2\pi i$ times the integral along the part in Y_j from P_j to Q_j of the residue of ω along Y, where we observe that ω restricts to zero on the other two parts of γ_j, viz., there are no non-zero holomorphic 2-forms living on any subset of $\mathbb{P}^1$.

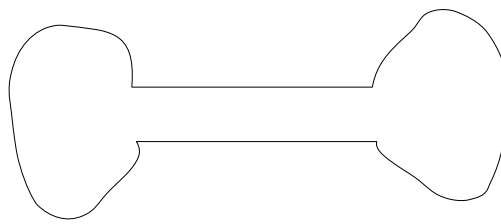


Figure 9.1: The cycle γ_j

Note that this integral is determined by the finite dimensional $\mathbb{Q}$-vector space of residues on Y. So pick P_j and Q_j sufficiently general in Y_j so that we can never end up in $\mathbb{Q}(2)$ with this integral except with the trivial residue. Then doing this for all $j = 1, ..., n$, we arrive at U' for which $\Gamma H^2(U', \mathbb{Q}(2)) = 0$, using the fact that $H^0(X', \Omega^2_{X'}) \cap H^2(X', \mathbb{Q}(2)) = 0$. We deduce:

Theorem 9.1 *Let*

$$t \in B := \left\{ (P_1, Q_1, ..., P_n, Q_n) \in \prod_{j=1}^{n} Y_j^2 \;\middle|\; \begin{array}{c} P_j \neq Q_j \\ \{P_j, Q_j\} \in Y_j \backslash \{Y_j \cap Z(Y)\} \\ j = 1, ..., n \end{array} \right\},$$

with corresponding family $\{U'_t\}_{t \in B}$. *Then* $\Gamma H^2(U'_t, \mathbb{Q}(2)) = 0$ *for a very general point* t *in* B. *Moreover, since* $\Gamma H^2(U, \mathbb{Q}(2))$ *injects into* $\Gamma Gr_0^W H^2(U', \mathbb{Q}(2))$, *matters can be arranged so that* $\Gamma Gr_0^W H^2(U'_t, \mathbb{Q}(2))$ *is non-trivial for such* t.

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Remark 9.2. (i) Let $t \in B$ be given such that $\Gamma H^2(U'_t,\mathbb{Q}(2)) = 0$, and put $U' := U'_t$, with $U' = X'\backslash Y'$. Then referring to Example 4.14, it follows that the Abel-Jacobi map is injective on the subgroup of cycles in $\mathrm{CH}^2(X',1;\mathbb{Q})$ supported on Y', modulo the image of the space of decomposables in $\mathrm{CH}^2(X',1;\mathbb{Q})$, again supported on Y'.

(ii) An interesting problem is to test the Beilinson-Hodge conjecture for those U'_t where $\Gamma H^2(U'_t,\mathbb{Q}(2)) \neq 0$. This line of enquiry will be pursued at another time.

10. Appendix: The Abel-Jacobi map revisited

The purpose of this appendix is to establish the commutativity of diagram (4.5) above. Due to its technical nature, the reader with pressing obligations can easily skip this without losing sight of the main results of this paper. We first digress by describing the Abel-Jacobi map

$$\lambda_k : \widetilde{E}_k^{-m+k,-k+1} \to J\big(Gr_{-k}^W\big),$$

where Gr_{-k}^W is described in (4.4). In the case $k = 1$, viz.,

$$\lambda_1 : \widetilde{E}_1^{-m+1,0} \to J\big(Gr_{-1}^W\big),$$

this is induced by the classical Abel-Jacobi map

$$\mathrm{CH}^{r-m+1}_{\mathrm{hom}}(Y^{[m-1]};\mathbb{Q}) \xrightarrow{\xi\mapsto\int_{\partial^{-1}\xi}(-)} J\big(H^{2r-2m+1}(Y^{[m-1]},\mathbb{Q}(r-m+1))\big).$$

