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Homomorphism and Dimension

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For Béla Bollobás on his 60th birthday

The dimension of a graph, that is, the dimension of its incidence poset, has become a major bridge between posets and graphs. Although allowing a nice characterization of planarity, this dimension behaves badly with respect to homomorphisms.

We introduce the *universal dimension* of a graph G as the maximum dimension of a graph having a homomorphism to G. The universal dimension, which is clearly homomorphism monotone, is related to the existence of some balanced bicolouration of the vertices with respect to some realizer.

Nontrivial new results related to the original graph dimension are subsequently deduced from our study of universal dimension, including chromatic properties, extremal properties and a disproof of two conjectures of Felsner and Trotter.

1. Introduction

Half a century ago, from the concept of dimension of a partial order [9], Dushnik started an investigation of the poset of one- and k-element subsets of $[n] = \{1, \ldots, n\}$ [8], and introduced a construction which has a natural interpretation in terms of abstract simplicial complexes, and which has been used or re-introduced in several contexts, such as mathematical economy [18], integer programming [2], and commutative algebra [3].

The dimension of the poset of one- and two-element subsets of $[n]$ (the complete graph incidence poset) has, in particular, been investigated by Spencer [20], Füredi, Hajnal, Rödl and Trotter [12], and more recently by Hoşten and Morris [13].

The study of the dimension of the incidence poset of graphs (other than complete ones) have attracted particular attention in the past few years, since W. Schnyder proved that the incidence poset of a graph has dimension at most 3 if and only if the graph is planar [19]. This theorem has been generalized by Brightwell and Trotter [6, 7] to a characterization of the dimension of the vertex, edge and face posets of planar graphs, and to a generalization to multigraphs.

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Although the generalization of Schnyder's result to the problem of the geometric realization of an abstract simplicial complex is quite natural [16] and may be viewed as the existence of embeddings of posets in Euclidean space with a general separation property [17], the property of the class of planar graphs to be minor closed disappears in dimension at least 4. The incidence poset dimension is not homomorphism monotone (any nonplanar graph with a homomorphism to K_4 has dimension 4, although K_4 has dimension 3). In this paper we introduce a new graph dimension, the *universal dimension* of a graph, which will make this connection clear. We will prove that this universal dimension differs by at most two from the incidence poset dimension, while being (by its definition) homomorphism monotone.

We will relate the universal dimension of a graph to the existence of realizers admitting a special bicolouration and will give some applications of our results, such as the non-bounding of chromatic numbers of graphs of dimension 4, or extremal results on graphs of incidence poset dimension d, in the framework investigated by Bollobás and Thomason [5]. We also disprove two conjectures of Felsner and Trotter.

2. Definitions and notation

In the following we will only consider finite simple loopless graphs with at least one edge.

Let $P = (X, \leqslant)$ be a finite poset. A *realizer* of P is a nonempty family $\mathscr{R} = (<_1, \ldots, <_t)$ of linear orders on X whose intersection $\bigcap_{i=1}^{t} <_i$ is P. The *dimension* of P is the minimum cardinality of a realizer of P [9].

Homomorphisms of graphs are adjacency-preserving maps: a map $f : V(H) \to V(G)$ is a homomorphism of the graph H to the graph G if $\{f(x), f(y)\} \in E(G)$ whenever $\{x, y\} \in E(H)$. We will denote by $H \to G$ the existence of a homomorphism of H to G. The *chromatic number* $\chi(H)$ of a graph H is thus the smallest integer n, such that H has a homomorphism to K_n.

Following the definition of [14], the *multiplication* $G^{\cdot\cdot}(W_1, \ldots, W_n)$ of a graph G with vertex set $V(G) = \{v_1, \ldots, v_n\}$ is defined by

$$
\begin{aligned}
& V(G^{\cdot\cdot}) = W_1 \cup \cdots \cup W_n, \\
\forall 1 \leqslant i \leqslant n, \quad & |W_i| \geqslant 1 \\
\forall 1 \leqslant i < j \leqslant n, \quad & W_i \cap W_j = \emptyset, \\
\forall 1 \leqslant i \leqslant j \leqslant n,\ \forall u \in W_i,\ \forall v \in W_j, \quad & \{u, v\} \in E(G^{\cdot\cdot}) \text{ if and only if } \{v_i, v_j\} \in E(G)
\end{aligned}
$$

The sets $W_1, \ldots, W_n$ are called the *multivertices* corresponding to vertices $v_1, \ldots, v_n$, respectively.

