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SMALL-BOUND ISOMORPHISMS OF FUNCTION SPACES

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Abstract

Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. For $i = 1, 2$, let K_i be a locally compact (Hausdorff) topological space and let $\mathcal{H}_i$ be a closed subspace of $C_0(K_i, \mathbb{F})$ such that each point of the Choquet boundary $\mathrm{Ch}_{\mathcal{H}_i} K_i$ of $\mathcal{H}_i$ is a weak peak point. We show that if there exists an isomorphism $T\colon \mathcal{H}_1 \to \mathcal{H}_2$ with $\|T\| \cdot \|T^{-1}\| < 2$, then $\mathrm{Ch}_{\mathcal{H}_1} K_1$ is homeomorphic to $\mathrm{Ch}_{\mathcal{H}_2} K_2$. We then provide a one-sided version of this result. Finally we prove that under the assumption on weak peak points the Choquet boundaries have the same cardinality provided $\mathcal{H}_1$ is isomorphic to $\mathcal{H}_2$.

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1. Introduction

We work within the framework of real or complex vector spaces and write $\mathbb{F}$ for the respective field $\mathbb{R}$ or $\mathbb{C}$. All normed linear spaces are tacitly assumed to be of dimension at least 1. Further, let $S_{\mathbb{F}}$ stand for the set $\{\lambda \in \mathbb{F};\ |\lambda| = 1\}$. For a compact (Hausdorff) space K, let $C(K, \mathbb{F})$ stand for the space of all continuous $\mathbb{F}$-valued functions on K, and for a locally compact (Hausdorff) space K, let $C_0(K, \mathbb{F})$ denote the space of all continuous $\mathbb{F}$-valued functions vanishing at infinity. We consider both these spaces endowed with the supremum norm. For a compact space K, we identify the dual space $(C(K, \mathbb{F}))^*$ with the space $\mathcal{M}(K, \mathbb{F})$ of all $\mathbb{F}$-valued Radon measures on K. Unless stated otherwise, we consider $\mathcal{M}(K, \mathbb{F})$ endowed with the weak* topology given by this duality. We further write $\mathcal{M}^1(K)$ for probability Radon measures on K and $\mathcal{M}^+(K)$ for positive Radon measures on K. Let ε_x stand for the Dirac measure at the point $x \in K$.

We start with the classical Banach–Stone theorem asserting that, given a pair of compact spaces K and L, they are homeomorphic provided $C(K, \mathbb{F})$ is isometric to $C(L, \mathbb{F})$ (see [15, Theorem 3.117]).

A remarkable generalization of the Banach–Stone theorem was given by Amir [3] and Cambern [7]. They showed that compact spaces K and S are homeomorphic if

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there exists an isomorphism $T\colon C(K,\mathbb{F}) \to C(S,\mathbb{F})$ with $\|T\| \cdot \|T^{-1}\| < 2$. Alternative proofs were given by Cohen [12] and Drewnowski [14].

The latest result in the direction of the Amir–Cambern theorem is due to Galego and Porto da Silva in [19] who proved the following theorem. If T is a function from $C_0(K,\mathbb{R})$ to $C_0(S,\mathbb{R})$, $T(0) = 0$, and both T and T^{-1} are bijective coarse $(M,1)$-quasi-isometries with $M < \sqrt{2}$, then K and S are homeomorphic and there exist a homeomorphism ϕ from S to K and a continuous function $\lambda\colon S \to \{-1,1\}$ such that for all $s \in S$ and $f \in C_0(K,\mathbb{R})$,

$$\|MTf(s) - \lambda(s)f(\phi(s))\| \le (M^2-1)\|f\| + \delta,$$

where δ does not depend on f or s.

Chu and Cohen provided in [10] a very nice generalization of the Amir–Cambern theorem in the context of affine continuous functions on compact convex sets. In order to explain their results we need a bit of terminology. By a compact convex set we mean a compact convex subset of a locally convex (Hausdorff) space. Let $\mathfrak{A}(X,\mathbb{F})$ be the space of all continuous $\mathbb{F}$-valued affine functions on a compact convex set X endowed with the supremum norm. If X is a compact convex set, for any $\mu \in \mathcal{M}^1(X)$ there exists a unique point $r(\mu) \in X$ such that $\mu(a) = a(r(\mu))$, $a \in \mathfrak{A}(X,\mathbb{C})$ (see [2, Proposition I.2.1]). We call $r(\mu)$ the *barycenter* of μ. If $\mu,\nu \in \mathcal{M}^+(X)$, then $\mu \prec \nu$ if $\mu(k) \le \nu(k)$ for each convex continuous function k on X. A measure $\mu \in \mathcal{M}^+(X)$ is *maximal* if μ is $\prec$-maximal.

By the Choquet–Bishop–de Leeuw representation theorem (see [2, Theorem I.4.8]), for each $x \in X$ there exists a maximal measure $\mu \in \mathcal{M}^1(X)$ with $r(\mu) = x$. If this measure is uniquely determined for each $x \in X$, the set X is called a *simplex*. It is called a *Bauer simplex* if, moreover, the set $\operatorname{ext} X$ of extreme points of X is closed. In this case, the space $\mathfrak{A}(X,\mathbb{F})$ is isometric to the space $C(\operatorname{ext} X,\mathbb{F})$ (see [2, Theorem II.4.3]). On the other hand, given a space $C(K,\mathbb{F})$, it is isometric to the space $\mathfrak{A}(\mathcal{M}^1(K),\mathbb{F})$ ([2, Corollary II.4.2]).

A reformulation of the result of Amir and Cambern for simplexes reads as follows. Given Bauer simplexes X and Y, the sets $\operatorname{ext} X$ and $\operatorname{ext} Y$ are homeomorphic, provided there exists an isomorphism $T\colon \mathfrak{A}(X,\mathbb{F}) \to \mathfrak{A}(Y,\mathbb{F})$ with $\|T\| \cdot \|T^{-1}\| < 2$.

The aforementioned Chu and Cohen proved in [10] that for compact convex sets X and Y, the sets $\operatorname{ext} X$ and $\operatorname{ext} Y$ are homeomorphic provided there exists an isomorphism $T\colon \mathfrak{A}(X,\mathbb{R}) \to \mathfrak{A}(Y,\mathbb{R})$ with $\|T\| \cdot \|T^{-1}\| < 2$ and one of the following conditions holds:

(i) X and Y are simplexes such that their extreme points are weak peak points;

(ii) X and Y are metrizable and their extreme points are weak peak points;

(iii) $\operatorname{ext} X$ and $\operatorname{ext} Y$ are closed and extreme points if X and Y are split faces.

A point $x \in X$ is a *weak peak point* if, given $\varepsilon \in (0,1)$ and an open set $U \subset X$ containing x, there exists a in the unit ball $B_{\mathfrak{A}(X,\mathbb{R})}$ of $\mathfrak{A}(X,\mathbb{R})$ such that $|a| < \varepsilon$ on $\operatorname{ext} X \setminus U$ and $a(x) > 1-\varepsilon$ (see [10, p. 73]).

In [29], it was shown that extreme points of X and Y are homeomorphic, provided there exists an isomorphism $T\colon \mathfrak{A}(X,\mathbb{R}) \to \mathfrak{A}(Y,\mathbb{R})$ with $\|T\| \cdot \|T^{-1}\| < 2$, extreme points are weak peak points and both $\operatorname{ext} X$ and $\operatorname{ext} Y$ are Lindelöf sets.

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In [13] the same result is proved without the assumption of the Lindelöf property, and in [31] the analogous result for the case of complex functions is provided. It turns out that this result is in a sense optimal since the bound 2 cannot be improved (see [11], where a pair of nonhomeomorphic compact spaces K_1, K_2 for which there exists an isomorphism $T\colon C(K_1,\mathbb{R}) \to C(K_2,\mathbb{R})$ with $\|T\|\cdot\|T^{-1}\| = 2$ is constructed) and the assumption on weak peak points cannot be omitted (see [20], where the author constructs for each $\varepsilon \in (0,1)$ a pair of simplexes X_1, X_2 such that $\operatorname{ext} X_1$ is not homeomorphic to $\operatorname{ext} X_2$ but there is an isomorphism $T\colon \mathfrak{A}(X_1,\mathbb{R}) \to \mathfrak{A}(X_2,\mathbb{R})$ with $\|T\|\cdot\|T^{-1}\| < 1+\varepsilon$).

As a corollary of the theorems for affine functions, results on self-adjoint function spaces were obtained in [31]. More precisely, given a pair of self-adjoint closed spaces $\mathcal{H}_i \subset C(K,\mathbb{C})$ containing constant functions and separating points of K_i, $i = 1,2$, their Choquet boundaries are homeomorphic provided points in the Choquet boundaries are weak peak points and there exists an isomorphism $T\colon \mathcal{H}_1 \to \mathcal{H}_2$ with $\|T\|\cdot\|T^{-1}\| < 2$ (see [31, Theorem 5.3]).