The Abel-Jacobi map

$$\Phi_k : \widetilde{E}_2^{-m+k,-k+1} \to J\big(Gr_{-k}^W\big),$$

is defined using the formula in [KLM]. By degeneration of the mixed Hodge complex spectral sequence at E_2 (Deligne), and a map of spectral sequences from Chow groups to Hodge cohomology, together with functoriality of the Abel-Jacobi map, the map Φ_k induces λ_k for all $k \geq 2$. So we need only describe the map Φ_k explicitly. Let $\square^m := (\mathbb{P}^1\backslash\{1\})^m$ with coordinates z_i and 2^m codimension one faces obtained by setting $z_i = 0,\infty$, and boundary maps $\partial = \sum(-1)^{i-1}(\partial_i^0 - \partial_i^\infty)$, where ∂_i^0, ∂_i^∞ denote the restriction maps to the faces $z_i = 0$, $z_i = \infty$ respectively. Here we adopt the notation in [KLM] adapted to the cubical description of $\mathrm{CH}^r(X,m;\mathbb{Q})$, with cycles lying in $z^r(X\times\square^m;\mathbb{Q})$, in general position with respect to the 2^m faces of $\square^m$ as well as the real part $[-\infty,0]^m \subset \square^m$.

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Recall the Tate twist $\mathbb{Q}(r)$, let $X/\mathbb{C}$ be smooth projective with $d = \dim X$, and put $\mathcal{D}_X^k :=$ sheaf of currents that act on compactly supported $\mathbb{C}$-valued C^∞ forms of degree $2d-k$. Note that

$$\mathcal{D}_X^\bullet = \bigoplus_{p+q=\bullet} \mathcal{D}_X^{p,q},$$

where $\mathcal{D}_X^{p,q}$ acts on corresponding $(d-p,d-q)$ forms. Let $\mathcal{C}_X^k(\mathbb{Q}(r))$ be the sheaf of Borel-Moore chains of real codimension k in X with $\mathbb{Q}(r)$ coefficients. One has an inclusion $\mathcal{C}_X^k(\mathbb{Q}(r)) \subset \mathcal{D}_X^k$. Now put

$$\mathcal{M}_\mathcal{D}^\bullet = \mathrm{Cone}\big\{\mathcal{C}_X^\bullet(X,\mathbb{Q}(r)) \bigoplus F^r\mathcal{D}_X^\bullet(X) \to \mathcal{D}_X^\bullet(X)\big\}[-1].$$

The cohomology of this complex at $\bullet = k$ is precisely the Deligne cohomology $H_\mathcal{D}^k(X,\mathbb{Q}(r))$ (see [KLM]).

We recall the cycle class map

$$\Psi_{r,m} : \mathrm{CH}^r(X,m;\mathbb{Q}) \to H_\mathcal{D}^{2r-m}(X,\mathbb{Q}(r)),$$

in terms of the cubical description of $\mathrm{CH}^r(X,m)$. Note that for $m \geq 1$,

$$H_\mathcal{D}^{2r-m}(X,\mathbb{Q}(r)) \simeq J\big(H^{2r-m-1}(X,\mathbb{Q}(r))\big),$$

and under this identification, $\Psi_{r,m}$ is the Abel-Jacobi map. On $\square^m$ we introduce the currents (see [KLM, p. 383])

$$\Omega_{\square^m} := \bigwedge_1^m d\log z_j$$

$$T_{\square^m} := T_{z_1} \cap \cdots \cap T_{z_m} = \int_{[-\infty,0]^m} (-) =: \delta_{[-\infty,0]^m},$$

where T_{z_j} is integration on $\big\{(z_1,\ldots,z_m) \in \square^m \;\big|\; z_j \in [-\infty,0]\big\}$.

$$R_{\square^m} := \log z_1 d\log z_2 \wedge \cdots \wedge d\log z_m - (2\pi\mathrm{i})\log z_2 d\log z_3 \wedge \cdots \wedge d\log z_m \cdot T_{z_1}$$

$$+\cdots+(-1)^{m-1}(2\pi\mathrm{i})^{m-1}\log z_m \cdot \{T_{z_1} \cap \cdots \cap T_{z_{m-1}}\},$$

where log has the principal branch. One has

$$dR_{\square^m} = \Omega_{\square^m} - (2\pi\mathrm{i})^m T_{\square^m} - 2\pi\mathrm{i} R_{\partial\square^m}.$$