A *property* of graphs $\mathscr{P}$ is an (infinite) class of labelled graphs closed under isomorphism. A property $\mathscr{P}$ is *monotone* if it is closed under taking subgraphs; it is *hereditary* if it is closed under taking induced subgraphs; it is *additive* if it is closed under disjoint unions of graphs. The *n-level* $\mathscr{P}^n$ of a property $\mathscr{P}$ is the set of graphs in $\mathscr{P}$ with vertex set $[n]$. The *speed* of $\mathscr{P}$, $|\mathscr{P}^n|$ is the number of graphs in the property on n vertices. The *size* of a monotone property $\mathscr{P}$ at level n is the maximum number of edges in a graph of $\mathscr{P}^n$, that is $e_{\mathscr{P}}(n) = \max\{|E(G)|, G \in \mathscr{P}^n\}$.

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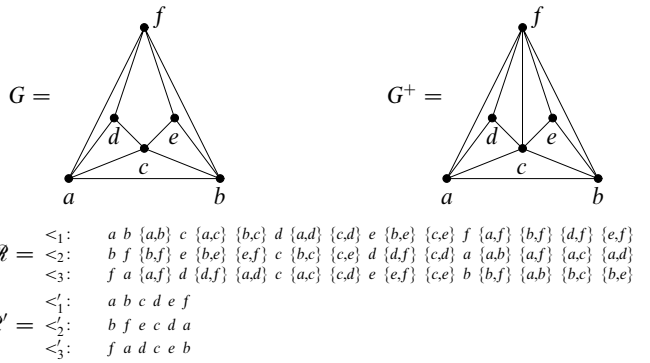


Figure 1. A graph G, a realizer $\mathscr{R}$ of P_G, a graph realizer $\mathscr{R}'$ of both G and G^+.

Definition 1. Let G be a graph. The *incidence poset* P_G of G is the height 2 poset having $V(G) \cup E(G)$ as its ground set, such that $a < b$ in P_G if $a \in V(G)$, $b \in E(G)$ and b is incident to a.

Definition 2. Let G be a graph. A *graph realizer* of G of size t is a family $\mathscr{R} = (<_1, \ldots, <_t)$ of linear orders on $V(G)$, such that:[1]

$$\forall x \neq y \in V(G), \quad \exists i \in [t],\ x <_i y$$
$$\forall \{x, y\} \in E(G),\ \forall z \in V(G) \setminus \{x, y\}, \quad \exists i \in [t],\ x <_i z \text{ and } y <_i z$$

The *dimension* $\dim G$ of G is the smallest size of a graph realizer of G.

As already noticed in [8] (in a more general framework), a graph realizer of G may be obtained from a realizer of P_G by reducing the ground set to $V(G)$ and, conversely, a realizer of P_G may be obtained from a graph realizer by inserting the edges of G in lexicographic order. Hence $\dim G = \dim P_G$. Notice that a graph G cannot be reconstructed from one of its graph realizers, although it can be reconstructed from one realizer of P_G (see Figure 1).

Remark 1. Let G be a graph and let H be a subgraph of G. Then, the restriction of the linear orders of a graph realizer of G to $V(H)$ defines a graph realizer of H. Thus:

$$H \subseteq G \Longrightarrow \dim H \leqslant \dim G. \tag{2.1}$$

Hence, the property $(\dim \leqslant d)$ of graphs with dimension at most d is monotone. It is also obviously additive.

[1] The first condition is missing in the definition of the dimension of a graph which appears in [11], causing the dimension of a graph to be possibly different from the dimension of its incidence poset.

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3. The universal dimension of a graph

We shall introduce a new dimension associated to a graph, which will be defined in order to be homomorphism monotone.

Definition 3. Let G be a graph. The *universal dimension* $\operatorname{udim} G$ is the supremum of the dimensions of the graphs having a homomorphism to G:

$$\operatorname{udim} G = \sup_{H \to G} \dim H \tag{3.1}$$

Remark 2. Thus, we have this extension of (2.1) for universal dimension:

$$H \to G \Longrightarrow \operatorname{udim}(H) \leqslant \operatorname{udim}(G). \tag{3.2}$$

Obviously, the property $(\operatorname{udim} \leqslant d)$ of graphs with universal dimension at most d is monotone and additive.

Remark 3. Obviously, $\operatorname{udim} K_1 = 1$. However, for any graph G with at least one edge, $K_{3,3} \to G$. According to Schnyder's theorem, $\dim K_{3,3} > 3$ as $K_{3,3}$ is not planar. Hence no graph has universal dimension 2 or 3, and a graph has universal dimension at least 4 if and only if it has at least one edge.