The aim of the present paper is to extend this result to the case of locally compact spaces and general function spaces. So we need only assume that, for $i = 1,2$, $\mathcal{H}_i$ is a closed subspace of $C_0(K_i,\mathbb{F})$ for some locally compact space K_i such that each point of the Choquet boundary $\operatorname{Ch}_{\mathcal{H}_i} K_i$ is a weak peak point. We recall that $x \in K_i$ is a *weak peak point* if for a given $\varepsilon \in (0,1)$ and a neighborhood U of x there exists a function $h \in B_{\mathcal{H}_i}$ such that $h(x) > 1-\varepsilon$ and $|h| < \varepsilon$ on $\operatorname{Ch}_{\mathcal{H}_i} K_i \setminus U$.

We show that if there exists an isomorphism $T\colon \mathcal{H}_1 \to \mathcal{H}_2$ with $\|T\|\cdot\|T^{-1}\| < 2$, the Choquet boundaries of K_1 and K_2 are homeomorphic. (We recall that, given a closed subspace $\mathcal{H} \subset C_0(K,\mathbb{F})$ for some locally compact space K, a point $x \in K$ is in the Choquet boundary $\operatorname{Ch}_{\mathcal{H}} K$ if the point evaluation functional $\phi(x)$ defined as $\phi(x)(h) = h(x)$, $h \in \mathcal{H}$, is an extreme point of $B_{\mathcal{H}^*}$.)

Thus the main result of our paper is the following theorem.

THEOREM 1.1. *For $i = 1,2$, let $\mathcal{H}_i$ be a closed subspace of $C_0(K_i,\mathbb{F})$ for some locally compact space K_i. Assume that each point of $\operatorname{Ch}_{\mathcal{H}_i} K_i$ is a weak peak point and let $T\colon \mathcal{H}_1 \to \mathcal{H}_2$ be an isomorphism satisfying $\|T\|\cdot\|T^{-1}\| < 2$. Then $\operatorname{Ch}_{\mathcal{H}_1} K_1$ is homeomorphic to $\operatorname{Ch}_{\mathcal{H}_2} K_2$.*

As mentioned above, the assumption of weak peak points cannot be omitted (see [20]), and the bound 2 is optimal (see [11]), so Theorem 1.1 gives in this sense a final result for scalar-valued function spaces. The key ingredient which made it possible to remove all the additional assumptions on the function spaces is the maximum principle for affine functions of the first Borel class (see Lemma 2.9).

We refer the reader to [25, 27] for results on function algebras in the spirit of the above theorem. The case of the vector-valued Banach–Stone type theorem is treated, for example, in [1, 5, 6, 8, 17, 26].

The next result considers isomorphisms that are not generally surjective. Jarosz showed in [24] that if K_1, K_2 are locally compact spaces, $A \subset C_0(K_1,\mathbb{C})$ is an extremely regular closed subspace and a not necessarily surjective isomorphism

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$T : A \to C_0(K_2, \mathbb{C})$ satisfies $\|T\| \cdot \|T^{-1}\| < 2$, then K_1 is a continuous image of a subset of K_2. An analogous result for function spaces reads as follows.

THEOREM 1.2. *For $i = 1, 2$, let $\mathcal{H}_i$ be a closed subspace of $C_0(K_i, \mathbb{F})$ for some locally compact space K_i. Assume that each point of $\mathrm{Ch}_{\mathcal{H}_1} K_1$ is a weak peak point and let $T : \mathcal{H}_1 \to \mathcal{H}_2$ be an into isomorphism satisfying $\|T\| \cdot \|T^{-1}\| < 2$. Then there exist a set $L \subset \mathrm{Ch}_{\mathcal{H}_2} K_2$ and a continuous surjective mapping $\varphi : L \to \mathrm{Ch}_{\mathcal{H}_1} K_1$.*

In [9], it is proved that locally compact spaces K_1, K_2 have the same cardinality provided $C_0(K_1, \mathbb{F})$ is isomorphic to $C_0(K_2, \mathbb{F})$. We generalize this result to the context of function spaces by proving the following theorem.

THEOREM 1.3. *For $i = 1, 2$, let $\mathcal{H}_i$ be a closed subspace of $C_0(K_i, \mathbb{F})$ for some locally compact space K_i. Assume that each point of $\mathrm{Ch}_{\mathcal{H}_i} K_i$ is a weak peak point and let $T : \mathcal{H}_1 \to \mathcal{H}_2$ be an isomorphism. Then the cardinality of $\mathrm{Ch}_{\mathcal{H}_1} K_1$ is equal to the cardinality of $\mathrm{Ch}_{\mathcal{H}_2} K_2$.*

It turns out that we can use the basic strategy of the proofs in [31] (which in turn are adapted from [10]); however, some adjustments have to be made. The outcome of these adjustments is even some simplification of the methods used in [31]. More precisely, we work directly with functions in $\mathcal{M}(K, \mathbb{F})^*$ instead of $\mathcal{H}^{**}$ (see Lemmas 2.6 and 2.7). This allows us to use the standard decomposition of measures and thus we can avoid, for example, [31, Lemma 2.5]. Also, the construction of a 'peaking' function a_x^{**} in Lemma 2.6 is simpler than that in [31, Lemma 2.4].

2. Auxiliary results

This section collects auxiliary lemmas needed for the proofs of the main results.

LEMMA 2.1. *Let $\mathcal{H}$ be a closed subspace of $C(K, \mathbb{F})$ for some compact space K and $\phi : K \to B_{\mathcal{H}^*}$ be the evaluation mapping. Then*

$$\operatorname{ext} B_{\mathcal{H}^*} = S_{\mathbb{F}} \cdot \phi(\mathrm{Ch}_{\mathcal{H}} K).$$

PROOF. By [16, Corollary 2.3.6],

$$\operatorname{ext} B_{\mathcal{H}^*} \subset \{\lambda\phi(x);\ |\lambda| = 1, x \in K\}.$$

Moreover, it is easy to see that if $s \in B_{\mathcal{H}^*}$ and $\lambda \in S_{\mathbb{F}}$, then $s \in \operatorname{ext} B_{\mathcal{H}^*}$ if and only in $\lambda s \in \operatorname{ext} B_{\mathcal{H}^*}$.

Thus if $s \in \operatorname{ext} B_{\mathcal{H}^*}$, then $s = \lambda\phi(x)$ for some $\lambda \in \mathbb{F}$ with $|\lambda| = 1$ and $x \in K$, and by the observation above it follows that $\phi(x) \in \operatorname{ext} B_{\mathcal{H}^*}$. On the other hand, each point $\lambda\phi(x)$, where $\lambda \in S_{\mathbb{F}}$ and $x \in \mathrm{Ch}_{\mathcal{H}} K$, is an extreme point of $B_{\mathcal{H}^*}$, again by the previous observation. □

The following lemma is a very particular result on representing functionals by means of measures carried by the Choquet boundary. We refer the reader to [4, 18, 21, 23, 32, 33] for related results.

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LEMMA 2.2. *Let $\mathcal{H}$ be a subspace of $C(K,\mathbb{F})$ for some compact space K. Then for any $s \in \mathcal{H}^*$ there exists a measure $\mu \in \mathcal{M}(\overline{\mathrm{Ch}_{\mathcal{H}} K},\mathbb{F})$ such that $\mu = s$ on $\mathcal{H}$ and $\|\mu\| = \|s\|$.*

PROOF. Let $s \in \mathcal{H}^*$ be given. We write $\mathcal{A} \subset C(\overline{\mathrm{Ch}_{\mathcal{H}} K},\mathbb{F})$ for the space $\{h|_{\overline{\mathrm{Ch}_{\mathcal{H}} K}};\ h \in \mathcal{H}\}$. By [16, Theorem 2.3.8], for each $h \in \mathcal{H}$ there exists $x \in \mathrm{Ch}_{\mathcal{H}} K$ such that $|h(x)| = \|h\|$. Thus the restriction mapping $r\colon \mathcal{H} \to \mathcal{A}$ given by $r(h) = h|_{\overline{\mathrm{Ch}_{\mathcal{H}} K}}$ is an isometric isomorphism and we can define $t \in \mathcal{A}^*$ by $t = s \circ r^{-1}$. Then $\|t\| = \|s\|$. Using the Hahn–Banach theorem, we find a measure

$$\mu \in (C(\overline{\mathrm{Ch}_{\mathcal{H}} K},\mathbb{F}))^* = \mathcal{M}(\overline{\mathrm{Ch}_{\mathcal{H}} K},\mathbb{F})$$

such that $\|\mu\| = \|t\|$ and $t = \mu$ on $\mathcal{A}$. Then $\|\mu\| = \|s\|$ and

$$\mu(h) = \int_{\overline{\mathrm{Ch}_{\mathcal{H}} K}} h\,d\mu = t(h|_{\overline{\mathrm{Ch}_{\mathcal{H}} K}}) = s(h), \quad h \in \mathcal{H}.$$

This finishes the proof. □

An important topological notion is that of a function of the first Borel class. Thus we recall that, given a pair of topological spaces K, L, a function $f\colon K \to L$ is *of the first Borel class* if $f^{-1}(U)$ is a countable union of differences of closed sets in K for any $U \subset L$ open (see [34] or [30, Definition 5.13]). We just mention that, if $L = \mathbb{R}$, any semicontinuous function $f\colon K \to \mathbb{R}$ is of the first Borel class.