We consider a cycle $\xi \in z^r(X \times \square^m)$ in general position. One considers the projections $\pi_1 : |\xi| \to X$, $\pi_2 : |\xi| \to \square^m$. We put

$$R_\xi = \pi_{1,*} \circ \pi_2^* R_{\square^m}, \quad \Omega_\xi = \pi_{1,*} \circ \pi_2^* \Omega_{\square^m}, \quad T_\xi = \pi_{1,*} \circ \pi_2^* T_{\square^m}.$$

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Note that when $m = 0$, $R_\xi = 0$ (classical case!). Correspondingly

$$dR_\xi = \Omega_\xi - (2\pi\mathrm{i})^m T_\xi - 2\pi\mathrm{i} R_{\partial\xi}.$$

Up to a normalizing constant, the cycle class map $\Psi_{r,m}$ is induced by

$$\xi \mapsto \big((2\pi\mathrm{i})^m T_\xi, \Omega_\xi, R_\xi\big).$$

Recall $d = \dim X$. The Abel-Jacobi map

$$\Phi_{r,m} : \mathrm{CH}^r_{\mathrm{hom}}(X,m;\mathbb{Q}) \to \frac{F^{d-r+1}H^{2d-2r+m+1}(X,\mathbb{C})^\vee}{H_{2d-2r+m+1}(X,\mathbb{Q}(d-r))},$$

where we described $J\big(H^{2r-m-1}(X,\mathbb{Q}(r))\big)$ using the Carlson isomorphism, is defined as follows. If $\xi \in \mathrm{CH}^r(X,m;\mathbb{Q})$, then by a moving lemma [K-L, Lemma 8.14], we can assume that ξ is in general position with respect to the real cube $[-\infty,0]^m \subset \square^m$. Furthermore, $m > 0$ implies $T_\xi = dT_\zeta$, i.e., $\pm\partial\zeta = \xi \cap \{X \times [-\infty,0]^m\}$, $\Omega_\xi = -dS$, for some $S \in F^r\mathcal{D}^{2r-m-1}_X(X)$, so up to a normalizing constant and for $\omega \in F^{d-r+1}H^{2d-2r+m+1}(X,\mathbb{C})$, we have

$$\Phi_{r,m}(\xi)(\omega) = S(\omega) + (-2\pi\mathrm{i})^m \int_\zeta \omega + R_\xi(\omega) = (-2\pi\mathrm{i})^m \int_\zeta \omega + R_\xi(\omega),$$

where the latter equality stems from Hodge type considerations.

Now referring to (4.5), we fulfill a promise made earlier, viz.,

Theorem 10.1 *The diagram* (4.5) *is commutative. Specifically,*

$$h_k \circ \alpha_k = \lambda_k \circ d_k.$$

Proof: We have to unravel the definitions. We use the simplicial complex $Y^{[\bullet]} \to Y \hookrightarrow X$ as a way of describing $W_j H^{2r-m}(U,\mathbb{Q}(r))$. Let

$$\mathcal{K}^{2r-2m+i}_{\mathbb{Q}}(Y^{[m-i]}) = \mathcal{C}^{2r-2m+i}(Y^{[m-i]},\mathbb{Q}(r-m+i))$$

$$\mathcal{K}^{2r-2m+i}_{\mathbb{C}}(Y^{[m-i]}) = \mathcal{D}^{2r-2m+i}(Y^{[m-i]})$$

$$\mathbb{D} = d \pm \mathrm{Gy}.$$

A class $\xi \in W_0 H^{2r-m}(U,\mathbb{Q}(r))$ is represented by a $\mathbb{D}$-closed $(m+1)$-tuple

$$\xi = (\xi_0,\xi_1,\ldots,\xi_m) \in \bigoplus_{i=0}^{m} \mathcal{K}^{2r-2m+i}_{\mathbb{Q}}(Y^{[m-i]}).$$