We shall introduce the concept of bicolourable graph realizer, motivated by Theorem 3.3, that will assert that the universal dimension G will be the smallest size of a bicolourable graph realizer of G (similar to the definition of the dimension of G as the smallest size of a graph realizer of G).

Definition 4. Let $\mathscr{R} = \{<_1, \ldots, <_t\}$ be a graph realizer of a graph G. A *balanced bicolouration* of G with respect to $\mathscr{R}$ is a map $\Gamma : V(G) \times [t] \to \{0, 1\}$ so that, for every $x \in V(G)$ and for every vertex y adjacent to x (in G), there exists $i, j \in [t]$ so that

$$\begin{cases} y <_i x \text{ and } \Gamma(x, i) = 0, \\ y <_j x \text{ and } \Gamma(x, j) = 1. \end{cases} \tag{3.3}$$

When such a map exists, $\mathscr{R}$ is said to be a *bicolourable graph realizer* of G.

Remark 4. Notice that any graph G has a bicolourable graph realizer of size $(\dim G + 2)$ obtained from a graph realizer of G by adding two dual linear orders.

Lemma 3.1. *Let G be a graph of order n, let $d > 3$ be an integer, let $a_1, \ldots, a_n$ be positive integers, and let $G^+ = G^{::}(W_1, \ldots, W_n)$ be a multiplication of G, where (for $1 \leqslant p \leqslant n$)*

$$|W_p| > (a_p - 1) \left(\sum_{i=1}^{p-1} a_i + 1 \right)^d.$$

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Then, if $\mathscr{R} = (<_1, \ldots, <_d)$ *is a graph realizer of* G^+, *there exist* $W'_1 \subseteq W_1, \ldots, W'_n \subseteq W_n$ *such that*

- *for any* $1 \leqslant i \leqslant n$, $|W'_i| = a_i$,
- *for any* $1 \leqslant i < j \leqslant n$, *no two elements of* W'_j *are separated by an element of* W'_i *in some linear order in* $\mathscr{R}$.

Proof. Consider the following assertion for $1 \leqslant p \leqslant n$.

Π_p: there exists $W'_1 \subseteq W_1, \ldots, W'_p \subseteq W_p$, such that

- for any $1 \leqslant i \leqslant p$, $|W'_i| = a_i$,
- for any $1 \leqslant i < j \leqslant p$, no two elements of W'_j are separated by an element of W'_i in some linear order in $\mathscr{R}$.

We shall prove by induction that Π_p holds for any $1 \leqslant p \leqslant n$. Obviously, Π_1 holds ($W'_1 = W_1$).

Assume Π_p holds for some $1 \leqslant p < n$. Then, the elements of $W'_1 \cup \cdots \cup W'_p$ determine at most $s_p = \left(\sum_{i=1}^{p} a_i\right) + 1$ intervals in each of the linear orders of $\mathscr{R}$. Colour any vertex x in W_{p+1} by the d-tuple corresponding to the indexes of the intervals in which x appear in $<_1, \ldots, <_d$. As $|W_{p+1}| > (a_{p+1} - 1)s_p^d$, a_{p+1} elements of W_{p+1} are coloured the same. Choosing these a_{p+1} elements for W'_{p+1}, the assertion Π_{p+1} clearly holds. □

Lemma 3.2. *Let* G *be a graph of order* n, *let* $d > 3$ *be an integer, and let* $G^+ = G^{\cdot:}(W_1, \ldots, W_n)$ *be the multiplication of* G *where* $|W_p| = (2n)^{d(d+2)}$ *(for* $1 \leqslant p \leqslant n$).

If $\mathscr{R} = (<_1, \ldots, <_d)$ *is a graph realizer of* G^+, *then there exists* $W'_1 \subseteq W_1, \ldots, W'_n \subseteq W_n$ *such that, for any* $1 \leqslant i \leqslant n$:

- $|W'_i| = 2$,
- *the elements of* W'_i *are consecutive in each of the restrictions of the linear orders in* $\mathscr{R}$ *to* $W'_1 \cup \cdots \cup W'_n$.

Proof. According to Lemma 3.1, there exists $W''_1 \subseteq W_1, \ldots, W''_n \subseteq W_n$, such that

- for any $1 \leqslant i \leqslant n$, $|W''_i| = (2i-1)^d + 1$,
- for any $1 \leqslant i < j \leqslant n$, no two elements of W''_j are separated by an element of W''_i in some linear order in $\mathscr{R}$.

