Math. Proc. Camb. Phil. Soc. (1985), **97**, 399
Printed in Great Britain

Projective modules over pullback rings

BY A. N. WISEMAN

Department of Engineering Mathematics, Loughborough University of Technology

(*Received* 3 *March* 1983; *revised* 3 *May* 1984)

Introduction

Given two homomorphisms of rings $j_1\colon R_1 \to R'$ and $j_2\colon R_2 \to R'$ we may construct a new ring R called the pullback of R_1 and R_2 over R', together with homomorphisms $i_1\colon R \to R_1$ and $i_2\colon R \to R_2$ such that $j_1 i_1 = j_2 i_2$. There are many instances where special cases of pullbacks have been studied, either to construct new examples of rings or to reduce various problems to those concerning much simpler rings. A few of these instances are given in the references. Thus a general study of the properties of pullbacks in terms of those of the component rings would seem to be useful.

This paper is motivated by the results used in [3] to determine the global dimension of a special case of pullbacks called F-rings, and by the extent to which these results have since been used. The main result of this paper is Theorem 3·1, where restrictions on the flat dimensions of the component rings allow us to give an upper bound for the global dimension of the pullback in terms of those of the components.

Essentially, the results of this paper are given in [3] for pullbacks of commutative rings with the underlying conditions that one of the given homomorphisms j_1, j_2 is a surjection; the other given homomorphism is an injection; the ring R_2 is flat as an R-module; and the flat dimension of R_1 over R is at most 1. The starting point in [3] is a result given by Vasconcelos in [6], 3·1, which, as we shall see in Theorem 2·4, is a result on the projectivity of modules over a special case of a pullback ring, in terms of the projectivity of certain modules over the component rings.

As suggested in [3], we shall bypass Vasconcelos' result by using the more general result of Milnor in [5], chapter 2. To do this we need to know when an R-module M is the pullback of $R_1 \otimes_R M$ and $R_2 \otimes_R M$ (with the obvious maps), this being the aim of Section 1. We see in Corollary 1·2 that it is sufficient for j_2 to be a surjection and for $R_2 \otimes_R M$ to be R_2-flat.

In Section 2 we first show that the pullback of modules of Section 1 satisfies the conditions of Milnor's result. This result then tells us that the projectivity of $R_1 \otimes_R M$ and $R_2 \otimes_R M$ (as R_1-, R_2-modules respectively) ensures the projectivity of the R-module M. In Theorem 2·3 we relax the conditions and only require i_1 to be a surjection.

To obtain the results on the global dimension of pullback rings in Section 3, we first examine when long exact sequences remain exact under the tensor product. We then calculate the projective dimension of certain modules over the pullback ring in terms of the projective dimensions of modules over the component rings. This in turn allows us to prove the main result, Theorem 3·1.

https://doi.org/10.1017/S0305004100062964 Published online by Cambridge University Press

1. *Preliminaries*

By a ring R, we mean an associative ring with identity, and by an R-module we mean a unitary, left R-module. A ring homomorphism sends the identity to the identity.

A commutative square of rings and ring homomorphisms is said to be a *pullback*

$$\begin{array}{ccc} R & \xrightarrow{i_1} & R_1 \\ {\scriptstyle i_2}\downarrow & & \downarrow{\scriptstyle j_1} \\ R_2 & \xrightarrow[j_2]{} & R' \end{array} \tag{1}$$

(or Cartesian, or a fibre product) if given $r_1 \in R_1$, $r_2 \in R_2$ with $j_1(r_1) = j_2(r_2)$ there is a unique element $r \in R$ such that $i_1(r) = r_1$ and $i_2(r) = r_2$.

We shall use the equivalent definition that diagram (1) is a pullback if the restriction of i_2 to $\operatorname{Ker} i_1$ is an isomorphism on to $\operatorname{Ker} j_2$, and j_1 induces an injection of $\operatorname{Coker} i_1$ into $\operatorname{Coker} j_2$.

Note that if j_2 is a surjection then so is i_1, but not conversely.

Given the lower right triangle of diagram (1), we can always find a ring R, and maps i_1, i_2 such that the resulting square is a pullback. The ring R is unique up to isomorphism, and the maps are unique up to composition with this isomorphism. The *standard pullback* is the ring

$$R = \{(r_1, r_2) \in R_1 \times R_2 : j_1(r_1) = j_2(r_2)\},$$

where the maps i_1 and i_2 are the restrictions of the projections of $R_1 \times R_2$ onto R_1 and R_2. Similarly, we may define pullbacks of modules over a given ring.