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With $W_j := W_j H^{2r-m}(U,\mathbb{Q}(r))$, consider the short exact sequence

$$0 \to W_{-1} \to W_0 \xrightarrow{\xi \mapsto \xi_0} Gr_0^W \to 0.$$

Let us first describe $h_1 \circ \alpha_1 : \widetilde{E}_1^{-m,0} \to J\left(Gr_{-1}^W\right)$. Let $\gamma \in \widetilde{E}_1^{-m,0}$ and $\xi_0 = \beta(\gamma)$. In this case $\xi_0 \in \Gamma\left(Gr_0^W\right)$ is in the image of $\xi \in W_0$. Likewise ξ_0 is in the image of some $\xi^{\mathbb{C}} \in F^0 W_{0,\mathbb{C}}$. The difference $\xi - \xi^{\mathbb{C}} \in W_{-1,\mathbb{C}}$ maps to a class in $J\left(W_{-1}\right)$ which defines the Abel-Jacobi image of γ in $J\left(W_{-1}\right)$. We can assume that $\xi - \xi^{\mathbb{C}}$ is represented by the $\mathbb{D}$-closed m-tuple:

$$(\xi_1 - \xi_1^{\mathbb{C}}, \ldots, \xi_m - \xi_m^{\mathbb{C}}) \in \bigoplus_{i=1}^{m} \mathcal{K}_{\mathbb{C}}^{2r-2m+i}(Y^{[m-i]}).$$

with $\mathrm{Gy}(\xi_0) = \partial\xi_1$. The corresponding value $h_1 \circ \alpha_1(\gamma) \in J\left(Gr_{-1}^W\right)$ is given by the Abel-Jacobi membrane integral $\int_{\xi_1}(-)$, as the Hodge contribution given by $\xi_1^{\mathbb{C}}$ is trivial for Hodge type reasons. This is easily seen to be precisely $\lambda_1 \circ d_1(\gamma)$, where we recall that $d_1 = \mathrm{Gy}$. Thus $\lambda_1 \circ d_1 = h_1 \circ \alpha_1$. So now suppose that $d_1(\gamma) = 0 \in \widetilde{E}_1^{-m+1,0}$, i.e. $\gamma \in \widetilde{E}_2^{-m,0}$. This means that $\mathrm{Gy}(\gamma) = \partial\zeta_1$, where $\zeta_1 \in z^{r-m+1}(Y^{[m-1]}, 1; \mathbb{Q})$. Then $d_2(\gamma) = \mathrm{Gy}(\zeta_1) \in \widetilde{E}_2^{-m+2,-1}$. (We comment in passing that using the aforementioned moving lemma in [K-L], we can assume that $\zeta_{1,\mathbb{R}} := \zeta \cap Y^{[m-1]} \times [-\infty, 0]$ is a proper intersection, and hence that $\partial\zeta_{1,\mathbb{R}} = \mathrm{Gy}(\xi_0)$.) Since $d_1(\gamma) = \mathrm{Gy}(\gamma)$ is a coboundary, it follows that $h_1 \circ \alpha_1(\gamma) = \lambda_1 \circ d_1(\gamma) = 0$, and hence after removing classes in $W_{-1} + F^0 W_{-1,\mathbb{C}}$, [specifically, $\partial(\zeta_{1,\mathbb{R}} - \xi_1) = \mathrm{Gy}(\xi_0) - \mathrm{Gy}(\xi_0) = 0$, and using $W_{-1} \twoheadrightarrow Gr_{-1}^W$, $d(\Omega_{\zeta_1} - \xi_1^{\mathbb{C}}) = \mathrm{Gy}(\xi_0^{\mathbb{C}}) - \mathrm{Gy}(\xi_0^{\mathbb{C}}) = 0$ and using $F^0 W_{-1,\mathbb{C}} \twoheadrightarrow F^0 Gr_{-1,\mathbb{C}}^W$], we can assume that $\xi - \xi^{\mathbb{C}}$ is represented by the $\mathbb{D}$-closed m-tuple

$$(2\pi \mathrm{i} T_{\zeta_1} - \Omega_{\zeta_1}, \xi_2 - \xi_2^{\mathbb{C}}, \ldots, \xi_m - \xi_m^{\mathbb{C}}) \in \bigoplus_{i=1}^{m} \mathcal{K}_{\mathbb{C}}^{2r-2m+i}(Y^{m-i]}),$$

where

$$d\xi_2 = (2\pi \mathrm{i}) T_{\mathrm{Gy}(\zeta_1)}, \quad d\xi_2^{\mathbb{C}} = \Omega_{\mathrm{Gy}(\zeta_1)}.$$