According to Lemma 3.1 again (considering the indexes of the W''_i in reverse order), there exists $W'_n \subseteq W''_n, \ldots, W'_1 \subseteq W''_1$, such that

- for any $1 \leqslant i \leqslant n$, $|W'_i| = 2$,
- for any $1 \leqslant i < j \leqslant n$, no two elements of W'_i are separated by an element of W'_j in some linear order in $\mathscr{R}$.

Altogether, the elements of W'_i form an interval of each of the restrictions of the linear order in $\mathscr{R}$ to $W'_1 \cup \cdots \cup W'_n$. □

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Theorem 3.3. *Let G be a graph of order n, let $d > 3$ be an integer, and let $G^+ = G^{::}(W_1, \dots, W_n)$ be the multiplication of G, where $|W_i| = (2n)^{d(d+2)}$. Then, the following assertions are equivalent:*

(1) *G has a bicolourable graph realizer of size d,*
(2) $\operatorname{udim} G \leqslant d$,
(3) $\dim G^+ \leqslant d$,

Thus, the universal dimension $\operatorname{udim} G$ *of a graph G is the smallest size of a bicolourable graph realizer of G.*

Proof. Assume $\mathscr{R} = (<_1, \dots, <_d)$ is a bicolourable graph realizer of G, with balanced bicolouration Γ. If $H \to G$, then there exists a multiplication $H^+ = G^{::}(W'_1, \dots, W'_n)$ of G such that $H \subseteq H^+$. Let $V(G) = \{v_1, \dots, v_n\}$ and let $W'_i = \{(v_i, 1), \dots, (v_i, k_i)\}$ for $1 \leqslant i \leqslant n$. In each linear order $<_j$ of $\mathscr{R}$, replace the vertex v_i by the sequence $(v_i, 1), \dots, (v_i, k_i)$ if $\Gamma(v_i, j) = 0$, or by the sequence $(v_i, k_i), \dots, (v_1, 1)$ if $\Gamma(v_i, j) = 1$. One easily checks that the so-obtained d linear orders realize the graph H^+ and thus they also realize H. Hence, (1) implies (2).

As $G^+ \to G$, we have $\dim G^+ \leqslant \operatorname{udim} G$. Thus, (2) implies (3).

Assume $\dim G^+ \leqslant d$. Let $\mathscr{R} = (<_1, \dots, <_d)$ be a graph realizer of G^+. According to Lemma 3.2, there exists $W'_1 \subseteq W_1, \dots, W'_n \subseteq W_n$, such that, for any $1 \leqslant i \leqslant n$, W_i is a two-element set $\{x_i, y_i\}$, and such that x_i and y_i are consecutive in each of the restriction of the linear orders in $\mathscr{R}$ to $W'_1 \cup \dots \cup W'_n$.

Let $\mathscr{R}' = (<'_1, \dots, <'_d)$, where $<'_i$ is the restriction of $<_i$ to $\{x_1, \dots, x_n\}$. According to Remark 1, $\mathscr{R}'$ is a graph realizer of G. Define $\Gamma(x_i, k)$ as 0 if $x_i <_k y_i$, and 1 otherwise. Let x_i be any vertex of G and let x_j be a neighbour of x_i in G. Then x_j is adjacent to both x_i and y_i in G^+. Hence, as x_i and y_i are always consecutive in the restriction of the linear orders to $W'_1 \cup \dots \cup W'_n$, there exists $\alpha, \beta \in [d]$, such that $x_j <_\alpha x_i <_\alpha y_i$ and $x_j <_\beta y_i <_\beta x_i$. Thus, Γ is a balanced bicolouration of G with respect to $\mathscr{R}'$. Hence (3) implies (1). □

Corollary 3.4. *Let $1 < k < d$ be integers. Assume G has a graph realizer $(<_1, \dots, <_d)$ such that two adjacent vertices are not comparable in $\bigcap_{i=1}^{k} <_i$ and not comparable in $\bigcap_{i=k+1}^{d} <_i$. Then $(<_1, \dots, <_d)$ is bicolourable and thus* $\operatorname{udim} G \leqslant d$.

Proof. Define $\Gamma(x, i) = 0$ if $i \leqslant k$, and 1 otherwise. Then, Γ is a balanced bicolouration of G with respect to $(<_1, \dots, <_d)$. □

Corollary 3.5. *For any graph G,* $\dim G \leqslant \operatorname{udim} G \leqslant \dim G + 2$

Proof. Let $(<_1, \dots, <_d)$ be a graph realizer of G and $<$ any linear order on $V(G)$. Then $(<_1, \dots, <_d, <, \overline{<})$ is a graph realizer of G to which Corollary 3.4 applies. □

We shall now bound the universal dimension of the complete graph of order t in terms of graph dimension of complete graphs.