LEMMA 2.3. *Let $L_1, \dots, L_n$ be compact convex sets in a locally convex space and $L = \mathrm{co}(L_1 \cup \cdots \cup L_n)$. Let $f\colon L \to \mathbb{F}$ be an affine function such that $f|_{L_i}$ is of the first Borel class for each $i = 1, \dots, n$. Then f is of the first Borel class on L.*

PROOF. Let

$$\Delta = \left\{\lambda \in [0,\infty)^n;\ \sum_{i=1}^{n} \lambda_i = 1\right\}$$

and

$$H = L_1 \times \cdots \times L_n \times \Delta.$$

Furthermore, let $g\colon H \to \mathbb{F}$ be defined as

$$g(x_1, \dots, x_n, \lambda) = \sum_{i=1}^{n} \lambda_i f(x_i), \quad (x_1, \dots, x_n, \lambda) \in H.$$

By the proof of [30, Theorem 5.10], functions of the first Borel class form a linear space and a product of a continuous function and a function of the first Borel class is again of the first Borel class. From this it follows that g is of the first Borel class on H. We consider a continuous surjection $\varphi\colon H \to L$ defined as

$$\varphi(x_1, \dots, x_n, \lambda) = \sum_{i=1}^{n} \lambda_i x_i, \quad (x_1, \dots, x_n, \lambda) \in H.$$

Since f is affine on L, we obtain $f \circ \varphi = g$. By [30, Theorem 5.16], f is of the first Borel class on L. □

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LEMMA 2.4. *Let K be a compact space and $f\colon K \to \mathbb{F}$ be a bounded function of the first Borel class. Then $\widehat{f}\colon \mathcal{M}(K,\mathbb{F}) \to \mathbb{F}$, defined as*

$$\widehat{f}(\mu) = \mu(f), \quad \mu \in \mathcal{M}(K,\mathbb{F}),$$

is of the first Borel class on any ball $rB_{\mathcal{M}(K,\mathbb{F})}$, $r > 0$.

PROOF. We provide a proof for the case of complex measures; the easier case of real measures would be done similarly. Assume first that f is real. By [30, Proposition 5.30], $\widehat{f}$ is of the first Borel class on $\mathcal{M}^1(K)$. Let $L_1 = 2\mathcal{M}^1(K)$, $L_2 = -2\mathcal{M}^1(K)$, $L_3 = 2i\mathcal{M}^1(K)$ and $L_4 = -2i\mathcal{M}^1(K)$. Since L_i is affinely homeomorphic to $\mathcal{M}^1(K)$ and $\widehat{f}$ is linear, $\widehat{f}$ is of the first Borel class on each L_i, $i = 1,\dots,4$. By the decomposition of a complex measure, we obtain

$$B_{\mathcal{M}(K,\mathbb{F})} \subset L = \operatorname{co}(L_1 \cup L_2 \cup L_3 \cup L_4).$$

By linearity, it is enough to prove that $\widehat{f}$ is of the first Borel class on L. But this follows from Lemma 2.3.

If $f\colon K \to \mathbb{C}$, we have $f = f_1 + if_2$, where f_1, f_2 are real bounded functions of the first Borel class. Then the function

$$\widehat{f}(\mu) = \mu(f) = \mu(f_1) + i\mu(f_2) = \widehat{f_1}(\mu) + i\widehat{f_2}(\mu), \quad \mu \in B_{\mathcal{M}(K,\mathbb{F})},$$

is of the first Borel class as well. (Indeed, since the functions $\mu \mapsto \widehat{f_1}$ and $\mu \mapsto i\widehat{f_2}(\mu)$ are of the first Borel class, their sum is easily seen to be of the first Borel class as well (see, for example, the proof of [30, Theorem 5.10]). □

LEMMA 2.5. *Let $\mathcal{H}$ be a closed subspace of $C(K,\mathbb{F})$ for some compact space K. Let $x \in K$ be a weak peak point. Then, for any*

$$\mu \in \mathcal{M}(\overline{\operatorname{Ch}_{\mathcal{H}} K},\mathbb{F}) \cap \mathcal{H}^\perp,$$

we have $\mu(\chi_{\{x\}}) = 0$.

PROOF. Let $\mu \in \mathcal{M}(\overline{\operatorname{Ch}_{\mathcal{H}} K},\mathbb{F}) \cap \mathcal{H}^\perp$ be arbitrary and $\varepsilon > 0$ be given. We write $\mu = \lambda\varepsilon_x + \nu$, where $\nu(\{x\}) = 0$. Let U be a closed neighborhood of x such that $|\nu|(U) < \varepsilon$. Using the assumption, we find $h \in B_{\mathcal{H}}$ with $h(x) > 1 - \varepsilon$ and $|h| < \varepsilon$ on $\operatorname{Ch}_{\mathcal{H}} K \setminus U$. Then $|h| \le \varepsilon$ on

$$\overline{\operatorname{Ch}_{\mathcal{H}} K} \setminus U \subset \overline{\operatorname{Ch}_{\mathcal{H}} K \setminus U},$$

and thus

$$\begin{aligned} |\mu(\chi_{\{x\}})| &= |\mu(\chi_{\{x\}}) - \mu(h)| \le |\lambda\varepsilon_x(\chi_{\{x\}} - h)| + |\nu(h)| \\ &\le |\lambda|(1 - h(x)) + \int_{\overline{\operatorname{Ch}_{\mathcal{H}} K} \cap U} |h|d|\nu| + \int_{\overline{\operatorname{Ch}_{\mathcal{H}} K} \setminus U} |h|d|\nu| \\ &\le |\lambda|\varepsilon + \varepsilon + \varepsilon\|\nu\| \le \varepsilon(1 + \|\mu\|). \end{aligned}$$

Hence $\mu(\chi_{\{x\}}) = 0$. □

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LEMMA 2.6. *Let $\mathcal{H}$ be a closed subspace of $C(K,\mathbb{F})$ for some compact space K and let $\pi\colon \mathcal{M}(\overline{\mathrm{Ch}_{\mathcal{H}} K},\mathbb{F}) \to \mathcal{H}^*$ be the restriction mapping. Let $x \in \mathrm{Ch}_{\mathcal{H}} K$ be a weak peak point. For each $\mu \in \mathcal{M}(\overline{\mathrm{Ch}_{\mathcal{H}} K},\mathbb{F})$ we define $\widehat{\chi_{\{x\}}}(\mu) = \mu(\chi_{\{x\}})$. Then there exists $a_x^{**} \in \mathcal{H}^{**}$ such that $a_x^{**} \circ \pi = \widehat{\chi_{\{x\}}}$ and $\|a_x^{**}\| = 1$.*

PROOF. The element $\widehat{\chi_{\{x\}}}$ is contained in $(\mathcal{M}(\overline{\mathrm{Ch}_{\mathcal{H}} K},\mathbb{F}))^*$. In order to find the required element $a_x^{**} \in \mathcal{H}^{**}$ it is enough to realize that for any $\mu \in \mathcal{M}(\overline{\mathrm{Ch}_{\mathcal{H}} K},\mathbb{F}) \cap \mathcal{H}^{\perp}$ we have $\mu(\chi_{\{x\}}) = 0$ (see Lemma 2.5). Thus we can define

$$a_x^{**}(s) = \widehat{\chi_{\{x\}}}(\mu), \quad \mu \in \pi^{-1}(s),\ s \in \mathcal{H}^*.$$

To prove that $\|a_x^{**}\| = 1$, for a given $s \in \mathcal{H}^*$ we use Lemma 2.2 to find a measure $\mu \in \mathcal{M}(\overline{\mathrm{Ch}_{\mathcal{H}} K},\mathbb{F})$ with $\|\mu\| = \|s\|$ and $\pi(\mu) = s$. Then

$$|a_x^{**}(s)| = |\widehat{\chi_{\{x\}}}(\mu)| \le \|\widehat{\chi_{\{x\}}}\|\|\mu\| = \|s\|.$$

Thus $\|a_x^{**}\| \le 1$. On the other hand, the Dirac measure ε_x is carried by $\overline{\mathrm{Ch}_{\mathcal{H}} K}$, thus, by the definition of a_x^{**},

$$|a_x^{**}(\pi(\varepsilon_x))| = |\widehat{\chi_{\{x\}}}(\varepsilon_x)| = 1.$$

This finishes the proof. □

Given a Banach space E with its dual E^*, we write $\langle\cdot,\cdot\rangle\colon E \times E^* \to \mathbb{F}$ for the duality mapping.