For the rest of this section, we suppose that diagram (1) is a pullback, and that j_2 (and hence i_1) is a surjection.

Let M be an R-module. The commutative square obtained from diagram (1), by tensoring with M over R, will be denoted by $\otimes M$. The aim of this section is to determine when $\otimes M$ is a pullback.

From the definition of a pullback and since j_2 is a surjection, we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \operatorname{Ker} i_1 & \xrightarrow{h_1} & R & \xrightarrow{i_1} & R_1 & \longrightarrow & 0 \\ & & {\scriptstyle \cong}\downarrow{\scriptstyle k_2} & & \downarrow{\scriptstyle i_2} & & \downarrow{\scriptstyle j_1} & & \\ 0 & \longrightarrow & \operatorname{Ker} j_2 & \xrightarrow[h_2]{} & R_2 & \xrightarrow[j_2]{} & R' & \longrightarrow & 0 \end{array} \tag{2}$$

where h_1 and h_2 are inclusions, and k_2 (the restriction of i_2 to $\operatorname{Ker} i_1$) is an isomorphism. Tensoring with M over R, we obtain the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} (\operatorname{Ker} i_1) \otimes M & \xrightarrow{h_1'} & R \otimes M & \xrightarrow{i_1'} & R_1 \otimes M & \longrightarrow & 0 \\ {\scriptstyle k_2'}\downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle i_2'} & & \downarrow{\scriptstyle j_1'} & & \\ (\operatorname{Ker} j_2) \otimes M & \xrightarrow[h_2']{} & R_2 \otimes M & \xrightarrow[j_2']{} & R' \otimes M & \longrightarrow & 0 \end{array} \tag{3}$$

where $f' = f \otimes \mathrm{Id}_M$ for an R-homomorphism f. By the exactness in diagram (3), we have the following commutative diagram with exact rows:

https://doi.org/10.1017/S0305004100062964 Published online by Cambridge University Press

$$\begin{array}{ccccc} (\operatorname{Ker} i_1) \otimes M & \xrightarrow{h_1'} & \operatorname{Ker} i_1' & \longrightarrow & 0 \\ \downarrow{\scriptstyle k_1'}\cong & & \downarrow{\scriptstyle i_2''|} & & \\ (\operatorname{Ker} j_2) \otimes M & \xrightarrow[h_2']{} & \operatorname{Ker} j_2' & \longrightarrow & 0 \end{array} \tag{4}$$

where $i_2''|$ is i_2'' restricted to $\operatorname{Ker} i_1'$.

PROPOSITION 1·1. *With the above notation, the diagram $\otimes M$ is a pullback if and only if $i_2''|$ is monic.*

Proof. Note that $\otimes M$ is the right-hand square of diagram (3). From diagram (3), $\operatorname{Coker} i_1'$ and $\operatorname{Coker} j_2'$ are both zero. From diagram (4) we see that $i_2''|$ maps $\operatorname{Ker} i_1'$ on to $\operatorname{Ker} j_2'$. From the definition of a pullback we have $\otimes M$ is a pullback (with pullback module $M \cong R \otimes M$) if and only if $i_2''|$ is monic.

COROLLARY 1·2. *If $R_2 \otimes M$ is R_2-flat then M is the pullback of $\otimes M$.*

Proof. The commutative diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & (\operatorname{Ker} j_2) \otimes_{R_2} (R_2 \otimes_R M) & \longrightarrow & R_2 \otimes_{R_2} (R_2 \otimes_R M) \\ & & \downarrow\cong & & \downarrow\cong \\ & & (\operatorname{Ker} j_2) \otimes_R M & \xrightarrow[h_2']{} & R_2 \otimes_R M \end{array}$$

has exact top row when $R_2 \otimes_R M$ is R_2-flat, so that h_2' is monic. From diagram (4) and by Proposition (1·1), the result follows.