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Theorem 3.6. *Let t be a positive integer. Then,*

$$\dim K_{2t+\operatorname{udim} K_t} \leqslant \operatorname{udim} K_t \leqslant \dim K_{t+1} + 1. \tag{3.4}$$

Proof. Let $d = \operatorname{udim} K_t$. In order to prove $\dim K_{2t+d} \leqslant \operatorname{udim} K_t$, we may assume that t is maximal, namely that $\operatorname{udim} K_{t+1} > d$. Consider a graph realizer $(<_1, \ldots, <_d)$ of K_t with a balanced bicolouration Γ. Assume some vertex x of K_t is the minimum of at least two linear orders in $\mathscr{R}$, for instance $<_1$ and $<_2$. As the value of $\Gamma(x,1)$ and $\Gamma(x,2)$ is free, we may impose $\Gamma(x,1) = 0$ and $\Gamma(x,2) = 1$. Then, consider the complete graph on $t+1$ vertices obtained by adding a vertex v. In the linear orders $<_i$ $(3 \leqslant i \leqslant d)$, replace x by xv if $\Gamma(x,i) = 0$ and by vx, otherwise. In $<_1$, replace x by vx and in $<_2$ replace x by xv. Extend Γ by setting $\Gamma(v,i) = \Gamma(x,i)$ for $1 \leqslant i \leqslant d$. Then one easily checks that we have obtained a bicolourable realizer of K_{t+1} of size d, contradicting the maximality assumption on t.

Hence, no vertex of K_t is minimal in more than one linear order. In each of the $<_i$, replace each vertex x with the sequence

- $(x,1)(x,2)$, if $\Gamma(x,i) = 0$ and x is minimal in no linear order,
- $(x,2)(x,1)$, if $\Gamma(x,i) = 1$ and x is minimal in no linear order,
- $(x,1)(x,2)(x,3)$, if $\Gamma(x,i) = 0$ and x is minimal in a linear order $<_j$ with $j \neq i$,
- $(x,3)(x,2)(x,1)$, if $\Gamma(x,i) = 1$ and x is minimal in a linear order $<_j$ with $j \neq i$,
- $(x,1)(x,3)(x,2)$, if x is minimal in $<_i$.

The so-obtained set of d linear orders is a graph realizer of K_{2t+d}. Thus, $\operatorname{udim} K_t \geqslant \dim K_{2t+d}$.

Let $<_1, \ldots, <_d$ be a graph realizer of K_{t+1}, and let z be the maximum of $<_1$. Let $<'_i$ be the restriction of $<_i$ to $V(K_{t+1}) - z$, and let $<'_{d+1}$ be the dual of $<'_1$. Colour the vertices 0 in $<'_1$ and $<'_{d+1}$, and 1 in all the other linear orders. Then, for any $x <'_1 y$, we have $y <'_{d+1} x$, so that each is less than the other coloured 0. Moreover, there exists $1 < i \leqslant d$ such that $y <'_i x$ (as $<_1 \cdots <_d$ is an antichain). So y is smaller than a x coloured 1. Assume x is not smaller than a y coloured 1. Then, y is never above $\{x,z\}$, contradicting the fact that $\{x,z\}$ is an edge. Thus, we have defined a bicolourable graph realizer of K_t of size $d+1$. Hence, $\operatorname{udim} K_t \leqslant \dim K_{t+1} + 1$. □

Remark 5. Let $p \geqslant 3$. If $\dim K_{p+1} > \dim K_p$, then $\dim K_{2(p-1)} = \operatorname{udim} K_{p-1} = \dim K_p + 1$.

The following strengthening of the upper bound of $\operatorname{udim} K_t$ has been proposed by the referee of this paper.

Proposition 3.7. *Let t be a positive integer. Then*

$$\operatorname{udim} K_t \leqslant \min\{r : D(r-2) \geqslant t+2\}, \tag{3.5}$$

where $D(n)$ is the nth Dedekind number. □

Proof. In [11], the authors prove that K_t has a realizer of type $(<_1, \overline{<_1}, <_3, \ldots, <_d)$ if and only if $D(d-2) \geqslant t$. Let $<_2 = \overline{<_1}$. Label the vertices of K_t with $1, \ldots, t$, according to the