Note that

$$dR_{\zeta_1} = \Omega_{\zeta_1} - (2\pi \mathrm{i}) T_{\zeta_1} - (2\pi \mathrm{i}) R_{\partial\zeta_1} = \Omega_{\zeta_1} - (2\pi \mathrm{i}) T_{\zeta_1},$$

as $R_{\partial\zeta_1} = 0$ ($\partial\zeta_1$ representing the classical case!). But $\partial \mathrm{Gy}(\zeta_1) = \mathrm{Gy}(\partial\zeta_1) = \mathrm{Gy}^2(\gamma) = 0$. Hence working modulo the coboundary

$$\mathbb{D} R_{\zeta_1} = dR_{\zeta_1} \pm R_{\mathrm{Gy}(\zeta_1)},$$

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we can assume that $\xi - \xi^{\mathbb{C}}$ is represented by the $\mathbb{D}$-closed $(m-1)$-tuple

$$(\xi_2 - \xi_2^{\mathbb{C}} + R_{\mathrm{Gy}(\zeta_1)}, \xi_3 - \xi_3^{\mathbb{C}}, \ldots, \xi_m - \xi_m^{\mathbb{C}}) \in \bigoplus_{i=2}^{m} \mathcal{K}_{\mathbb{C}}^{2r-2m+i}(Y^{m-i]}).$$

So modulo $\mathbb{D}$-coboundary, $h_2 \circ \alpha_2(\gamma)$ is represented by the d-closed current $\xi_2 - \xi_2^{\mathbb{C}} + R_{\mathrm{Gy}(\zeta_1)}$, which is precisely $\lambda_2 \circ d_2(\gamma)$. Hence $h_2 \circ \alpha_2 = \lambda_2 \circ d_2$. The general case $h_k \circ \alpha_k = \lambda_k \circ d_k$ proceeds in a similar fashion. □

11. Appendix by M. Asakura

The purpose of this section is to prove the following.

Theorem 11.1 *Let U be a smooth complex variety. Then the cokernel of the $d\log$ map*

$$\mathrm{CH}^2(U,2) \to H^2(U,\mathbb{Z}(2))$$

is torsion free.

As an immediate consequence, we have:

Corollary 11.2 *$d\log : \mathrm{CH}^2(U,2) \to \Gamma H^2(U,\mathbb{Z}(2))$ is surjective $\Leftrightarrow$ $d\log \otimes \mathbb{Q} : \mathrm{CH}^2(U,2;\mathbb{Q}) \to \Gamma H^2(U,\mathbb{Q}(2))$ is surjective.*

Proof: It is enough to show that the cokernel of

$$\mathrm{CH}^2(U,2) \otimes \mathbb{Z}_l \to H^2_{\text{ét}}(U,\mathbb{Z}_l(2))$$

is torsion free for each prime l (indeed by comparison $H^2_{\text{ét}}(U,\mathbb{Z}_l(2)) \simeq H^2(U,\mathbb{Z}) \otimes \mathbb{Z}_l(2)$). The key ingredient is Suslin's exact sequence ([Su, Cor.23.4])

$$0 \to \mathrm{CH}^2(U,2)/l^n \to H^2_{\text{ét}}(U,\mathbb{Z}/l^n(2)) \to \mathrm{CH}^2(U,1)[l^n] \to 0, \tag{11.3}$$

where $M[n]$ denotes the n-torsion part of M. Taking the inductive limit of (11.3), we arrive at

$$0 \to \mathrm{CH}^2(U,2) \otimes \mathbb{Q}_l/Z_l \to H^2_{\text{ét}}(U,\mathbb{Q}_l/\mathbb{Z}_l(2)) \to \mathrm{CH}^2(U,1)[l^\infty] \to 0.$$

Therefore $\mathrm{CH}^2(U,1)[l^\infty]$ is of cofinite rank, i.e.