2. *Pullbacks of projectives*

The aim of this section is to show that, for our purposes, the pullback of projective modules is projective. For the moment, we still suppose that diagram (1) is a pullback and that j_2 is a surjection. Let M_k be an R_k-module ($k = 1, 2$) and let

$$h\colon R' \otimes_{R_1} M_1 \to R' \otimes_{R_2} M_2$$

be an R'-isomorphism. Let $M(M_1, M_2, h)$ denote the pullback of

$$\begin{array}{ccc} & & M_1 \\ & & \downarrow{\scriptstyle hj_1''} \\ M_2 & \xrightarrow[j_2'']{} & R' \otimes_{R_2} M_2 \end{array} \tag{5}$$

where $j_k''(m_k) = 1 \otimes m_k$, for $m_k \in M_k$. An elementary proof of the following result, due to Milnor, is given in ([5], chapter 2).

THEOREM 2·1. *If M_k is R_k-projective ($k = 1, 2$) then $M(M_1, M_2, h)$ is R-projective.*

Recall that the lower right-hand triangle of $\otimes M$ is

$$\begin{array}{ccc} & & R_1 \otimes_R M \\ & & \downarrow{\scriptstyle j_1'} \\ R_2 \otimes_R M & \xrightarrow[j_2']{} & R' \otimes_R M \end{array} \tag{6}$$

for an R-module M, where $j_k' = j_k \otimes \mathrm{Id}_M$.

https://doi.org/10.1017/S0305004100062964 Published online by Cambridge University Press

By the associativity of the tensor product, we have R'-isomorphisms

$$f_k\colon R' \otimes_{R_k} (R_k \otimes_R M) \to R' \otimes_R M,$$

where $f_k(r' \otimes (r_k \otimes m)) = r'r_k \otimes m = r'j_k(r_k) \otimes m$.

We also have R_k-homomorphisms

$$g_k\colon R_k \otimes_R M \to R' \otimes_{R_k} (R_k \otimes M),$$

where $g_k(x) = 1_{R'} \otimes x$, for $x \in R_k \otimes_R M$.

If $h = f_2^{-1} f_1$, the diagram

$$\begin{array}{ccc} & & R_1 \otimes_R M \\ & & \Big\downarrow{\scriptstyle hg_1} \\ R_2 \otimes_R M & \xrightarrow[g_2]{} & R' \otimes_{R_2} (R_2 \otimes_R M) \end{array} \tag{7}$$

conforms to the format of diagram (5).

LEMMA 2·2. *Diagram* (6) *and diagram* (7) *have the same pullback.*

Proof. The standard pullback of diagram (7) is

$$N = \{(x, y) \in (R_1 \otimes_R M) \times (R_2 \otimes_R M)\colon hg_1(x) = g_2(y)\}.$$

It is routine to check that $j_k' = f_k g_k$, so that $hg_1(x) = g_2(y)$ implies that $j_1'(x) = j_2'(y)$. Hence N is the standard pullback of diagram (6).

We now come to the main result of this section.

THEOREM 2·3. *Given a pullback diagram* (1) *in which* i_1 *is a surjection, an* R*-module* M *is* R*-projective if and only if both* $R_1 \otimes_R M$ *and* $R_2 \otimes_R M$ *are projective* (*over* R_1 *and* R_2 *respectively*).

Proof. If M is R-projective, it is well known that $R_k \otimes_R M$ is R_k-projective ($k = 1, 2$). Now assume that $R_1 \otimes_R M$ and $R_2 \otimes_R M$ are projective.

If j_2 is a surjection then the pullback of diagram (7) is R-projective by Theorem 2·1. By Lemma 2·2 and Corollary 1·2 this pullback is M, as required.

If now only i_1 is a surjection, we have $j_1(R_1) \subset j_2(R_2)$ by the commutativity of diagram (1). From the definition, it is easy to see that the diagram

$$\begin{array}{ccc} R & \xrightarrow{i_1} & R_1 \\ {\scriptstyle i_2}\big\downarrow & & \big\downarrow{\scriptstyle j_1} \\ R_2 & \xrightarrow[j_2]{} & j_2(R_2) \end{array}$$

is a pullback, and the result follows from the case when j_2 is a surjection.

The following result is the generalization to non-commutative rings of a result of Vasconcelos ([6], 3·1). This result was the starting point for Greenberg's study of the global dimension of F-rings, which led to his classification of the local rings of global dimension two in [3]. Here we see that the first part follows easily from our general results on pullbacks. In fact the first part is equivalent to Theorem 2·3 when one of the maps j_1, j_2 is a surjection and the other is an injection. The proof given here, of the second part of the following result, illustrates the use of the results on pullbacks together with some elementary algebra.

https://doi.org/10.1017/S0305004100062964 Published online by Cambridge University Press

THEOREM 2·4. *Let R be a subring of the ring R_2 and let I be a (two-sided) ideal of R. Let M be an R-module. Then M is R-projective if and only if $R_2 \otimes M$ is R_2-projective and M/IM is R/I-projective, in the following cases:*

(i) *I is also an ideal of R_2;*

(ii) *I is nilpotent and R is contained in the centre of R_2.*

Proof. (i) In this case, R is the pullback of R/I and R_2 over R_2/I with the obvious maps, so the result is immediate from Theorem 2·3.