LEMMA 2.7. *Let $\mathcal{H}$ be a closed subspace of $C(K,\mathbb{F})$ for some compact space K and $\pi\colon \mathcal{M}(\overline{\mathrm{Ch}_{\mathcal{H}} K},\mathbb{F}) \to \mathcal{H}^*$ be the restriction mapping. Let $\widehat{f} \in (\mathcal{M}(\overline{\mathrm{Ch}_{\mathcal{H}} K},\mathbb{F}))^*$ and $a^{**} \in \mathcal{H}^{**}$ satisfy $\widehat{f} = a^{**} \circ \pi$.*

(a) *Then for any $s \in \mathcal{H}^*$ and $\mu \in \pi^{-1}(s)$,*

$$\langle s, a^{**}\rangle = \langle \mu, \widehat{f}\rangle.$$

(b) *For any $r > 0$, if $\widehat{f}$ is of the first Borel class on $rB_{\mathcal{M}(\overline{\mathrm{Ch}_{\mathcal{H}} K},\mathbb{F})}$, then a^{**} is of the first Borel class on $(rB_{\mathcal{H}^*}, \text{weak}^*)$.*

PROOF. (a) Given $s \in \mathcal{H}^*$ and $\mu \in \pi^{-1}(s)$, we realize that the dual mapping $\pi^*\colon \mathcal{H}^{**} \to (\mathcal{M}(\overline{\mathrm{Ch}_{\mathcal{H}} K},\mathbb{F}))^*$ satisfies $\pi^*(a^{**}) = a^{**} \circ \pi = \widehat{f}$. Thus

$$\langle s, a^{**}\rangle = \langle \pi(\mu), a^{**}\rangle = \langle \mu, \pi^*(a^{**})\rangle = \langle \mu, \widehat{f}\rangle.$$

(b) For any $r > 0$, the mapping $\pi\colon rB_{\mathcal{M}(\overline{\mathrm{Ch}_{\mathcal{H}} K},\mathbb{F})} \to rB_{\mathcal{H}^*}$ is a weak*-weak* continuous surjection (see Lemma 2.2). By [22, Theorem 10] (see also [30, Theorem 5.26(d)]), if $\widehat{f}$ is of the first Borel class on $rB_{\mathcal{M}(\overline{\mathrm{Ch}_{\mathcal{H}} K},\mathbb{F})}$, then a^{**} is of the first Borel class on $rB_{\mathcal{H}^*}$. □

LEMMA 2.8. *Let $\mathcal{A}$ be a closed subspace of $C_0(K,\mathbb{F})$ for some locally compact space K. Let $J = K \cup \{\alpha\}$ be the one-point compactification of K, where α is the point at infinity. Let*

$$\mathcal{B} = \{g \in C(J,\mathbb{F});\ g|_K \in \mathcal{A} \ \&\ g(\alpha) = 0\}.$$

Then $\mathcal{B}$ is a closed subspace of $C(J,\mathbb{F})$ isometric to $\mathcal{A}$ such that $\mathrm{Ch}_{\mathcal{A}} K = \mathrm{Ch}_{\mathcal{B}} J$.

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Proof. Clearly, any function $a \in \mathcal{A}$ has a unique extension $b_a \in \mathcal{B}$ and the mapping $\beta : a \mapsto b_a$ is an isometric isomorphism. Thus $\mathcal{B}$ is a closed subspace of $C(J, \mathbb{F})$.

Further, let $\phi_K \colon K \to \mathcal{A}^*$ and $\phi_J \colon J \to \mathcal{B}^*$ be the evaluation mappings, and we denote by ι the canonical inclusion of K into J. Then $\phi_K = \beta^* \circ \phi_J \circ \iota$. Since β^* is an isometric isomorphism, it preserves the extreme points of the unit ball. Hence, for a point $x \in K$,

$$x \in \mathrm{Ch}_{\mathcal{A}} K \Leftrightarrow \phi_K(x) \in \operatorname{ext} B_{\mathcal{A}^*} \Leftrightarrow \phi_J(\iota(x)) \in \operatorname{ext} B_{\mathcal{B}^*} \Leftrightarrow \iota(x) \in \mathrm{Ch}_{\mathcal{B}} J.$$

Moreover, since $\phi_J(\alpha) = 0$, the point α does not belong to the Choquet boundary of $\mathcal{B}$. This finishes the proof. □

Lemma 2.9. *Let $f \colon X \to \mathbb{F}$ be an affine function of the first Borel class on a compact convex set. Then*

$$\sup_{x \in X} |f(x)| = \sup_{x \in \operatorname{ext} X} |f(x)|.$$

Proof. The assertion follows from [13, Corollary 1.5] since any function of the first Borel class has the point of continuity property (see [28, Theorem 2.3]). □

3. Proof of Theorem 1.1

The main strategy of the proof originated in the proofs in [10]; however, we need to add some extra arguments provided by the previous lemmas. In particular, Lemma 2.7 allows us to work directly with measures.

Proof of Theorem 1.1. We first assume that the spaces K_1, K_2 are compact. Secondly, we suppose that there exists $c \in \mathbb{R}$ such that $1 < c < 2$, $\|T\| < 2$ and $\|T^{-1}\| < 1/c$, otherwise we would multiply T by a suitable constant.

Claim 1. For any $a^{**} \in \mathcal{H}_1^{**} \setminus \{0\}$ and $b^{**} \in \mathcal{H}_2^{**} \setminus \{0\}$, we have $\|T^{**}a^{**}\| > c\|a^{**}\|$ and $\|(T^{-1})^{**}b^{**}\| > (1/2)\|b^{**}\|$.

This follows from the equalities $\|T^{**}\| = \|T\|$ and $\|(T^{-1})^{**}\| = \|T^{-1}\|$.

For $i = 1, 2$, let $\pi_i \colon \mathcal{M}(\overline{\mathrm{Ch}_{\mathcal{H}_i} K_i}, \mathbb{F}) \to \mathcal{H}_i^*$ be the restriction mapping and let $\phi_i \colon K_i \to B_{\mathcal{H}_i^*}$ be the evaluation mapping. For each $x \in \mathrm{Ch}_{\mathcal{H}_1} K_1$ we consider the function $\chi_{\{x\}}$. Let $\widehat{\chi_{\{x\}}} \colon \mathcal{M}(\overline{\mathrm{Ch}_{\mathcal{H}_1} K_1}) \to \mathbb{F}$ be defined as in Lemma 2.4 and let $a_x^{**} \in \mathcal{H}_1^{**}$ satisfy $a_x^{**} \circ \pi_1 = \widehat{\chi_{\{x\}}}$ (see Lemma 2.6). Then a_x^{**} is of the first Borel class on $rB_{\mathcal{H}_1^*}$ for any $r > 0$ (see Lemmas 2.4 and 2.7(b)). Analogously we define for $y \in \mathrm{Ch}_{\mathcal{H}_2} K_2$ the function $\widehat{\chi_{\{y\}}}$ and the element $b_y^{**} \in \mathcal{H}_2^{**}$.

We define mappings ρ_1 and ρ_2 as follows:

$$\rho_1(x) = \{y \in \mathrm{Ch}_{\mathcal{H}_2} K_2;\ |\langle \phi_1(x), (T^{-1})^{**} b_y^{**} \rangle| > \tfrac{1}{2}\}, \quad x \in \mathrm{Ch}_{\mathcal{H}_1} K_1;$$
$$\rho_2(y) = \{x \in \mathrm{Ch}_{\mathcal{H}_1} K_1;\ |\langle \phi_2(y), T^{**} a_x^{**} \rangle| > c\}, \quad y \in \mathrm{Ch}_{\mathcal{H}_2} K_2.$$

Claim 2. ρ_1 and ρ_2 are at most single-valued.

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Let $x \in \mathrm{Ch}_{\mathcal{H}_1} K_1$ be such that there exist distinct points $y_1, y_2 \in \mathrm{Ch}_{\mathcal{H}_2} K_2$ with

$$|\langle (T^{-1})^* \phi_1(x), b_{y_i}^{**} \rangle| = |\langle \phi_1(x), (T^{-1})^{**} b_{y_i}^{**} \rangle| > \tfrac{1}{2}, \quad i = 1, 2.$$

Using Lemma 2.2, we find a measure $\nu_x \in \mathcal{M}(\overline{\mathrm{Ch}_{\mathcal{H}_2} K_2}, \mathbb{F})$ with $\pi_2(\nu_x) = (T^{-1})^* \phi_1(x)$ and $\|\nu_x\| = \|(T^{-1})^* \phi_1(x)\|$. We write

$$\nu_x = \lambda_1 \varepsilon_{y_1} + \lambda_2 \varepsilon_{y_2} + \mu,$$

where $\lambda_1, \lambda_2 \in \mathbb{F}$, $\mu(\{y_1\}) = \mu(\{y_2\}) = 0$. Using Lemma 2.7(a), we obtain

$$\begin{aligned} \tfrac{1}{2} < |\langle (T^{-1})^* \phi_1(x), b_{y_i}^{**} \rangle| &= |\langle \nu_x, \widehat{\chi_{\{y_i\}}} \rangle| = |\nu_x(\chi_{\{y_i\}})| \\ &= |(\lambda_1 \varepsilon_{y_1} + \lambda_2 \varepsilon_{y_2} + \mu)(\chi_{\{y_i\}})| = |\lambda_i|, \quad i = 1, 2. \end{aligned}$$

Since $y_1 \neq y_2$, using the inequality $\|(T^{-1})^*\| = \|T^{-1}\| < 1/c < 1$, we obtain

$$1 > \|(T^{-1})^* \phi_1(x)\| = \|\nu_x\| \geq |\lambda_1| + |\lambda_2| > 1,$$

which is a contradiction.

Analogously we show that $\rho_2(y)$ is at most single-valued for each $y \in \mathrm{Ch}_{\mathcal{H}_2} K_2$.

Claim 3.1. $\|(T^{-1})^{**} b_y^{**}\| = \sup_{x \in \mathrm{Ch}_{\mathcal{H}_1} K_1} |\langle \phi_1(x), (T^{-1})^{**} b_y^{**} \rangle|$ for $y \in \mathrm{Ch}_{\mathcal{H}_2} K_2$ and $\|T^{**} a_x^{**}\| = \sup_{y \in \mathrm{Ch}_{\mathcal{H}_2} K_2} |\langle \phi_2(y), T^{**} a_x^{**} \rangle|$ for $x \in \mathrm{Ch}_{\mathcal{H}_1} K_1$.