$$\mathrm{CH}^2(U,1)[l^\infty] \simeq \left(\mathbb{Q}_l/\mathbb{Z}_l\right)^{\oplus} \oplus N,$$

where N is a finite group, hence

$$\mathrm{CH}^2(U,1)[l^n] \simeq \left(\mathbb{Z}/l^n\right)^{\oplus} \oplus N[l^n].$$

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Now take the projective limit of (11.3), with homomorphism

$$\mathrm{CH}^2(U,1)[l^{n+1}] \to \mathrm{CH}^2(U,1)[l^n]$$

given by multiplication by l. Hence we see that

$$\varprojlim_n \big(\mathrm{CH}^2(U,1)[l^n]\big) \simeq \big(\mathbb{Z}_l\big)^{\oplus},$$

and accordingly

$$0 \to \varprojlim_n \big(\mathrm{CH}^2(U,2)/l^n\big) \to H^2_{\text{ét}}(U,\mathbb{Z}_l(2)) \to \big(\mathbb{Z}_l\big)^{\oplus} \to 0. \tag{11.4}$$

It remains to show that

$$\mathrm{CH}^2(U,2)\otimes\mathbb{Z}_l \to \varprojlim_n \big(\mathrm{CH}^2(U,2)/l^n\big) =: M$$

is surjective. To do this, first observe that M is a finitely generated $\mathbb{Z}_l$-module (as is the case for $H^2_{\text{ét}}(U,\mathbb{Z}_l(2))$ and $M \subset H^2_{\text{ét}}(U,\mathbb{Z}_l(2))$), and so it is enough to show that

$$\mathrm{CH}^2(U,2)/l \to M/l$$

is surjective by Nakayama's lemma (indeed, we will have bijectivity in this case). There is the obvious map $a : M/l \to \mathrm{CH}^2(U,2)/l$; moreover the composite

$$\mathrm{CH}^2(U,2)/l \to M/l \xrightarrow{a} \mathrm{CH}^2(U,2)/l$$

is the identity. So it suffices to show that the map a is injective. Tensoring (11.4) with $\mathbb{Z}/l$, together with (11.3), we arrive at the commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \to & M/l & \to & H^2_{\text{ét}}(U,\mathbb{Z}_l(2))\otimes\mathbb{Z}/l & \to & \big(\mathbb{Z}/l\big)^{\oplus} & \to & 0 \\
 & & \downarrow a & & \downarrow b & & & & \\
0 & \to & \mathrm{CH}^2(U,2)/l & \to & H^2_{\text{ét}}(U,\mathbb{Z}/l(2)). & & & &
\end{array}$$

Since the map b is injective, the same holds for a, and we are done. □

Remark 11.5. (i) $\mathrm{CH}^2(U,2)\otimes\mathbb{Z}_l$ maps surjectively to the l-adic completion of $\mathrm{CH}^2(U,2)$, but never injectively. Indeed, the source is not a finitely generated $\mathbb{Z}_l$-module, albeit the target is.

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(ii) Regarding the injectivity of b in the above proof, here is a simple proof. From the exact sequence

$$0 \to \mathbb{Z}/l^{n-1}(r) \xrightarrow{\cdot l} \mathbb{Z}/l^n(r) \to \mathbb{Z}/l(r) \to 0$$

of étale sheaves, we have the long exact sequence

$$\cdots \to H^i_{\text{ét}}(U,\mathbb{Z}/l^{n-1}(r)) \to H^i_{\text{ét}}(U,\mathbb{Z}/l^n(r)) \to H^i_{\text{ét}}(U,\mathbb{Z}/l(r)) \to \cdots.$$

Take the projective limit on n. Then the exactness remains true because all terms are finite groups. We thus have

$$\cdots \to H^i_{\text{ét}}(U,\mathbb{Z}_l(r)) \xrightarrow{\cdot l} H^i_{\text{ét}}(U,\mathbb{Z}_l(r)) \to H^i_{\text{ét}}(U,\mathbb{Z}/l(r)) \to \cdots.$$

and hence $H^i_{\text{ét}}(U,\mathbb{Z}_l(r))/l \to H^i_{\text{ét}}(U,\mathbb{Z}/l(r))$ is injective.

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