(ii) We first prove the case when $I^2 = 0$. Although R is not a pullback in any obvious way, we may form the pullback of

$$\begin{array}{ccc} & & R/I \\ & & \downarrow{\scriptstyle j_1} \\ R_2 & \xrightarrow[j_2]{} & R_2/R_2 I \end{array} \tag{8}$$

where j_1 and j_2 are the natural maps. The ring

$$S = \{(\bar{r}, r_2) \in (R/I) \times R_2 : j_1(\bar{r}) = j_2(r_2)\}$$

is the standard pullback of diagram (8), where $\bar{r}$ is the natural image of r in R/I. There is a ring homomorphism $f: R \to S$, where $f(r) = (\bar{r}, r)$. Clearly f is an injection and

$$f(I) = \{(0, r) : r \in I\}.$$

If $(\bar{a}, b) \in S$ and $(0, c) \in f(I)$, then $(\bar{a}, b)(0, c) = (0, bc)$. Since $j_1(\bar{a}) = j_2(b)$, we have $b = a + r$, where $r \in R_2 I$. Then $bc = ac + rc = ac$, since $rc \in R_2 I^2 = 0$. Hence $(0, bc) \in f(I)$ and $f(I)$ is also an ideal of S. Thus we may regard R as a subring of S, with common ideal I.

By case (i) of this theorem, M is R-projective if and only if $S \otimes_R M$ is S-projective and M/IM is R/I-projective. Applying Theorem 2·3 to the pullback diagram (8), we have $S \otimes_R M$ is S-projective if and only if $R_2 \otimes_S (S \otimes_R M)$ ($\cong R_2 \otimes_R M$) is R_2-projective, and $(R/I) \otimes_S (S \otimes_R M)$ ($\cong (R/I) \otimes_R M$) is R/I-projective, and the result follows.

We now complete the proof by induction. Suppose that the result holds for all pairs of rings $R \subset R_2$ with an ideal I such that $I^n = 0$ and with R contained in the centre of R_2. Let R be a ring contained in the centre of a ring R_2 and let I be an ideal of R such that $I^{n+1} = 0$.

Let $J = R \cap (R_2 I^n)$ and let $K = (R \cap R_2 I)/J$. Then K is an ideal of R/J such that $K^n = 0$ and R/J is contained in the centre of $R_2/(R_2 I^n)$. By the inductive hypothesis, $(R/J) \otimes_R M$ is R/J-projective if and only if $(R_2/R_2 I^n) \otimes_{R/J} (R/J) \otimes_R M$ is $R_2/R_2 I^n$-projective and $((R/J)/K) \otimes_{R/J} (R/J) \otimes_R M$ is $(R/J)/K$-projective. Note that $(R/J)/K \simeq R/(R \cap R_2 I)$.

If $(R/I) \otimes_R M$ is R/I-projective then

$$(R/(R \cap R_2 I)) \otimes_R M \simeq (R/(R \cap R_2 I)) \otimes_{R/I} (R/I) \otimes_R M$$

is $R/(R \cap R_2 I)$-projective. If $R_2 \otimes_R M$ is R_2-projective then

$$(R_2/R_2 I^n) \otimes_R M \simeq (R_2/R_2 I^n) \otimes_R R_2 \otimes_R M$$

is $R_2/R_2 I^n$-projective. Hence if $(R/I) \otimes_R M$ is R/I-projective and $R_2 \otimes_R M$ is R_2-projective then $(R/J) \otimes_R M$ is R/J-projective.

https://doi.org/10.1017/S0305004100062964 Published online by Cambridge University Press

Since $I^{n+1} = 0$ we have $J^2 = 0$, so by the case $n = 2$, M is R-projective if and only if $(R/J) \otimes_R M$ is R/J-projective and $R_2 \otimes_R M$ is R_2-projective.

Thus if $(R/I) \otimes_R M$ is R/I-projective and $R_2 \otimes_R M$ is R_2-projective then M is R-projective. The converse is well known.