First we show that the element $(T^{-1})^{**} b_y^{**} \in \mathcal{H}_1^{**}$ is of the first Borel class on $B_{\mathcal{H}_1^*}$.

Indeed, we know that b_y^{**} is of the first Borel class on any ball in $\mathcal{H}_2^*$, in particular on $2B_{\mathcal{H}_2^*}$. Since $(T^{-1})^*$ is a weak*-weak* homeomorphism, $(T^{-1})^*(B_{\mathcal{H}_1^*}) \subset 2B_{\mathcal{H}_2^*}$ and $(T^{-1})^{**} b_y^{**} = b_y^{**} \circ (T^{-1})^*$, it follows that $(T^{-1})^{**} b_y^{**}$ is of the first Borel class on $B_{\mathcal{H}_1^*}$ as well.

Now, by Lemma 2.9, we have

$$\|(T^{-1})^{**} b_y^{**}\| = \sup_{a^* \in \mathrm{ext}\, B_{\mathcal{H}_1^*}} |\langle a^*, (T^{-1})^{**} b_y^{**} \rangle|,$$

and by Lemma 2.1 this is equal to

$$\sup_{a^* \in S_{\mathbb{F}} \cdot \phi_1(\mathrm{Ch}_{\mathcal{H}_1} K_1)} |\langle a^*, (T^{-1})^{**} b_y^{**} \rangle| = \sup_{x \in \mathrm{Ch}_{\mathcal{H}_1} K_1} |\langle \phi_1(x), (T^{-1})^{**} b_y^{**} \rangle|.$$

The second part of Claim 3.1 can be proven in the same way.

Let L_1 and L_2 denote the domain of ρ_1 and ρ_2, respectively.

Claim 3.2. The mappings $\rho_1 \colon L_1 \to \mathrm{Ch}_{\mathcal{H}_2} K_2$ and $\rho_2 \colon L_2 \to \mathrm{Ch}_{\mathcal{H}_1} K_1$ are surjective.

Let $y \in \mathrm{Ch}_{\mathcal{H}_2} K_2$ be given. Since $\|b_y^{**}\| = 1$, we get

$$\tfrac{1}{2} = \tfrac{1}{2} \|b_y^{**}\| < \|(T^{-1})^{**} b_y^{**}\|,$$

the inequality being true by Claim 1. But by Claim 3.1 the last term is equal to

$$\sup_{x \in \mathrm{Ch}_{\mathcal{H}_1} K_1} |\langle \phi_1(x), (T^{-1})^{**} b_y^{**} \rangle|.$$

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Thus there exists $x \in \mathrm{Ch}_{\mathcal{H}_1} K_1$ such that $|\langle \phi_1(x), (T^{-1})^{**} b_y^{**}\rangle| > \frac{1}{2}$, that is, $\rho_1(x) = y$, and thus ρ_1 is surjective.

The fact that ρ_2 is surjective follows similarly from the inequalities

$$c = c\|a_x^{**}\| < \|T^{**} a_x^{**}\| = \sup_{y \in \mathrm{Ch}_{\mathcal{H}_2} K_2} |\langle \phi_2(y), T^{**} a_x^{**}\rangle|$$

for $x \in \mathrm{Ch}_{\mathcal{H}_1} K_1$.

Claim 4. We have $L_1 = \mathrm{Ch}_{\mathcal{H}_1} K_1$, $L_2 = \mathrm{Ch}_{\mathcal{H}_2} K_2$, $\rho_2(\rho_1(x)) = x$, $x \in \mathrm{Ch}_{\mathcal{H}_1} K_1$, and $\rho_1(\rho_2(y)) = y$, $y \in \mathrm{Ch}_{\mathcal{H}_2} K_2$.

Let $y \in \mathrm{Ch}_{\mathcal{H}_2} K_2$ be given. We denote

$$d = \|(T^{-1})^{**} b_y^{**}\|.$$

Then, by Claim 1 and since $c > 1$, we obtain that $d > \max\{\frac{1}{2}, d/c\}$. Thus, by Claim 3.1, there exists $x \in \mathrm{Ch}_{\mathcal{H}_1} K_1$ such that $|\langle \phi_1(x), (T^{-1})^{**} b_y^{**}\rangle| > \max\{\frac{1}{2}, d/c\}$. Then $\rho_1(x) = y$.

By Claim 1 we have $\|T^{**} a_x^{**}\| > c$, thus by the second part of Claim 3.1 there exists $\widehat{y} \in \mathrm{Ch}_{\mathcal{H}_2} K_2$ such that $|\langle \phi_2(\widehat{y}), T^{**} a_x^{**}\rangle| > c$. Then $\rho_2(\widehat{y}) = x$. We assume that $\widehat{y} \neq y$ and seek a contradiction.

Using Lemma 2.2, we find $\mu_{\widehat{y}} \in \mathcal{M}(\overline{\mathrm{Ch}_{\mathcal{H}_1} K_1}, \mathbb{F})$ such that $\|\mu_{\widehat{y}}\| = \|T^* \phi_2(\widehat{y})\|$ and $\pi_1(\mu_{\widehat{y}}) = T^* \phi_2(\widehat{y})$. We write

$$\mu_{\widehat{y}} = \lambda \varepsilon_x + \mu, \quad \lambda \in \mathbb{F}, \quad \mu \in \mathcal{M}(\overline{\mathrm{Ch}_{\mathcal{H}_1} K_1}, \mathbb{F}), \quad \text{with } \mu(\{x\}) = 0.$$

Then

$$\begin{aligned}\langle \lambda \phi_1(x) + \pi_1(\mu), (T^{-1})^{**} b_y^{**}\rangle &= \langle \pi_1(\mu_{\widehat{y}}), (T^{-1})^{**} b_y^{**}\rangle = \langle T^* \phi_2(\widehat{y}), (T^{-1})^{**} b_y^{**}\rangle \\ &= \langle \phi_2(\widehat{y}), T^{**} (T^{-1})^{**} b_y^{**}\rangle = \langle \phi_2(\widehat{y}), b_y^{**}\rangle.\end{aligned}$$

By Lemma 2.7(a) this is equal to

$$\langle \varepsilon_{\widehat{y}}, \widehat{\chi_y}\rangle = \chi_y(\widehat{y}) = 0,$$

since $y \neq \widehat{y}$. Moreover, we have

$$c < |\langle \phi_2(\widehat{y}), T^{**} a_x^{**}\rangle| = |\langle T^* \phi_2(\widehat{y}), a_x^{**}\rangle| = |\langle \pi_1(\mu_{\widehat{y}}), a_x^{**}\rangle| = |\langle \lambda \varepsilon_x + \mu, \chi_x\rangle| = |\lambda|,$$

where the third equality holds again by Lemma 2.7(a). Since

$$\|\mu\| + |\lambda| = \|\mu_{\widehat{y}}\| = \|T^* \phi_2(\widehat{y})\| < 2\|\phi_2(\widehat{y})\| = 2,$$

we obtain $\|\mu\| < 2 - c$. Putting everything together, we get

$$\begin{aligned}d < |\lambda| d/c < |\lambda| |\langle \phi_1(x), (T^{-1})^{**} b_y^{**}\rangle| &= |\langle \lambda \phi_1(x), (T^{-1})^{**} b_y^{**}\rangle| \\ = |\langle \pi_1(\mu), (T^{-1})^{**} b_y^{**}\rangle| \leq d\|\mu\| \leq d(2 - c) < d,\end{aligned}$$

which is a contradiction. Thus we proved that $L_2 = \mathrm{Ch}_{\mathcal{H}_2} K_2$ and $\rho_1(\rho_2(y)) = y$, $y \in \mathrm{Ch}_{\mathcal{H}_2} K_2$. Since ρ_2 is surjective by Claim 3.2, it follows that $L_1 = \mathrm{Ch}_{\mathcal{H}_1} K_1$.

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Thus if $x \in \mathrm{Ch}_{\mathcal{H}_1} K_1 = L_1$, by Claim 3.2 we find $y \in \mathrm{Ch}_{\mathcal{H}_2} K_2$ with $\rho_2(y) = x$ and obtain

$$\rho_2(\rho_1(x)) = \rho_2(\rho_1(\rho_2(y))) = \rho_2(y) = x.$$

So far we have proved that $\rho_1 \colon \mathrm{Ch}_{\mathcal{H}_1} K_1 \to \mathrm{Ch}_{\mathcal{H}_2} K_2$ is a bijection, with ρ_2 being its inverse. We now use the assumption on weak peak points to check that ρ_1 is a homeomorphism.

Claim 5. The mapping ρ_2 is continuous.

Let $F \subset \mathrm{Ch}_{\mathcal{H}_1} K_1$ be a nonempty closed set and let $F = \mathrm{Ch}_{\mathcal{H}_1} K_1 \cap H$ for some closed set $H \subset K_1$. Obviously we may assume that $F \neq \mathrm{Ch}_{\mathcal{H}_1} K_1$. We want to prove that $\rho_2^{-1}(F)$ is closed in $\mathrm{Ch}_{\mathcal{H}_2} K_2$.