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linear order $<_1$. Let $1 < x < y < t$. As $\{1, y\}$ is an edge of K_t, there exists $1 \leqslant i_x \leqslant d$, such that x is greater than 1 and y with respect to $<_{i_x}$. Hence $i_x > 2$. Similarly, as $\{x, n\}$ is an edge of K_t, there exists $2 < i_y \leqslant t$ such that y is greater than n and x with respect to $<_{i_y}$, and we have

$$\begin{cases} x <_1 y \text{ and } \Gamma(y,1) = 0, \\ x <_{i_y} y \text{ and } \Gamma(y,i_y) = 1, \\ y <_2 x \text{ and } \Gamma(x,2) = 0, \\ y <_{i_x} x \text{ and } \Gamma(x,i_x) = 1. \end{cases}$$

The restriction of $(<_1, \overline{<_1}, <_3, \ldots, <_d)$ to $\{2, \ldots, t-1\}$ thus defines with Γ a bicoloured realizer of K_{t-2}. □

The last part proof of Theorem 3.6 actually shows that every graph G has a vertex x such that $\operatorname{udim}(G - x) \leqslant \dim G + 1$. One may ask if such a deletion is actually necessary.

Problem 1. Let G be a graph with dimension at least 3. Is it true that $\operatorname{udim} G \leqslant \dim G + 1$?

Notice that the restriction $\dim G > 2$ is necessary, as $\dim K_2 = 2$ and $\operatorname{udim} K_2 = 4$. Moreover, according to the Schnyder's theorem and the 4-colour theorem, any graph of dimension 3 is planar and has universal dimension at most 4 (as $\operatorname{udim} K_4 = 4$). Thus, the answer is 'yes' for the special case where $\dim G = 3$.

4. Applications

4.1. Chromatic number

We shall now prove that graphs with universal dimension 4 have no bounded chromatic number. For this we will make use of the following Theorem of Kříž and Nešetřil.

Theorem 4.1 ([15]). *For each $k > 0$, there exists a finite set X and linear orderings $<_1, <_2$ on X such that the Hasse diagram H of $<_1 <_2$ has chromatic number at least k.*

Lemma 4.2. *Let $k \geqslant 2$ be an integer, X a finite set and $<_1, \ldots, <_k$ linear orders on X. For $I \subseteq [k]$, denote P_I the partial order $(\bigcap_{i \in I} <_i) \cap (\bigcap_{i \in [k] \setminus I} \overline{<_i})$, and $H(P_I)$ its Hasse diagram.*

Then, the graph G with vertex set X and edge set $\bigcup_{I \subseteq [k]} E(H(P_I))$ has $(<_1, \ldots, <_k, \overline{<_k}, \ldots, \overline{<_1})$ as graph realizer, and thus has universal dimension at most $2k$.

Proof. Define $<_{2k+1-i} = \overline{<_i}$, for $1 \leqslant i \leqslant k$ and $\mathscr{R} = (<_1, \ldots, <_{2k})$. Let $I \subseteq [k]$, let $\{x, y\}$ be an edge of $H(P_I)$ and let $z \notin \{x, y\}$. Without loss of generality we may assume that x is smaller than y with respect to P_I. Assume z is not greater than both x and y with respect to some $<_i$ $(1 \leqslant i \leqslant 2k)$. Then, z is also never smaller than both x and y with respect to some $<_i$ $(1 \leqslant i \leqslant 2k)$, as it would be greater than both x and y with respect to $<_{2k+1-i}$. Thus z lies between x and y in all the $<_i$. Hence $x < z < y$ with respect to P_I,

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contradicting $\{x, y\} \in H(P_I)$. We conclude that z is greater than x and y in some $<_i$, and thus that $\mathscr{R}$ is a graph realizer of G.

This graph realizer may be rewritten as $(<_1, \overline{<_1}, <_2, \ldots, <_k, \overline{<_2}, \ldots, \overline{<_k})$. As two vertices are not comparable in $<_1 \cap \overline{<_1}$ and not comparable in $\left(\bigcap_{i=2}^{k} <_i\right) \cap \left(\bigcap_{i=2}^{k} \overline{<_i}\right)$, Corollary 3.4 applies and thus $\operatorname{udim} G \leqslant 2k$. □

Corollary 4.3. *Let $P = (X, \leqslant)$ be a poset of dimension at least 2 and let $H(P)$ be its Hasse diagram. Then $H(P)$ has a graph realizer of the form $(<_1, \ldots, <_{\dim P}, \overline{<_1}, \ldots, \overline{<_{\dim P}})$ and thus* $\operatorname{udim}(H(P)) \leqslant 2 \dim P$.