3. *Global dimension*

The aim of this section is to give an upper bound for the global dimension of a pullback ring in terms of the global dimensions of the component rings. For a ring R, $\mathrm{l.gl.dim}\, R$ will denote the left global dimension of R, the projective dimension of an R-module M will be denoted by $\mathrm{pd}_R M$ and the flat (or weak homological) dimension of M will be denoted by $\mathrm{fd}_R M$. The main result of this section is Theorem 3.1, the proof of which will be given further on in the section.

THEOREM 3·1. *Let diagram* (1) *be a pullback in which i_1 is a surjection. If $\mathrm{fd}_R R_k$ is at most n for $k = 1, 2$ then*

$$\mathrm{l.gl.dim}\, R \leqslant \max\{\mathrm{l.gl.dim}\, R_1, \mathrm{l.gl.dim}\, R_2\} + n.$$

That we must impose conditions on the component rings is demonstrated by the example due to Osofsky in [**6**], p. 29, in which the component rings have finite global dimension whilst that of the pullback is infinite. The nature of these conditions is suggested by the following lemma in which we examine when long exact sequences remain exact under the tensor product.

LEMMA 3·2. *Let $h\colon R \to R'$ be a homomorphism of rings and let M be an R-module such that $\mathrm{Tor}_m^R(R', M) = 0$ whenever $m \geqslant 1$. Given any exact sequence of R-modules*

$$0 \to M_n \to M_{n-1} \to \ldots \xrightarrow{f_1} M_0 \to M \to 0$$

with M_i R-projective for $i = 0, \ldots, n-1$, the sequence obtained by tensoring it with R' over R is exact.

Proof. The proof is by induction on n and we sketch the inductive step. Consider the two exact sequences

$$0 \to M_{n+1} \to M_n \to \ldots \to M_1 \to \mathrm{Im}\, f_1 \to 0$$

and

$$0 \to \mathrm{Im}\, f_1 \to M_0 \to M \to 0.$$

The second of these sequences remains exact when tensored with R', since

$$\mathrm{Tor}_m^R(R', M) = 0.$$

In addition, the long exact sequence in Tor obtained from it gives $\mathrm{Tor}_1^R(R', \mathrm{Im}\, f_1) = 0$, since M_0 is projective and $\mathrm{Tor}_m^R(R', M) = 0$. Thus by the inductive hypothesis, the first sequence above remains exact when tensored with R'. The result follows by putting the two tensored sequences together in the obvious way.

PROPOSITION 3·3. *Let diagram* (1) *be a pullback with i_1 a surjection. If M is an R-module such that $\mathrm{Tor}_m^R(R_k, M) = 0$ for $m \geqslant 1$ and $k = 1, 2$, then*

$$\mathrm{pd}_R M = \max\{\mathrm{pd}_{R_1} R_1 \otimes_R M, \mathrm{pd}_{R_2} R_2 \otimes_R M\}.$$

Proof. Given an exact sequence of R-modules

$$0 \to L \to P_{m-1} \to \ldots \to P_0 \to M \to 0$$

https://doi.org/10.1017/S0305004100062964 Published online by Cambridge University Press

with the P_i projective, the sequence obtained by tensoring it with R_k is exact by Lemma 3·2. If the maximum is finite and equal to m then $R_k \otimes_R L$ is R_k-projective. By Theorem 2·3, L is R-projective and hence $\mathrm{pd}_R M$ is at most m. Note that if $pd_R M$ is infinite then so is the maximum.

If $\mathrm{pd}_R M$ is finite and equal to m then we may suppose that L is projective. Thus by Lemma 3·2, if we tensor the above sequence with R_k we obtain a projective resolution for $R_k \otimes_R M$. Hence the maximum is at most m. Note that if the maximum is finite then so is $\mathrm{pd}_R M$.

COROLLARY 3·4. *Let diagram* (1) *be a pullback with* i_1 *a surjection.*

(i) *If* $\mathrm{fd}_R R_k = 0$ *for* $k = 1, 2$ *then*

$$\mathrm{pd}_R M = \max\{\mathrm{pd}_{R_1} R_1 \otimes_R M, \mathrm{pd}_{R_2} R_2 \otimes_R M\}$$

for all R-modules M.