To this end, we construct for each $x \in \mathrm{Ch}_{\mathcal{H}_1} K_1 \setminus F$ a function $h_x \in \mathcal{H}_1$ as follows. Fix $x \in \mathrm{Ch}_{\mathcal{H}_1} K_1 \setminus F$ and $y \in \mathrm{Ch}_{\mathcal{H}_2} K_2$ with $\rho_2(y) = x$. Let V be a closed neighborhood of x with $V \cap H = \emptyset$. Using Lemma 2.2, we write $T^*\phi_2(y) = \pi_1(\mu_x)$, where $\mu_x \in \mathcal{M}(\overline{\mathrm{Ch}_{\mathcal{H}_1} K_1}, \mathbb{F})$ satisfies $\|\mu_x\| = \|T^*\phi_2(y)\|$. Let $\mu_x = \lambda\varepsilon_x + \nu$, where $\lambda \in \mathbb{F}$ and $\nu \in \mathcal{M}(\overline{\mathrm{Ch}_{\mathcal{H}_1} K_1}, \mathbb{F})$ satisfies $\nu(\{x\}) = 0$. Since $\rho_2(y) = x$,

$$c < |\langle \phi_2(y), T^{**} a_x^{**} \rangle| = |T^*\phi_2(y), a_x^{**}\rangle| = |\langle \pi_1(\mu_x), a_x^{**}\rangle| = |\mu_x(\chi_{\{x\}})| = |\lambda|.$$

Let $\varepsilon > 0$ satisfy

$$\varepsilon < \min\left\{\frac{|\lambda| - c}{|\lambda| + 3}, c - 1\right\}.$$

We select a compact set $A \subset \overline{\mathrm{Ch}_{\mathcal{H}_1} K_1}$ such that $x \notin A$ and $|\nu|(\overline{\mathrm{Ch}_{\mathcal{H}_1} K_1} \setminus A) < \varepsilon$ and let $U \subset V$ be a closed neighborhood of x satisfying $U \cap A = \emptyset$. Using the assumption on weak peak points, we select $h_x \in B_{\mathcal{H}_1}$ with $h_x(x) > 1 - \varepsilon$ and $|h_x| < \varepsilon$ on $\mathrm{Ch}_{\mathcal{H}_1} K_1 \setminus U$. Then $|h_x| \leq \varepsilon$ on $F \cup A$ since $|h_x| \leq \varepsilon$ on

$$\overline{\mathrm{Ch}_{\mathcal{H}_1} K_1 \setminus U} \supset \overline{\mathrm{Ch}_{\mathcal{H}_1} K_1} \setminus U \supset F \cup A,$$

where the first inclusion is true since U is closed.

We now claim that

$$\rho_2^{-1}(F) = \bigcap_{x \in \mathrm{Ch}_{\mathcal{H}_1} K_1 \setminus F} \{z \in \mathrm{Ch}_{\mathcal{H}_2} K_2;\ |Th_x(z)| \leq c\}. \tag{3-1}$$

Indeed, let $y \in \mathrm{Ch}_{\mathcal{H}_2} K_2 \setminus \rho_2^{-1}(F)$ be given. Then we consider the function h_x, where $x = \rho_2(y) \in \mathrm{Ch}_{\mathcal{H}_1} K_1 \setminus F$. Using the inequality $\|\mu_x\| \leq 2$, we then have

$$\begin{aligned}
|Th_x(y)| &= |\langle Th_x, \phi_2(y)\rangle| = |\langle h_x, T^*\phi_2(y)\rangle| = |\mu_x(h_x)| = |\lambda h_x(x) + \nu(h_x)| \\
&\geq |\lambda|(1 - \varepsilon) - \int_{\overline{\mathrm{Ch}_{\mathcal{H}_1} K_1}} |h_x| d|\nu| \\
&= |\lambda|(1 - \varepsilon) - \int_{\overline{\mathrm{Ch}_{\mathcal{H}_1} K_1} \setminus A} |h_x| d|\nu| - \int_A |h_x| d|\nu| \\
&\geq |\lambda|(1 - \varepsilon) - \varepsilon - \varepsilon\|\nu\| \geq |\lambda|(1 - \varepsilon) - 3\varepsilon > c,
\end{aligned}$$

which shows the inclusion '$\supset$' in (3-1).

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For the proof of the reverse inclusion, we select $z \in \rho_2^{-1}(F)$. Then $\rho_2(z) \in F$ and, by the definition of ρ_2,

$$c < |\langle \phi_2(z), T^{**} a^{**}_{\rho_2(z)} \rangle| = |\langle T^* \phi_2(z), a^{**}_{\rho_2(z)} \rangle|.$$

Using Lemma 2.2, let $T^*\phi_2(z) = \pi_1(\mu_z)$, where $\mu_z \in \mathcal{M}(\overline{\mathrm{Ch}_{\mathcal{H}_1} K_1}, \mathbb{F})$ satisfies $\|\mu_z\| = \|T^*\phi_2(z)\|$. We write $\mu_z = \lambda \varepsilon_{\rho_2(z)} + \mu$, where $\lambda \in \mathbb{F}$ and $\mu \in \mathcal{M}(\overline{\mathrm{Ch}_{\mathcal{H}_1} K_1}, \mathbb{F})$ satisfies $\mu(\{\rho_2(z)\}) = 0$. Then, by Lemma 2.7(a),

$$c < |\langle \pi_1(\mu_z), a^{**}_{\rho_2(z)} \rangle| = |\mu_z(\chi_{\{\rho_2(z)\}})| = |\lambda|,$$

and thus

$$2 > \|T^*\phi_2(z)\| = \|\mu_z\| = |\lambda| + \|\mu\| > c + \|\mu\|.$$

From these estimates it follows that, for each $x \in \mathrm{Ch}_{\mathcal{H}_1} K_1 \setminus F$,

$$|T h_x(z)| = |\langle T h_x, \phi_2(z) \rangle| = |\langle h_x, T^* \phi_2(z) \rangle| = |\langle h_x, \pi_1(\mu_z) \rangle| = \left| \int_{\overline{\mathrm{Ch}_{\mathcal{H}_1} K_1}} h_x \, d\mu_z \right|$$

$$= \left| \lambda h_x(\rho_2(z)) + \int_{\overline{\mathrm{Ch}_{\mathcal{H}_1} K_1}} h_x \, d\mu \right| \le |\lambda h_x(\rho_2(z))| + \|\mu\|.$$

Moreover, using the previous computations and the facts that $|\lambda| \le \|\mu_z\| \le 2$, $\rho_2(z) \in F$ and $|h_x| \le \varepsilon$ on F, we obtain that

$$|\lambda h_x(\rho_2(z))| + \|\mu\| \le |\lambda|\varepsilon + (2 - c) \le 2\varepsilon + (2 - c) < c,$$

by the choice of ε. Hence

$$z \in \{u \in \mathrm{Ch}_{\mathcal{H}_2} K_2;\ |T h_x(u)| \le c\}$$

and (3-1) is verified.

By (3-1), $\rho_2^{-1}(F)$ is a closed subset of $\mathrm{Ch}_{\mathcal{H}_2} K_2$, and thus ρ_2 is continuous.

Analogously, we would verify that ρ_1 is continuous.

This finishes the proof for the compact case. Now we assume that K_1, K_2 are locally compact and consider their one-point compactifications $J_i = K_i \cup \{\alpha_i\}$, where, for $i = 1, 2$, α_i is the point representing infinity. The spaces $\mathcal{H}_i$ are then closed subspaces of $C(J_i, \mathbb{F})$ satisfying $h(\alpha_i) = 0$, $h \in \mathcal{H}_i$. Since $\mathrm{Ch}_{\mathcal{H}_i} K_i = \mathrm{Ch}_{\mathcal{H}_i} J_i$ by Lemma 2.8, the assumption on weak peak points for $\mathrm{Ch}_{\mathcal{H}_i} J_i$ is satisfied. Thus the compact case implies the existence of a homeomorphism between $\mathrm{Ch}_{\mathcal{H}_1} J_1$ and $\mathrm{Ch}_{\mathcal{H}_2} J_2$. Hence the theorem follows. □

4. Proof of Theorem 1.2

Proof of Theorem 1.2. We follow the proof of Theorem 1.1. Again we may assume that the spaces K_1, K_2 are compact (see Lemma 2.8), and we may consider $1 < c < 2$ and T such that $\|T\| < 2$ and $\|Th\| > c\|h\|$, $h \in \mathcal{H}_1 \setminus \{0\}$. Then, as before, we have

$$\|T^{**} h^{**}\| > c\|h^{**}\|, \quad h^{**} \in \mathcal{H}_1^{**} \setminus \{0\}.$$

For $x \in \mathrm{Ch}_{\mathcal{H}_1} K_1$, let $\widehat{\chi_{\{x\}}}$ and a_x^{**} be as in the proof of Theorem 1.1. Again we define

$$\rho(y) = \{x \in \mathrm{Ch}_{\mathcal{H}_1} K_1;\ |\langle \phi_2(y), T^{**} a_x^{**} \rangle| > c\}, \quad y \in \mathrm{Ch}_{\mathcal{H}_2} K_2.$$

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Claim 1. ρ is at most single-valued.