Theorem 4.4. *For any integer $k \geqslant 2$, there exists a graph G_k with chromatic number at least k, universal dimension 4, and having a realizer of the form $(<_1, <_2, \overline{<_1}, \overline{<_2})$.*

Proof. The existence of a graph G_k with chromatic number at least k, universal dimension at most 4, and having a realizer of the form $(<_1, <_2, \overline{<_1}, \overline{<_2})$ is a direct consequence of Theorem 4.1 and Corollary 4.3. According to Remark 3, G_k has universal dimension at least 4 and thus universal dimension exactly 4. □

Corollary 4.5. *There exists no integer k, such that any graph G of dimension 4 having a realizer of the form $(<_1, <_2, \overline{<_1}, \overline{<_2})$ has a vertex partition into at most k sets $V(G) = V_1 \cup \cdots \cup V_k$, each inducing a planar graph.*

Proof. Assume $V(G)$ has a partition in k sets, each inducing a planar graph. Then $\chi(G) \leqslant 4k$. As graphs of dimension 4 having a realizer of the form $(<_1, <_2, \overline{<_1}, \overline{<_2})$ have no bounded chromatic number, the result follows. □

Felsner and Trotter proposed the following two conjectures, which are obviously disproved by Corollary 4.5.

Conjecture 4.6 ([11]). *Let $G = (V, E)$ be a graph. Then* $\dim G \leqslant 4$ *if and only if the vertex set V can be partitioned into four parts so that each part induces an outerplanar graph.*

Conjecture 4.7 ([11]). *Let $G = (V, E)$ be a graph. Then* $\dim G \leqslant 4$ *and G has a realizer of the form $(<_1, <_2, <_3, \overline{<_3})$ if and only if the vertex set V can be partitioned into two parts so that each part induces an outerplanar graph.*

Problem 2. Does there exist a graph G of universal dimension 4 having no graph realizer of the form $(<_1, <_2, \overline{<_1}, \overline{<_2})$?

4.2. Extremal graphs

An (r, s)-*colouring* of a graph G is a colouring of the vertices of G such that each of the first s ones induces a complete graphs and each of the $r - s$ remaining ones induces an empty graph. The *colouring number* $r(\mathscr{P})$ is the maximal r for which there exists an s, $0 \leqslant s \leqslant r$, such that every (r, s)-colourable graph has property $\mathscr{P}$ (see [4]).

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The speed $|(\dim \leqslant d)^n|$ of the property $(\dim \leqslant d)$ is the number of graphs of order n having dimension at most d. These numbers are related to the colouring number of the property $(\dim \leqslant d)$ by the following theorem.

Theorem 4.8 (Bollobás and Thomason [5]). *Let $\mathscr{P}$ be a hereditary property of graphs and let $\mathscr{P}^n$ be the set of graphs in $\mathscr{P}$ with vertex set $[n]$. Then*

$$|\mathscr{P}^n| = 2^{(1-1/r+o(1))n^2/2},$$

where $r = r(\mathscr{P})$ is the colouring number of $\mathscr{P}$.

The colouring number of the property $(\dim \leqslant d)$ (and hence its speed) is given by the following simple formula.

Theorem 4.9. *Let $d \geqslant 4$ be an integer, and let t be the greatest integer such that* $\operatorname{udim} K_t = d$. *Then the colouring number of the property* $(\dim \leqslant d)$ *is*

$$r(\dim \leqslant d) = t.$$

Thus, the number of labelled graphs of order n having dimension at most d is $2^{(1-1/t+o(1))n^2/2}$.

Proof. Any $(t,0)$-colourable graph is t-chromatic and thus has dimension at most d (as $\operatorname{udim} K_t = d$). Hence, $r(\dim \leqslant d) \geqslant t$.

There exists a $(t+1)$-chromatic graph with dimension strictly greater than d, as $\operatorname{udim} K_{t+1} > d$. Thus, there exists a $(t+1,0)$ colourable graph with universal dimension strictly greater than d. Moreover, as complete graphs are $(t+1,s)$-colourable for any $s \geqslant 1$ but have unbounded dimension, we deduce $r(\dim \leqslant d) < t+1$. □

The size $e_{\dim \leqslant d}(n)$ (resp.$e_{\operatorname{udim} \leqslant d}(n)$) of the property $(\dim \leqslant d)$ (resp. $(\operatorname{udim} \leqslant d)$) is the maximum number of edges a graph G can have when it has n vertices and dimension at most d (resp. universal dimension at most d). It is proved in [1] that $\lim_{p\to\infty} e_{\dim \leqslant 4}(n)/n^2 = 3/8$.

We shall show that $e_{\dim \leqslant d}(n)$ is closely related to the universal dimension of complete graphs.