(ii) *If* $\mathrm{fd}_R R_k \leqslant 1$ *for* $k = 1, 2$ *then*

$$\mathrm{pd}_R I = \max\{\mathrm{pd}_{R_1} R_1 \otimes_R I, \mathrm{pd}_{R_2} R_2 \otimes_R I\}$$

for all left ideals I of R.

Proof. If $\mathrm{fd}_R R_k = 0$ then $\mathrm{Tor}_m^R(R_k, M) = 0$ for $m \geqslant 1$ and if $\mathrm{fd}_R R_k \leqslant 1$ then $\mathrm{Tor}_m^R(R_k, I) = 0$ for $m \geqslant 1$ and for all ideals I of R. Hence the result follows from Proposition 3·3.

We now come to the proof of Theorem 3·1.

Proof of Theorem 3·1. Suppose that $\mathrm{fd}_R R_k \leqslant n$ for $k = 1, 2$. Given a projective resolution of an R-module M

$$\longrightarrow P_r \xrightarrow{f_r} P_{r-1} \xrightarrow{f_{r-1}} \cdots \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \longrightarrow 0,$$

consider the short exact sequences for $t \geqslant 1$,

$$0 \to \mathrm{Im}\, f_t \to P_{t-1} \to \mathrm{Im}\, f_{t-1} \to 0.$$

It is well known that

$$\mathrm{pd}_R(\mathrm{Im}\, f_{t-1}) \leqslant 1 + \mathrm{pd}_R(\mathrm{Im}\, f_t).$$

Hence, noting that $\mathrm{Im}\, f_0 = M$, we have

$$\mathrm{pd}_R M \leqslant n + \mathrm{pd}_R(\mathrm{Im}\, f_n).$$

Using the long exact sequence in Tor of the short exact sequences above and recalling that the P_t are projective, we have

$$\mathrm{Tor}_s^R(R_k, \mathrm{Im}\, f_t) \cong \mathrm{Tor}_{s+1}^R(R_k, \mathrm{Im}\, f_{t-1})$$

for $k = 1, 2$ and for $s \geqslant 1$. Hence if $m \geqslant 1$, we have

$$\mathrm{Tor}_m^R(R_k, \mathrm{Im}\, f_n) \cong \mathrm{Tor}_{m+n}^R(R_k, M)$$

the expression on the right being zero because $\mathrm{fd}_R R_k \leqslant n$. Thus we may apply Proposition 3·3 to $\mathrm{Im}\, f_n$ so that

$$\mathrm{pd}_R M \leqslant n + \max\{\mathrm{pd}_{R_1}(R_1 \otimes_R (\mathrm{Im}\, f_n)), \mathrm{pd}_{R_2}(R_2 \otimes_R (\mathrm{Im}\, f_n))\}.$$

The result now follows by taking the supremum of $\mathrm{pd}_R M$ over all R-modules M.

That we have obtained upper bounds in Theorem 3·1 and cannot guarantee equality

https://doi.org/10.1017/S0305004100062964 Published online by Cambridge University Press

is clearer if the proof of the case $n = 0$ is obtained by using Corollary 3·4 (i). On taking the supremum of the projective dimensions of all R-modules, we are not necessarily taking the supremum of the projective dimensions of all R_k-modules. An example is given in [3], 4·7, where the global dimension of the pullback is less than the upper bound of Theorem 3·1.

These results are part of the author's doctoral dissertation. The research was carried out at the University of Sheffield, and was supported by a Science Research Council award. The author is very grateful to his supervisors, A. J. Douglas and P. Vámos, for all their assistance.

REFERENCES

[1] J. E. CARRIG. The homological dimensions of symmetric algebras. *Trans. Amer. Math. Soc.* **236** (1978), 275–285.
[2] D. E. DOBBS. On the global dimension of $D + M$. *Canad. Math. Bull.* **18** (1975), 657–660.
[3] B. GREENBERG. Global dimension of cartesian squares. *J. Algebra* **32** (1974), 31–43.
[4] B. GREENBERG. Coherence in cartesian squares. *J. Algebra* **50** (1978), 12–25.
[5] J. MILNOR. *Introduction to Algebraic K-theory*. Annals Math. Studies, vol. 72 (University Press, Princeton, 1971).
[6] W. V. VASCONCELOS. *Rings of Dimension Two*. Lecture Notes in Pure and Applied Mathematics, vol. 22 (Marcel Dekker, 1976).

https://doi.org/10.1017/S0305004100062964 Published online by Cambridge University Press