Indeed, let $x_1, x_2 \in \mathrm{Ch}_{\mathcal{H}_1} K_1$ be distinct such that $|\langle \phi_2(y), T^{**} a^{**}_{x_i}\rangle| > c$ for some $y \in \mathrm{Ch}_{\mathcal{H}_2} K_2$, $i = 1, 2$. Using Lemma 2.2, we find $\mu \in \mathcal{M}(\overline{\mathrm{Ch}_{\mathcal{H}_1} K_1}, \mathbb{F})$ with $\pi_1(\mu) = T^* \phi_2(y)$ and $\|\mu\| = \|T^* \phi_2(y)\|$. We write

$$\mu = \lambda_1 \varepsilon_{x_1} + \lambda_2 \varepsilon_{x_2} + \nu,$$

where $\lambda_i \in \mathbb{F}$ and $\nu(\{x_i\}) = 0$, $i = 1, 2$. By Lemma 2.7,

$$c < |\langle T^* \phi_2(y), a^{**}_{x_i}\rangle| = |\langle \mu, \chi_{\{x_i\}}\rangle| = |\lambda_i|, \quad i = 1, 2.$$

Thus

$$2 > \|T^* \phi_2(y)\| = \|\mu\| \geq |\lambda_1| + |\lambda_2| > 2c$$

yields a contradiction with the inequality $c > 1$.

Let L denote the domain of ρ.

Claim 2. ρ is surjective.

Assume that for some $x \in \mathrm{Ch}_{\mathcal{H}_1} K_1$ we have $c \geq |\langle \phi_2(y), T^{**} a^{**}_x\rangle|$, $y \in \mathrm{Ch}_{\mathcal{H}_2} K_2$. Then we have, as in the proof of Theorem 1.1,

$$\begin{aligned} c &\geq \sup_{y \in \mathrm{Ch}_{\mathcal{H}_2} K_2} |\langle \phi_2(y), T^{**} a^{**}_x\rangle| = \sup_{s \in S_{\mathbb{F}} \cdot \phi_2(\mathrm{Ch}_{\mathcal{H}_2} K_2))} |\langle s, T^{**} a^{**}_x\rangle| \\ &= \sup_{s \in B_{\mathcal{H}_2^*}} |\langle s, T^{**} a^{**}_x\rangle| = \|T^{**} a^{**}_x\| > c\|a^{**}_x\| \geq c, \end{aligned}$$

which is a contradiction.

Claim 3. $\rho \colon L \to \mathrm{Ch}_{\mathcal{H}_1} K_1$ is continuous.

Let $F \subset \mathrm{Ch}_{\mathcal{H}_1} K_1$ be a closed set and let $F = \mathrm{Ch}_{\mathcal{H}_1} K_1 \cap H$ for some closed set $H \subset K_1$. We may assume that $F \neq \mathrm{Ch}_{\mathcal{H}_1} K_1$. We want to prove that $\rho^{-1}(F)$ is closed in L.

To this end, we construct for each $x \in \mathrm{Ch}_{\mathcal{H}_1} K_1 \setminus F$ and $y \in \rho^{-1}(x)$ a function $h_{x,y} \in \mathcal{H}_1$ as follows. Let V be a closed neighborhood of x with $V \cap H = \emptyset$. Using Lemma 2.2, we write

$$T^* \phi_2(y) = \pi_1(\mu) \quad \text{and} \quad \mu = \lambda \varepsilon_x + \nu,$$

where $\mu \in \mathcal{M}(\overline{\mathrm{Ch}_{\mathcal{H}_1} K_1}, \mathbb{F})$ satisfies $\|\mu\| = \|T^* \phi_2(y)\|$, $\lambda \in \mathbb{F}$ and $\nu(\{x\}) = 0$. Then $|\lambda| > c$, since $\rho(y) = x$. Let $\varepsilon > 0$ satisfy

$$\varepsilon < \min\left\{\frac{|\lambda| - c}{3 + |\lambda|}, c - 1\right\}.$$

Let $A \subset \overline{\mathrm{Ch}_{\mathcal{H}_1} K_1} \setminus \{x\}$ be a compact set satisfying $|\nu|(\overline{\mathrm{Ch}_{\mathcal{H}_1} K_1} \setminus A) < \varepsilon$. Then there exists a function $h_{x,y} \in \mathcal{H}_1$ such that

$$\|h_{x,y}\| \leq 1, \quad h_{x,y}(x) > 1 - \varepsilon \quad \text{and} \quad |h_{x,y}| \leq \varepsilon \quad \text{on } F \cup A.$$

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Now we claim that

$$\rho^{-1}(F) = \bigcap_{x\in \mathrm{Ch}_{\mathcal{H}_1} K_1\setminus F} \bigcap_{y\in\rho^{-1}(x)} \{z \in L;\ |Th_{x,y}(z)| \le c\}. \tag{4-1}$$

Indeed, if $y \in L \setminus \rho^{-1}(F)$, then we consider the function $h_{x,y}$, where $x = \rho(y) \in \mathrm{Ch}_{\mathcal{H}_1} K_1 \setminus F$. Then, as above, we use Lemma 2.2 to write

$$T^*\phi_2(y) = \pi_1(\mu) \quad \text{and} \quad \mu = \lambda\varepsilon_x + \nu.$$

By the choice of the function $h_{x,y}$ we have

$$\begin{aligned} |Th_{x,y}(y)| &= |\langle h_{x,y}, T^*\phi_2(y)\rangle| = |\langle h_{x,y}, \lambda\varepsilon_x + \nu\rangle| \\ &\ge |\lambda|(1-\varepsilon) - \int_{\overline{\mathrm{Ch}_{\mathcal{H}_1} K_1\setminus A}} |h_{x,y}|d|\nu| - \int_A |h_{x,y}|d|\nu| \\ &\ge |\lambda|(1-\varepsilon) - \varepsilon - \varepsilon\|\nu\| > c. \end{aligned}$$

Hence

$$y \notin \bigcap_{x\in \mathrm{Ch}_{\mathcal{H}_1} K_1\setminus F} \bigcap_{u\in\rho^{-1}(x)} \{z \in L;\ |Th_{x,u}(z)| \le c\},$$

which shows the inclusion '$\supset$' in (4-1).

For the proof of the reverse inclusion, we select $z \in \rho^{-1}(F)$ and let $x \in \mathrm{Ch}_{\mathcal{H}_1} K_1 \setminus F$ and $y \in \rho^{-1}(x)$ be arbitrary. Then $\rho(z) \in F$ and, by the definition of ρ,

$$c < |\langle \phi_2(z), T^{**}a^{**}_{\rho(z)}\rangle| = |\langle T^*\phi_2(z), a^{**}_{\rho(z)}\rangle|.$$

By Lemma 2.2, let

$$T^*\phi_2(z) = \pi_1(\mu_z) \quad \text{and} \quad \mu = \lambda\varepsilon_{\rho(z)} + \nu_z,$$

where $\mu_z \in \mathcal{M}(\overline{\mathrm{Ch}_{\mathcal{H}_1} K_1}, \mathbb{F})$ satisfies $\|\mu_z\| = \|T^*\phi_2(z)\|$, $\lambda \in \mathbb{F}$ and $\nu_z(\{\rho(z)\}) = 0$. Then

$$c < |\langle \pi_1(\mu_z), a^{**}_{\rho(z)}\rangle| = |\langle \lambda\varepsilon_{\rho(z)} + \nu_z, \chi_{\{\rho(z)\}}\rangle| = |\lambda|,$$

and thus

$$2 > \|T^*\phi_2(z)\| = \|\mu_z\| = |\lambda| + \|\nu_z\| > c + \|\nu_z\|.$$

From these estimates it follows that

$$|Th_{x,y}(z)| = |\langle Th_{x,y}, \phi_2(z)\rangle| = |\langle h_{x,y}, T^*\phi_2(z)\rangle| = |\langle h_{x,y}, \pi_1(\mu_z)\rangle| = \left|\int_{\overline{\mathrm{Ch}_{\mathcal{H}_1} K_1}} h_{x,y}\, d\mu_z\right|$$

$$= \left|\lambda h_{x,y}(\rho_2(z)) + \int_{\overline{\mathrm{Ch}_{\mathcal{H}_1} K_1}} h_{x,y}\, d\nu_z\right| \le |\lambda h_{x,y}(\rho_2(z))| + \|\nu_z\|.$$

Moreover, by the previous computations and since $|\lambda| \le \|\mu\| \le 2$, $\rho_2(z) \in F$ and $|h_{x,y}| \le \varepsilon$ on F, we obtain that

$$|\lambda h_{x,y}(\rho_2(z))| + \|\nu_z\| \le |\lambda|\varepsilon + (2-c) \le 2\varepsilon + (2-c) < c,$$

by the choice of ε. Hence

$$z \in \{u \in L;\ |Th_{x,y}(u)| \le c\}$$

and (4-1) is verified.

By (4-1), $\rho^{-1}(F)$ is a closed subset of L, and thus ρ is continuous. This finishes the proof. □

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5. Proof of Theorem 1.3

We start the proof of Theorem 1.3 with the following result imitating [31, Lemma 3.1] for the case of function spaces.