Theorem 4.10. *Let $d \geqslant 4$ be an integer, and let t be the greatest integer such that* $\operatorname{udim} K_t = d$. *Then:*

$$\left(1-\frac{1}{t}\right)\binom{n}{2} + O(n) \leqslant e_{\operatorname{udim} \leqslant d}(n) \leqslant e_{\dim \leqslant d}(n) \leqslant \left(1-\frac{1}{t}\right)\binom{n}{2} + o(n^2).$$

Proof. As $\operatorname{udim}(K_{t+1}) > d$, the graph $G = K_{t+1}^+$, defined in Theorem 3.3, is such that $\dim G > d$.

As G is $(t+1)$-chromatic, the maximum number of edges possessed by a graph of order n not having G as a subgraph is, according to the Erdős–Stone theorem [10],

$$\operatorname{ex}(n;G) = \left(1-\frac{1}{t}\right)\frac{p^2}{2} + o(n^2).$$

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Table 1. Order of complete graphs which are extremal for dimension and universal dimension, respectively.

t	$\max\{n : \dim K_n \leqslant t\}$	$\max\{n : \operatorname{udim} K_n \leqslant t\}$
4	12	4
5	81	between 30 and 38
6	2646	between 1088 and 1320
7	1422564	between 583507 and 711273

Hence, $e_{\dim\leqslant d}(n) \leqslant \operatorname{ex}(n; G)$. As $\dim G \leqslant \operatorname{udim} G$, $e_{\operatorname{udim}\leqslant d}(n) \leqslant e_{\dim\leqslant d}(n)$.

Moreover, the complete balanced t-partite graph is t-colourable and has universal dimension at most d. This graph has $\left(1 - \frac{1}{t}\right)\binom{n}{2} + O(n)$ edges, thus concluding the proof. □

The upper bounds appearing in Table 1 result from Theorem 3.6.

Using a technique similar to that of [13] and selecting pairs of consecutive antichains (in lexicographic order) which form a cover, we build a realizer of K_{2n}, where vertices $2k-1$ and $2k$ $(1 \leqslant k \leqslant n)$ are consecutive in each of the linear orders. This enumeration technique allows us to build a bicoloured graph realizer of K_n, leading to the lower bounds shown in Table 1 (notice that the validity of the computed bicoloured graph realizer has been checked by computer). For instance, as evidence for $\operatorname{udim} K_{30} = 5$, we reproduce here the computed 5-balanced bicolouration of a graph realizer of K_{30}:

$<_1$: 17 15 16 5 1 2 3 4 7 6 12 8 9 10 11 14 13 21 22 18 19 20 28 27 25 23 24 26 29 30
$<_2$: 20 18 19 21 22 6 7 2 1 4 3 5 13 14 9 8 11 10 12 15 16 17 30 29 26 24 23 25 27 28
$<_3$: 26 25 23 24 28 27 30 29 7 6 5 3 4 1 2 14 13 12 10 11 8 9 17 16 15 20 19 18 22 21
$<_4$: 29 30 27 28 23 24 25 26 21 22 18 19 20 15 16 17 8 9 10 11 12 13 14 1 2 3 4 5 6 7
$<_5$: 30 29 28 27 26 25 24 23 22 21 20 19 18 17 16 15 14 13 12 11 10 9 8 7 6 5 4 3 2 1

5. Conclusion

The universal dimension of a graph appears to be a nontrivial connection between poset dimensions and graph homomorphisms. In particular, for $d \geqslant 4$, no finite family $\mathscr{F}$ of finite graphs exists for a characterization like $G \to H \in \mathscr{F} \iff \operatorname{udim} G \leqslant d$. However, it is possible that such a characterization could exist for a fixed chromatic number.

Problem 3. For any $k \geqslant 2$ and any $d \geqslant 4$, does there exist a finite k-chromatic graph $U_{d,k}$ of universal dimension d such that any k-chromatic graph of universal dimension d has a homomorphism to $U_{d,k}$?

An additive hereditary property is *reducible* if there exists two hereditary properties $\mathscr{P}_1$ and $\mathscr{P}_2$, such that a graph G belongs to $\mathscr{P}$ if and only if there exists a partition (V_1, V_2) of G, so that $G[V_1] \in \mathscr{P}_1$ and $G[V_2] \in \mathscr{P}_2$ (see [14]). The question of determining whether or not (udim $\leqslant d$) is reducible leads to the following question.

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Problem 4. Let G be an extremal graph of universal dimension d (*i.e.*, $\operatorname{udim} G = d$, but the addition of any edge to G increases the universal dimension). Then, is the complementary graph of G disconnected?

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