LEMMA 5.1. *Let K be a compact space, let $\mathcal{H}$ be a subspace of $C(K,\mathbb{F})$ of finite dimension and let each point of $\mathrm{Ch}_{\mathcal{H}} K$ be a weak peak point. Then $\mathrm{Ch}_{\mathcal{H}} K$ is finite and $\mathcal{H}$ is isometrically isomorphic to $\ell^\infty(\mathrm{Ch}_{\mathcal{H}} K,\mathbb{F})$.*

PROOF. Let $x \in \mathrm{Ch}_{\mathcal{H}} K$ be given. We show that there exists a function $h_x \in B_{\mathcal{H}}$ such that $h_x(x) = 1$ and $h_x = 0$ on $\mathrm{Ch}_{\mathcal{H}} K \setminus \{x\}$. To this end we consider the net $\{h_{U,\varepsilon}\}$ in $B_{\mathcal{H}}$, where U is a neighborhood of x, $\varepsilon \in (0,1)$ and $h_{U,\varepsilon}$ is a function satisfying $h_{U,\varepsilon}(x) > 1-\varepsilon$ and $|h_{U,\varepsilon}| < \varepsilon$ on $\mathrm{Ch}_{\mathcal{H}} K \setminus U$. We consider the partial order on the set of pairs (U,ε) given by $(U_1,\varepsilon_1) \le (U_2,\varepsilon_2)$ provided $U_2 \subset U_1$ and $\varepsilon_2 < \varepsilon_1$. Since $B_{\mathcal{H}}$ is compact in the norm topology, the net $\{h_{U,\varepsilon}\}$ possesses a cluster point $h_x \in B_{\mathcal{H}}$. Then $h_x(x) = 1$ and $h_x = 0$ on $\mathrm{Ch}_{\mathcal{H}} K \setminus \{x\}$.

From this observation it now follows that $\mathrm{Ch}_{\mathcal{H}} K$ is finite. Indeed, assuming that $\mathrm{Ch}_{\mathcal{H}} K$ contains an infinite set $\{x_n;\, n \in \mathbb{N}\}$, the functions $\{h_{x_n};\, n \in \mathbb{N}\}$ are linearly independent, which contradicts the assumption.

It remains to prove that the restriction mapping $r\colon \mathcal{H} \to \ell^\infty(\mathrm{Ch}_{\mathcal{H}} K,\mathbb{F})$ is a surjective isometry. It is an isometry by the maximum principle [16, Theorem 2.3.8]. It is surjective, since any $f \in \ell^\infty(\mathrm{Ch}_{\mathcal{H}} K,\mathbb{F})$ can be written as

$$f = \sum_{x \in \mathrm{Ch}_{\mathcal{H}} K} f(x) h_x|_{\mathrm{Ch}_{\mathcal{H}} K},$$

and thus $h = \sum_{x \in \mathrm{Ch}_{\mathcal{H}} K} f(x) h_x$ satisfies $r(h) = f$. □

PROOF OF THEOREM 1.3. Using Lemma 2.8, we may assume that K_1, K_2 are compact. If $\mathcal{H}_1$ (and thus also $\mathcal{H}_2$) is finite-dimensional, we obtain from Lemma 5.1 the equality

$$\begin{aligned} |\mathrm{Ch}_{\mathcal{H}_1} K_1| &= \dim(\ell^\infty(\mathrm{Ch}_{\mathcal{H}_1} K_1,\mathbb{F})) = \dim \mathcal{H}_1 = \dim \mathcal{H}_2 \\ &= \dim(\ell^\infty(\mathrm{Ch}_{\mathcal{H}_2} K_2,\mathbb{F})) = |\mathrm{Ch}_{\mathcal{H}_2} K_2|. \end{aligned}$$

From now on we may thus assume that both spaces are infinite-dimensional.

Let $\pi_1\colon \mathcal{M}(\overline{\mathrm{Ch}_{\mathcal{H}_1} K_1},\mathbb{F}) \to \mathcal{H}_1^*$ be the restriction mapping. For each $x \in \mathrm{Ch}_{\mathcal{H}_1} K_1$ we consider the function $\chi_{\{x\}}$. Let $\widehat{\chi_{\{x\}}}\colon \mathcal{M}(\overline{\mathrm{Ch}_{\mathcal{H}_1} K_1}) \to \mathbb{F}$ be defined as in Lemma 2.4 and let $a_x^{**} \in \mathcal{H}_1^{**}$ satisfy $a_x^{**} \circ \pi_1 = \widehat{\chi_{\{x\}}}$ (see Lemma 2.6). Then a_x^{**} is of the first Borel class on $rB_{H_1^*}$ for any $r > 0$ (see Lemmas 2.4 and 2.7(b)).

For a fixed point $y \in \mathrm{Ch}_{\mathcal{H}_2} K_2$, let $\lambda_y(x) = \langle T^*\phi_2(y), a_x^{**}\rangle$. We claim that the set

$$X_y = \{x \in \mathrm{Ch}_{\mathcal{H}_1} K_1;\, \lambda_y(x) \neq 0\}$$

is at most countable. Indeed, let $s = T^*\phi_2(y)$ and $\mu \in \mathcal{M}(\overline{\mathrm{Ch}_{\mathcal{H}_1} K_1},\mathbb{F})$ extend s. Let $x \in \mathrm{Ch}_{\mathcal{H}_1} K_1$ be arbitrary. Using Lemma 2.7, we have

$$\mu(\{x\}) = \mu(\chi_{\{x\}}) = \langle \mu, \widehat{\chi_{\{x\}}}\rangle = \langle T^*\phi_2(y), a_x^{**}\rangle.$$

Since $\|\mu\| < \infty$, $\mu(\{x\}) \neq 0$ for at most countably many $x \in \mathrm{Ch}_{\mathcal{H}_1} K_1$.

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We now prove that for each $x \in \mathrm{Ch}_{\mathcal{H}_1} K_1$ there exists $y \in \mathrm{Ch}_{\mathcal{H}_2} K_2$ such that $x \in X_y$. To this end, we assume the contrary. Let $x \in \mathrm{Ch}_{\mathcal{H}_1} K_1$ be such that

$$\langle T^* \phi_2(y), a_x^{**} \rangle = 0, \quad y \in \mathrm{Ch}_{\mathcal{H}_2} K_2.$$

Lemma 2.9 then yields

$$\begin{aligned} 0 &= \sup_{y \in \mathrm{Ch}_{\mathcal{H}_2} K_2} |\langle T^* \phi_2(y), a_x^{**} \rangle| = \sup_{y \in \mathrm{Ch}_{\mathcal{H}_2} K_2} |\langle \phi_2(y), T^{**} a_x^{**} \rangle| \\ &= \sup_{s \in S_{\mathbb{F}} \cdot \phi_2(\mathrm{Ch}_{\mathcal{H}_2} K_2)} |\langle s, T^{**} a_x^{**} \rangle| = \sup_{s \in B_{\mathcal{H}_2^*}} |\langle s, T^{**} a_x^{**} \rangle| = \|T^{**} a_x^{**}\| \neq 0, \end{aligned}$$

a contradiction.

Now both the spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ are infinite-dimensional, and thus the sets $\mathrm{Ch}_{\mathcal{H}_1} K_1$ and $\mathrm{Ch}_{\mathcal{H}_2} K_2$ are infinite. Indeed, if $\mathrm{Ch}_{\mathcal{H}_1} K_1$ were finite, by the maximum principle (see [16, Theorem 2.3.8]) we would obtain that the space $\mathcal{H}_1 \subset \ell^\infty(\mathrm{Ch}_{\mathcal{H}_1} K_1, \mathbb{F})$ is finite-dimensional.

Now, since we have $\mathrm{Ch}_{\mathcal{H}_1} K_1 = \bigcup_{y \in \mathrm{Ch}_{\mathcal{H}_2} K_2} X_y$, we get $|\mathrm{Ch}_{\mathcal{H}_1} K_1| \leq |\mathrm{Ch}_{\mathcal{H}_2} K_2|$.

By reversing the role of K_1 and K_2 we obtain the converse inequality, which concludes the proof. □

The following question, arising from comparison of Theorems 1.1 and 1.2, was suggested by the anonymous reviewer of the paper: What if we assume that T is as in Theorem 1.1 (in particular, onto) and the assumption on weak peak points is imposed only on $\mathcal{H}_1$? Then by Theorem 1.2, there exist a set $L \subset \mathrm{Ch}_{\mathcal{H}_2} K_2$ and a continuous surjective mapping $\phi\colon L \to \mathrm{Ch}_{\mathcal{H}_1} K_1$. But can one say something more? In particular, must it be that $L = \mathrm{Ch}_{\mathcal{H}_2} K_2$? More precisely, we pose the following question.

Question 5.2. For $i = 1, 2$, let $\mathcal{H}_i$ be a closed subspace of $C_0(K_i, \mathbb{F})$ for some locally compact space K_i. Assume that each point of $\mathrm{Ch}_{\mathcal{H}_1} K_1$ is a weak peak point and let $T\colon \mathcal{H}_1 \to \mathcal{H}_2$ be an onto isomorphism satisfying $\|T\| \cdot \|T^{-1}\| < 2$. Does it follow that $\mathrm{Ch}_{\mathcal{H}_1} K_1$ is a continuous image of $\mathrm{Ch}_{\mathcal{H}_2} K_2$?

We suspect that the answer to the above question is no, but we do not know of any counterexample.

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